Volume Conjecture and Asymptotic Expansion of q-Series

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CONTENTS

We consider the "volume conjecture," which states that an asymptotic limit of Kashaev's invariant (or, the colored Jones type invariant) of knot $\mathcal{K}$ gives the hyperbolic volume of the complement of knot $\mathcal{K}$. In the first part, we analytically study an asymptotic behavior of the invariant for the torus knot, and propose identities concerning an asymptotic expansion of q-series which reduces to the invariant with q being the N-th root of unity. This is a generalization of an identity recently studied by Zagier. In the second part, we show that "volume conjecture" is numerically supported for hyperbolic knots and links (knots up to 6-crossing, Whitehead link, and Borromean rings).

1. INTRODUCTION

In [Kashaev 95, Kashaev 97], Kashaev defined an invariant $\langle\mathcal{K}\rangle_N$ for knot $\mathcal{K}$ using a quantum dilogarithm function at the N-th root of unity, and proposed the stimulating conjecture that for a hyperbolic knot $\mathcal{K}$ an asymptotic limit $N \to \infty$ of the invariant $\langle\mathcal{K}\rangle_N$ gives a hyperbolic volume of a knot complement,

$$\lim_{N\to\infty} \frac{2\,\pi}{N} \log\bigl|\langle\mathcal{K}\rangle_N\bigr| = v_3 \cdot \bigl\|S^3 \setminus \mathcal{K}\bigr\|, \tag{2–8}$$

where v_3 is the hyperbolic volume of the regular ideal tetrahedron, and $\|\cdot\|$ denotes the Gromov norm. It was later proved [Murakami and Murakami 01] that Kashaev's invariant coincides with a specific value of the colored Jones polynomial. In several attempts since then, a geometrical aspect to relate Kashaev's R-matrix with an ideal octahedron in the three-dimensional hyperbolic space has been clarified (see, e.g., [Thurston 99, Yokota 00, Hikami 01]). Furthermore, a relationship with the Chern–Simons invariant was pointed out [Murakami et al. 02].

In this paper, we are interested in an explicit form of Kashaev's invariant for the knot $\mathcal{K}$. In general, this invariant can be regarded as a reduction of certain q-series. In [Zagier 01], Zagier derived a *strange identity*

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for a q-series, $F^{(3,2)}(q) = \sum_{n=0}^{\infty}(q)_n$, which was originally introduced in [Stoimenow 98] as an upper bound of the number of linearly independent Vassiliev invariants. He showed that $F^{(3,2)}(\mathrm{e}^{-t})$ is related to the *half-differential* of the Dedekind η-function. From our viewpoint, $F^{(3,2)}(q)$ with $q \to \mathrm{e}^{2\pi\mathrm{i}/N}$ is nothing but Kashaev's invariant for the trefoil, $\langle 3_1\rangle_N = F^{(3,2)}(\mathrm{e}^{2\pi\mathrm{i}/N})$. This motivates us to study an asymptotic expansion of the q-series which, when q is the N-th root of unity, reduces to Kashaev's invariant for the torus knot. We introduce the q-series $F^{(2m+1,2)}(q)$ as a generalization of Zagier's q-series and prove an identity,

$$\begin{aligned} &F^{(2m+1,2)}(\mathrm{e}^{2\pi\mathrm{i}/N}) \\ &\quad\simeq \frac{2}{\sqrt{2\,m+1}}\, N^{\frac{3}{2}}\, \mathrm{e}^{\frac{\pi\mathrm{i}}{4}-\frac{\pi\mathrm{i}}{N}\,\frac{(2m-1)^2}{4(2m+1)}} \\ &\quad\times \sum_{j=0}^{m-1}(-1)^j\,(m-j)\,\sin\left(\frac{2\,j+1}{2\,m+1}\pi\right)\,\mathrm{e}^{-N\pi\mathrm{i}\frac{(2j+1)^2}{4(2m+1)}} \\ &\quad+\mathrm{e}^{-\frac{\pi\mathrm{i}}{N}\,\frac{(2m-1)^2}{4(2m+1)}}\sum_{n=0}^{\infty}\frac{T_n^{(2m+1,2)}}{n!}\,\left(\frac{\pi}{4\,(2\,m+1)\,N\,\mathrm{i}}\right)^n, \end{aligned} \tag{3–15}$$

and then propose a conjecture,

$$\begin{aligned} &F^{(2m+1,2)}(\mathrm{e}^{-t}) \\ &\quad= \mathrm{e}^{\frac{(2m-1)^2}{8(2m+1)}t}\sum_{n=0}^{\infty}\frac{T_n^{(2m+1,2)}}{n!}\left(\frac{t}{2^3\,(2\,m+1)}\right)^n. \end{aligned} \tag{3–24}$$

Here $F^{(2m+1,2)}(q)$, respectively T-number $T_n^{(2m+1,2)}$, are defined in Equations (3–22), respectively (3–10).

In Section 2, we review the volume conjecture and explain how to construct Kashaev's invariant from the enhanced Yang–Baxter operator. In Section 3, we study analytically in detail Kashaev's invariant for the torus knot. We consider an asymptotic expansion of the invariant following Kashaev-Tirkkonen [Kashaev and Tirkkonen 00], and derive an asymptotic formula for q-series with $q \to \mathrm{e}^{2\pi\mathrm{i}/N}$. In Section 4, we study numerically an asymptotic behavior of invariants for hyperbolic knots and links. We use PARI/GP [PARI 00], and show that there is a universal logarithmic correction to invariants. We then propose a conjecture as an extension of Equation (2–8),

$$\log\bigl|\langle\mathcal{K}\rangle_N\bigr| \sim v_3\cdot\bigl\|S^3\setminus\mathcal{K}\bigr\|\cdot\frac{N}{2\,\pi}+\frac{3}{2}\,\#(\mathcal{K})\cdot\log N+O(N^0), \tag{4–1}$$

where $\#(\mathcal{K})$ is the number of prime factors of a knot considered as a connected-sum of prime knots.

