

Tangle Embeddings and Quandle Cocycle Invariants

Kheira Ameer, Mohamed Elhamdadi, Tom Rose, Masahico Saito, and Chad Smudde

CONTENTS

1. Introduction
 2. Preliminaries
 3. Boundary Monochromatic Colorings and the Cocycle Invariants of Tangles
 4. Quandle Cocycle Invariants as Obstructions to Tangle Embeddings
 5. Embedding Disjoint Tangles
 6. Computational Aspects and Concluding Remarks
- Acknowledgments
References

To study embeddings of tangles in knots, we use quandle cocycle invariants. Computations are carried out for tables of knots and tangles to investigate which tangles may or may not embed in knots in the tables.

1. INTRODUCTION

A *tangle* is a pair (B, A) , where A is a set of properly embedded arcs in a 3-ball B . A tangle will have four endpoints throughout this article unless otherwise specified. A tangle T is *embedded* in a link (or a knot) L if there is an embedded ball B in 3-space such that T is equivalent to the pair $(B, B \cap L)$. All maps are assumed to be smooth. Tangles are represented by diagrams in a manner similar to knot diagrams.

Tangle embeddings have been studied by several authors recently. In [Krebes 99], the determinant was used in relation to evaluations of the Jones polynomial, which have been further investigated in [Chung and Lin 06, Krebes et al. 00, Przytycki et al. 05]. Topological interpretations of the results in [Krebes 99] were considered in [Przytycki et al. 05, Ruberman 00]. Tangles have also been used to study DNA recombinations [Ernst and Summers 90].

In this article, we present a method of using quandle cocycle invariants as obstructions to embedding oriented tangles into oriented knots (see Remark 4.5 for more on orientation), and examine their effectiveness as obstructions by looking at the table of tangles presented in [Kanenobu et al. 03] and the knot table in [Cha and Livingston 08]. Quandles are self-distributive sets with additional properties (see below for details). They have been used in the study of knots since the 1980s. A cohomology theory of quandles has been developed, and their cocycles have been used as state-sum invariants of knots and knotted surfaces [Carter et al. 03]. Quandles were also used to investigate tangles in [Darcy and Navarraz-Madsen 06, Niebrzydowski 06].

2000 AMS Subject Classification: 57M25

Keywords: Knots, tangles, quandles, cocycle invariants, tangle embedding

Krebes proved in [Krebes 99] that if a tangle T can be embedded in a link L , then the greatest common divisor of the determinant of the numerator $N(T)$ and the denominator $D(T)$ (which we denote by $\text{Kr}(T)$) divides the determinant of L . The method we present in this paper gives stronger obstructions than Krebes’s method. For example, while the tangle $T(6_3)$ in the table of tangles in Figure 6 has Krebes’s invariant $\text{Kr}(T(6_3)) = 1$, which does not give any obstructions, the quandle cocycle invariants are able to completely determine whether the tangle $T(6_3)$ embeds in the knots in the table up to nine crossings (see Proposition 4.4).

We focus on the effectiveness of quandle cocycle invariants as obstructions. For each tangle, we first find Alexander quandles that color the given tangle. Using a 3-cocycle of this quandle, we calculate the cocycle invariant of the tangle. This invariant is then compared to the cocycle invariant of knots to detect the knots that do not embed the given tangle. We also used both 2-cocycles and 3-cocycles for embedding of disjoint tangles. It will be shown that the invariants often provide effective obstructions when a given tangle has nontrivial colorings by quandles.

This paper is organized as follows. After a review of preliminary material in Section 2, colorings of tangles are defined in Section 3, and the tangles in Figure 6 that have nontrivial colorings by Alexander quandles are listed. The main theorem is presented in Section 4. For tangles listed in Section 3, it is examined which tangles may or may not embed in knots from the knot table. In Section 5, embeddings of multiple disjoint copies of tangles are discussed. Part of the results are based on the work in Kheira Ameer’s [Ameer 07] doctoral dissertation.

2. PRELIMINARIES

2.1 Tangles and Their Operations

The conventions described in this subsection are commonly found in the literature (see, for example, [Adams 94, Murasugi 96]).

The four endpoints of a given tangle diagram T are located at four corners of a circle in a plane at angles $\pi/4$, $3\pi/4$, $5\pi/4$, and $7\pi/4$ when the circle is placed with the origin as its center. These endpoints are labeled NE, NW, SW, and SE, respectively, representing the directions of a compass.

The addition $T_1 + T_2$ of two tangles T_1, T_2 is another tangle defined from the original two as depicted in Figure 1. There are two ways of closing the endpoints of a tangle,

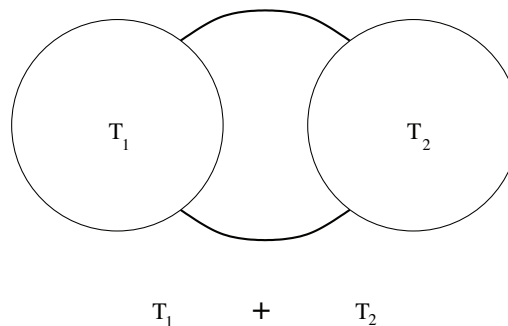


FIGURE 1. Addition of tangles.

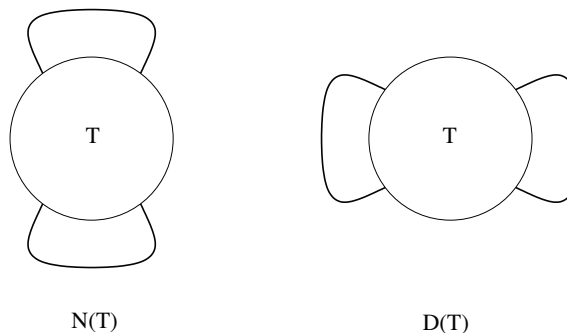


FIGURE 2. Closures (numerator $N(T)$ and denominator $D(T)$) of tangles.

called closures: the *numerator* $N(T)$ and *denominator* $D(T)$ of a tangle T , defined as depicted in Figure 2.

There is a family of “trivial” or “rational” tangles, some of which are depicted in Figure 3. These tangles are obtained from the trivial tangle of two vertical straight arcs by successively twisting endpoints vertically and horizontally. The notation $R(n_1, n_2, \dots, n_k)$ indicates that the tangle is obtained from $R(0)$ by first performing n_1 half-twists about the vertical axis, then n_2 twists horizontally, and repeating this process k times according to a given sequence of integers n_1, \dots, n_k . See again [Adams 94] or [Murasugi 96] for more details.

