On the Folkman Number $f(2, 3, 4)$

Andrzej Dudek and Vojtěch Rödl

CONTENTS

Let $f(2, 3, 4)$ denote the smallest integer n such that there exists a K_4-free graph of order n for which any 2-coloring of its edges yields at least one monochromatic triangle. It is well known that such a number must exist. For a long time, the best known upper bound, provided by J. Spencer, was $f(2, 3, 4) < 3 \cdot 10^9$. Recently, L. Lu announced that $f(2, 3, 4) < 10{,}000$. In this note, we will give a computer-assisted proof showing that $f(2, 3, 4) < 1000$. To prove this, we will generalize an idea of Goodman's, giving a necessary and sufficient condition for a graph G to yield a monochromatic triangle for every edge coloring.

1. INTRODUCTION

Let $\mathcal{F}(r, k, l)$, $k < l$, be a family of K_l-free graphs with the property that if $G \in \mathcal{F}(r, k, l)$, then every r-coloring of the edges of G must yield at least one monochromatic copy of K_k. It was shown in [Folkman 70] that $\mathcal{F}(2, k, l) \neq \emptyset$. The general case, i.e., $\mathcal{F}(r, k, l) \neq \emptyset$, $r \geq 2$, was settled in the positive by J. Nešetřil and the second author [Nešetřil and Rödl 76].

Let $f(r, k, l) = \min_{G \in \mathcal{F}(r,k,l)} |V(G)|$. The problem of determining the numbers $f(r, k, l)$ in general includes the classical Ramsey numbers and thus is not easy. In this note we focus on the case $r = 2$ and $k = 3$. We will write $G \to \triangle$ and say that G *arrows a triangle* if every 2-coloring of G yields a monochromatic triangle. Since the Ramsey number $R(3, 3)$ is equal to 6, clearly $f(2, 3, l) = 6$ for $l > 6$.

The value of $f(2, 3, 6) = 8$ was determined by R. Graham [Graham 68], and $f(2, 3, 5) = 15$ by K. Piwakowski, S. Radziszowski, and S. Urbański [Piwakowski et al. 99]. In the remaining case, the upper bounds on $f(2, 3, 4)$ obtained in [Folkman 70] and [Nešetřil and Rödl 76] are extremely large (iterated tower). Consequently, in 1975, P. Erdős [Erdős 75] offered \$100 for proving or disproving that $f(2, 3, 4) < 10^{10}$. Applying Goodman's idea [Goodman 59] (of counting triangles in a graph and in its complement) for random graphs, P. Frankl and the second author [Frankl and Rödl 86] came relatively close to the desired bound, showing that $f(2, 3, 4) < 8 \times 10^{11}$.

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This result was improved by J. Spencer [Spencer 88], who refined the argument and proved $f(2,3,4) < 3\times 10^9$, giving a positive answer to Erdős's question. Subsequently, F. Chung and R. Graham [Chung and Graham 98] conjectured that $f(2,3,4) < 10^6$ and offered \$100 for a proof or disproof. Recently, L. Lu [Lu 08] showed that $f(2,3,4) < 10{,}000$. (A weaker result, $f(2,3,4) < 1.3\times 10^5$, also answering Chung and Graham's question, was independently found in an earlier version of this paper [Dudek 08]).

All these proofs are based on a modification of Goodman's idea that explores the local property of every vertex neighborhood in a graph (see Corollary 2.2).

In this note, we will present a K_4-free graph G_{941} of order 941 and give a computer-assisted proof that $G_{941} \in \mathcal{F}(2,3,4)$. This yields $f(2,3,4) \le 941$. To prove it, we will develop a technique that is a generalization of ideas from [Goodman 59, Nešetřil and Rödl 76, Spencer 88]. More precisely, for every graph G we will construct a graph H with the property that G arrows a triangle if and only if the maxcut of H is less than twice number of triangles in G.

2. COMPUTER-ASSISTED PROOF OF $f(2,3,4) < 1000$

2.1 Counting Blue and Red Triangles

In order to find an upper bound on the number $f(2,3,4)$, we will use an idea of [Goodman 59]. For any blue–red coloring of G, let $T_{\mathrm{BR}}(v)$, $T_{\mathrm{BB}}(v)$, and $T_{\mathrm{RR}}(v)$ count the numbers of triangles containing vertex v for which two edges incident to v are colored respectively blue–red, blue–blue, and red–red. Also, let T_{blue} (T_{red}) be the number of blue (red) monochromatic triangles.