2. KASHAEV'S INVARIANT AND VOLUME CONJECTURE

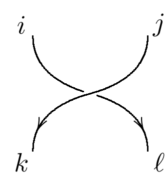
The quantum invariant of knot $\mathcal{K}$ can be constructed once we have the enhanced Yang–Baxter operator [Turaev 88], (R,μ,α,β), satisfying

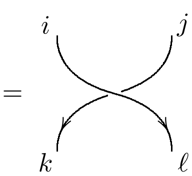
$$(R\otimes 1)(1\otimes R)(R\otimes 1) = (1\otimes R)(R\otimes 1)(1\otimes R), \tag{2–1}$$

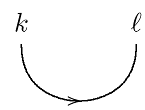
$$\bigl(\mu\otimes\mu\bigr)\,R = R\,\bigl(\mu\otimes\mu\bigr), \tag{2–2}$$

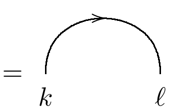
$$\mathrm{Tr}_2\bigl(R^{\pm 1}\,(1\otimes\mu)\bigr) = \alpha^{\pm 1}\,\beta. \tag{2–3}$$

The operators $R^{\pm 1}$ and $\mu^{\pm 1}$ are usually depicted as follows;

$R^{ij}_{k\ell} =$ 

$\left(R^{-1}\right)^{ij}_{k\ell} =$ 

$\mu^k_\ell =$ 

$\left(\mu^{-1}\right)^k_\ell =$ 

When the knot $\mathcal{K}$ is given as a closure of a braid ξ with n strands, the invariant $\tau_1(\mathcal{K})$ is computed as for the (1,1)-tangle of knot $\mathcal{K}$ as

$$\tau_1(K) = \alpha^{-w(\xi)}\,\beta^{-n}\,\mathrm{Tr}_{2,\ldots,n}\Bigl(b_R(\xi)\,(1\otimes\mu^{\otimes(n-1)})\Bigr). \tag{2–4}$$

Here we have associated the homomorphism $b_R(\xi)$ by replacing the braid group $\sigma_i^{\pm 1}$ in ξ with $R^{\pm 1}$, and $w(\xi)$ denotes a writhe (a sum of exponents).

Kashaev's invariant is originally defined by use of the quantum dilogarithm function with a deformation parameter being the N-th root of unity [Fadeev and Kashaev 94],

$$\omega = \exp\left(2\,\pi\,\mathrm{i}/N\right). \tag{2–5}$$

The invariant is then defined as follows: We use the q-product,

$$(\omega)_n = \prod_{i=1}^{n}(1-\omega^i),$$

$$(\omega)_n^{\,*} = \prod_{i=1}^{n}(1-\omega^{-i}).$$

Theorem 2.1. [Kashaev 95, Kashaev 97] (See also [Murakami and Murakami 01].) *Kashaev's invariant $\langle \mathcal{K} \rangle_N$ for the knot $\mathcal{K}$ is defined by Equation (2–4) with the following R and μ matrices;*

$$R^{ij}_{k\ell} = \frac{N\,\omega^{1-(k-j+1)(\ell-i)}}{(\omega)_{[\ell-k-1]}\,(\omega)^*_{[j-\ell]}\,(\omega)_{[i-j]}\,(\omega)^*_{[k-i]}} \cdot \theta \begin{bmatrix} i & j \\ k & \ell \end{bmatrix}, \tag{2–6a}$$

$$(R^{-1})^{ij}_{k\ell} = \frac{N\,\omega^{-1+(\ell-i-1)(k-j)}}{(\omega)^*_{[\ell-k-1]}\,(\omega)_{[j-\ell]}\,(\omega)^*_{[i-j]}\,(\omega)_{[k-i]}} \cdot \theta \begin{bmatrix} i & j \\ k & \ell \end{bmatrix}, \tag{2–6b}$$

$$\mu^k_\ell = -\delta_{k,\ell+1}\,\omega^{\frac{1}{2}}. \tag{2–6c}$$

Here $[x] \in \{0,1,\dots,N-1\}$ modulo N, and

$$\theta \begin{bmatrix} i & j \\ k & \ell \end{bmatrix} = 1, \text{ if and only if } \begin{cases} i \le k < \ell \le j, \\ j \le i \le k < \ell, \\ \ell \le j \le i \le k \ (\text{with } \ell < k), \\ k < \ell \le j \le i. \end{cases}$$

In [Murakami and Murakami 01], it was shown that this invariant coincides with a specific value of the colored Jones polynomial, the invariant of knot $\mathcal{K}$ colored by the irreducible $\mathrm{SU}(2)_q$-module of dimension N with a parameter $q \to \exp(2\pi\,\mathrm{i}/N)$.

Theorem 2.2. [Murakami and Murakami 01] *Kashaev's invariant $\langle \mathcal{K} \rangle_N$ coincides with the colored Jones polynomial at the N-th root of unity, whose R-matrix is given by*

$$R^{ij}_{k\ell} = \sum_{n=0}^{\min(N-1-i,j)} \delta_{\ell,i+n}\,\delta_{k,j-n}\,(-1)^{i+j+n} \times \frac{(\omega)^*_{i+n}\,(\omega)_j}{(\omega)_i{}^*\,(\omega)_{j-n}\,(\omega)_n{}^*}\,\omega^{ij+\frac{1}{2}(i+j-n)}, \tag{2–7a}$$

$$\left(R^{-1}\right)^{ij}_{k\ell} = \sum_{n=0}^{\min(N-1-j,i)} \delta_{\ell,i-n}\,\delta_{k,j+n}\,(-1)^{i+j+n} \times \frac{(\omega)_i{}^*\,(\omega)_{j+n}}{(\omega)^*_{i-n}\,(\omega)_j\,(\omega)_n}\,\omega^{-ij-\frac{1}{2}(i+j-n)}, \tag{2–7b}$$

$$\mu^k_\ell = -\delta_{k,\ell}\,\omega^{k+\frac{1}{2}}. \tag{2–7c}$$

In this article, we focus on the following stimulating conjecture.

Conjecture 2.3. [Kashaev 97, Murakami and Murakami 01] *The asymptotic behavior of Kashaev's invariant gives the hyperbolic volume of the knot complement of knot $\mathcal{K}$;*

$$\lim_{N\to\infty} \frac{2\pi}{N} \log\bigl|\langle \mathcal{K} \rangle_N\bigr| = v_3 \cdot \bigl\|S^3 \setminus \mathcal{K}\bigr\|, \tag{2–8}$$

where v_3 is the hyperbolic volume of the regular ideal tetrahedron, and $\|\cdot\|$ denotes the Gromov norm.

A mathematically rigorous proof of this conjecture has not been established yet (only thea case of the figure-eight knot was proved (see, e.g., [Murakami 00]). However, several geometrical studies have been done; the relationship between Kashaev's R-matrix and the ideal hyperbolic octahedron has been established [Thurston 99, Yokota 00, Hikami 01], and it was found that the saddle point equation of the invariant coincides with the hyperbolicity consistency condition.

As it is well known [Neumann and Zagier 85, Yoshida 85] that the hyperbolic volume is closely related to the Chern–Simons invariant, we also propose a complexification of Conjecture 2.3.

Conjecture 2.4. [Murakami et al. 02, Baseilhac and Benedetti 01]

$$\lim_{N\to\infty} \frac{2\pi}{N} \log\bigl(\langle \mathcal{K} \rangle_N\bigr) = v_3 \cdot \bigl\|S^3 \setminus \mathcal{K}\bigr\| + \mathrm{i}\ \mathrm{CS}(\mathcal{K}), \tag{2–9}$$

where CS *denotes the Chern–Simons invariant,*

$$\mathrm{CS}(\mathcal{M}) = 2\pi^2\ \mathrm{cs}(\mathcal{M}),$$

$$\mathrm{cs}_{\mathcal{M}}(A) = \frac{1}{8\pi^2} \int_{\mathcal{M}} \mathrm{Tr}\left(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A\right).$$

3. TORUS KNOTS AND q-SERIES

3.1 Invariant of the Torus Knot

We consider the (m,p)-torus knot, where we suppose that m and p are coprime integers. The knot is expressed in terms of generators of Artin's braid group as

$$\xi = \left(\sigma_1\,\sigma_2 \cdots \sigma_{m-1}\right)^p.$$

Hereafter, we denote it as $\mathrm{Trs}(m,p)$. For $(m,p) = (3,2)$ and $(5,2)$ case, they are called the trefoil knot and the Solomon's Seal knot, respectively (see Figure 1).

