

L^p Spectral Radius Estimates for the Lamé System on an Infinite Sector

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We prove, using interval analysis methods, that the L^2 , L^4 , and L^8 spectral radii of the traction double layer potential operator associated with the Lamé system on an infinite sector in \mathbb{R}^2 are within 2.5×10^{-3} , 10^{-2} , and 10^{-2} , respectively, from a certain conjectured value that depends explicitly on the aperture of the sector and the Lamé moduli of the system. We also indicate how to extend these results to L^p for entire intervals of p , $p \geq 2$.

1. INTRODUCTION

Estimates of the spectral radius of bounded linear operators associated with boundary value problems

$$\begin{cases} Lu = 0 & \text{in } \Omega, \\ Bu = f & \text{on } \partial\Omega, \end{cases} \quad (1-1)$$

where L is an elliptic operator and B the boundary conditions, are important in establishing convergence of iterative solution methods. More specifically, solving (1-1) often means to invert $I+K$, where K is a singular integral operator associated with the system, and in order to establish convergence of the Neumann series of this inverse, $\rho(K, L^p) < 1$ must be satisfied, where $\rho(K, L^p)$ denotes the spectral radius of the operator K on L^p . See, for example, [Dahlberg et al. 88, Kupradze 65, Lewis 90, Mitrea 04] for more details in the case of elastostatics.

Elasticity is the theory of mechanics of solid bodies that deform elastically under external influences; elastostatics treats equilibrium states in elasticity. For the basic notions of elasticity, see, for example [Landau and Lifschitz 86, Marsden and Hughes 94]. The traction problem of elastostatics is to describe equilibrium states of surface interactions. This problem is modeled by equation (1-2) below. We define an infinite sector of aperture θ , that is, the interior of the unbounded region determined by θ .

Definition 1.1. $\Omega_\theta := \{(x, y) \in \mathbb{R}^2 : \exists r > 0, \phi \in (0, \theta) \text{ s.t. } x = r \cos \phi, y = r \sin \phi\}$.

2000 AMS Subject Classification: Primary 35J25, 47-04; Secondary 45E05, 65G20

Keywords: Lamé system, traction conormal derivative, spectral radius, interval analysis, computer-aided proof

Consider the system of elastostatics on an infinite sector such as $\Omega_\theta \subseteq \mathbb{R}^2$,

$$\begin{cases} \mu \Delta \vec{u} + (\lambda + \mu) \nabla \operatorname{div} \vec{u} = \vec{0} \text{ in } \Omega_\theta, \\ \frac{\partial \vec{u}}{\partial N_\mu} = \vec{f} \in L^p(\partial\Omega_\theta), \end{cases} \quad (1-2)$$

where the Lamé moduli μ, λ satisfy $\mu > 0$ and $\lambda + \mu \geq 0$, and

$$\frac{\partial \vec{u}}{\partial N_\mu} = \mu(\nabla \vec{u} + (\nabla \vec{u})^t) \cdot N + \lambda(\operatorname{div} \vec{u})N$$

denotes the traction conormal derivative. Let K be the so-called traction double layer potential operator, which is a singular integral operator naturally associated with this system.

In [Mitrea 04], it is proved that the spectral radius of the Lamé system, that is, the spectral radius of K , on an infinite sector is less than one if p is large. Mitrea and Tucker extend this in [Mitrea and Tucker 07] to $p \geq 2$ by proving that the spectral radius for $p = 2$ is described, within a small error, by an explicit formula and then estimating this formula. Their estimate is done by combining analytic methods and validated numerics.

The explicit formula is interesting in itself, and since the analytic part of the proof of the estimate is done for all $1 < p < \infty$ with a much better estimate than for the computer-assisted part, it is desirable to improve the computer-assisted part of the proof. In this paper we construct a different algorithm for this part of the proof, which improves the estimate for $p = 2$ and also extends it to other values of p .

