A Hardy–Ramanujan Formula for Lie Algebras

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CONTENTS

We study certain combinatoric aspects of the set of all unitary representations of a finite-dimensional semisimple Lie algebra $\mathfrak{g}$. We interpret the Hardy–Ramanujan–Rademacher formula for the integer partition function as a statement about $\mathfrak{su}_2$, and explore in some detail the generalization to other Lie algebras. We conjecture that the number $\mathrm{Mod}(\mathfrak{g}, d)$ of $\mathfrak{g}$-modules in dimension d is given by $(\alpha/d)\exp(\beta d^\gamma)$ for $d \gg 1$, which (if true) has profound consequences for other combinatorial invariants of $\mathfrak{g}$-modules. In particular, the fraction $\mathcal{F}_1(\mathfrak{g}, d)$ of d-dimensional $\mathfrak{g}$-modules that have a one-dimensional submodule is determined by the generating function for $\mathrm{Mod}(\mathfrak{g}, d)$. The dependence of $\mathcal{F}_1(\mathfrak{g}, d)$ on d is complicated and beautiful, depending on the congruence class of d mod n and on generating curves that resemble a double helix within a given congruence class. We also consider the total number of repeated irreducible summands in the direct sum decomposition as a function on the space of all $\mathfrak{g}$-modules in a fixed dimension, and plot its histogram. This is related to the concept (used in quantum information theory) of *noiseless subsystem*. We identify a simple function that is conjectured to be the asymptotic form of the aforementioned histogram, and verify numerically that this is correct for $\mathfrak{su}_n$.

1. INTRODUCTION

Let d be a positive integer, let $\mathfrak{g}$ be a Lie algebra, and let $\mathcal{R}(\mathfrak{g}, d)$ denote the (finite) set of all d-dimensional modules of $\mathfrak{g}$, up to equivalence. The set $\mathcal{R}(\mathfrak{g}, d)$ has a great deal of internal structure; of particular relevance for the current study is the natural map

$$\mathfrak{p} : \mathcal{R}(\mathfrak{g}, d) \to \mathcal{P}(d), \tag{1–1}$$

where $\mathcal{P}(d)$ denotes the set of all integer partitions of d. The map (1–1) is defined by taking the dimensions of the irreducibles in the direct sum decomposition of a representation as elements of a partition. Thus if $R \in \mathcal{R}(\mathfrak{g}, d)$ is a representation, and $R \cong \bigoplus_{i=1}^n R_i$, where R_i is irreducible, then $\mathfrak{p}(R) = (p_1, \ldots, p_n)$, where $p_i = \dim R_i$. This notation is sometimes undesirable because it allows for repetitions, such as $p_1 = p_2$.

2000 AMS Subject Classification: Primary 17B15; Secondary 05A99

Keywords: Enumerative combinatorics, representation theory

1058-6458/2007 $ 0.50 per page

A *multiset* is a set for which repeated elements are counted. To formally specify a multiset, one must give a pair (A, m), where A is a set and $m : A \to \mathbb{N}$. For each $a \in A$ the number $m(a)$ is called the *multiplicity* of a. We will also make use of the *total multiplicity*,

$$M_T(A) = \sum_{a \in A} m(a).$$

An integer partition is naturally a multiset, as is the set of dimensions of irreducible components of a representation. The multiplicity of each summand in a direct sum is important, but not the order in which the summands are specified. We will use square-bracket notation for multisets of integers; the integer partition $5 = 3 + 1 + 1$ therefore consists of the multiset $A = [1, 3]$ with $m(1) = 2$, $m(3) = 1$. If $A = [p_1, \ldots, p_n]$, we will sometimes use the notation $n_i = m(p_i)$.

Representation theory of $\mathfrak{su}_2$ reduces to the theory of integer partitions. For each integer $d \geq 1$, there is precisely one irreducible representation of $\mathfrak{su}_2$ of dimension d, where the half-integer $\frac{1}{2}(d-1)$ is called *spin* in physics. For Lie algebras other than $\mathfrak{su}_2$, there may exist more than one irreducible representation in a given dimension, or none at all.

Definition 1.1. Let $\xi_{\mathfrak{g}}(d)$ denote the number of irreducible $\mathfrak{g}$-modules in dimension d. Let

$$\mathrm{Mod}(\mathfrak{g}, d) := |\mathcal{R}(\mathfrak{g}, d)| \tag{1–2}$$

denote the total number of $\mathfrak{g}$-modules in dimension d.

In the appendix (Section 5), we give an explicit formula for $\xi_{\mathfrak{su}_3}(d)$ that is particularly amenable to computation, and an algorithm to calculate $\xi_{\mathfrak{g}}(d)$ for other $\mathfrak{g}$. Note that $\xi_{\mathfrak{su}_2}(d) = 1 \ \forall d$, while this will not hold for any other Lie algebra. This implies that the map (1–1) is bijective, a property that also is unique to $\mathfrak{su}_2$. For $k \geq 1$, let $\mathcal{P}(d)_k$ denote the set of partitions of d that contain k at least once. There is a clear bijection $\mathcal{P}(d)_k \leftrightarrow \mathcal{P}(d-k)$. Hence if k/d is small, we obtain by the Hardy–Ramanujan formula

$$\frac{\mathcal{P}(d)_k}{\mathcal{P}(d)} \sim \frac{d}{d-k}\, e^{\pi\sqrt{2/3}\left((d-k)^{1/2} - d^{1/2}\right)}. \tag{1–3}$$

Equation (1–3) is the asymptotic fraction of $\mathfrak{su}_2$ representations that contain a k-dimensional irreducible subrepresentation. For all k, d in the allowed range, (1–3) is a concave function increasing monotonically to 1 as d increases; see Figure 1.