Using results from quantum groups, the explicit form of the colored Jones polynomial of the torus knot is obtainable.

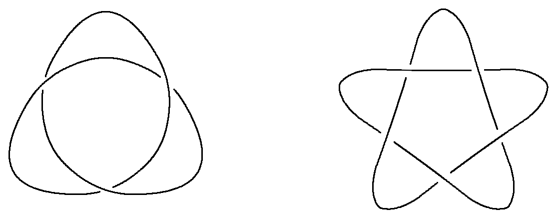


FIGURE 1. Trefoil Knot and Solomon's Seal Knot.

Proposition 3.1. [Morton 95, Rosso and Jones 93] *The colored Jones polynomial $J_{\mathcal{K}}(h;N)$ for $\mathcal{K} = \mathrm{Trs}(m,p)$ is given by*

$$
\begin{aligned}
&2\,\mathrm{sh}\left(\frac{N\,h}{2}\right)\frac{J_{\mathcal{K}}(h;N)}{J_{\mathcal{O}}(h;N)} \\
&\quad = \sum_{\epsilon=\pm 1}\sum_{r=-(N-1)/2}^{(N-1)/2} \epsilon\, \mathrm{e}^{h\,m\,p\,r^2 + h\,r(m+\epsilon\,p) + \frac{1}{2}\,\epsilon\,h}, \qquad (3\text{–}1)
\end{aligned}
$$

where a parameter q is set to be $q = \exp(h)$. Unknot is denoted by $\mathcal{O}$, and we have

$$
J_{\mathcal{O}}(h;N) = \frac{\mathrm{sh}(N\,h/2)}{\mathrm{sh}(h/2)}.
$$

By use of the relationship between the colored Jones polynomial and Kashaev's invariant (Theorem 2.2), we can give an asymptotic expansion of the invariant of the torus knot.

Proposition 3.2. [Kashaev and Tirkkonen 00] *For the torus knot $\mathcal{K} = \mathrm{Trs}(m,p)$ with m and p being coprime, Kashaev's invariant is represented by the following integral:*

$$
\begin{aligned}
&\langle \mathrm{Trs}(m,p)\rangle_N \\
&\quad = \left(\frac{m\,p\,N}{2}\right)^{3/2} \mathrm{e}^{\pi \mathrm{i}(N+\frac{1}{N}) - \frac{\pi \mathrm{i}}{2N}(\frac{m}{p}+\frac{p}{m}) - \frac{\pi \mathrm{i}}{4}} \\
&\quad \times \int_{\mathscr{C}} \mathrm{d}z\, \mathrm{e}^{mpN\pi(z+\frac{\mathrm{i}}{2}z^2)}\, z^2\, \frac{\mathrm{sh}(m\,\pi\,z)\,\mathrm{sh}(p\,\pi\,z)}{\mathrm{sh}(m\,p\,\pi\,z)}. \qquad (3\text{–}2)
\end{aligned}
$$

Proof: We follow [Kashaev and Tirkkonen 00]. As Kashaev's invariant is defined for the (1,1)-tangle of knots due to Theorem 2.2, we have for $\mathcal{K} = \mathrm{Trs}(m,p)$ that

$$
\langle \mathrm{Trs}(m,p)\rangle_N = \mathrm{e}^{\pi \mathrm{i}(N+\frac{1}{N})} \lim_{h\to 2\pi \mathrm{i}/N} \frac{J_{\mathcal{K}}(h;N)}{J_{\mathcal{O}}(h;N)}. \qquad (3\text{–}3)
$$

We rewrite the r.h.s. using the Gauss integral formula

$$
\sqrt{\pi\,h}\,\mathrm{e}^{h w^2} = \int_{\mathscr{C}} \mathrm{d}z\, \exp\left(-\frac{z^2}{h} + 2\,w\,z\right),
$$

where a path $\mathscr{C}$ is to be chosen by the convergence condition. We apply the Gaussian integral formula to Equation (3–1) and get

$$
\begin{aligned}
&2\,\mathrm{e}^{\frac{h}{4}(\frac{m}{p}+\frac{p}{m})}\,\mathrm{sh}\left(\frac{N\,h}{2}\right)\frac{J_{\mathcal{K}}(h;N)}{J_{\mathcal{O}}(h;N)} \\
&= \sum_{\epsilon=\pm 1}\sum_{r=-(N-1)/2}^{(N-1)/2} \epsilon\, \mathrm{e}^{h\,m\,p\left(r+\frac{m+\epsilon p}{2mp}\right)^2} \\
&= \sum_{\epsilon=\pm 1}\epsilon \sum_{r=-(N-1)/2}^{(N-1)/2} \frac{1}{\sqrt{\pi\,h\,m\,p}} \int_{\mathscr{C}} \mathrm{d}z\, \mathrm{e}^{-\frac{z^2}{hmp} + z(2r+\frac{1}{p}+\frac{\epsilon}{m})} \\
&= \frac{2}{\sqrt{\pi\,h\,m\,p}} \int_{\mathscr{C}} \mathrm{d}z\, \mathrm{e}^{-\frac{z^2}{hmp}+\frac{z}{p}}\, \frac{\mathrm{sh}(N\,z)\,\mathrm{sh}(\frac{z}{m})}{\mathrm{sh}(z)}.
\end{aligned}
$$

Summing the integrand with one replacing $z \to -z$, we have

$$
= \frac{2}{\sqrt{\pi\,h\,m\,p}} \int_{\mathscr{C}} \mathrm{d}z\, \mathrm{e}^{-\frac{z^2}{hmp}}\, \frac{\mathrm{sh}(N\,z)\,\mathrm{sh}(\frac{z}{m})\,\mathrm{sh}(\frac{z}{p})}{\mathrm{sh}\,z}.
$$

Decomposing $\mathrm{sh}(N\,z)$ into $(\mathrm{e}^{Nz} - \mathrm{e}^{-Nz})/2$ and using an invariance under $z \to -z$, we see that

$$
\begin{aligned}
&= \frac{2}{\sqrt{\pi\,h\,m\,p}} \int_{\mathscr{C}} \mathrm{d}z\, \mathrm{e}^{-\frac{z^2}{hmp}+Nz}\, \frac{\mathrm{sh}(\frac{z}{m})\,\mathrm{sh}(\frac{z}{p})}{\mathrm{sh}\,z} \\
&= \sqrt{\frac{m\,p}{\pi\,h}} \int_{\mathscr{C}} \mathrm{d}z\, \mathrm{e}^{mp(Nz-\frac{z^2}{h})}\, \frac{2\,\mathrm{sh}(m\,z)\,\mathrm{sh}(p\,z)}{\mathrm{sh}(m\,p\,z)}.
\end{aligned}
$$