2. PROBLEM

The problem at hand is to find an enclosure of the spectrum of K , which is described implicitly by relation (2-2) in Theorem 2.2, and explicitly by the four curves $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4$ in the same theorem. We will describe an algorithm that encloses the spectrum in later sections. In order to formulate our main theorem and the conjectured spectral radius we need to introduce the following expressions for $\alpha \in [0, \pi]$, $\kappa \in [0, 1]$, and $x \in (0, 1)$. Let

$$\begin{aligned} S(\alpha, x, \kappa) &:= \left\{ \sin^2(\alpha x) + \kappa^2 \cos^2(\alpha x) \right. \\ &\quad \left. - (\kappa \cos(\pi x) - (1 - \kappa)x \sin \alpha \sin(\alpha x))^2 \right\}^{1/2} \\ &\quad + (1 - \kappa)x \sin \alpha \cos(\alpha x) \Big| \cdot \frac{1}{\sin(\pi x)}. \end{aligned}$$

Our main result is a rigorous estimate of the spectral radius $\rho(K, L^p(\partial\Omega_\theta))$ of K in terms of S . Note that

blackboard boldface letters denote interval enclosures of the corresponding variables and functions; for instance, \mathbb{A} is an enclosure of A , $A \in \mathbb{A}$.

Theorem 2.1. *Let $\Omega_\theta \subset \mathbb{R}^2$ be an infinite sector of aperture $\theta \in (0, 2\pi)$ and let K be the traction double layer potential operator associated with (1-2), and set $\kappa := \mu/(2\mu + \lambda) \in [\frac{1}{40}, \frac{9}{10}]$. Then for $\theta \in [\frac{\pi}{200}, 2\pi - \frac{\pi}{200}]$, we have*

$$S(|\pi - \theta|, p, \kappa) \leq \rho(K; L^p(\partial\Omega_\theta)) \leq S(|\pi - \theta|, p, \kappa) + \epsilon,$$

for the following values of p and ϵ :

p	2	4	8	[2, 4]	[4, 8]
ϵ	0.0025	0.01	0.01	0.1	0.2

The lower bound is known analytically, as in [Mitrea and Tucker 07], and the upper bound is proved using rigorous numerics, to be introduced in Section 4. It is conjectured in [Mitrea and Tucker 07] that Theorem 2.1 actually holds with $\epsilon = 0$ and $1 < p < \infty$.

Note that the algorithm seems to work for any specific $p \in [2, 8]$ with $\epsilon = 0.01$, although running the program for all values of the parameter makes it excessively slow. In fact, Theorem 2.1 is harder to verify for larger p due to the denominator D in (2-4), so if one constructs an algorithm that works for some p^* , then such an algorithm should also work for any p , $2 \leq p \leq p^*$.

The spectrum is described explicitly using auxiliary functions. Therefore, let

$$\begin{aligned} A_\kappa &:= (1 - \kappa)z \sin \theta, & B &:= \sin((\pi - \theta)z), & (2-1) \\ C &:= \cos((\pi - \theta)z), & D &:= \sin(\pi z), & E := \cos(\pi z), \end{aligned}$$

where $\theta, \kappa \in \mathbb{R}$ and $z \in \mathbb{C}$.

The spectrum of the operator K is described by the following theorem [Mitrea 04]. A typical spectrum is illustrated in Figure 1.

Theorem 2.2. *Let $\Omega_\theta \subset \mathbb{R}^2$ be an infinite sector of aperture $\theta \in (0, 2\pi)$ and let K be the traction double layer potential operator associated with the system (1-2) with Lamé moduli λ, μ satisfying the conditions in (1-2). Then for every $1 < p < \infty$, we have*

$$\begin{aligned} \sigma(K; L^p(\partial\Omega_\theta)) &= \left\{ w \in \mathbb{C} : (wD \pm A_\kappa C)^2 = Q_{\kappa, \mp} \right. & (2-2) \\ &\quad \left. \text{for some } z \in \frac{1}{p} + i\mathbb{R} \right\} \cup \{\kappa, -\kappa\}, \end{aligned}$$

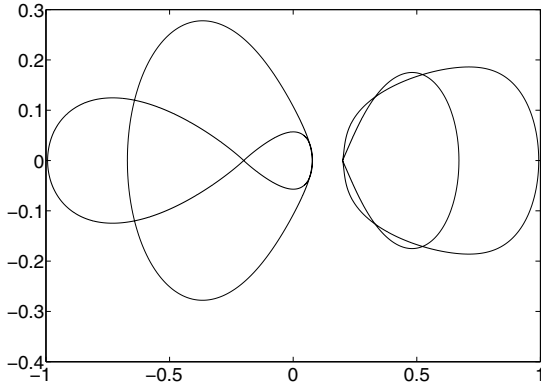


FIGURE 1. A typical spectrum, here with $\kappa = 0.2$, $\theta = 1$, $x = 0.5$.

where $\kappa = \frac{\mu}{2\mu + \lambda}$ and A_κ, B, C, D are as in (2-1). Also

$$Q_{\kappa, \pm} := B^2 + \kappa^2 C^2 - (\kappa E \pm A_\kappa B)^2. \tag{2-3}$$