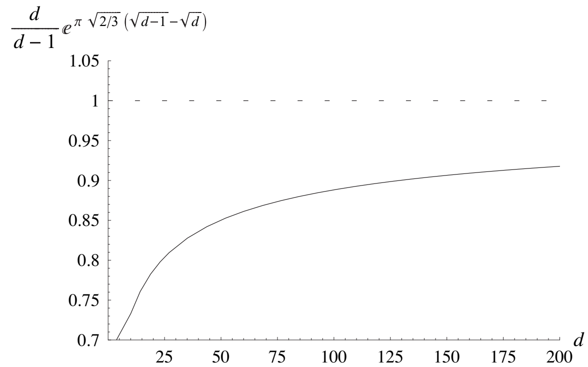

FIGURE 1. The asymptotic fraction of d-dimensional $\mathfrak{su}_2$-modules that contain a 1-dimensional submodule, as a function of d.

After developing the necessary tools, we will consider which aspects of this behavior generalize to $\mathfrak{su}_n$. For $n > 2$, the fraction of d-dimensional $\mathfrak{su}_n$-modules that contain a one-dimensional submodule depends strongly on the congruence class of d mod n. For a fixed congruence class, the points align themselves along a smooth curve as soon as d is moderately large, and these curves generically have multiple maxima and minima. There is just one qualitative feature of Figure 1 that seems to generalize: the fact that the curve approaches 1 as $d \to \infty$. We will return to this point in Section 3.1.

2. THE NUMBER OF $\mathfrak{g}$-MODULES IN A FIXED DIMENSION

2.1 The Exact Formula

In this section, we will compute (1–2) by several different methods.

One formula for (1–2) is in terms of partitions. Given an integer partition p, we say that a representation R *has shape* p if $\mathfrak{p}(R) = p$. A naive guess for the number of representations with shape $\mathfrak{p}$ might be $\prod_{i=1}^{n} \xi_{\mathfrak{g}}(p_i)$. This guess is correct if and only if $\mathfrak{p}$ does not contain repetitions (i.e., every element has multiplicity one). For example, if $m(p_1) > 1$ and $\xi_{\mathfrak{g}}(p_1) > 1$, then the naive guess is high. These cases may be counted using a standard combinatorial function, which we now explain.

Writing p as a multiset, $p = [p_1, \ldots, p_n]$ with multiplicities $n_i = m(p_i)$, we consider the $k_i = \xi_{\mathfrak{g}}(p_i)$ different p_i-dimensional irreducible representations as "letters" in an alphabet. The set of distinct n_i-fold direct sums of these irreducible representations is in one-to-one correspondence with the set of words of length n_i in an alphabet with k_i letters, and the size of this set is

$$S(n_i, k_i) := \binom{n_i + k_i - 1}{n_i}.$$

Therefore, one anticipates an exact formula for $\mathrm{Mod}(\mathfrak{g}, d)$ that sums $\prod_i S(n_i, \xi_\mathfrak{g}(p_i))$ over partitions, and the precise version is (2–2). Unfortunately, (2–2) is not numerically efficient; a more efficient computation will be obtained using generating functions in the following section.

Given a set H of nonnegative integers, let $\mathcal{P}(H, d) \subset \mathcal{P}(d)$ denote the set of partitions of d with parts in H. Let

$$\mathfrak{D}(g) = {\xi_\mathfrak{g}}^{-1}(\mathbb{N} \setminus \{0\}), \tag{2–1}$$

so $\mathfrak{D}(\mathfrak{g})$ is the (infinite) set of possible dimensions of irreducible $\mathfrak{g}$-modules. For example,

$$\mathfrak{D}(\mathfrak{su}_3) = \{1, 3, 6, 8, 10, 15, 21, 24, 27, 28, 35, 36, \dots\},$$
$$\mathfrak{D}(\mathfrak{su}_4) = \{1, 4, 6, 10, 15, 20, 35, 36, 45, 50, 56, \dots\}.$$

Elements of $\mathfrak{D}(\mathfrak{su}_n)$ for any n can be computed using the method given in the appendix.

Theorem 2.1. *Assume that each partition p of d is expressed as a multiset $p = [p_1, \dots, p_n]$ with multiplicity vector $\vec{n}$ (so the p_i are all distinct and $d = \sum_{i=1}^n n_i p_i$). Then*

$$\mathrm{Mod}(\mathfrak{g}, d) = \sum_{p \in \mathcal{P}(\mathfrak{D}(\mathfrak{g}), d)} \prod_{i=1}^{|p|} \binom{n_i + \xi_\mathfrak{g}(p_i) - 1}{n_i}. \tag{2–2}$$

Note that $|p|$ denotes the length of the multiset, i.e., the number of unique elements.

When $\xi_{\mathfrak{su}_3} \neq 0$, its most frequently assumed value is $\xi_{\mathfrak{su}_3} = 2$, in which case the binomial coefficient in (2–2) simplifies to $\binom{n_i+1}{n_i} = n_i + 1$. In that case, the partitions that contribute the most are those that maximize the product $\prod_i (n_i + 1)$, which is the same as maximizing $\prod n_i$. Therefore, the largest terms in the sum (2–2) are those that do not contain singlets;[1] however, the terms that do contain singlets are more numerous. The competition between these two types of terms determines the fraction of representations that contain a singlet, which we will analyze. Unfortunately, the computing time necessary to evaluate the right-hand side of (2–2) has superpolynomial growth as a function of d. In the next section we discuss a polynomial-time algorithm.

[1] A *singlet* is a one-dimensional subrepresentation.

2.2 Generating Function for Mod $\mathfrak{g}d$

Although (2–2) gives one way to compute $\mathrm{Mod}(\mathfrak{g}, d)$, this is by far not the most efficient way. In fact, it is not necessary to list the partitions contributing to the sum in (2–2). The following result, which was contributed by the journal referee, improves upon the algorithm originally suggested by the author.