To obtain Kashaev's invariant $\langle \mathrm{Trs}(m,p)\rangle_N$ defined in Equation (3–3), we differentiate the above integral with respect to h, and we obtain Equation (3–2). □

Proposition 3.3. *An asymptotic expansion of Kashaev's invariant for $\mathcal{K} = \mathrm{Trs}(m,p)$ is given by*

$$
\begin{aligned}
&\langle \mathrm{Trs}(m,p)\rangle_N \\
&\quad \simeq \left(\frac{m\,p\,N}{2}\right)^{3/2} \mathrm{e}^{\pi \mathrm{i} N + \frac{\pi \mathrm{i}}{N}\left(1-\frac{1}{2}(\frac{p}{m}+\frac{m}{p})\right) - \frac{\pi \mathrm{i}}{4}} \\
&\qquad \times \mathrm{Res}(m,p) \\
&\quad + (-1)^{(m+1)(p+1)}\, \mathrm{e}^{\pi \mathrm{i} N(1+\frac{1}{2}mp) + \frac{\pi \mathrm{i}}{N}\left(1-\frac{1}{2}(\frac{p}{m}+\frac{m}{p})\right)} \\
&\qquad \times \sum_{n=0}^{\infty} \frac{T_n^{(m,p)}}{n!}\left(\frac{\pi}{2\,m\,p\,N\,\mathrm{i}}\right)^n. \qquad (3\text{–}4)
\end{aligned}
$$

Here we have set

$$\begin{aligned}\mathrm{Res}(m,p) &= \frac{2\,\mathrm{i}}{(m\,p)^3}\sum_{n=1}^{mp-1}(-1)^{n+1}\,n^2\,\mathrm{sh}\left(\frac{n\,\pi}{p}\,\mathrm{i}\right)\mathrm{sh}\left(\frac{n\,\pi}{m}\,\mathrm{i}\right)\\ &\quad\times\mathrm{e}^{N\pi\mathrm{i}(n-\frac{n^2}{2mp})},\end{aligned} \tag{3–5}$$

and the T-series is given by

$$\frac{\mathrm{sh}(m\,w)\,\mathrm{sh}(p\,w)}{\mathrm{sh}(m\,p\,w)}=\sum_{n=0}^{\infty}\frac{T_n^{(m,p)}}{(2\,n+1)!}\,(-1)^n\,w^{2n+1}. \tag{3–6}$$

Proof: We use an integral representation (3–2) of the invariant. When we shift the path $\mathscr{C}$ to $\mathscr{C}+\mathrm{i}$, we get

$$\begin{aligned}\langle\mathrm{Trs}(m,p)\rangle_N &= \left(\frac{m\,p\,N}{2}\right)^{3/2}\mathrm{e}^{\pi\mathrm{i}(N+\frac{1}{N})-\frac{\pi\mathrm{i}}{2N}(\frac{m}{p}+\frac{p}{m})-\frac{\pi\mathrm{i}}{4}}\\ &\quad\times\Bigg(\mathrm{Res}(m,p)\\ &\quad+\int_{\mathscr{C}+\mathrm{i}}\mathrm{d}z\,\mathrm{e}^{mpN\pi(z+\frac{\mathrm{i}}{2}z^2)}\,z^2\,\frac{\mathrm{sh}(m\,\pi\,z)\,\mathrm{sh}(p\,\pi\,z)}{\mathrm{sh}(m\,p\,\pi\,z)}\Bigg).\end{aligned}$$

Here, the first term, $\mathrm{Res}(m,p)$, comes from residues of the integral at $z=\frac{n}{m\,p}\,\pi\,\mathrm{i}$ for $n=1,2,\ldots,mp-1$, and it is computed as Equation (3–5). In the second term, we introduce $z=w+\mathrm{i}$, and using a fact that the even functions only survive in the integrand, we get

$$\begin{aligned}&\langle\mathrm{Trs}(m,p)\rangle_N\\ &\quad=\left(\frac{m\,p\,N}{2}\right)^{3/2}\mathrm{e}^{\pi\mathrm{i}(N+\frac{1}{N})-\frac{\pi\mathrm{i}}{2N}(\frac{m}{p}+\frac{p}{m})-\frac{\pi\mathrm{i}}{4}}\\ &\quad\times\Big(\mathrm{Res}(m,p)+2\,\mathrm{i}\,(-1)^{mp+m+p}\mathrm{e}^{\frac{1}{2}mpN\pi\mathrm{i}}\\ &\quad\times\int_{\mathscr{C}}\mathrm{d}w\,\mathrm{e}^{\frac{1}{2}\mathrm{i}mpN\pi w^2}\,w\,\frac{\mathrm{sh}(m\,\pi\,w)\,\mathrm{sh}(p\,\pi\,w)}{\mathrm{sh}(m\,p\,\pi\,w)}\Big).\end{aligned} \tag{3–7}$$

Substituting the expansion (3–6) into an integrand, we recover Equation (3–4). □

Remark 3.4. The T-numbers can be written in terms of the L-series. The left-hand side of Equation (3–6) is expanded as

$$\frac{\mathrm{sh}(m\,w)\,\mathrm{sh}(p\,w)}{\mathrm{sh}(m\,p\,w)}=\frac{1}{2}\sum_{n=0}^{\infty}\chi_{2mp}(n)\,\mathrm{e}^{-nw}, \tag{3–8}$$

where $\chi_{2mp}(n)$ is a periodic function modulo $2\,m\,p$:

$n \mod 2\,m\,p$	$\chi_{2mp}(n)$
$m\,p-m-p$	1
$m\,p-m+p$	-1
$m\,p+m-p$	-1
$m\,p+m+p$	1
others	0

(3–9)

We apply the Mellin transformation to Equations (3–6) and (3–8), $\frac{1}{2}\sum_{n=0}^{\infty}\chi_{2mp}(n)\,\mathrm{e}^{-nw}\simeq\sum_{n=0}^{\infty}\frac{T_n^{(m,p)}}{(2n+1)!}\,(-1)^n\,w^{2n+1}$. The left-hand side is integrated as

$$\begin{aligned}&\frac{1}{2}\sum_{n=0}^{\infty}\chi_{2mp}(n)\int_0^{\infty}w^{s-1}\,\mathrm{e}^{-nw}\,\mathrm{d}w\\ &\quad=\frac{\Gamma(s)}{2}\sum_{n=0}^{\infty}\chi_{2mp}(n)\,\frac{1}{n^s}\\ &\quad=\frac{\Gamma(s)}{2}\,L(s,\chi_{2mp}),\end{aligned}$$