In particular, the spectrum is described by four curves:

$$\sigma(K; L^p(\partial\Omega_\theta)) = \bigcup_{i=1}^4 \Sigma_i(\theta, p, \kappa).$$

Above $[-\infty, \infty] \ni y \mapsto \Sigma_i(\theta, p, \kappa)(y) \in \mathbb{C}$ is a parametric closed curve in the complex plane, given by

$$\begin{aligned} \Sigma_1(\theta, p, \kappa)(y) &:= \frac{\sqrt{Q_{\kappa,-}} - A_\kappa C}{D}, \\ \Sigma_2(\theta, p, \kappa)(y) &:= \frac{-\sqrt{Q_{\kappa,-}} - A_\kappa C}{D}, \\ \Sigma_3(\theta, p, \kappa)(y) &:= \frac{\sqrt{Q_{\kappa,+}} + A_\kappa C}{D}, \\ \Sigma_4(\theta, p, \kappa)(y) &:= \frac{-\sqrt{Q_{\kappa,+}} + A_\kappa C}{D}, \end{aligned} \tag{2-4}$$

where A_κ, B, C, D, E are evaluated at $z = \frac{1}{p} + iy$, $y \in \mathbb{R}$.

Note that the problem has symmetries, both in the parameter region, that is, (κ, θ, x, y) -space, and in the spectrum. Therefore, we need to consider only $y \geq 0$ in the parameter region and $\Im(z) \geq 0$ in the spectrum. Furthermore, it is proved in [Mitrea and Tucker 07] that Theorem 2.1 holds with $\epsilon = 10^{-6}$ in the region $|y| \geq 10^4$ for all $1 < p < \infty$. Thus, Theorem 2.1 needs to be proved only for the compact region

$$(\theta, \kappa, x, y) \in \left(\left[\frac{\pi}{200}, \pi \right], \left[\frac{1}{40}, \frac{9}{10} \right], \left[\frac{1}{8}, \frac{1}{2} \right], [0, 10000] \right),$$

which corresponds to $2 \leq p \leq 8$.

3. STRATEGY

The approach in [Mitrea and Tucker 07] used in earlier algorithms to solve this problem was to enclose the four curves (2-4) in Theorem 2.2 that explicitly describe the spectrum. There are several problems with enclosing these curves directly, primarily the square root, and for $p \neq 2$ the denominator D . The problem with the square root is that it is not Lipschitz continuous at the origin, which increases the width of the enclosures; that is, it increases the effect of earlier errors when it is implemented using interval arithmetic, to be introduced below.

Thus, avoiding this formulation is likely to improve the run time of a computer-assisted proof and thereby allow for better estimates, that is, smaller ϵ in Theorem 2.1. The denominator D has absolute value less than 1 for $p \neq 2$ and y small, which in interval analysis, like the square root, increases the effect of earlier errors. This drastically increases the complexity of the computer-assisted proof, especially since we believe that the spectral radius is attained for $y = 0$.

Instead, we want to use the description of the spectrum given by (2-2) and try to enclose this set by implicitly solving for ω . Interval analysis provides us with the possibility to exclude sets that are not in the spectrum. This is done by first enclosing the ranges of the functions appearing in (2-1) and (2-3), that is, A_κ, C, D , and $Q_{\kappa, \pm}$, and then partitioning a suitable compact subset of the complex plane that is known to contain the spectrum.

For each box \mathbb{W} in this partition $(\mathbb{W}D \pm A_\kappa)^2 \cap Q_{\kappa, \pm}$ is calculated, and if this intersection is empty, then it has been proved rigorously that the box \mathbb{W} does not intersect the spectrum of the traction double layer potential operator K .

In order to implement this on a computer we must be able to find the suitable compact search region described above. This is possible because the curves describing the spectrum are connected and we know that some part of each curve lies inside the circle with radius S . Thus, if we can prove that there exists a circle with radius $S + \epsilon$ that does not intersect the spectrum, then the spectral radius must be smaller than this radius. The algorithms that we use to construct such a circle, and to verify that it does not intersect the spectrum, are described in Section 5.

4. METHODS

The proof is computer-assisted and is based on interval analysis. Interval analysis yields rigorous results for continuous problems, taking both discretization and round-

ing errors into account. For a thorough introduction to interval analysis we refer to [Alefeld and Herzberger 83, Moore 66, Moore 79, Neumaier 90, Petković and Petković 98].