Theorem 2.2. *$\mathrm{Mod}(\mathfrak{g}, d)$ is the coefficient of q^d in the power series expansion of*

$$\mathcal{M}_\mathfrak{g}(q) := \prod_{k \geq 1} \frac{1}{(1 - q^k)^{\xi_\mathfrak{g}(k)}}. \tag{2–3}$$

The product over $k \geq 1$ could be replaced by a product over $k \in \mathfrak{D}(\mathfrak{g})$, since $\xi_\mathfrak{g}(k) = 0$ for all $k \notin \mathfrak{D}(\mathfrak{g})$. The computation of the power series itself can be done in time

$$\sum_{k=0}^{d} (d - k)\xi_\mathfrak{g}(k) \leq d \sum_{k=0}^{d} \xi_\mathfrak{g}(k).$$

The reciprocal can be done in time $O(d \log d)$.

2.3 Asymptotics for Large d

Theorem 2.2 gives a means of evaluating $\mathrm{Mod}(\mathfrak{g}, d)$ *exactly* that scales polynomially with d, but it unfortunately does not give a closed-form expression for $\mathrm{Mod}(\mathfrak{g}, d)$. Therefore, it is also interesting to determine the asymptotic behavior of the coefficients of (2–3) at large d.

The Hardy–Ramanujan formula is

$$\mathrm{Mod}(\mathfrak{su}_2, d) = |\mathcal{P}(d)| \underset{d \gg 1}{\sim} \frac{1}{4d\sqrt{3}} \exp\left(\pi\sqrt{2d/3}\right). \tag{2–4}$$

For a proof, see [Hardy and Wright 79, Andrews 98]. The Hardy–Ramanujan formula is the simplest case of Meinardus's theorem [Meinardus 54], a more general result that gives asymptotic behavior of the coefficients of a large class of generating functions. Specifically, if

$$\sum_{n=0}^{\infty} r(n) q^n = \prod_{m \geq 1} (1 - q^m)^{-a_m}, \quad a_m \geq 0,$$

then, under certain assumptions on the Dirichlet series $D(s)$ associated to the sequence $\{a_m\}$, Meinardus's theorem gives an asymptotic formula (2–6) for $r(n)$, valid for large n.

The following conjecture is the "Hardy–Ramanujan formula for Lie algebras," from which the title of this paper is drawn.

Conjecture 2.3. $\mathrm{Mod}(\mathfrak{g}, d)$ *is given, asymptotically for* $d \gg 1$, *by a formula similar to the Hardy–Ramanujan formula* (2–4). *Specifically,*

$$\mathrm{Mod}(\mathfrak{g}, d) \sim \frac{\alpha}{d} \exp(\beta\, d^{\gamma}) \tag{2–5}$$

for positive real n-dependent constants $\alpha, \beta > 0$ *and* $0 < \gamma < 1$.

Proof Proof (sketch): Meinardus [Meinardus 54] applied a saddle-point method to obtain the asymptotic value of the power-series coefficients $r(n)$ of the function $f(q) = \prod_{n=1}^{\infty}(1-q^n)^{-a_n}$, where the a_n are real, nonnegative numbers. His formula is

$$r(n) = \alpha n^{\kappa} \exp\left(\beta n^{\gamma}\right) \cdot [1 + O(n^{-k_1})], \tag{2–6}$$

where

$$\gamma = \frac{\sigma}{\sigma+1}, \tag{2–7}$$
$$\beta = (1+\frac{1}{\sigma})\,[A\,\Gamma(\sigma+1)\,\zeta(\sigma+1)]^{1/(\sigma+1)}\ ,$$

where σ is the convergence abscissa of

$$D(s) = \sum_{n=1}^{\infty} a_n n^{-s},$$

A is the residue of $D(s)$ at $s = 1$, and κ, C, k_1 are explicitly given in terms of σ, $D(0)$, $D'(0)$. The validity of (2–6) requires that $D(s)$ and $\sum_1^{\infty} a_n q^n$ satisfy certain function-theoretic conditions, as discussed in [Andrews 98]. We *assume* that for semisimple Lie algebras, the sequence $a_n = \xi_{\mathfrak{g}}(n)$ leads to the fulfillment of the relevant function-theoretic conditions. Then (2–6) is precisely our conjectured relation for $\mathrm{Mod}(\mathfrak{g}, n)$ with $\kappa = -1$. □

2.4 Numerical Evidence

We now give strong numerical evidence in support of Conjecture 2.3. For computational purposes, let us restrict attention to $\mathfrak{g} = \mathfrak{su}_n$, and check the conjecture for various values of n and ranges of d. Note that in every plot of $\mathrm{Mod}(\mathfrak{su}_n, d)$ as a function of d, for *small* values of d, the points will appear in "horizontal groups" of n points. For example, the number of $(7+k)$-dimensional representations of $\mathfrak{su}_7$ is the same as the number of 7-dimensional ones, for $0 < k < 7$. Each of the $(7+k)$-dimensional representations is either completely reducible into singlets, or else takes the form of the one nontrivial 7-dimensional representation, plus a direct sum of k singlets. On the other hand, this reasoning breaks down for $d/n \gg 1$, and in that region, the curve joining the points will become smooth.

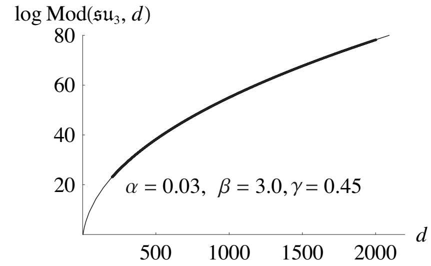

FIGURE 2. Graph of $\log \mathrm{Mod}(\mathfrak{su}_3, d)$ for $d/2 \in [10^2, 10^3]$, fitted by the curve $\log(\frac{\alpha}{d}) + \beta\, d^{\gamma}$ with parameter values as shown.

The $\mathfrak{su}_3$ values for (α, β, γ), approximately 0.03, 3.0, and 0.45 (see Figure 2), are close to the values 0.14, 2.57, and 0.5 given by Hardy–Ramanujan for $\mathfrak{su}_2$. Even though (2–7) shows that β and γ are not truly independent, we have treated them as independent parameters in our curve-fitting, since we are not able to determine A, the residue of the Dirichlet series. If Conjecture 2.3 is true, one can estimate the convergence abscissa of $D(s) = \sum \xi_{\mathfrak{g}}(n) n^{-s}$ by fitting a value of γ, and then estimate its residue A at $s = 1$ by fitting β.