In [Kanenobu et al. 03], *prime* tangles (with crossing number at most seven) are classified, and a table of their diagrams is given. A prime tangle is a tangle that satisfies two conditions: (1) any 2-sphere in the 3-ball B intersecting T transversely in two points bounds a ball with an unknotted arc as the intersection with T , (2) there is no properly embedded disk in B that separates T . The table consists of a single 5-crossing tangle followed by four 6-crossing tangles, and 18 tangles of 7 crossings. Some multiple-component tangles were also classified in [Kanenobu et al. 03]. The tangles are named in a scheme

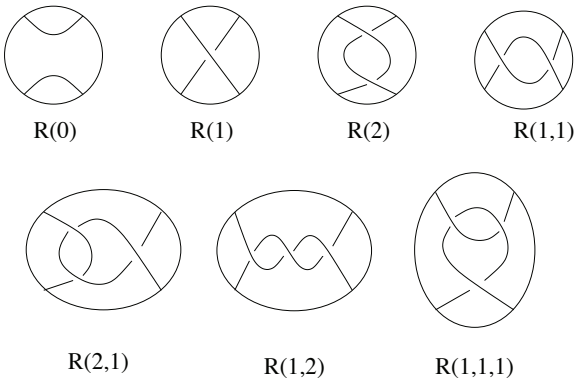


FIGURE 3. Some rational tangles.

similar to knots by integers with subscripts. Some of the tangles are presented in Figure 6.

2.2 Quandles, Colorings, and Cocycle Invariants

A *quandle* X is a set with a binary operation $(a, b) \mapsto a * b$ such that

- (I) For any $a \in X$, $a * a = a$.
- (II) For any $a, b \in X$, there is a unique $c \in X$ such that $a = c * b$.
- (III) For any $a, b, c \in X$, we have $(a * b) * c = (a * c) * (b * c)$.

A *rack* is a set with a binary operation that satisfies (II) and (III). Racks and quandles have been studied in, for example, [Brieskorn 88, Fenn and Rourke 92, Joyce 82, Matveev 82].

The following are typical examples of quandles. A group G with conjugation as the quandle operation, $a * b = bab^{-1}$, is a quandle. Any $\mathbb{Z}[t, t^{-1}]$ -module M is a quandle with $a * b = ta + (1 - t)b$, $a, b \in M$, which is called an *Alexander quandle*. Let n be a positive integer and for elements $i, j \in \mathbb{Z}_n$, define $i * j \equiv 2j - i \pmod n$. Then $*$ defines a quandle structure called the *dihedral quandle* R_n .

Let X be a fixed quandle. Let K be a given oriented classical knot or link diagram, and let \mathcal{R} be the set of (over-)arcs. The normals (normal vectors) are given in such a way that the ordered pair (tangent, normal) agrees

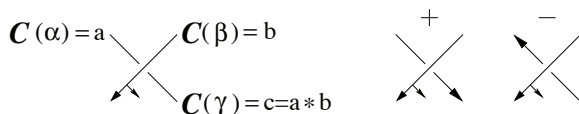


FIGURE 4. Quandle relation at a crossing, and signs of crossings.

with the orientation of the plane; see Figure 4 (left). A (quandle) *coloring* \mathcal{C} is a map $\mathcal{C} : \mathcal{R} \rightarrow X$ such that at every crossing, the relation depicted in Figure 4 holds. Specifically, let β be the over-arc at a crossing, and let α and γ be the under-arcs, such that the normal of the over-arc points from α to γ . Then $\mathcal{C}(\alpha) * \mathcal{C}(\beta) = \mathcal{C}(\gamma)$ holds. The (ordered) colors $(\mathcal{C}(\alpha), \mathcal{C}(\beta))$ are called *source colors*. Let $\text{Col}_X(K)$ denote the set of colorings of a knot diagram K by a quandle X .

Let K be a knot diagram in the plane. Let X be a finite quandle and A an abelian group. Let $\phi : X \times X \rightarrow A$ be a quandle 2-cocycle, which can be regarded as a function satisfying the 2-cocycle condition

$$\phi(x, y) - \phi(x, z) + \phi(x * y, z) - \phi(x * z, y * z) = 0,$$

$\forall x, y, z \in X$ and $\phi(x, x) = 0 \forall x \in X$. Let \mathcal{C} be a coloring of a given knot diagram K by X .

The *Boltzmann weight* $B(\mathcal{C}, \tau) = B_\phi(\mathcal{C}, \tau)$ at a crossing τ of K is then defined by $B(\mathcal{C}, \tau) = \epsilon(\tau)\phi(x_\tau, y_\tau)$, where the pair (x_τ, y_τ) consists of the source colors at τ and $\epsilon(\tau)$ is the sign (± 1) of τ . The signs at a crossing are depicted in Figure 4 (right). Then the 2-cocycle invariant $\Phi(K) = \Phi_\phi(K)$ in a multiset form is defined by

$$\Phi_\phi(K) = \left\{ \sum_{\tau} B(\mathcal{C}, \tau) \mid \mathcal{C} \in \text{Col}_X(K) \right\}.$$

Definition 2.1. A multiset is a pair (S, m) , where S is a set and m is a function that assigns to each element a in S a positive integer (called the multiplicity, meaning the number of occurrences) of a .

For example, $\{0, 0, 1, 1, 1\}$ represents a multiset (S, m) where $S = \{0, 1\}$, $m(0) = 2$, and $m(1) = 3$. This is also denoted in this paper by $\{\sqcup_2 0, \sqcup_3 1\}$.

Let $\theta : X \times X \times X \rightarrow A$ be a quandle 3-cocycle, which can be regarded as a function satisfying

$$\begin{aligned} &\theta(x, z, w) - \theta(x, y, w) + \theta(x, y, z) - \theta(x * y, z, w) \\ &+ \theta(x * z, y * z, w) - \theta(x * w, y * w, z * w) = 0, \\ &\forall x, y, z, w \in X, \end{aligned}$$

and $\theta(x, x, y) = 0 = \theta(x, y, y) \forall x, y \in X$.

Let \mathcal{C} be a coloring of arcs and regions of a given diagram K . Specifically, for a coloring \mathcal{C} , there is a coloring of regions that extends \mathcal{C} as depicted in Figure 5. Suppose that two regions R_1 and R_2 are separated by an arc colored by y and the normal of the arc points from R_1 to R_2 . If R_1 is colored by x , then R_2 receives the color $x * y$. Let (x_τ, y_τ, z_τ) (called the *ordered triple*

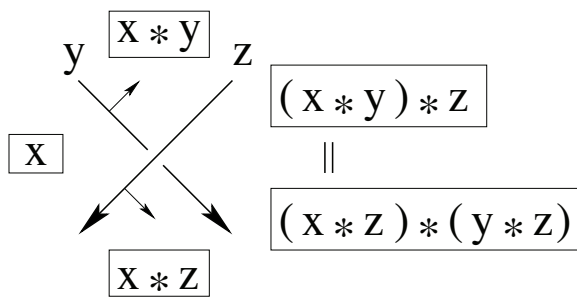


FIGURE 5. Region colors at a crossing.

of colors at a crossing τ) be the colors near a crossing τ such that x is the color of the region (called the source region) from which both orientation normals of over- and under-arc point, y is the color of the under-arc (called the source under-arc) from which the normal of the over-arc points, and z is the color of the over-arc (see Figure 5). The weight in this case is defined by $B(\mathcal{C}, \tau) = \epsilon(\tau)\phi(x_\tau, y_\tau, z_\tau)$. The 3-cocycle invariant is defined in a similar way to the 2-cocycle invariant by the multiset $\Phi_\theta(K) = \{\sum_\tau B(\mathcal{C}, \tau) \mid \mathcal{C} \in \text{Colr}_X(K)\}$, where $\text{Colr}_X(K)$ denotes the set of colorings of the regions of K by X .