Let \mathcal{R} denote the set of closed intervals. For any element $\mathbb{A} \in \mathcal{R}$, we adopt the notation $\mathbb{A} = [\underline{\mathbb{A}}, \overline{\mathbb{A}}]$, where $\underline{\mathbb{A}}, \overline{\mathbb{A}} \in \mathbb{R}$. Arithmetic operations and function evaluations for a large class of functions can be extended to the interval setting. In particular, functions $f : \mathbb{C} \rightarrow \mathbb{C}$ acting in the complex plane can be extended, although finding good-quality enclosures of the elementary functions is much more complicated than in the real-valued setting.¹

One of the main reasons for passing to interval arithmetic is that this approach provides a simple way of enclosing the range of a function. If F is an interval extension of f , then

$$\text{Range}(f, \mathbb{A}) \subseteq F(\mathbb{A}).$$

Another key feature of interval arithmetic is that interval extensions of functions are *inclusion monotonic*, that is, if $\mathbb{A} \subseteq \mathbb{B}$, then $f(\mathbb{A}) \subseteq f(\mathbb{B})$.

5. IMPLEMENTATION

The objective is to prove that Theorem 2.1 holds with ϵ as small as possible and the set of parameters \mathbb{P} as wide as possible. Of course, the conjecture is that Theorem 2.1 should hold for all positive ϵ and $\mathbb{P} = (1, \infty)$. In [Mitrea and Tucker 07], the authors show that Theorem 2.1 holds for the thin interval $\mathbb{P} = p = 2$ and $\epsilon = 0.01$. The algorithm proposed here enables us to improve this result, although anything but a thin \mathbb{P} turns out to be very hard computationally. We work on the global parameter domain

$$(\theta, \kappa, x, y) \in \left(\left[\frac{\pi}{200}, \pi \right], \left[\frac{1}{40}, \frac{9}{10} \right], \left[\frac{1}{8}, \frac{1}{2} \right], [0, 10000] \right),$$

which corresponds to $\mathbb{P} = [2, 8]$. The actual computations using $\epsilon = 0.01$, however, were made only for thin intervals, with p fixed as 2, 4, or 8. With the full parameter region $p \in [2, 8]$, only $\epsilon = 0.2$ was achieved, although when we restricted attention to $\mathbb{P} = [2, 4]$, $\epsilon = 0.1$ was possible. Describing the algorithms, we concentrate on the case in which \mathbb{P} is a thin interval; adaptation to the general case is straightforward.

¹We thank Markus Neher and Ingo Eble for developing CoStLy—Complex interval Standard functions Library—and for their valuable assistance with interfacing it to the CXSC library [CXSC 05].

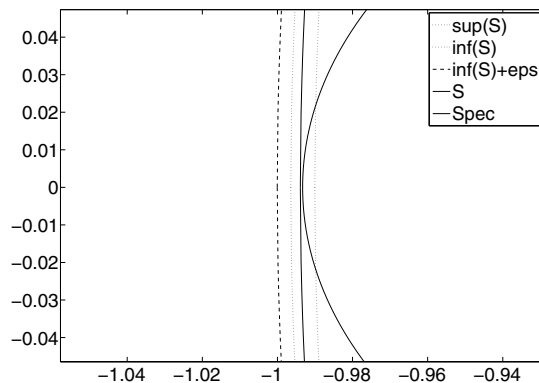


FIGURE 2. Piece of the spectrum and the four circles with radii $\underline{S}, \underline{S} + \epsilon, \overline{S}, S$, respectively at $\kappa = 0.2, \theta = 1, x = 0.5, \epsilon = 0.01$.

The aim of the algorithm is to prove that

$$\rho < \underline{S} + \epsilon,$$

and since $\underline{S} \leq S$ always holds due to the properties of interval arithmetic, Theorem 2.1 is proved. The main problem with this approach is that in order for a circle with radius $\underline{S} + \epsilon$ to have empty intersection with the spectrum, we must have $\text{width}(\mathbb{S}) < \eta$, where $\eta < \epsilon$, and preferably \mathbb{S} should be as narrow as possible; see Figure 2.

Experiments reveal that the computational time increases dramatically with decreasing η . The results presented here were achieved using $\eta = 0.8 \times \epsilon$, which turns out to be a good compromise. Since S does not depend on y , the enclosure \mathbb{S} of S is achieved by adaptively splitting in the κ and θ directions of the parameter region. This algorithm is presented as Algorithm 5.1, whose output is shown in Figure 3, which also illustrates the increased complexity in reducing ϵ .