while the right-hand side is

$$\begin{aligned}&\int_0^{\infty}\Big(\sum_{n=0}^{N-1}\frac{T_n^{(m,p)}}{(2\,n+1)!}\,(-1)^n\,w^{2n+1}\\ &\qquad+O(w^{2N+1})\Big)\,w^{s-1}\,\mathrm{d}w\\ &\quad=\sum_{n=0}^{N-1}\frac{T_n^{(m,p)}}{(2\,n+1)!}\,(-1)^n\,\frac{1}{2\,n+s+1}+R_{2N+1}(s),\end{aligned}$$

with $R_N(s)$ holomorphic in $\Re(s)>-N$. Comparing the residues at $s=-2\,n-1$, we find that the T-numbers $T_n^{(m,p)}$ can be given in terms of the associated L-series as

$$\begin{aligned}T_n^{(m,p)}&=\frac{1}{2}\,(-1)^{n+1}\,L(-2\,n-1,\chi_{2mp})\\ &=\frac{1}{2}\,(-1)^n\,\frac{(2\,m\,p)^{2n+1}}{2\,n+2}\\ &\qquad\times\sum_{a=1}^{2mp}\chi_{2mp}(a)\,B_{2n+2}\left(\frac{a}{2\,m\,p}\right),\end{aligned} \tag{3–10}$$

where $B_n(x)$ is the Bernoulli polynomial. It is noted that the T-number with $(m,p)=(3,2)$ is called the Glaisher T-number [Sloane 02]. A table of T-numbers is given in Table 1.

We now give an explicit form of the invariant for the $(2\,m+1,2)$-torus knot (for $m\geq 1$) by use of Kashaev's R-matrix in Theorem 2.1.

Lemma 3.5. *Kashaev's invariant for the* $(2m+1,2)$*-torus knot is given explicitly as follows:*

n	0	1	2	3	4	5
$T_n^{(3,2)}$	1	23	1681	257543	67637281	27138236663
$T_n^{(5,2)}$	1	71	14641	6242711	4555133281	5076970085351
$T_n^{(7,2)}$	1	143	58081	48571823	69471000001	151763444497103
$T_n^{(9,2)}$	1	239	160801	222359759	525750911041	1898604115708079
$T_n^{(11,2)}$	1	359	361201	746248439	2635820840161	14219082731542919
$T_n^{(13,2)}$	1	503	707281	2041111463	10069440665761	75868751534107223
$T_n^{(15,2)}$	1	671	1256641	4828434911	31713479172481	318124890738776351
$T_n^{(17,2)}$	1	863	2076481	10248374303	86458934113921	1113984641517368543
$T_n^{(19,2)}$	1	1079	3243601	19997487719	210737173733281	3391720107333707159
$T_n^{(21,2)}$	1	1319	4844401	36486145079	469706038871521	9234991712596896839

TABLE 1. T-numbers.

- *Trefoil* 3_1 *($m = 1$):*

$$\langle \mathrm{Trs}(3,2)\rangle_N = \sum_{a=0}^{N-1} (\omega)_a, \tag{3–11a}$$

- *Solomon's Seal Knot* 5_1 *($m = 2$):*

$$\langle \mathrm{Trs}(5,2)\rangle_N = \sum_{\substack{a,b=0\\ 0\le a+b\le N-1}}^{N-1} \omega^{-ab}(\omega)_{a+b}, \tag{3–11b}$$

- *$(2\,m+1, 2)$-torus knot ($m > 2$):*

$$\begin{aligned}
&\langle \mathrm{Trs}(2\,m+1,2)\rangle_N \\
&= N \sum_{1\le a_{2m-2}\le\cdots\le a_1\le N-1} (-1)^{\sum_{j=1}^{2m-2} a_j} \\
&\quad\times \frac{\omega^{\frac{1}{2}\sum_{j=1}^{2m-2} a_j(a_j-1)}}{\prod_{j=1}^{2m-3}(\omega)_{a_j-a_{j+1}}} \\
&= \sum_{0\le c_{2m-2}\le\cdots\le c_2\le N-c_1-1\le N-2} (-1)^{\sum_{j=3}^{2m-2} c_j} \\
&\quad\times \omega^{-c_1c_2+\frac{1}{2}\sum_{j=3}^{2m-2} c_j(c_j+1)}\, \frac{(\omega)_{c_1+c_2}}{\prod_{j=2}^{2m-3}(\omega)_{c_j-c_{j+1}}} \\
&= \sum_{\substack{a_1,a_2,\dots,a_{2m-2}=0\\ 0\le a_1+a_2+\cdots+a_{2m-2}\le N-1}}^{N-1} \frac{(\omega)_{a_1+a_2+\cdots+a_{2m-2}}}{\prod_{j=2}^{2m-3}(\omega)_{a_j}} \\
&\quad\times (-1)^{\sum_{j=3}^{2m-2} j\,a_j} \\
&\quad\times \omega^{-a_1a_2+\sum_{j=3}^{2m-2}\left(\frac{j}{2}-1-a_1\right)a_j+\frac{1}{2}\sum_{j=3}^{2m-2}(a_j+a_{j+1}+\cdots+a_{2m-2})^2}.
\end{aligned} \tag{3–11c}$$

Proof: This is a tedious but straightforward computation. The following identities are useful [Yokota 00, Murakami and Murakami 01]:

$$(\omega)^*_{[i-1]}\,(\omega)_{[-i]} = N, \tag{3–12}$$

$$\begin{aligned}
&\sum_{k\in[\ell,m]} \frac{\omega^{-(m-\ell+1)k}}{(\omega)_{[m-k]}\,(\omega)^*_{[k-\ell]}} \\
&\qquad = (-1)^{[m-\ell]}\,\omega^{([m-\ell]+1)([m-\ell]-2m)/2},
\end{aligned} \tag{3–13}$$

$$\sum_{k\in[i,j]} \frac{\omega^{-k(i-j)}}{(\omega)_{[i-k]}\,(\omega)^*_{[k-j]}} = \delta_{i,j}. \tag{3–14}$$

□

Recalling a result of Proposition 3.3, we obtain an asymptotic expansion for the above set of the ω-series.

Corollary 3.6. *We have an asymptotic expansion for the ω-series with limit $N\to\infty$:*

- *Trefoil ($m = 1$):*

$$\begin{aligned}
\sum_{a=0}^{N-1}(\omega)_a &\simeq N^{\frac{3}{2}} \exp\left(\frac{\pi\,\mathrm{i}}{4} - \frac{\pi\,\mathrm{i}\,N}{12} - \frac{\pi\,\mathrm{i}}{12\,N}\right) \\
&\quad + \mathrm{e}^{-\frac{\pi \mathrm{i}}{12N}} \sum_{n=0}^{\infty} \frac{T_n^{(3,2)}}{n!}\left(\frac{\pi}{12\,\mathrm{i}\,N}\right)^n.
\end{aligned} \tag{3–15a}$$