If the quandle X is finite, the invariant as a multiset can be written by an expression similar to those for the state-sums; if a given multiset of group elements is $\{\sqcup_{m_1} g_1, \dots, \sqcup_{m_\ell} g_\ell\}$, then we use the polynomial notation $m_1 u^{g_1} + \dots + m_\ell u^{g_\ell}$, where u is a formal symbol. For example, the multiset value of the invariant for a trefoil with the Alexander quandle $X = \mathbb{Z}_2[t, t^{-1}]/(t^2 + t + 1)$ with the same coefficient group $A = X$ and a certain 2-cocycle is $\{\sqcup_4(0), \sqcup_{12}(t + 1)\}$, and is denoted by $4 + 12u^{(t+1)}$, where we use the convention $u^0 = 1$ and exponential rules apply.

For computing the invariants, one needs an explicit formula for cocycles. Polynomial expressions were used first in [Mochizuki 03], and investigated closely including higher-dimensional cocycles in [Ameur 07].

3. BOUNDARY MONOCHROMATIC COLORINGS AND THE COCYCLE INVARIANTS OF TANGLES

We use quandle cocycle invariants as obstructions to embedding tangles in knots. We first define cocycle invariants for tangles.

Definition 3.1. Let T be a tangle and X a quandle. A (boundary-monochromatic) coloring $\mathcal{C} : \mathcal{A} \rightarrow X$ is a map from the set of arcs in a diagram of T to X satisfying the

Quandle	Tangle Colored
$\mathbb{Z}_p[t, t^{-1}]/(t^2 - t + 1)$	6 ₂ , 6 ₃ , 7 ₁₇ (NW In, SW In).
$\mathbb{Z}_2[t, t^{-1}]/(t^2 + t + 1)$	6 ₂ , 6 ₃ , 7 ₁₇ (NW In, SW In), 7 ₄ (NW In, NE In), 7 ₅ (NW In, NE In), 7 ₆ (NW In, NE In), 7 ₇ (NW In, NE In).
R_3	6 ₂ , 6 ₃ , 7 ₁₆ , 7 ₁₇ .
R_5	7 ₁₃ , 7 ₁₈ .
R_7	7 ₁₅ .

TABLE 1. Tangles with nontrivial colorings

same quandle coloring condition as for knot diagrams at each crossing such that the (four) boundary points of the tangle diagram receive the same element of X .

For a coloring \mathcal{C} of a tangle diagram T , region colorings are defined in a similar manner as in the knot case.

Denote by $\text{Col}_x(T)$ and $\text{Col}_X(T)$ the set of boundary-monochromatic colorings of T with the boundary color $x \in X$ and the set of all boundary-monochromatic colorings, respectively. Let $\Phi(T, x) = \sum_{\mathcal{C} \in \text{Col}_x(T)} \prod_\tau B(\mathcal{C}, \tau)$. Then the cocycle invariant for a tangle T is defined by $\Phi_\phi(T) = \sum_{x \in X} \Phi(T, x)$. The invariants for region colorings are defined in a similar manner, by taking the sum over all colorings of regions as well as colorings of diagrams.

It can be proved in a way similar to the case of a knot that the number of colorings $|\text{Col}_X(T)|$ does not depend on the choice of a diagram of T . If a diagram D_1 of T has a coloring \mathcal{C}_1 , and a diagram D_2 is obtained from D_1 by a Reidemeister move, then there is a unique coloring \mathcal{C}_2 of D_2 induced from \mathcal{C}_1 such that the colors stay the same except where the move is performed. Given two diagrams D_1 and D_2 of a tangle T , there is one-to-one correspondence between the set of colorings of D_1 and the set of colorings of D_2 , and the cocycle invariant is well defined.

Table 1 summarizes the tangles in the tangle table that have nontrivial boundary monochromatic colorings by some Alexander quandles. These are found through computing by hand, with occasional assistance from the computer software Maple. Specifically, variables $x_i, i = 1, 2, \dots$, are assigned to the arcs of tangle diagrams as indicated in Figure 6. Coloring conditions of the form $x_k = tx_i + (1 - t)x_j$ are imposed at crossings, giving rise to a system of linear equations with coefficients in $\mathbb{Z}[t, t^{-1}]$ that is solved to find which Alexander quandles give nontrivial colorings of the tangles given in