Although it is not possible to show them all here, plots of $\mathrm{Mod}(\mathfrak{su}_n, d)$ as a function of d were found to be extremely similar to Figures 2 and 3 (verifying Conjecture 2.3) for all n in the range $3 \le n \le 50$. The main difference between the plots for various values of n comes through the dependence of (α, β, γ) on n.

3. STATISTICS OF THE SET OF ALL

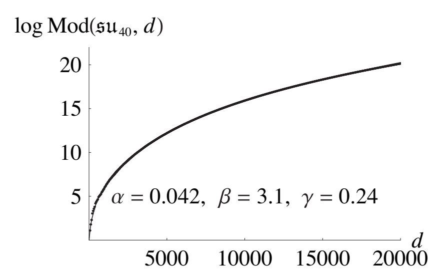

FIGURE 3. Graph of $\log[\mathrm{Mod}(\mathfrak{su}_{40}, d)]$ for $d \in [0, 2 \times 10^4]$, fitted by the curve $\log(\frac{\alpha}{d}) + \beta\, d^{\gamma}$ with parameter values as shown. The agreement between the points and the curve is sufficiently close that it is not possible to distinguish them visually.

REPRESENTATIONS

3.1 Statistics of Submodules

A *singlet* in a $\mathfrak{g}$-module is a one-dimensional (i.e., trivial) submodule. Singlets in a representation correspond to decoherence-free subspaces in quantum computation [Lidar et al. 98].

Consider $\mathfrak{g} = \mathfrak{su}_3$ in dimension 6. There are 11 partitions of 6, only 4 of which correspond to direct sum decompositions of representations of $\mathfrak{su}_3$:

$p = \{6\}$	2 representations,
$p = \{3, 3\}$	3 representations,
$p = \{3, 1, 1, 1\}$	2 representations,
$p = \{1, \dots, 1\}$	1 representation.

Therefore in $d = 6$, three of eight total representations contain a singlet, or 37.5%. By contrast, for $d = 5$, because there are no representations of $\mathfrak{su}_3$ in dimensions 2, 4, and 5, only two partitions contribute. These are $5 = 3+1+1$ and $5 = 1+\cdots+1$, giving two representations and one representation respectively, and 100% of the representations contain a singlet.

In Figure 4, we plot the fraction of $\mathfrak{su}_3$ representations that contain a singlet, as a function of d. For $d > 25$, one can clearly distinguish three series in Figure 4 that all seem to converge to 1. These three series correspond respectively to the cases $d \equiv 0, 1, 2 \pmod 3$. Within each series, the points are very regular and seem to line themselves up along a smooth curve, which at first seems mysterious.

Definition 3.1. Let $\mathrm{Mod}_k(\mathfrak{g}, d)$ denote the number of $\mathfrak{g}$-modules in dimension d that contain at least one k-dimensional submodule, and define $\mathcal{F}_k(\mathfrak{g}, d) := \mathrm{Mod}_k(\mathfrak{g}, d)/\mathrm{Mod}(\mathfrak{g}, d)$.

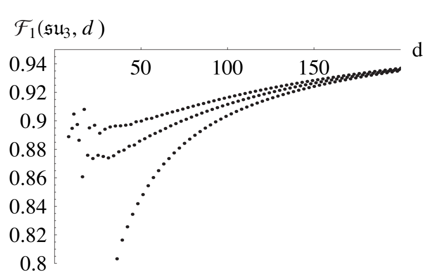

FIGURE 4. Fractions $\mathcal{F}_1(\mathfrak{su}_3, d)$ of $\mathfrak{su}_3$ representations that contain a singlet versus d. There are three distinct curves corresponding to the three congruence classes of d mod 3. Note a qualitative similarity to Figure 1.

Theorem 3.2. *If* $\xi_{\mathfrak{g}}(k) = 1$*, then*

$$\mathrm{Mod}_k(\mathfrak{g}, d) = \mathrm{Mod}(\mathfrak{g}, d-k).$$

In particular,

$$\mathrm{Mod}_1(\mathfrak{g}, d) = \mathrm{Mod}(\mathfrak{g}, d-1)\,. \tag{3–1}$$

Further, the generating function for $\mathrm{Mod}_1(\mathfrak{g}, d)$ *is*

$$\sum_{d=1}^{\infty} \mathrm{Mod}_1(\mathfrak{g}, d) q^d = q\mathcal{M}_{\mathfrak{g}}(q) = \frac{q}{\prod_{k\geq 1}(1-q^k)^{\xi_{\mathfrak{g}}(k)}}\,. \tag{3–2}$$

Proof: Let $V_d = A_{d-k} \oplus B_k$, where dimensions are indicated by subscripts. Define $\phi(V_d) = A_{d-k}$, which is well defined because even if V_d has several k-dimensional submodules, $\xi_{\mathfrak{g}}(k) = 1$ implies that they are all unitarily equivalent. Note that $\phi : \mathcal{R}_k(\mathfrak{g}, d) \to \mathcal{R}(\mathfrak{g}, d-k)$ is one-to-one and onto, so it follows that $\mathrm{Mod}_k(\mathfrak{g}, d) = \mathrm{Mod}(\mathfrak{g}, d-k)$. Equation (3–1) is the special case $k = 1$ of this, and (3–1) implies (3–2). □

Remark 3.3. $\mathrm{Mod}_k(\mathfrak{g}, d) < \xi_{\mathfrak{g}}(k)\mathrm{Mod}(\mathfrak{g}, d-k)$ if $\xi_{\mathfrak{g}}(k) > 1$, so the assumption $\xi_{\mathfrak{g}}(k) = 1$ cannot be relaxed.