The sum $\sum_{v\in V(G)} T_{\mathrm{BR}}(v)$ counts two times the number of nonmonochromatic triangles. This is because each such triangle is counted once for two different vertices. On the other hand, the sum $\sum_{v\in V(G)} \big(T_{\mathrm{BB}}(v) + T_{\mathrm{RR}}(v)\big)$ counts three times the number of monochromatic triangles and once the number of nonmonochromatic triangles. Hence,

$$\sum_{v\in V(G)} T_{\mathrm{BR}}(v) = 2\sum_{v\in V(G)} \big(T_{\mathrm{BB}}(v) + T_{\mathrm{RR}}(v)\big) - 6\big(T_{\mathrm{blue}} + T_{\mathrm{red}}\big). \qquad (2\text{–}1)$$

Consequently, $G \to \triangle$ if and only if for every edge coloring of G,

$$\sum_{v\in V(G)} T_{\mathrm{BR}}(v) < 2\sum_{v\in V(G)} \big(T_{\mathrm{BB}}(v) + T_{\mathrm{RR}}(v)\big). \qquad (2\text{–}2)$$

Denote by $N(v)$ the set of neighbors of a vertex $v \in V$ and let $G[N(v)]$ be a subgraph of G induced on $N(v)$. Moreover, for a given cut $C \subset V(G)$, let

$$M_C(G) = \big\{\{x,y\} \in E(G) \mid x \in C \text{ and } y \in V \setminus C\big\},$$

and let

$$M(G) = \max_{C\subset V} M_C(G),$$

i.e., $M(G)$ is the value corresponding to the solution of the maxcut problem for G.

Proposition 2.1. [Frankl and Rödl 86, Spencer 88] *Let $G = (V,E)$ be a graph that satisfies*

$$\sum_{v\in V(G)} M(G[N(v)]) < \frac{2}{3}\sum_{v\in V(G)} \big|E(G[N(v)])\big|. \qquad (2\text{–}3)$$

Then $G \to \triangle$.

An easy consequence of Proposition 2.1 is the following corollary.

Corollary 2.2. *Let $G = (V,E)$ be a graph that satisfies*

$$M(G[N(v)]) < \frac{2}{3}\big|E(G[N(v)])\big| \qquad (2\text{–}4)$$

for every vertex $v \in V(G)$. Then $G \to \triangle$.

Note that in particular, Corollary 2.2 gives a sufficient condition for a K_4-free graph to be in $\mathcal{F}(2,3,4)$. We will extend this idea and give a necessary and sufficient condition for a graph G to yield a monochromatic triangle for every edge coloring. More precisely, for every graph $G = (V,E)$ with $t_\triangle = t_\triangle(G)$ triangles, we construct a graph H with $|E|$ vertices such that $G \to \triangle$ if and only if the maxcut of H is less than $2t_\triangle$.

Let G be a graph with the vertex set $V(G) = \{1,2,\ldots,n\}$. For every vertex $i \in V(G)$, let G_i be a graph with

$$V(G_i) = \big\{\{i,j\} \mid j \in N(i)\big\}$$

and

$$E(G_i) = \big\{\{\{i,j\},\{i,k\}\} \mid ijk \text{ is a triangle in } G\big\}.$$

Clearly, G_i is isomorphic to the subgraph $G[N(i)]$ of G induced on the neighborhood $N(i)$.

Now we define a graph H as follows. Let

$$V(H) = E(G)$$

and

$$E(H) = \bigcup_{i \in V(G)} E(G_i).$$

In other words, H is a graph whose set of vertices is the set of edges of G such that e and f are adjacent in H if e and f belong to a triangle in G. Clearly $|V(H)| = |E(G)|$ and $|E(H)| = 3t_\triangle(G)$. Moreover, observe that there is a one-to-one correspondence between blue–red colorings of edges of G and bipartitions of vertices of H. Let C be a cut with the partition $V(H) = B \cup R$. Since the edges between B and R correspond to nonmonochromatic triangles in G, we conclude that the value corresponding to the cut C equals

$$M_C(H) = \sum_{i \in V(G)} T_{\mathrm{BR}}(i). \tag{2–5}$$

Counting the edges that lie entirely in B or in R yields

$$\begin{aligned}\sum_{i \in V(G)} \big(T_{\mathrm{BB}}(i) + T_{\mathrm{RR}}(i)\big) &= |E(H)| - M_C(H) \qquad (2–6)\\ &= \big(3t_\triangle - M_C(H)\big).\end{aligned}$$

By (2–1) we have that

$$\sum_{i \in V(G)} T_{\mathrm{BR}}(i) \le 2 \sum_{i \in V(G)} \big(T_{\mathrm{BB}}(i) + T_{\mathrm{RR}}(i)\big),$$

and by (2–2), $G \to \triangle$ if and only if the strict inequality holds for every edge coloring of G.

Consequently, (2–5) and (2–6) yield that $G \to \triangle$ if and only if

$$M_C(H) < 2\big(3t_\triangle - M_C(H)\big)$$

for every cut of H. Consequently, we have the following theorem.