- *Solomon's Seal Knot ($m = 2$):*

$$\begin{aligned}
&\sum_{0\le a+b\le N-1} \omega^{-ab}\,(\omega)_{a+b} \\
&\quad\simeq \frac{2}{\sqrt{5}}\,N^{\frac{3}{2}}\,\mathrm{e}^{\frac{\pi}{4}\mathrm{i}-\frac{9\mathrm{i}\pi}{20N}}\left(2\,a\,\mathrm{e}^{-\frac{N\pi\mathrm{i}}{20}} - b\,\mathrm{e}^{-\frac{9N\pi\mathrm{i}}{20}}\right) \\
&\qquad + \mathrm{e}^{-\frac{9\mathrm{i}\pi}{20N}} \sum_{n=0}^{\infty} \frac{T_n^{(5,2)}}{n!}\left(\frac{\pi}{20\,\mathrm{i}\,N}\right)^n,
\end{aligned} \tag{3–15b}$$

where

$$a=\sin\left(\frac{\pi}{5}\right)=\frac{\sqrt{5}}{2}\sqrt{\frac{1}{2}\big(1-\frac{1}{\sqrt{5}}\big)},$$

$$b=\sin\left(\frac{2\pi}{5}\right)=\frac{\sqrt{5}}{2}\sqrt{\frac{1}{2}\big(1+\frac{1}{\sqrt{5}}\big)}.$$

- *$(2m+1,2)$-torus knot ($m>2$):*

$$\begin{aligned}&\sum_{\substack{a_1,a_2,\dots,a_{2m-2}=0\\0\leq a_1+a_2+\cdots+a_{2m-2}\leq N-1}}^{N-1}\frac{(\omega)_{a_1+a_2+\cdots+a_{2m-2}}}{\prod_{j=2}^{2m-3}(\omega)_{a_j}}(-1)^{\sum_{j=3}^{2m-2}j\,a_j}\\&\times\omega^{-a_1a_2+\sum_{j=3}^{2m-2}\left(\frac{j}{2}-1-a_1\right)a_j+\frac{1}{2}\sum_{j=3}^{2m-2}(a_j+a_{j+1}+\cdots+a_{2m-2})^2}\\&\simeq\frac{2}{\sqrt{2m+1}}N^{\frac{3}{2}}\,\mathrm{e}^{\frac{\pi\mathrm{i}}{4}-\frac{\pi\mathrm{i}}{N}\frac{(2m-1)^2}{4(2m+1)}}\sum_{j=0}^{m-1}(-1)^j\,(m-j)\\&\quad\times\sin\left(\frac{2j+1}{2m+1}\pi\right)\mathrm{e}^{-N\pi\mathrm{i}\frac{(2j+1)^2}{4(2m+1)}}\\&\quad+\mathrm{e}^{-\frac{\pi\mathrm{i}}{N}\frac{(2m-1)^2}{4(2m+1)}}\sum_{n=0}^{\infty}\frac{T_n^{(2m+1,2)}}{n!}\left(\frac{\pi}{4\,(2m+1)\,N\,\mathrm{i}}\right)^n.\end{aligned}\tag{3–15c}$$

Remark 3.7. Equation (3–15a) was conjectured in [Zagier 01] (according to this reference, it was due to Kontsevich), and it was discussed that a power exponent 3/2 of $N^{3/2}$ which appeared on the right-hand side is related with a weight of the "nearly modular function." Namely, we define

$$\Phi^{(2m+1)}(\alpha)=\mathrm{e}^{\frac{(2m-1)^2}{4(2m+1)}\pi\mathrm{i}\alpha}\,F^{(2m+1,2)}(\mathrm{e}^{2\pi\mathrm{i}\alpha}),\tag{3–16}$$

where $F^{(2m+1,2)}(q)$ will be defined in Equation (3–22). Then, from Equation (3–15), we have the modular transformation property,

$$\Phi^{(3)}\big(\frac{1}{N}\big)+\big(-\mathrm{i}\,N\big)^{\frac{3}{2}}\,\Phi^{(3)}(-N)=\sum_{n=0}^{\infty}\frac{T_n^{(3,2)}}{n!}\left(\frac{\pi}{12\,\mathrm{i}\,N}\right)^n.\tag{3–17}$$

A generalization of this property will be discussed below.

Remark 3.8. The torus knot is not hyperbolic, and we have $\|S^3\setminus\mathrm{Torus}\|=0$. In view of complexification of the volume conjecture (Conjecture 2.4), Equation (3–15a) shows

$$\mathrm{CS}(\mathrm{Trefoil})=-\frac{\pi^2}{6}.\tag{3–18}$$

Equations (3–15b) and (3–15c) indicate a decomposition into several terms labelled by flat connections, and we have

$$\begin{aligned}&\mathrm{CS}\big(\mathrm{Trs}(2m+1,2)\big)\\&\quad=\left\{-\frac{(2j+1)^2}{2\,(2m+1)}\,\pi^2\,\middle|\,j=0,1,\dots,m-1\right\}.\end{aligned}\tag{3–19}$$

This decomposition may be explained as follows. The fundamental group of $S^3\setminus\mathrm{Trs}(m,p)$ has a presentation

$$\pi_1\big(S^3\setminus\mathrm{Trs}(m,p)\big)=\big\langle x,y\,|\,x^m=y^p\big\rangle.\tag{3–20}$$

As was discussed in [Klassen 91], there are $(m-1)\,(p-1)/2$ disjoint irreducible representations, $\rho:\pi_1(S^3\setminus\mathrm{Trs}(m,p))\to SU(2)$, up to conjugacy. This corresponds to a decomposition in Equation (3–5). Especially, in the case of $\mathrm{Trs}(2m+1,2)$, we have m representations in which the eigenvalues of $\rho(y)$, respectively $\rho(x)$, are given by $\exp(\pm\pi\,\mathrm{i}/2)$, respecively $\exp(\pm\frac{2j+1}{2m+1}\pi\,\mathrm{i})$ with $j=0,1,\dots,m-1$. The Chern–Simons invariant may be computed by considering a path of representation along a line of [Kirk and Klassen 93].

For our later discussion, we comment on asymptotics of the invariant which simply follows from Equation (3–4).

Corollary 3.9. *For the torus knot $\mathcal{K}=\mathrm{Trs}(m,p)$, we have in the limit $N\to\infty$ that*

$$\log\big|\langle\mathrm{Trs}(m,p)\rangle_N\big|\sim\frac{3}{2}\log N.\tag{3–21}$$

3.2 q-Series

We define the q-series based on Kashaev's invariant of the $(2m+1,2)$-torus knot which was given in Equation (3–11).