Remark 3.4. There is also a simple direct proof of (3–2): Let $t_2(q) = \prod_{k\geq 2}(1-q^k)^{-\xi_{\mathfrak{g}}(k)}$, which is the generating function for the number of d-dimensional modules whose irreducible submodules have size *at least* 2. The difference $\mathcal{M}_{\mathfrak{g}}(q) - t_2(q)$ gives the number that contain a singlet. However, $t_2(q) = (1-q)\mathcal{M}_{\mathfrak{g}}(q)$, which implies (3–2).

Theorem 3.2 has several applications: We can compute $\mathcal{F}_1(\mathfrak{g}, d)$ from $\mathcal{M}_{\mathfrak{g}}(q)$ by power series expansion. Moreover, if the coefficients of $\mathcal{M}_{\mathfrak{g}}(q)$ increase at an increasing rate, then $\lim_{d\to\infty} \mathcal{F}_1(\mathfrak{g}, d) = 1$.

Corollary 3.5. *If Conjecture 2.3 is true, then* $\lim_{d\to\infty} \mathcal{F}_1(\mathfrak{g}, d) = 1$.

Proof: Assume Conjecture 2.3. Then for $d \gg 1$, we have

$$\log \mathcal{F}_1(\mathfrak{g}, d) \sim \log\left(\frac{d}{d-1}\right) + \beta[(d-1)^\gamma - d^\gamma]. \tag{3–3}$$

Clearly $\log(\frac{d}{d-1}) \to 0$. One may see that

$$(d-1)^\gamma - d^\gamma \approx -\gamma d^{\gamma-1} + O(d^{\gamma-2}) \to 0$$

for large d, since $0 < \gamma < 1$. □

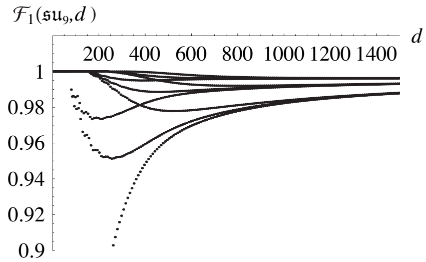

FIGURE 5. Plot of $\mathcal{F}_1(\mathfrak{su}_9, d)$. For $d > 1000$, the points arrange themselves into three groups. The lower group consists of points with $d \equiv 1 \pmod 3$, while the middle group has $d \equiv 2 \pmod 3$, and the upper group has $d \equiv 0 \pmod 3$.

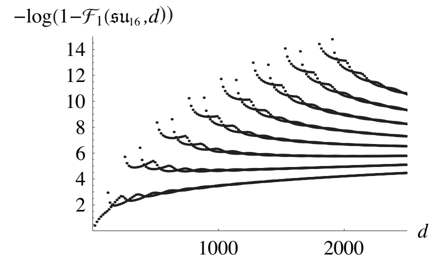

FIGURE 7. The plot shows $-\log(1 - \mathcal{F}_1(\mathfrak{g}, d))$ as a function of d for $\mathfrak{su}_{16}$. The coordinate transformation $y \to -\log(1 - y)$ allows us to zoom in on the region very near 1 while still showing the points below this region. Points at 8 on the vertical axis are actually at $1 - e^{-8} \approx 0.9997$. Every point is part of a helical pair.

In fact, the proof shows more: if a weaker form of Conjecture 2.3 holds with *any* value of the exponent κ from (2–6), then $\lim_{d\to\infty} \mathcal{F}_1(\mathfrak{g}, d) = 1$.

Let us now discuss the special case $\mathfrak{g} = \mathfrak{su}_n$, which has many interesting features. We have strong evidence that $\mathcal{F}_1(\mathfrak{g}, d) \to 1$ as d increases, but interestingly, the *rate of approach* depends strongly on the congruence class of d mod n. For each congruence class $k \in \mathbb{Z}/n\mathbb{Z}$, it is visually clear that the points

$$\Gamma_k^{(n)} := \{(d, \mathcal{F}_1(\mathfrak{su}_n, d)) : d \equiv k \pmod n\}$$

lie along a smooth curve $\gamma_k^{(n)}(d)$ that approaches 1 as $d \to \infty$; see Figure 5. This approach is not monotonic; in fact, numerics suggest that $\gamma_k^{(n)}$ undergoes some form of damped oscillation, and moreover, for sufficiently large even n, the γ_k occur in *helical pairs*, where the name "helical" is inspired by the geometry of Figure 6. The helical pairs occur *only* for $n \in 2\mathbb{Z}$; if n is odd, the graph of $\mathcal{F}_1(\mathfrak{su}_n, d)$ will contain solitary oscillating curves, as in Figure 5.

We have conjectured that $\mathcal{F}_1(\mathfrak{g}, d) \to 1$ as d increases, and empirically the convergence is quite rapid in some cases. This has the unfortunate consequence that most of the structure of Figure 6 is hidden in the region very close to 1. In order to see all points, some coordinate transformation is needed. It is useful to apply the function $-\log(1 - x)$ to the data, thus plotting $-\log(1 - \mathcal{F}_1(\mathfrak{g}, d))$. For example, when this function equals 10, the data point is at $1 - e^{-10}$, so this coordinate transformation enables us to see the intricate structure at very small separation from 1. The resulting plot confirms that for the examples studied where n is even, all points lie in helical pairs. If n is odd, as shown in Figure 8, the helical pairs do not form.

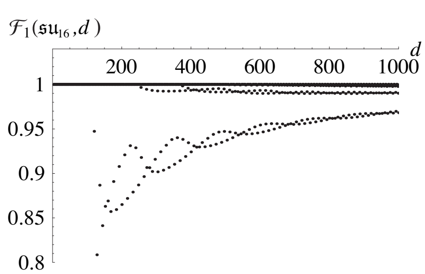

FIGURE 6. Plot of $\mathcal{F}_1(\mathfrak{su}_{16}, d)$. In the center of the plot, between 0.85 and 0.95, one can observe the helical pair of curves $\gamma_{15}^{(16)}$ and $\gamma_7^{(16)}$.