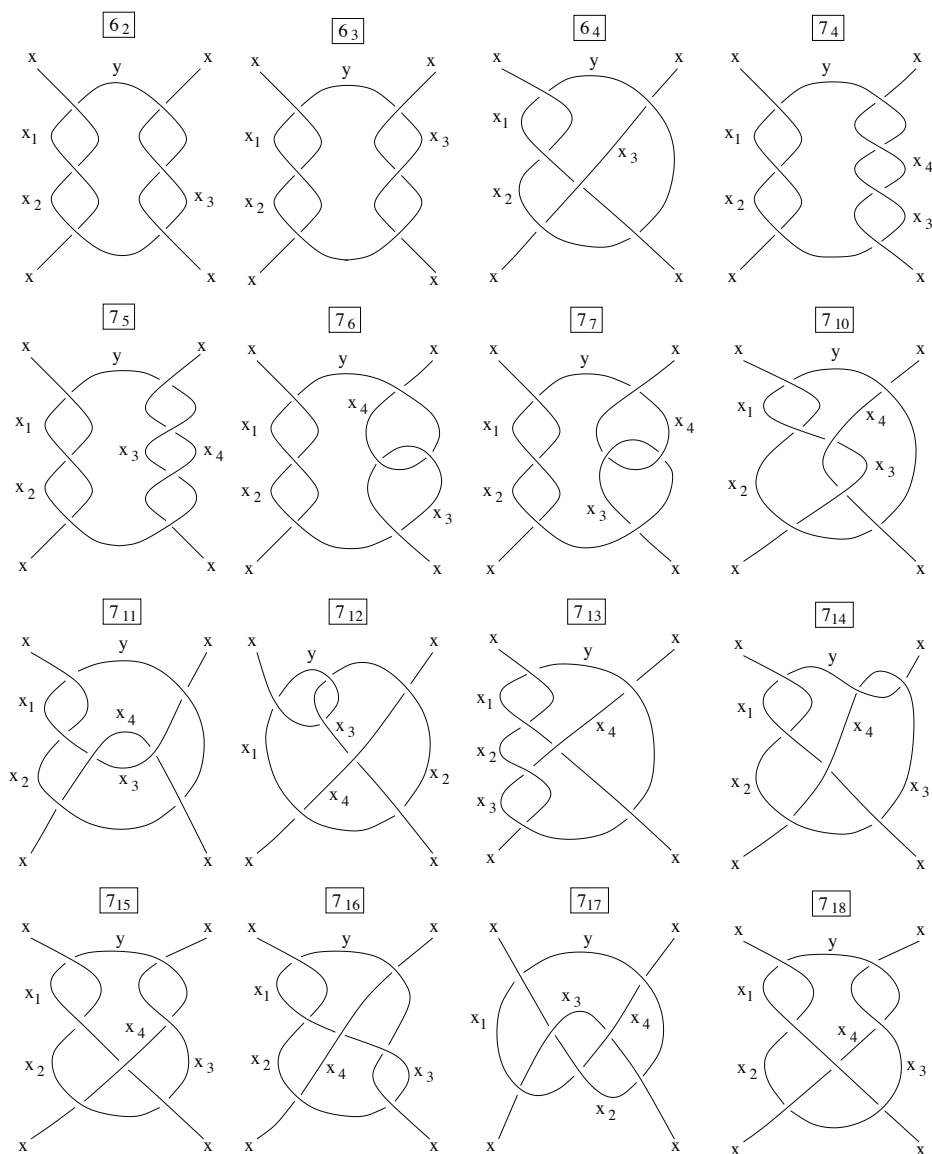


FIGURE 6. Tangles with nontrivial colorings by Alexander quandles.

Figure 6. See Section 6 for more details on computations. This list is similar to the original list in [Kanenobu et al. 03], but excludes the tangles that color trivially with any Alexander quandle. We compare with the knot table, so the tangles with closed components are also excluded.

4. QUANDLE COCYCLE INVARIANTS AS OBSTRUCTIONS TO TANGLE EMBEDDINGS

The quandle 2- and 3-cocycle invariants are defined for tangles in a manner similar to the knot case using the set of boundary monochromatic colorings, and denoted by $\Phi_\phi(T)$. We use the multiset version of the invariant.

Definition 4.1. The inclusion of multisets is denoted by \subset_m . Specifically, if an element x is repeated n times in a multiset, call n the multiplicity of x , then $M \subset_m N$ for multisets M, N means that if $x \in M$, then $x \in N$ and the multiplicity of x in M is less than or equal to the multiplicity of x in N .

Theorem 4.2. Let T be a tangle and X a quandle. Suppose that T embeds in a link L . Then we have the inclusion $\Phi_\phi(T) \subset_m \Phi_\phi(L)$.

Proof: Suppose that a diagram of T embeds in a diagram of L . By abuse of language, we call the diagrams them-

selves respectively T and L . For a coloring \mathcal{C} of T , let x be the color of the boundary points. Then there is a unique coloring \mathcal{C}' of L such that the restriction of \mathcal{C}' to T is \mathcal{C} and all the arcs of L outside T automatically receive the color x . Then the contribution of $\sum_{\tau \in T} B(\mathcal{C}, \tau)$ to $\Phi_\phi(T)$ is equal to the contribution $\sum_{\tau \in L} B(\mathcal{C}', \tau)$ to $\Phi_\phi(L)$, and the theorem follows. The same argument works for region colorings and 3-cocycle invariants. \square

In Table 2, a summary is presented for the tangles that have a nontrivial coloring by Alexander quandles. In the left column of the table, the tangles that appear in Table 1 are listed. In the middle column, knots that we found to embed a given tangle are listed. The third column lists the knots for which we could not exclude the possibility of embedding the given tangle using cocycle invariants. The tangles are specified by the notation $T(6_2)$, for example, for the tangle numbered 6_2 , to distinguish them from knots. We note that there are 84 knots in the table up to (and including) 9-crossing knots. For the tangle $T(6_3)$, for example, all except 3 out of 84 knots are detected by the cocycle invariants as not allowing an embedding of the tangle. It is checked by hand that these remaining three do embed it.

To demonstrate how we obtain these results, we state and prove the following.

Proposition 4.3. *The tangle $T(6_2)$ with the orientation of the NW arc inward and the SW arc outward does not embed in the knots in the table up to nine crossings except, possibly, for $8_{18}, 9_{29}, 9_{38}$.*

Proof: We exhibit a method for determining the invariant from the table in [Smudde 2008], in which computations of the cocycle invariants were based on the knot table in [Cha and Livingston 08]. The tangle $T(6_2)$ can be written as the addition of two tangles $R(3) + R(3)$, and is colored nontrivially by the quandle $Z_p[t]/(t^2 - t + 1)$. We note that the closure $D(R(3))$ is the left-hand trefoil, that is, the mirror of the (right-hand) trefoil given in [Cha and Livingston 08].

For $p = 2$, the table of quandle cocycle invariants in [Smudde 2008] gives $16 + 48u^t$ as the invariant for the trefoil with the 3-cocycle $\phi(x, y, z) = (x - y)(y - z)^2$, with values also in $Z_2[t]/(t^2 - t + 1)$. This implies that any nontrivial coloring contributes t to the invariant. Its mirror has the same property. With two copies, any nontrivial coloring of the tangle contributes $2t = 0$ when $p = 2$. Hence the value of the invariant of the tangle is 64. Examining the table, we see that this does not embed in

knots up to nine crossings except for the following possibilities:

$$8_5, 8_{10}, 8_{15}, 8_{18}, 8_{19}, 8_{20}, 8_{21}, 9_{16}, 9_{22}, 9_{24}, 9_{25}, 9_{28}, 9_{29}, 9_{30}, 9_{36}, 9_{38}, 9_{39}, 9_{40}, 9_{41}, 9_{42}, 9_{43}, 9_{44}, 9_{45}, 9_{49}.$$

For $p = 3$, the invariant table gives $243 + 486u^{(2t+2)}$ as the invariant for the trefoil. This implies that 486 nontrivial colorings contribute $2t + 2$ to the invariant. Its mirror contributes $t + 1$. With two copies, 486 nontrivial colorings of the tangle contribute $2t + 2$. Hence the value of the invariant of the tangle is $243 + 486u^{(2t+2)}$. It follows by examining the table that this does not embed in knots up to nine crossings except for $3_1, 8_{18}, 9_2, 9_4, 9_{29}, 9_{34}, 9_{38}$.

For $p = 5$, the table gives

$$625 + 3750u^{(t+3)} + 3750u^{(4t+2)} + 3750u^{(3t+4)} + 3750u^{(2t+1)}$$

as the invariant for the trefoil. As in the previous cases, the value of the invariant of this tangle is

$$625 + 3750u^{(3t+4)} + 3750u^{(2t+1)} + 3750u^{(4t+2)} + 3750u^{(t+3)}$$

(for example, for the contribution $t + 3$ of the trefoil, the mirror contributes $4t + 2$, its double contributes $3t + 4$).