- Trefoil ($m=1$):

$$F^{(3,2)}(q)=\sum_{n=0}^{\infty}(q)_n,\tag{3–22a}$$

- Solomon's Seal Knot ($m=2$):

$$F^{(5,2)}(q)=\sum_{a,b=0}^{\infty}q^{-ab}\,(q)_{a+b},\tag{3–22b}$$

- $(2\,m+1,2)$-Torus Knot ($m>2$):

$$
\begin{aligned}
&F^{(2m+1,2)}(q)\\
&=\sum_{\substack{0\le c_1<\infty\\ 0\le c_{2m-2}\le\cdots\le c_2<\infty}} (-1)^{\sum_{j=3}^{2m-2} c_j}\, q^{-c_1 c_2+\frac{1}{2}\sum_{j=3}^{2m-2} c_j(c_j+1)}\\
&\qquad\times\frac{(q)_{c_1+c_2}}{\prod_{j=2}^{2m-3}(q)_{c_j-c_{j+1}}}\\
&=\sum_{a_1,\dots,a_{2m-2}=0}^{\infty}\frac{(q)_{a_1+a_2+\cdots+a_{2m-2}}}{\prod_{j=2}^{2m-3}(q)_{a_j}}\,(-1)^{\sum_{j=3}^{2m-2} j\,a_j}\\
&\times q^{-a_1 a_2+\sum_{j=3}^{2m-2}(\frac{j}{2}-1-a_1)a_j+\frac{1}{2}\sum_{j=3}^{2m-2}(a_j+a_{j+1}+\cdots+a_{2m-2})^2}.
\end{aligned}
\tag{3–22c}
$$

In this section, we use the following notation:

$$(x)_n=(x;q)_n=\prod_{i=1}^{n}(1-x\,q^{i-1}).$$

Note that generally the q-series functions $F^{(2m+1,2)}(q)$ do not converge in any open set, but in the limit $q\to\omega\equiv\exp(2\,\pi\,\mathrm{i}/N)$ the functions reduce to the invariant of the torus knot:

$$F^{(2m+1,2)}(\omega)=\big\langle \mathrm{Trs}(2\,m+1,2)\big\rangle_N. \tag{3–23}$$

Collecting these observations, we propose the following conjecture on the asymptotic expansion of the q-series. We have numerically checked the validity of this conjecture for several n and m.

Conjecture 3.10. *We have the asymptotic expansions of the q-series $F^{(2m+1,2)}(q)$ defined in Equation* (3–22) *as* ($q=\mathrm{e}^{-t}$)

$$
\begin{aligned}
&F^{(2m+1,2)}(\mathrm{e}^{-t})\\
&=\mathrm{e}^{\frac{(2m-1)^2}{8(2m+1)}t}\sum_{n=0}^{\infty}\frac{T_n^{(2m+1,2)}}{n!}\left(\frac{t}{2^3\,(2\,m+1)}\right)^n,
\end{aligned}
\tag{3–24}
$$

where the T-number is defined by Equation (3–10) *(or Equation* (3–6)*).*

This conjecture is proved in [Zagier 01] for the case $m=1$ as follows.

Theorem 3.11. [Zagier 01] *Conjecture 3.10 for $m=1$ is correct.*

$$
\begin{aligned}
&\sum_{n=0}^{\infty}(1-\mathrm{e}^{-t})\,(1-\mathrm{e}^{-2t})\cdots(1-\mathrm{e}^{-nt})\\
&\qquad=\mathrm{e}^{t/24}\sum_{n=0}^{\infty}\frac{T_n^{(3,2)}}{n!}\left(\frac{t}{24}\right)^n.
\end{aligned}
\tag{3–25}
$$

Proof: We outline a proof following [Zagier 01] (see also [Andrews et al. 01] for a generalization of this identity). We define a function $S(x)$ by

$$
\begin{aligned}
S(x)&=\sum_{n=0}^{\infty}(x)_{n+1}\,x^n\\
&=(x\,q)_\infty+(1-x)\sum_{n=0}^{\infty}\big((x\,q)_n-(x\,q)_\infty\big)\,x^n.
\end{aligned}
\tag{3–26}
$$

The subtraction of $(x\,q)_\infty$ in the summation is to avoid divergence in the limit $x\to1$, and the second equality is proved using the Euler identity,

$$\sum_{n=0}^{\infty}\frac{x^n}{(q)_n}=\frac{1}{(x)_\infty}.$$

We can check that it solves the q-difference equation,

$$S(x)=1-q\,x^2-q^2\,x^3\,S(q\,x). \tag{3–27}$$

On the other hand, we can easily see that a function

$$S(x)=\sum_{n=1}^{\infty}\chi_{12}(n)\,x^{\frac{1}{2}(n-1)}\,q^{\frac{1}{24}(n^2-1)}, \tag{3–28}$$

also solves the same q-difference equation (3–27). Here $\chi_{12}(n)$ is the Dirichlet character which follows from Equation (3–9) with $(m,p)=(3,2)$:

$n \mod 12$	1	5	7	11	others
$\chi_{12}(n)$	1	-1	-1	1	0

It is remarked that $S(x=1)$ coincides with the Dedekind η-function,

$$(q)_\infty=\sum_{n=1}^{\infty}\chi_{12}(n)\,q^{\frac{1}{24}(n^2-1)}, \tag{3–29}$$

where the equality follows from the Jacobi triple product identity. Thus, from Equations (3–26) and (3–28), we find that

$$
\begin{aligned}
&(x\,q)_\infty+(1-x)\sum_{n=0}^{\infty}\big((x\,q)_n-(x\,q)_\infty\big)\,x^n\\
&\qquad=\sum_{n=0}^{\infty}\chi_{12}(n)\,x^{\frac{1}{2}(n-1)}\,q^{\frac{1}{24}(n^2-1)}.
\end{aligned}
\tag{3–30}
$$

By differentiating with respect to x and setting $x\to1$, we get

$$
\begin{aligned}
&(q)_\infty\cdot\left(\frac{1}{2}-\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\right)-\sum_{n=0}^{\infty}\big((q)_n-(q)_\infty\big)\\
&\qquad=\frac{1}{2}\sum_{n=0}^{\infty}n\,\chi_{12}(n)\,q^{\frac{1}{24}(n^2-1)}.
\end{aligned}
\tag{3–31}
$$

Thus, in the limit $t \to 0$, we obtain

$$-2\,\mathrm{e}^{-t/24}\,F^{(3,2)}(\mathrm{e}^{-t}) \simeq \sum_{n=0}^{\infty} n\,\chi_{12}(n)\,\mathrm{e}^{-\frac{1}{24}n^2 t}, \qquad (3\text{–}32)$$

because $(q)_\infty$ induces an infinite order of t when we set $q = \mathrm{e}^{-t}$. Applying the Mellin transformation to an equality $\sum_{n=0}^{\infty} n\,\chi_{12}(n)\,\mathrm{e}^{-\frac{1}{24}n^2 t} \sim \sum_{n=0}^{\infty} \gamma_n\, t^n$, we get

$$\gamma_n = \frac{(-1)^n}{24^n\, n!}\, L(-2\,n-1, \chi_{12}).$$

By use of the relationship (3–10) between the L-series and the T-numbers, we find

$$\gamma_n = -2\,\frac{T_n^{(3,2)}}{24^n\, n!},$$

which proves Equation (3–25). □

Remark 3.12. From Equation (3–29), the right-hand side of Equation (3–25) is regarded as a "half-differential" of the Dedekind η-function [Zagier 01].