3.2 Noiseless Subsystems

Let $R : \mathfrak{g} \to \mathrm{End}(\mathcal{H})$ be a unitary representation of a semisimple Lie algebra $\mathfrak{g}$ on a Hilbert space $\mathcal{H}$. In this situation, the representation is completely reducible; let us write

$$\mathcal{H} = \Big(\bigoplus_{i=1}^{n_1} V_1^{(i)}\Big) \oplus \cdots \oplus \Big(\bigoplus_{j=1}^{n_r} V_r^{(j)}\Big), \qquad (3\text{–}4)$$

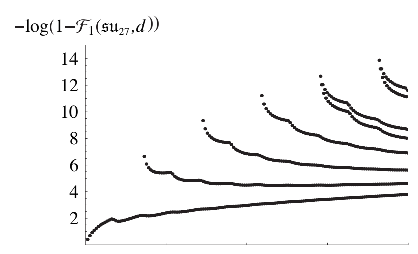

FIGURE 8. Plot of $-\log(1 - \mathcal{F}_1(\mathfrak{su}_{27}, d))$. The lowest curve is $d \equiv 26 \pmod{27}$, the next is $d \equiv 25$, etc. However, the congruence class does not always change by one for neighboring curves!

where each $V_i^{(j)}$ is irreducible, and

$$V_i^{(j)} \cong V_k^{(\ell)} \;\Leftrightarrow\; i = k,$$

where $\cong$ denotes isomorphism as $\mathfrak{g}$-modules. Possibly conjugating R with a unitary matrix and fixing convenient bases, we may assume that for each $x \in \mathfrak{g}$, and for each j, $R(x)$ is represented by the same matrix on each of the n_j isomorphic summands $V_j^{(1)}, \ldots, V_j^{(n_j)}$. If this conjugation has been performed, we say that R is *repetitive*.

Definition 3.6. A unitary representation R is said to admit a *noiseless subsystem of size* N if R is repetitive and if $N = M_T(\mathfrak{p}(R))$. In the notation of (3–4), this is the statement that $N = \sum_{j=1}^{r} n_j$, where $\vec{n}$ is the multiplicity vector of the multiset $\mathfrak{p}(R)$, and $r = |\mathfrak{p}(R)|$.

The terminology of Definition 3.6 is borrowed from quantum information theory; in that application, N is the dimension of a vector space of quantum states that will not decohere after interacting with a second system when the interaction Hamiltonian is a special type of Hamiltonian determined by R (see [Ritter 05b] for details on the application of representation theory to quantum information theory). The idea of noiseless subsystems, introduced to quantum information theory by Knill, Laflamme, and Viola [Knill et al. 00], has had a profound impact. Experimental realizations were given in [Viola et al. 01, Fortunato et al. 03, Lidar and Whaley 03], while further theoretical investigations include [Zanardi 01, Rasetti and Zanardi 00, Ritter 05a].

The representation R lifts to a unique associative algebra homomorphism $\tilde{R}$ of the universal enveloping algebra $\mathcal{U}_\mathfrak{g}$ by the universal property

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\;i\;} & \mathcal{U}_\mathfrak{g} \\ & {\scriptstyle R}\searrow & \downarrow{\scriptstyle \tilde{R}} \\ & & \mathrm{End}(\mathcal{H}) \end{array} \qquad (3\text{–}5)$$

The action of $\tilde{R}$ is simply to convert the tensor product to matrix multiplication, i.e., $\tilde{R}(x \otimes y) = R(x) \cdot R(y)$, etc. The property discussed in Definition 3.6 is equivalent to the statement that

$$\mathcal{A} := \tilde{R}(\mathcal{U}_\mathfrak{g}) \cong \bigoplus_{i=1}^{r} \mathrm{End}(V_i) \otimes I_{n_i},$$

where I_{n_i} denotes the $n_i \times n_i$ identity matrix.

Theorem 3.7. *The number of $\mathfrak{g}$-modules in dimension d that admit an N-dimensional noiseless subsystem is the coefficient of $t^N q^d$ in the power series*

$$\mathcal{N}_{\mathfrak{g},\,d}(t, q) := \prod_{k \geq 1} (1 - tq^k)^{-\xi_\mathfrak{g}(k)} \in \mathbb{Z}[t][[q]]\,.$$

3.3 The Distribution of Total Multiplicity

Given any bounded integer-valued invariant $\mathcal{I}(R)$ defined for representations R with $\dim(R) = d$, one can plot a histogram for $\mathcal{I}(R)$, i.e., the statistical distribution of the likelihood that a (uniformly distributed) random d-dimensional module R has $\mathcal{I}(R)$ equal to each possible value in its range.

Aside from its physical application to the theory of quantum decoherence, the total multiplicity function provides a mathematically interesting integer-valued invariant to probe the internal structure of the set of all d-dimensional representations of a Lie algebra.

Let $f_{n,d}(N)$ denote the fraction of d-dimensional $\mathfrak{su}_n$-modules that admit an N-dimensional noiseless subsystem. By theorem 3.7, one has

$$f_{n,d}(N) = \frac{\mathrm{Coeff}(\mathcal{N}_{\mathfrak{g},\,d}(q),\, t^N q^d)}{\mathrm{Mod}(\mathfrak{su}_n, d)}. \qquad (3\text{–}6)$$

Formula (3–6) allows one to plot the resulting histograms. The results are illuminating.

First note from the definition of $f_{d,n}(N)$ that

$$\sum_{N=1}^{d} f_{n,d}(N) = 1,$$

since every representation is counted once. Therefore, we may view $f_{n,d}$ as the probability distribution function for a random variable, and if a smooth curve is drawn interpolating the points $(N, f_{n,d}(N))$, then any such curve must subtend unit area.

Figure 9 shows the exact values of $f_{3,100}(N)$ together with a fit to the *inverted beta* statistical distribution, defined by

$$f^{(\alpha,\beta)}(t) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, t^{\alpha-1}(t+1)^{-\alpha-\beta}, \qquad (3\text{–}7)$$

where $\alpha, \beta > 0$. Note that $\int_0^\infty f^{(\alpha,\beta)}(x)dx = 1$ and $f^{(\alpha,\beta)}(x)$ has a maximum at $(\alpha-1)/(\beta+1)$.