From the table, this does not embed in knots up to nine crossings except for

$$3_1, 8_3, 8_5, 8_{11}, 8_{15}, 8_{18}, 8_{19}, 8_{21}, 9_1, 9_5, 9_6, 9_{16}, 9_{19}, 9_{23}, 9_{28}, 9_{29}, 9_{38}, 9_{40}.$$

For $p = 7$, the trefoil has 117649 as the invariant value, and so does the tangle. From the table this does not embed in knots up to nine crossings except for

$$3_1, 8_5, 8_{10}, 8_{11}, 8_{15}, 8_{18}, 8_{19}, 8_{20}, 8_{21}, 9_1, 9_6, 9_{16}, 9_{23}, 9_{28}, 9_{29}, 9_{38}, 9_{40}.$$

Combining all these facts, we deduce that this tangle does not embed in knots of up to nine crossings except for perhaps $8_{18}, 9_{29}, 9_{38}$. \square

We have not been able to determine whether the tangle $T(6_2)$ actually embeds in these three knots that the invariant failed to exclude. In the next example, however, we were able to determine completely the embedding problem for knots of up to nine crossings.

Proposition 4.4. *The knots in the table up to nine crossings in which the tangle $T(6_3)$ embeds are exactly $8_{10}, 8_{20}, 9_{24}$. Here, the orientation of the tangle is such that the endpoint NW is oriented inward and the SW endpoint is oriented outward.*

Tangle	Embeds in	May Embed in
6 ₂ (NW In, SW Out)	(8 ₅) [*] = $N(T(6_2) + R(-2))$	8 ₁₈ , 9 ₂₉ , 9 ₃₈ .
6 ₂ (NW In, SW In)	(3 ₁) = $N(T(6_2) + R(-1))$	3 ₁ , 7 ₄ , 7 ₇ , 8 ₁₈ , 9 ₁₀ , 9 ₂₉ , 9 ₃₅ , 9 ₃₇ , 9 ₃₈ , 9 ₄₆ , 9 ₄₈ .
6 ₃ (NW In, SW Out)	(8 ₁₀) = $N(T(6_3) + R(2, 1))^*$ (8 ₂₀) = $N(T(6_3) + R(2))^*$ (9 ₂₄) = $N(T(6_3) + R(2, 2))^*$	8 ₁₀ , 8 ₂₀ , 9 ₂₄ .
7 ₄ (NW In, NE In)	(4 ₁) = $N(T(7_4) + R(-1))$	3 ₁ , 4 ₁ , 7 ₂ , 7 ₃ , 8 ₁ , 8 ₄ , 8 ₁₁ , 8 ₁₃ , 8 ₁₈ , 9 ₁ , 9 ₆ , 9 ₁₂ , 9 ₁₃ , 9 ₁₄ , 9 ₂₁ , 9 ₂₃ , 9 ₃₅ , 9 ₃₇ , 9 ₄₀ . Same as 7 ₄ (NW In, NE In).
7 ₅ (NW In, NE In)	(7 ₃) [*] = $N(T(7_5) + R(-1))$	8 ₅ , 8 ₁₀ , 8 ₁₅ , 8 ₁₈ - 8 ₂₁ , 9 ₁₆ , 9 ₂₂ , 9 ₂₄ , 9 ₂₅ , 9 ₂₈ - 9 ₃₀ , 9 ₃₆ , 9 ₃₈ , 9 ₃₉ , 9 ₄₁ - 9 ₄₅ , 9 ₄₉ .
7 ₆ (NW In, NE In)		Same as 7 ₆ (NW In, NE In).
7 ₇ (NW In, NE In)		
7 ₁₃ (NW In, NE Out)	(7 ₄) = $N(T(7_{13}))$ (8 ₁₆) = $N(T(7_{13}) + R(1))$ (9 ₃₉) = $N(T(7_{13}) + R(1, 1))$ (9 ₄₉) = $N(T(7_{13}) + R(-1, -1))$	4 ₁ , 7 ₄ , 9 ₂₄ , 9 ₃₇ , 9 ₃₉ , 9 ₄₀ , 9 ₄₉ .
7 ₁₅ (NW In, SW In)	(5 ₂) = $N(T(7_{15}) + R(-1))$	5 ₂ , 8 ₁₆ , 9 ₄₁ , 9 ₄₂ .
7 ₁₅ (NW In, SW Out)	(7 ₇) = $D(T(7_{15}))$ (9 ₄₁) = $N(T(7_{15}) + R(2))$	7 ₁ , 7 ₇ , 8 ₅ , 9 ₄ , 9 ₁₂ , 9 ₄₁ .
7 ₁₆ (NW In, NE In)	(7 ₇) [*] = $D(T(7_{16}))$	8 ₅ , 8 ₁₅ , 8 ₁₈ , 8 ₁₉ , 8 ₂₁ , 9 ₂ , 9 ₄ , 9 ₁₁ , 9 ₁₅ , 9 ₁₆ , 9 ₂₈ , 9 ₃₄ , 9 ₃₇ , 9 ₄₀ , 9 ₄₆ , 9 ₄₇ . Same as 6 ₂ (NW In, SW In).
7 ₁₆ (NW In, NE Out)	(7 ₄) = $N(T(7_{16}))$	8 ₁₈ , 9 ₄₀ .
7 ₁₇ (NW In, SW In)	(8 ₁₈) = $N(T(7_{17}) + R(1))$	Same as 7 ₁₆ (NW In, NE In).
7 ₁₇ (NW In, SW Out)		
7 ₁₈ (NW In, SW In)	(8 ₂₁) = $N(T(7_{18}) + R(1))$	5 ₁ , 8 ₁₈ , 8 ₂₁ , 9 ₂ , 9 ₁₂ , 9 ₂₃ , 9 ₃₁ , 9 ₄₀ , 9 ₄₉ . Same as 7 ₁₈ (NW In, SW In).
7 ₁₈ (NW In, SW Out)	(5 ₁) = $D(T(7_{18}))$	

TABLE 2. Summary of the results.

Proof: The tangle $T(6_3)$ is written as the addition $R(3) + R(-3)$. Hence it is colored nontrivially by

$$Z_p[t]/(t^2 - t + 1)$$

for any $p \in \mathbb{Z}$ (we use only primes), as well as the dihedral quandle R_3 . For the quandle $Z_p[t]/(t^2 - t + 1)$ we used the 3-cocycle $f(x, y, z) = (x - y)(y - z)^p$. The colors of the source region for these two copies of the trefoil diagrams ($R(3)$ and $R(-3)$) coincide. The signs of the crossings are opposite. Hence the invariant is trivial, $(p^2)^3$ copies of 0, for $Z_p[t]/(t^2 - t + 1)$. Note that even with the trivial invariant value $(p^2)^3$ for T , the cocycle invariant can give stronger obstruction than the numbers of colorings if a knot has a nontrivial invariant value and has a smaller constant than $(p^2)^3$. For $p = 5$, in particular, from the calculations in [Smudde 2008], Theorem 4.2 implies that this tangle may embed, among knots in the table up to nine crossings, only in 8₁₀, 8₁₂, 8₁₈, 8₂₀, 9₂₄. The invariant with R_3 further excludes 8₁₂ and 8₁₈. Therefore the tangle may embed only in 8₁₀, 8₂₀, and 9₂₄.