Theorem 2.3. *Let G be a graph. Then there exists a graph H of order $|E(G)|$ with $M(H) \le 2t_\triangle(G)$ such that $G \to \triangle$ if and only if $M(H) < 2t_\triangle(G)$.*

2.2 Approximating the Maxcut

Since Theorem 2.3 requires an assumption regarding the maxcut of the graph H, we will approximate it with Proposition 2.4 below. The proof of this proposition for regular graphs can be found in [Krivelevich and Sudakov 06]. Along the lines of their proof one can obtain the following easy generalization, which we present here.

Proposition 2.4. *Let $H = (V, E)$ be a graph of order n. Let $\lambda_{\min} = \lambda_{\min}(H)$ be the smallest eigenvalue of the adjacency matrix of H. Then*

$$M(H) \le \frac{|E(H)|}{2} - \frac{\lambda_{\min}|V(H)|}{4}.$$

Proof: Let $A = (a_{ij})$ be the adjacency matrix of $H = (V, E)$ with average degree d and $V = \{1, 2, \ldots, n\}$. Let $\mathbf{x} = (x_1, \ldots, x_n)$ be any vector with coordinates ± 1. Then

$$\begin{aligned}\sum_{\{i,j\} \in E} (x_i - x_j)^2 &= \sum_{i=1}^{n} d_i x_i^2 - \sum_{i \ne j} a_{ij} x_i x_j\\ &= \sum_{i=1}^{n} d_i - \sum_{i \ne j} a_{ij} x_i x_j\\ &= nd - \mathbf{x}^T A \mathbf{x}.\end{aligned}$$

By the Rayleigh–Ritz ratio (see, e.g., [Horn and Johnson 85, Theorem 4.2.2]), for any vector $\mathbf{z} \in \mathbb{R}^n$, we have $\mathbf{z}^T A \mathbf{z} \ge \lambda_{\min}\|\mathbf{z}\|^2$, where by $\|\cdot\|$ we denote the Euclidean norm. Therefore,

$$\begin{aligned}\sum_{\{i,j\} \in E} (x_i - x_j)^2 &= nd - \mathbf{x}^T A \mathbf{x}\\ &\le nd - \lambda_{\min}\|\mathbf{x}\|^2 \qquad (2–7)\\ &= nd - \lambda_{\min} n.\end{aligned}$$

Let $V = V_1 \cup V_2$ be an arbitrary partition of V into two disjoint subsets and let $e(V_1, V_2)$ be the number of edges in the bipartite subgraph of H with bipartition (V_1, V_2). For every vertex $i \in V$, set $x_i = 1$ if $i \in V_1$ and $x_i = -1$ if $i \in V_2$. Note that for every edge $\{i, j\}$ of H, $(x_i - x_j)^2 = 4$ if this edge has its endpoints in the distinct parts of the above partition and is zero otherwise. Now using (2–7), we conclude that

$$\begin{aligned}e(V_1, V_2) &= \frac{1}{4} \sum_{\{i,j\} \in E} (x_i - x_j)^2\\ &\le \frac{1}{4}(dn - \lambda_{\min} n)\\ &= \frac{|E|}{2} - \frac{\lambda_{\min}|V|}{4},\end{aligned}$$

which completes the proof. □

2.3 Numerical Results

Let G be a circulant graph defined as follows:

$$V(G_{941}) = \mathbb{Z}_{941}$$

and

$$E(G_{941}) = \big\{\{x, y\} \mid x - y = \alpha^5 \bmod 941\big\},$$

i.e., the set of edges consists of those pairs of vertices x and y that differ by a fifth residue of 941. Equivalently,

$$V(G_{941}) = \{0, 1, \ldots, 940\}$$

and

$$E(G_{941}) = \big\{\{x, y\} \;\big|\; |x - y| \in D \text{ or } 941 - |x - y| \in D\big\},$$

where D is a distance set defined by

$$\begin{aligned} D = \{&1, 12, 15, 32, 34, 37, 40, 42, 44, 46, 50, 52, 54, 55, 65, 73, 83, 93,\\ &97, 112, 114, 116, 118, 119, 122, 123, 131, 140, 142, 144, 145,\\ &147, 153, 154, 161, 167, 172, 175, 178, 180, 182, 189, 191, 198,\\ &202, 207, 215, 218, 223, 225, 234, 243, 248, 251, 254, 278, 281,\\ &282, 293, 302, 304, 310, 311, 317, 318, 323, 328, 339, 341, 380,\\ &384, 386, 389, 392, 399, 402, 403, 406, 408, 410, 413, 418, 419,\\ &427, 428, 431, 437, 444, 447, 451, 454, 461, 466, 467\}. \end{aligned}$$