Algorithm 5.1.

```
paramList += initialDomain;
    // Add one box to the search list.
while ( !IsEmpty(paramList) ) {
    param = Pop(searchList);
    // The current parameter box.
    R = spectralRadius(param);
    // Compute the conjectured spectral radius.
    ETA = 0.8*EPS;
    // Set the value of ETA
    if ( diam(S) < ETA)
        // Save those parameter boxes that yield...
        SThinList += p;
        // ...thin enclosures of S
    else
```

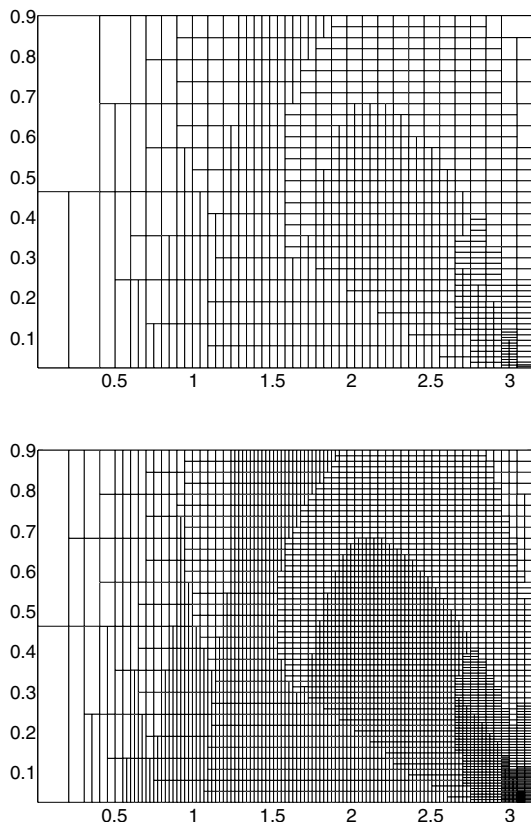


FIGURE 3. The initial partition of the (θ, κ) -plane with $x = \frac{1}{2}$. The figures on the top and bottom correspond to $\epsilon = 0.1$ and $\epsilon = 0.05$, respectively

```

splitAndStore(param, paramList);
    // Divide the parameter space
}

```

Algorithm 5.1 yields the division of κ and θ in the parameter region needed in order to reduce the width of the enclosure of S . A box from this splitting is used together with a small piece of \mathbb{Y} as input for the main algorithm. For this parameter region, a coarse covering of the semicircle with radius $\underline{S} + \epsilon$ is constructed. In fact, it is covered by only one box having side lengths $2 \times (\underline{S} + \epsilon + \delta)$ and $(\underline{S} + \epsilon + \delta)$. Here $\delta = 10^{-14}$ is a small number that is added to ensure that the semicircle of interest is inside the cover. Each box in the cover is split into four, and those that either are within the required accuracy or have empty intersection with the spectrum are removed. This is repeated a few times or until we arrive at a cover of maximum size (= 50).

Having refined the box sizes in the spectral plane, we turn our attention to the parameter space and split adap-

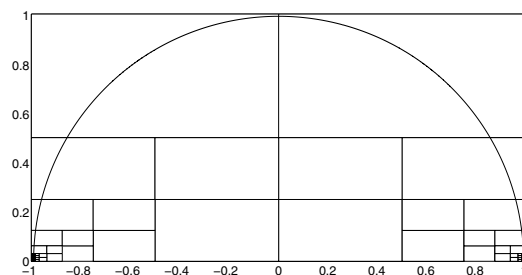


FIGURE 4. Partition of the cover of the circle $\underline{S} + \epsilon$ at $\mathbb{Y} = [0, 10^{-3}]$, $x = \frac{1}{2}$, and $\epsilon = 0.01$.

tively, trying to minimize the width of the enclosure of AC in Theorem 2.2. That is, we try to minimize the width of all the entities that occur in the implicit description of the spectrum. The entire process is repeated until the cover is empty. Algorithm 5.2 is repeated until Theorem 2.1 has been verified for all $y \in \mathbb{Y}$ and for all the boxes from the partition constructed by Algorithm 5.1.

There is a good reason to use only one box for the initial covering: it is computationally much faster to adaptively split only that part of the cover that was not removed by an earlier step and check the conditions for removal there than to make a very fine covering of the circle and thereafter check the removal conditions only once.

The splitting of \mathbb{Y} was done manually, with small steps for small y and larger for large y . More specifically, the first interval was taken as $\mathbb{Y} = [0, 10^{-3}]$ and the last one as $\mathbb{Y} = [10, 10000]$. A partition of the cover is illustrated in Figure 4.