On the other hand, it is seen that

$$\begin{aligned} (8_{10}) &= N(T(6_3) + R(2, 1))^*, \\ (8_{20}) &= N(T(6_3) + R(2))^*, \\ (9_{24}) &= N(T(6_3) + R(2, 2))^*, \end{aligned}$$

where K^* denotes the mirror image of a knot K , and R denotes the rational tangles. Note that this tangle $T(6_3)$ is equivalent to its mirror. Therefore we have shown that the tangle $T(6_3)$ does indeed embed in these three knots. □

Remark 4.5. In general, the orientation needs to be specified to define the quandle cocycle invariants. (In our case only the dihedral quandles can be used for the invariant without specifying the orientations [Satoh 07].) Furthermore, the mirror images of a given knot in the table may be different. Thus all of our results are stated for oriented tangles and oriented knots, and do not include their mirror images. Our conventions for specifying

orientations of tangles have already been explained. For knots in the table, we used Livingston’s table [Cha and Livingston 08], which includes particular choices of mirrors if a knot is not amphicheiral. For the orientations, we used the braid form in [Cha and Livingston 08] for our calculations, so that the orientations are specified by downward orientations of the braids.

5. EMBEDDING DISJOINT TANGLES

In this section we discuss embeddings of disjoint unions of tangles in knots. We prove two propositions that will be used as obstructions to embedding disjoint unions of tangles and give some examples.

Let $C = \sum_{i=1}^k m_i u^{c_i}$, $D = \sum_{j=1}^\ell n_j u^{d_j}$ be polynomial expressions of multiset values of the invariants, where $m_i, n_j \in \mathbb{Z}_+$, $c_i, d_j \in A$, where A is the coefficient abelian group. Then we define $C \times D = \sum_{i,j} m_i n_j u^{c_i+d_j}$. Let $|X|$ denote the number of elements of a quandle X .

The quandle cocycle invariants are defined for a disjoint union of tangles $T_1 \sqcup \dots \sqcup T_k$ in a manner similar to invariants of tangles, by requiring that all the boundary points of T_1, \dots, T_k receive the same color. Let ϕ be a 2-cocycle of a quandle X and define

$$\Phi_\phi(T_1 \sqcup \dots \sqcup T_k) = \sum_{x_j \in X} \prod_{i=1}^k \Phi_\phi(T_i, x_j).$$

Proposition 5.1. *Let ϕ be a 2-cocycle. Let T_1, \dots, T_k be a disjoint union of tangles such that for all $i = 1, \dots, k$, the condition $\Phi_\phi(T_i, x) = \Phi_\phi(T_i, y)$ holds for all $x, y \in X$. Then*

$$\Phi_\phi(T_1 \sqcup \dots \sqcup T_k) = \frac{1}{|X|^{k-1}} \Phi_\phi(T_1) \times \dots \times \Phi_\phi(T_k).$$

Furthermore, if a disjoint union of T_1, \dots, T_k embeds in a link L , then

$$\Phi_\phi(T_1 \sqcup \dots \sqcup T_k) \subset_m \Phi_\phi(L).$$

Proof: We compute

$$\Phi_\phi(T_1 \sqcup \dots \sqcup T_k) = \sum_{x_j \in X} \prod_{i=1}^k \Phi(T_i, x_j) = |X| \prod_{i=1}^k \Phi(T_i, x)$$

for any fixed $x \in X$, since $\Phi(T_i, x) = \Phi(T_i, y)$ for all $x, y \in X$. The condition also implies that $\Phi(T_i, x) = \frac{1}{|X|} \Phi_\phi(T_i)$ for all $i = 1, \dots, k$. Hence

$$\Phi_\phi(T_1 \sqcup \dots \sqcup T_k) = \frac{1}{|X|^{k-1}} \Phi_\phi(T_1) \times \dots \times \Phi_\phi(T_k).$$

Thus by the same argument as the proof of Theorem 4.2, if $T_1 \sqcup \dots \sqcup T_k$ embeds in a link L , then

$$\frac{1}{|X|^{k-1}} \Phi_\phi(T_1) \times \Phi_\phi(T_2) \times \dots \times \Phi_\phi(T_k) \subset_m \Phi_\phi(L).$$

□

Example 5.2. For the following examples, we consider the quandle $X = \mathbb{Z}_2[t, t^{-1}]/(t^2 + t + 1)$, and the 2-cocycle $f(x, y) = (x - y)^2 y$. The invariant values for this quandle are available in [Smudde 2008] (here we used knots up to nine crossings). It is seen that the following tangles satisfy the condition required in Proposition 5.1 by direct calculation. Alternatively, either the triviality of the invariant or the property that only the trivial colorings make trivial contributions to the invariant implies the condition required.

(a) We compute $\Phi_f(T(6_2) \sqcup T(6_2)) = \frac{1}{4}(16 \times 16) = 64$ (Proposition 4.3). Using [Smudde 2008], we compare this invariant to the cocycle invariant of knots in the knot table, and conclude that $T(6_2) \sqcup T(6_2)$ does not embed in any knot in the knot table up to nine crossings.

The invariant value of $T(6_3)$ is $\Phi_f(T(6_3)) = 16$ by an argument similar to those used for 3-cocycles in the proof of Proposition 4.4. By Theorem 5.1, $\Phi_f(T(6_3) \sqcup T(6_3)) = \frac{1}{4}(16 \times 16) = 64$. Hence $T(6_3) \sqcup T(6_3)$ does not embed in any knot in the knot table up to nine crossings.

The disjoint union $T(6_2) \sqcup T(6_3)$ also has the same invariant value 64; hence the same conclusion holds.

(b) The invariant of the tangle $T(7_5)$ with orientation (NW In, NE In) is $\Phi_f(T(7_5)) = 4 + 12u^{(t+1)}$. This can be seen from the fact that $T(7_5)$ embeds in the knot $(7_3)^*$ and the number of colorings by this quandle is the same for $T(7_5)$ and $(7_3)^*$, so that by an argument similar to the proof of Proposition 4.3, the tangle has the same invariant value as $(7_3)^*$ (see [Smudde 2008]). Hence by Theorem 5.1, we obtain

$$\Phi_f(T(7_5) \sqcup T(7_5)) = \frac{1}{4}(4 + 12u^{(t+1)})^2 = 40 + 24u^{(t+1)}.$$

Using [Smudde 2008], we compare this invariant to the cocycle invariant of knots in the table, and we conclude that $T(7_5) \sqcup T(7_5)$ does not embed in any knot in the table up to nine crossings.