Conjecture 3.8. *For all $n \geq 2$, $\exists D$ such that $f_{n,d}$ is well approximated by the inverted beta function* (3–7) *for all $d > D$. The accuracy of this approximation increases as d is increased with n fixed.*

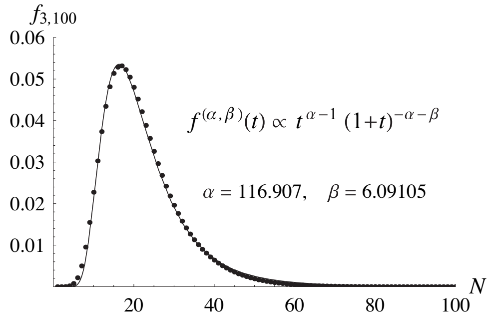

FIGURE 9. Points represent exact computations of $f_{3,100}(N)$, the fraction of $\mathfrak{su}_3$-representations in dimension $d = 100$ that admit an N-dimensional noiseless subsystem. The points are fitted to an inverted beta distribution (3–7) with $\alpha \sim 116.907$ and $\beta \sim 6.091$.

Since the distribution $f_{n,d}$ is not mathematically known to take the form (3–7), it is of interest to determine how close the agreement is. It is important that the points in Figure 9 represent computations of exact values. One possible measure of accuracy is the average deviation of the model from the "data," divided by the total integral of the data (which is normalized to unity in this case):

$$\Delta_f := \frac{1}{d}\sum_{i=1}^{d} |y_i - f(x_i)|.$$

For the example of Figure 9, $\Delta_f = 5.2 \times 10^{-4}$, providing numerical evidence in favor of Conjecture 3.8.

It is of obvious interest to check that $f_{n,d}(x)$ has the functional form (3–7) for $n > 3$. We have verified this up to at least $n = 10$, though one may have to increase d in order to force the points $(x, f_{n,d}(x))$ to approximate a smooth curve. The relevant points and their approximating curves are shown in Figures 9 and 10.

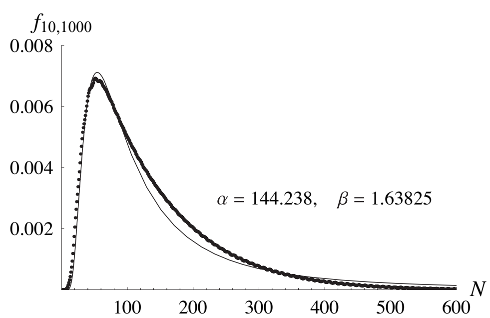

FIGURE 10. Points represent exact computations of $f_{10,1000}(N)$, the fraction of $\mathfrak{su}_{10}$-representations in dimension $d = 1000$ that admit an N-dimensional noiseless subsystem. The points are fitted to an inverted beta distribution (3–7).

4. CONCLUSIONS AND OUTLOOK

The classification of all real and complex simple Lie algebras was complete as of Cartan's famous 1914 paper [Cartan 14]; it is refreshing to find a situation in which these well-understood algebras still hold some mysteries.

We considered the following questions for unitary representations of a semisimple Lie algebra $\mathfrak{g}$. How many representations exist, up to equivalence, for fixed dimension d? How does the number of representations grow with d? Suppose that we reduce each representation into irreducibles. What fraction of d-dimensional $\mathfrak{g}$-modules contain a one-dimensional submodule? What is the statistical distribution function that describes the frequencies of integer-valued invariants of the reduction (such as the total multiplicity)?

One appreciates how difficult these questions are by noting that in the case of the simplest nontrivial Lie algebra $\mathfrak{g} = \mathfrak{su}_2$, the questions posed above reduce to highly nontrivial questions about integer partitions. In particular, in the $n = 2$ case, the "how many?" question is solved by the Hardy–Ramanujan–Rademacher explicit formula (see [Andrews 98] for an exposition). The questions for other Lie algebras are certainly of equal or greater difficulty, and they have not been solved explicitly. Even the $d \gg 1$ asymptotics (the Hardy–Ramanujan approximation to the Rademacher formula) are not known for $\mathfrak{g} \neq \mathfrak{su}_2$. For this problem, we offer Conjecture 2.3 and much supporting numerical evidence.

There is a generating function $\mathcal{M}_{\mathfrak{g}}(q)$ given by (2–3) for the exact number of $\mathfrak{g}$-modules in dimension d. However, this generating function is given in terms of $\xi_{\mathfrak{g}}(d)$, which for most algebras is itself quite a complicated function. Nonetheless, $\xi_{\mathfrak{g}}(d)$ can be calculated exactly by computers for d up to a few million using Algorithm 5.3, which counts Young tableaux. Using $\mathcal{M}_{\mathfrak{g}}(q)$, we were able to plot some of the aforementioned statistics. The statistic $\mathcal{F}_1(\mathfrak{su}_n, d)$ (the fraction of representations containing a singlet) has a particularly interesting structure, with the behavior depending strongly on the congruence class of d mod n. Each congruence class gives a smooth curve, and these curves form helical pairs when n is even.

In our view, the main open problem is to prove Conjecture 2.3, the "Hardy–Ramanujan formula for Lie algebras." Conjecture 2.3 was verified by the author for $\mathfrak{g} = \mathfrak{su}_n$ for all $n \leq 50$ and $d \leq 2 \times 10^4$. A proof of Conjecture 2.3 would follow by standard techniques if one had good analytic control over $\xi_{\mathfrak{g}}(d)$, since the latter is the only place where the particular Lie algebra enters the generating function. Ultimately, however, Conjecture 2.3 is only an asymptotic formula, and will never explain the helical pairs seen in Figures 6 and 7, since in the region where the generalized Hardy–Ramanujan formula applies, the helical structure is already suppressed.