Algorithm 5.2.

```

makeBox(paramList, SthinList, x, y);
    //Make a list of param's for a y piece
while(!IsEmpty(paramList) {
    //Verify Thm 2.1 for this parameter set.
    paramBox=Pop(paramList);
    spectralVerify(paramBox);
}

spectralVerify(paramBox) {
    S=spectralRadius(paramBox);
    //Calc the conj spectral radius
    circleList+=coverCircle(\inf(S)+EPS+DELTA));
    //Initate the cover
    parcelList+=Parcel(paramBox, circleList);
    //Param's and cover together
}

```

```

while(!IsEmpty(parcelList)) {
  checkParcel=Pop(parcelList);
  i=0;
  do {
    checkParcel.splitAndStoreCoverList();
    //Split the cover
    checkParcel.conjecture(EPS);
    //Remove if conj. is EPS true
    i++;
  }
  while(i<10 && checkParcel.coverSize() < 50 );

  if(checkParcel.coverSize() !=0 )
    //Adapt split the param space
    parcelList+=checkParcel.splitParam();
}
}

conjecture(EPS) {
  //Remove boxes that are either
  while( !IsEmpty(coverList) ) {
    //inside of the allowed circle
    w=Pop(coverList);
    //or do not intersect the spectrum
    if( (Sup(abs(w)) >=
      Inf(spectralRadius(checkParcel)) + EPS)
      && spectralHit(w) )
      returnList+=w;
  }
  coverList=returnList;
}

spectralHit(w) {
  lhsPos = sqr(w*D + AC);
  //Calculate left side of (2-2)
  if ( Intersect(lhsPos, Qneg) )
    //if intersection is nonempty
    return true;
  //there is spectrum in box
  lhsNeg = sqr(w*D - AC);
  //Same for negative left
  if ( Intersect(lhsNeg, Qpos) )
    return true;
  return false;
}

```

The complete source files to the program can be found as supplementary material at the authors' website. ²

6. RESULTS

The algorithms described above were implemented using C++ with the CXSC toolbox [CXSC 05, Hammer et al. 95]. The program was executed on two dual 3.2-GHz Intel Xeon processors, with a total of 3072 MB RAM.

²<http://www.math.uu.se/~johnson>.

\mathbb{P}	ϵ	Run Time	#Boxes from Alg. 5.1
2	0.01	3 h	90587
2	0.0025	70 h 31 min	1449293
4	0.01	29 h 14 min	587439
8	0.01	228 h 12 min	3486342
[2, 4]	0.1	16 h 16 min	399129
[2, 8]	0.2	16 h 58 min	656358

TABLE 1. Values of p and ϵ where Theorem 2.1 holds.

Elastic Material	κ	Elastic Material	κ
Glass	1/3	Copper	0.2426
Steel	0.3106	Aluminum	0.2407
Iron	0.3059	Lead	0.0467
Nickel	0.2833	Rubber	0.0283
Bronze	0.2754		

TABLE 2. The constant κ for some common materials.

The run times increased dramatically with increased p and decreased ϵ . The reason for this is the huge increase of the initial partition of (κ, θ) -space required in order to get width $(\mathbb{S}) < \eta$. For a comparison, see Figure 3. The computations prove Theorem 2.1 with

$$(\theta, \kappa, x, y) \in \left(\left[\frac{\pi}{200}, \pi \right], \left[\frac{1}{40}, \frac{9}{10} \right], \mathbb{P}, [0, 10000] \right),$$

for the values of ϵ and ranges of \mathbb{P} in Table 1.

The first result is stated for comparison with the method used in [Mitrea and Tucker 07], which took 16 hours, using five processors for the same result. This indicates that the method used here is an improvement. It is not only faster, but also more robust, in the sense that it allows us to extend the results to other L^p -spaces. The run time increases dramatically with increased p , and although Theorem 2.1 has been proved only for $p = 2, 4$, and 8, the algorithm can handle any $p \in [2, 8]$.

The limits of the considered κ domain $\left[\frac{1}{40}, \frac{9}{10} \right]$ are imposed by the program. It does not work for a larger domain. For κ small, one can see from Figure 3 that smaller and smaller boxes are needed as κ decreases. This is, however, from an engineering perspective not a serious limitation, as one can see from the range of materials in Table 2 [Ciarlet 88, p. 129].

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Received November 7, 2006; accepted January 23, 2008.