(c) Again by Theorem 5.1,

$$\Phi_f(T(6_2) \sqcup T(7_5)) = \frac{1}{4}16(4 + 12u^{(t+1)}) = 16 + 48u^{(t+1)}.$$

We find that $T(6_2) \sqcup T(7_5)$ does not embed in any knot in the knot table up to nine crossings with the possible exceptions of 8_{18} and 9_{40} .

Since the invariant value for $T(6_3) \sqcup T(7_5)$ is the same as $T(6_2) \sqcup T(7_5)$, we obtain the same conclusion.

Let ψ be a 3-cocycle of a quandle X with coefficient group A . Denote by $\Phi_\psi(T, x, s)$ the 3-cocycle invariant with the boundary color $x \in X$ and the color of the leftmost region $s \in X$. Then the 3-cocycle invariant for the disjoint union of tangles $\sqcup_{i=1}^k T_i$ is defined if the T_i 's satisfy the condition $\Phi_\psi(T_i, x, s) = \Phi_\psi(T_i, x', s')$ for all $x, x', s, s' \in X$ for all $i = 1, \dots, k$, and is defined in this case by

$$\begin{aligned} \Phi_\psi(T_1 \sqcup \dots \sqcup T_k) &= \sum_{x_j \in X, s \in X} \prod_{i=1}^k \Phi_\psi(T_i, x_j, s) \\ &= |X| \sum_{x_j \in X} \prod_{i=1}^k \Phi_\psi(T_i, x_j, s) \end{aligned}$$

for a fixed $s \in X$. Note that the invariant does not depend on the fixed region color s because of the above assumption. Then the same argument as the proof of the preceding theorem can be applied to show the following.

Proposition 5.3. *Let ψ be a 3-cocycle. Let T_1, \dots, T_k be a disjoint union of tangles such that for all $i = 1, \dots, k$, the condition $\Phi_\psi(T_i, x, s) = \Phi_\psi(T_i, x', s')$ holds for all $x, x', s, s' \in X$. Then we have*

$$\Phi_\psi(T_1 \sqcup \dots \sqcup T_k) = \frac{1}{|X|^{2(k-1)}} \Phi_\psi(T_1) \times \dots \times \Phi_\psi(T_k).$$

Furthermore, if a disjoint union of T_1, \dots, T_k embeds in a link L , then

$$\Phi_\psi(T_1 \sqcup \dots \sqcup T_k) \subset_m \Phi_\psi(L).$$

Example 5.4. We used dihedral quandles R_p with Mochizuki's cocycle [Mochizuki 03]

$$\psi(x, y, z) = \frac{1}{p}(x - y)[(2z^p - y^p) - (2z - y)^p] \pmod{p}.$$

The invariant values are available in [Smudde 06]¹ for knots with up to 12-crossings for $p = 3$ and $p = 5$. By arguments similar to those in Example 5.2, it is seen that the following tangles satisfy the condition required in Proposition 5.3.

- (a) For the dihedral quandle R_3 and the Mochizuki 3-cocycle ψ , the tangle $T(6_2)$ satisfies the condition in Theorem 5.3. This is because $T(6_2)$ is the sum of two copies of part of the trefoil diagrams, and the trefoil has the property that any nontrivial coloring gives the same nontrivial contribution to the cocycle invariant. Since $\Phi_\psi(T(6_2)) = 9(1 + 2u)$, by Theorem 5.1 we obtain

$$\begin{aligned} \Phi_\psi(T(6_2) \sqcup T(6_2)) &= \frac{1}{3^2} 81(1 + 4u + 4u^2) \\ &= 9 + 36u + 36u^2. \end{aligned}$$

Using [Smudde 06],² we compare this invariant to the cocycle invariant of knots in the knot table, and we find that $T(6_2) \sqcup T(6_2)$ does not embed in any knot in the knot table up to 11 crossings (there are 801) except for, possibly, 8_{18} and $11a_{314}$. From the invariant value, the number of colorings of $T(6_2) \sqcup T(6_2)$ is 81, and among 801 knots in the table up to 11 crossings, there are 40 with at least 81 colorings. Hence the number of colorings alone can exclude all but 40 knots, but the cocycle invariant is able to exclude all but 2.

- (b) $\Phi_\psi(T(6_3) \sqcup T(6_3)) = \frac{1}{3^2} 3^3 3^3 = 81$ with R_3 , and $T(6_3) \sqcup T(6_3)$ does not embed in any knot in the knot table up to 11 crossings, except possibly 10_{99} ; hence the cocycle invariant excludes all 801 knots but one.
- (c) For the quandle R_5 and tangles $T(7_{13})$ and $T(7_{18})$, both have invariant $25(1 + 2u + 2u^3)$; hence $T(7_{13}) \sqcup T(7_{18})$ has invariant $25(5 + u + 4u^2 + 2u^3 + 4u^4)$. Thus it does not embed in any knot in the knot table up to 11 crossings, except possibly 10_{103} , 10_{155} , $11a_{317}$, $11n_{148}$.

6. COMPUTATIONAL ASPECTS AND CONCLUDING REMARKS

In this section we explain briefly how computations were performed for tangles and knots in the table, and comment on how the remaining problems and new problems raised in the paper can be explored using computer programs.

For tangles, two main methods were used: (1) Find Alexander quandles that color a given tangle nontrivially

¹(<http://shell.cas.usf.edu/quandle/Invariants/DihInv/>).

²([http://shell.cas.usf.edu/quandle/Invariants/DihInv/Dih_Z_3mod\(t+1\)3cocinv.txt](http://shell.cas.usf.edu/quandle/Invariants/DihInv/Dih_Z_3mod(t+1)3cocinv.txt)).

by hand (or Maple-assisted) calculations. (2) Take closures of the tangle, find the resulting knot in the table, and use the computational results in [Smudde 2008] to determine the cocycle invariants. See the proof of Proposition 4.3 for more details of the argument (2). Here we exhibit a hand calculation (1) as an example for the tangle $T(7_{17})$ with orientation NW in, SW out.

Let x be the color of the boundary arcs, and let y, x_1, x_2, x_3, x_4 be the colors of the arcs as depicted in Figure 6. From the crossings adjacent to the NW, SW, SE, NE endpoints, respectively, we obtain the relations

$$\begin{aligned}x_1 &= t^{-1}y + (1 - t^{-1})x = x + t^{-1}(y - x), \\x_3 &= tx + (1 - t)x_1 = x + (t^{-1} - 1)(y - x), \\x_2 &= ty + (1 - t)x = x + t(y - x), \\x_4 &= tx + (1 - t)y = y + t(x - y).\end{aligned}$$

From the remaining three crossings, we obtain

$$\begin{aligned}x_2 &= tx + (1 - t)x_3, \\x_4 &= tx_1 + (1 - t)x_2, \\x_3 &= tx + (1 - t)x_4.\end{aligned}$$

The system gives solutions $t = 2$ and $p = 3$, so that we consider the quandle $\mathbb{Z}_3[t, t^{-1}]/(t - 2) = R_3$ that colors the tangle $T(7_{17})$ nontrivially.