The second main open problem suggested here is the validity of Conjecture 3.8, which has been verified for $n \leq 10$ in Figures 9 and 10. If Conjecture 3.8 is false, it would be intriguing to know why the agreement observed is so good. It would also be interesting to understand the location of the maximum of $f_{n,d}$.

5. APPENDIX: THE NUMBER OF IRREDUCIBLE $\mathfrak{su}_n$-MODULES FOR GENERAL n

We describe how to efficiently compute the function $\xi_{\mathfrak{g}}(d)$. For $\mathfrak{g} = \mathfrak{su}_3$, a closed-form expression is possible, and so we discuss that case first, before giving a method for general $\mathfrak{su}_n$.

5.1 The $\mathfrak{su}_3$ Case

Young diagrams for $\mathfrak{su}_3$ are characterized by row lengths n_1, n_2 with $n_1 \geq 0$ and $0 \leq n_2 \leq n_1$. The fundamental representation is $\lambda = (1, 0)$, while the adjoint is $\lambda = (2, 1)$. The representations (n_1, n_2) and $(n_1, n_1 - n_2)$ are conjugate to each other and have the same dimension. Representations of the form $(2n, n)$ are self-conjugate.

The dimension given by the Weyl character formula [Fulton and Harris 91] is then

$$\dim(n_1, n_2) = \frac{1}{2}(n_1 + 2)(n_2 + 1)(n_1 - n_2 + 1) = \frac{xy(x-y)}{2},$$

where $x = n_1 + 2$, and $y = n_2 + 1$.

We now compute the total number N_d of irreducible representations of $\mathfrak{su}_3$ with dimension less than d, by finding, for each fixed n_1, the number of n_2 that satisfy $\dim(n_1, n_2) \leq d$, and then summing over n_1. Expressing the sum in terms of $x = n_1 + 2$ gives simpler notation. We give the result as Theorem 5.1, omitting the lengthy but straightforward proof. For a real number $\gamma \in \mathbb{R}$, we let $\lfloor \gamma \rfloor$ denote the greatest integer less than or equal to γ. Similarly, $\lceil \gamma \rceil$ denotes the least integer greater than or equal to γ.

Theorem 5.1. *Let N_d denote the total number of irreducible representations of $\mathfrak{su}_3$ with dimension less than d. Then N_d is given* exactly *by the finite sum*

$$N_d = \frac{1}{2} l_d (l_d - 1) - \sum_{x = k_d + 1}^{l_d} \left(\lfloor y_+(x) \rfloor - \lceil y_-(x) \rceil + 1 \right), \quad (5\text{–}1)$$

where $k_d := \lfloor 2\sqrt[3]{d} \rfloor$, $l_d := \lfloor \frac{1}{2}(1 + \sqrt{1 + 8d}) \rfloor$, and

$$y_{\pm}(x) = \frac{x-2}{2} \pm \frac{1}{2}\left(x^2 - \frac{8d}{x} \right)^{1/2}.$$

Corollary 5.2. *The exact number $\xi_{\mathfrak{su}_3}(d)$ of irreducible representations in dimension d is given by*

$$\xi_{\mathfrak{su}_3}(d) = N_{d+\frac{1}{2}} - N_{d-\frac{1}{2}}. \quad (5\text{–}2)$$

For determining the number of irreducible representations of dimension d, equations (5–1) and (5–2) provide a radical computational speedup over the naive algorithm of enumerating all possible Young diagrams and computing the dimension of each. These equations may be implemented with an optimized program in languages designed for numerical computation.

5.2 The Case of General $n \geq 3$.

Let us now discuss the computation for general $n \geq 3$. Let $\lambda = (n_1, n_2, \ldots, n_\ell)$ denote a Young tableau with n_i boxes in the ith row. Let $\dim(\lambda)$ denote the dimension of the $\mathfrak{su}_n$-representation corresponding to this tableau, as computed from the Weyl character formula (see [Fulton and Harris 91]).

The number $\xi_{\mathfrak{su}_n}(d)$ of irreducible $\mathfrak{su}_n$-modules in dimension d equals the number of $(n_1, \ldots, n_\ell) \in \mathbb{N}^\ell$ that satisfy the proper inequalities for a Young tableau,

$$n_1 \geq n_2 \geq \cdots,$$

and that lie on the affine variety in $\mathbb{R}^\ell$ defined by

$$\dim(\lambda) = d.$$

While closed-form expressions for $\dim(\lambda)$ are not known, we will give one simple, if somewhat naive, computer algorithm to calculate $\xi_{\mathfrak{su}(n)}(d)$. While improvements on this algorithm are possible, in the intended application it performed well, and the speed bottleneck lies elsewhere.

Algorithm 5.3.

1. Iterate the following from $n_1 = 1$ to ∞ until a return statement is reached.

2. Let $\mathcal{S}_{n_1}$ denote the (finite) set of Young tableaux for $\mathfrak{su}_n$ with n_1 boxes in the first row. Let
$$\Xi_{n_1}(d) := \left| \{ \lambda \in \mathcal{S}_{n_1} \ : \ \dim(\lambda) = d \} \right|,$$
where $|A|$ denotes the cardinality of a set A.
3. If $\min_{\lambda \in \mathcal{S}_{n_1}} \dim(\lambda) > d$, then return the value
$$\xi_{\mathfrak{su}(n)}(d) = \sum_{i=1}^{k} \Xi_i(d).$$

An implementation of Algorithm 5.3 gave the values of $\mathfrak{D}(\mathfrak{su}_n)$ listed in Section 2.1.

ACKNOWLEDGMENTS

I am very grateful to the journal referee for very helpful comments that dramatically improved the paper. I gratefully acknowledge helpful discussions with Lisa Carbone, Noam Elkies, and Gregg Zuckerman. I am indebted to Cameron Freer, Arthur Gaer, and the Harvard Mathematics Department for an account on a cluster of fast machines, where the computations involving large values of n and d were performed.

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Received October 28, 2005; accepted in revised form November 20, 2006.