One can check that G_{941} is a K_4-free, 188-regular graph with $|V(G_{941})| = 941$, $|E(G_{941})| = 88{,}454$, and $t_\triangle(G_{941}) = 707{,}632$. Then, the graph H corresponding to G_{941} in Theorem 2.3 is 48-regular with $|V(H)| = 88{,}454$, $|E(H)| = 3t_\triangle(G_{941}) = 2{,}122{,}896$. Moreover, using in Matlab the function `eigs` for real, symmetric, and sparse matrices with the option `sa`, we get $\lambda_{\min}(H) \geq -15.196$. Thus, Proposition 2.4 implies

$$\begin{aligned} M(H) &\leq \frac{|E(H)|}{2} - \frac{\lambda_{\min}(H)|V(H)|}{4} \leq 1{,}397{,}484.746\\ &< 1{,}415{,}264 = 2t_\triangle(G_{941}). \end{aligned}$$

Consequently, Theorem 2.3 yields the main result of this note.

Theorem 2.5. *The Folkman number $f(2, 3, 4)$ is less than or equal to 941.*

Remark 2.6. For given numbers n and r, let $G(n, r)$ be a circulant graph with vertex set

$$V(G(n, r)) = \mathbb{Z}_n$$

and edge set

$$E(G(n, r)) = \big\{\{x, y\} \mid x \neq y \text{ and } x - y = \alpha^r \bmod n\big\}.$$

Note that $G(n, r)$ is well defined, i.e., the graph is undirected if -1 is an rth residue of n. In particular, $G_{941} = G(941, 5)$. By exhaustive search we found that G_{941} is the smallest graph among all graphs $G(n, r)$ for which our technique works that belongs to the family $\mathcal{F}(2, 3, 4)$.

$G(n,r)$	ρ
$G(127,3)$	0.030884
$G(281,4)$	0.042306
$G(313,4)$	0.040612
$G(337,4)$	0.034517
$G(353,4)$	0.037667
$G(457,4)$	0.030386
$G(541,5)$	0.049676
$G(571,5)$	0.044144
$G(701,5)$	0.029507
$G(769,6)$	0.044195
$G(937,6)$	0.048529
$\boldsymbol{G(941,5)}$	-0.012728

TABLE 1. Candidates for membership and one member of $\mathcal{F}(2, 3, 4)$.

For a given K_4-free graph $G(n, r)$, let H be a graph that corresponds to $G(n, r)$ from Theorem 2.3. Let $\alpha = \frac{|E(H)|}{2} - \frac{\lambda_{\min}(H)|V(H)|}{4}$ and $\beta = 2t_\triangle(G(n, r))$. In view of Theorem 2.3 and Proposition 2.4, if $\alpha < \beta$, then $G(n, r) \rightarrow \triangle$, and so $G(n, r) \in \mathcal{F}(2, 3, 4)$. Obviously the converse is not true, since α is only an approximation on $M(H)$. We define a parameter $\rho = \frac{\alpha - \beta}{\alpha}$ to get an estimate of how "close" $G(n, r)$ is to the property $\mathcal{F}(2, 3, 4)$. In Table 1 we have listed all (up to isomorphism) K_4-free graphs $G(n, r)$ with $n \leq 941$ and $\rho < 0.05$.

3. CONCLUDING REMARKS

Recently, S. P. Radziszowski and Xu Xiaodong suggested [Radziszowski and Xiaodong 07] that the graph $G_{127} = G(127, 3)$, considered in [Hill and Irving 82], belongs to the family $\mathcal{F}(2, 3, 4)$. One can check that $t_\triangle(G_{127}) = 9779$. Let H be a graph from Theorem 2.3 that corresponds to G_{127}. Using a semidefinite program with polyhedral relaxations [Rendl et al. 07a, Rendl et al. 07b], we obtained an upper bound on $M(H) \leq 19558 = 2t_\triangle(G_{127})$. Note that $2t_\triangle(G_{127})$ is also the straightforward upper bound from Theorem 2.3. This coincidence between numerical and theoretical bounds may suggest that $G_{127} \nrightarrow \triangle$. However, the question whether $G_{127} \in \mathcal{F}(2, 3, 4)$ remains open.

A related interesting question is to find a reasonable upper bound for $f(3, 3, 4)$. We tried to find another argument that would ensure the existence of relatively small K_4-free graphs. Such a construction for 2-colors was considered in an earlier version of our paper [Dudek 08]. The existence of a reasonably small graph G that yields a monochromatic triangle under every 3-coloring is an open question that we are currently trying to address.

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Andrzej Dudek, Department of Mathematics and Computer Science, Emory University, Mathematics and Science Center, 400 Dowman Drive, Atlanta, GA 30322 (adudek@mathcs.emory.edu)

Vojtěch Rödl, Department of Mathematics and Computer Science, Emory University, Mathematics and Science Center, 400 Dowman Drive, Atlanta, GA 30322 (rodl@mathcs.emory.edu)

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