For invariants for knots in the table, closed braid forms in [Cha and Livingston 08] were used. computer (Maple and C++) programs were written that perform the following procedures: All possible colors are assigned at the top arcs (top colors) of a braid, the colors at the bottom arcs are computed according to coloring rules, and compared with the top colors. If they agree, it gives a coloring of the closed braid, and the cocycle value is computed. Programs and outputs can be found in [Smudde 06].³ The tables of quandle cocycle invariants for knots in the table up to nine crossings that were used for the tangle embeddings studied in this paper are found in [Smudde 2008].

The following problems remain to be explored computationally. Although invariants for knots in the table are computed systematically, the size of the quandle that the developed programs can handle is very limited, since the number of colorings increases exponentially compared to the size of the quandle. More efficient calculations are desirable. The invariants for tangles are sometimes computed case by case from the table in [Kanenobu et al. 03] by hand. It is of interest, then, to develop a method of

computing tangle invariants by computer programs. In particular, a theory of partially closed braid forms may be useful.

ACKNOWLEDGMENTS

We would like to thank J.S. Carter and S. Satoh for valuable conversations. M. S. was supported in part by NSF grants DMS #0301089 and #0603876. We are also grateful to the referees for valuable comments and suggestions.

REFERENCES

- [Adams 94] C. Adams *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots*. Providence: American Mathematical Society, 1994.
- [Ameur 07] K. Ameur. “Polynomial Quandle Cocycles, Their Knot Invariants and Applications.” PhD thesis, University of South Florida, 2007.
- [Brieskorn 88] E. Brieskorn. “Automorphic Sets and Singularities.” *Contemporary Math.* 78 (1988), 45–115.
- [Carter et al. 03] J. S. Carter, D. Jelsovsky, S. Kamada, L. Langford, and M. Saito. “Quandle Cohomology and State-Sum Invariants of Knotted Curves and Surfaces.” *Trans. Amer. Math. Soc.* 355 (2003), 3947–3989.
- [Cha and Livingston 08] J. C. Cha and C. Livingston. “Knot-Info: Table of Knot Invariants.” Available online (<http://www.indiana.edu/~knotinfo/>), 2008.
- [Chung and Lin 06] J.-W. Chung and X.-S. Lin. “On n -Punctured Ball Tangles.” *J. Knot Theory Ramifications* 15 (2006), 715–748.
- [Darcy and Navarra-Madsen 06] I. K. Darcy and J. Navarra-Madsen. “Coloring n -String Tangles.” Preprint, arXiv:math.GT/0609587, 2006.
- [Ernst and Sumners 90] C. Ernst D. W. Sumners. “A Calculus for Rational Tangles: Applications to DNA Recombination.” *Math. Proc. Cambridge Philos. Soc.* 108 (1990), 489–515.
- [Fenn and Rourke 92] R. Fenn and C. Rourke. “Racks and Links in Codimension Two.” *J. Knot Theory Ramifications* 1 (1992), 343–406.
- [Joyce 82] D. Joyce. “A Classifying Invariant of Knots, the Knot Quandle.” *J. Pure Appl. Alg.* 23 (1982), 37–65.
- [Kanenobu et al. 03] T. Kanenobu, H. Saito, and S. Satoh. “Tangles with Up to Seven Crossings.” *Interdisciplinary Information Sciences* 9 (2003), 127–140.
- [Krebes 99] D. A. Krebes. “An Obstruction to Embedding 4-Tangles in Links.” *J. Knot Theory Ramifications* 8 (1999), 321–352.
- [Krebes et al. 00] D. A. Krebes, D. S. Silver, and S. G. Williams. “Persistent Invariants of Tangles.” *J. Knot Theory Ramifications* 9 (2000), 471–477.
- [Matveev 82] S. Matveev. “Distributive Groupoids in Knot Theory” (Russian). *Math. USSR-Sbornik* 47 (1982), 73–83.

³(<http://shell.cas.usf.edu/quandle/Invariants/database/database.php>).

- [Mochizuki 03] T. Mochizuki. “Some Calculations of Cohomology Groups of Finite Alexander Quandles.” *J. Pure Appl. Algebra* 179 (2003), 287–330.
- [Murasugi 96] K. Murasugi. *Knot Theory and Its Applications*. Boston: Birkhäuser, 1996.
- [Niebrzydowski 06] M. Niebrzydowski. “On Colored Quandle Longitudes and Its Applications to Tangle Embeddings and Virtual Knots.” *J. Knot Theory Ramifications* 15 (2006) 1049–1059.
- [Przytycki et al. 05] J. H. Przytycki, D. S. Silver, and S. G. Williams. “3-Manifolds, Tangles and Persistent Invariants.” *Math. Proc. Cambridge Philos. Soc.* 139 (2005), 291–306.
- [Ruberman 00] D. Ruberman. “Embedding Tangles in Links.” *J. Knot Theory Ramifications* 9 (2000), 523–530.
- [Satoh 07] S. Satoh. “A Note on the Shadow Cocycle of a Knot with a Base Point.” *J. Knot Theory Ramifications* 16 (2007), 959–967.
- [Smudde 06] C. Smudde. “Computer Programs and Outputs of Quandle Cocycle Invariants for Knot Table.” Available online (<http://shell.cas.usf.edu/quandle/>), 2006.
- [Smudde 2008] C. Smudde. “Quandle Cocycle Invariants for Knot Table Used for Tangle Embedding,” excerpts from [Smudde 06]. Available online (<http://shell.cas.usf.edu/quandle/Applications/AERSS-Table.pdf>), 2008.

Kheira Ameur, Department of Mathematics and Statistics, University of South Florida, 4202 E. Fowler Ave., PHY 114, Tampa, FL 33620 (ameur@mail.usf.edu)

Mohamed Elhamdadi, Department of Mathematics and Statistics, University of South Florida, 4202 E. Fowler Ave., PHY 114, Tampa, FL 33620 (emohamed@math.usf.edu)

Tom Rose, Sarasota High School, 1000 South School Ave, Sarasota, FL 34237 (tomrr49@hotmail.com)

Masahico Saito, Department of Mathematics and Statistics, University of South Florida, 4202 E. Fowler Ave., PHY 114, Tampa, FL 3362 (saito@math.usf.edu)

Chad Smudde, Department of Mathematics, Pensacola Junior College, 1000 College Blvd., Pensacola, FL 32504 (csmudde@pjc.edu)

Received January 22, 2008; accepted in revised form July 3, 2008.