Remark 3.13. Conjecture 3.10 is formally derived as follows: Equation (3–25), i.e., a proof of Conjecture 3.10 in the case $m = 1$, suggests that we may apply a naive analytic continuation

$$N \longleftrightarrow \frac{2\,\pi}{\mathrm{i}\,t}, \qquad (3\text{–}33)$$

in the integral (3–7), i.e., we may set

$$\begin{aligned} F^{(2m+1,2)}&(\mathrm{e}^{-t}) \\ &\simeq \mathrm{i}\left(\frac{2\,(2\,m+1)\,\pi}{t}\right)^{3/2} \mathrm{e}^{\frac{(2m-1)^2}{8(2m+1)}t} \\ &\times \int_{\mathscr{C}} \mathrm{d}w\,\mathrm{e}^{\frac{2(2m+1)\pi^2}{t}w^2}\, w\,\frac{\mathrm{sh}(2\,\pi\,w)}{\mathrm{ch}((2\,m+1)\,\pi\,w)}. \end{aligned} \qquad (3\text{–}34)$$

In fact, substituting the expansion (3–8) with $(m,p) \to (2m+1, 2)$,

$$\begin{aligned} \frac{\mathrm{sh}(2\,x)}{\mathrm{ch}((2\,m+1)\,x)} &= \sum_{n=0}^{\infty} \chi_{8m+4}(n)\,\mathrm{e}^{-n\,x} \\ &= 2\sum_{n=0}^{\infty} (-1)^n\,\frac{T_n^{(2m+1,2)}}{(2\,n+1)!}\,x^{2n+1}, \end{aligned} \qquad (3\text{–}35)$$

$n \mod (8\,m+4)$	$2\,m-1$	$2\,m+3$	$6\,m+1$	$6\,m+5$	others
$\chi_{8m+4}(n)$	1	-1	-1	1	0

we obtain the right-hand side of Equation (3–24). Using the Mellin transformation, we also see that

$$\begin{aligned} F^{(2m+1,2)}&(\mathrm{e}^{-t}) \\ &\sim -\frac{1}{2}\sum_{n=0}^{\infty} n\,\chi_{8m+4}(n)\,\mathrm{e}^{-\frac{t}{8(2m+1)}(n^2-(2m-1)^2)}. \end{aligned} \qquad (3\text{–}36)$$

It is noted that the right-hand side is now a "half-differential" of the infinite q-product defined by

$$\begin{aligned} &\sum_{n=1}^{\infty} \chi_{8m+4}(n)\, q^{\frac{1}{8(2m+1)}(n^2-(2m-1)^2)} \\ &= (q, q^{2m}, q^{2m+1}; q^{2m+1})_\infty \\ &= \sum_{k=-\infty}^{\infty} (-1)^k\, q^{(m+\frac{1}{2})k^2+(m-\frac{1}{2})k} \\ &= (q)_\infty \cdot \sum_{n_{m-1}\geq\cdots\geq n_1\geq 0} \frac{q^{{n_1}^2+\cdots+n_{m-1}^2+n_1+\cdots+n_{m-1}}}{(q)_{n_{m-1}-n_{m-2}}\cdots(q)_{n_2-n_1}\,(q)_{n_1}}, \end{aligned} \qquad (3\text{–}37)$$

where the last equality is the Gordon–Andrews identity, a generalization of the Rogers–Ramanujan identity ($m = 2$).

Conjecture 3.10 suggests that there should be a q-series identity as a generalization of Zagier's identity (3–31), which we hope to report in a future publication [Hikami 02].

Remark 3.14. We consider an expansion of the q-series with $q \to 1-x$, and define $a_n^{(2m+1)}$ as coefficients of x^n:

$$F^{(2m+1,2)}(1-x) = \sum_{n=0}^{\infty} a_n^{(2m+1)}\, x^n. \qquad (3\text{–}38)$$

To calculate $a_n^{(2m+1)}$ from $T_n^{(2m+1)}$, we also define $b_n^{(2m+1)}$ following [Zagier 01] by

$$F^{(2m+1,2)}(\mathrm{e}^{-t}) = \sum_{n=0}^{\infty} \frac{b_n^{(2m+1)}}{n!}\, t^n. \qquad (3\text{–}39)$$

It is easy to see that a series $b_n^{(2m+1)}$ is written in terms of $T_n^{(2m+1,2)}$ as

$$b_n^{(2m+1)} = \left(\frac{(2\,m-1)^2}{8\,(2\,m+1)}\right)^n \sum_{k=0}^{n} \binom{n}{k}\, \frac{T_n^{(2m+1,2)}}{(2\,m-1)^{2k}}. \qquad (3\text{–}40)$$

Some of the computed terms are given in Table 2.

We use the nonnegative Stirling numbers of the first kind [Goldberg et al. 72] defined by

$$\prod_{j=0}^{n-1} (x+j) = \sum_{m=0}^{n} s(n,m)\, x^m. \qquad (3\text{–}41)$$

It is known that we have

$$\frac{t^m}{m!} = \sum_{n=m}^{\infty} s(n,m)\,\frac{(1-\mathrm{e}^{-t})^n}{n!}. \qquad (3\text{–}42)$$

n	0	1	2	3	4	5	6	7
$b_n^{(3)}$	1	1	3	19	207	3451	81663	2602699
$b_n^{(5)}$	1	2	10	104	1870	51632	2027470	107354144
$b_n^{(7)}$	1	3	21	303	7581	291903	16004541	1184112303
$b_n^{(9)}$	1	4	36	664	21276	1050664	73939356	7024817944
$b_n^{(11)}$	1	5	55	1235	48235	2906315	249689275	28969703915
$b_n^{(13)}$	1	6	78	2064	95082	6762216	686010858	94007233704
$b_n^{(15)}$	1	7	105	3199	169785	13919647	1628324985	257347060159
$b_n^{(17)}$	1	8	136	4688	281656	26150768	3465278776	620465295248
$b_n^{(19)}$	1	9	171	6579	441351	45771579	6776104311	1355621381739
$b_n^{(21)}$	1	10	210	8920	660870	75714880	12384774150	2737845857680

TABLE 2.

n	0	1	2	3	4	5	6	7
$a_n^{(3)}$	1	1	2	5	15	53	217	1014
$a_n^{(5)}$	1	2	6	23	109	621	4149	31851
$a_n^{(7)}$	1	3	12	62	402	3162	29308	312975
$a_n^{(9)}$	1	4	20	130	1070	10738	127316	1741705
$a_n^{(11)}$	1	5	30	235	2345	28623	413441	6896695
$a_n^{(13)}$	1	6	42	385	4515	64911	1105573	21759966
$a_n^{(15)}$	1	7	56	588	7924	131124	2572640	58354762
$a_n^{(17)}$	1	8	72	852	12972	242820	5392464	138497502
$a_n^{(19)}$	1	9	90	1185	20115	420201	10419057	298862100
$a_n^{(21)}$	1	10	110	1595	29865	688721	18859357	597554925

TABLE 3.

Using this identity, we obtain the a-series from the b-series as

$$a_n^{(2m+1)} = \frac{1}{n!}\sum_{k=1}^{n} s(n,k)\, b_k^{(2m+1)}. \tag{3–43}$$

Some of the a-series are given in Table 3.

Table 3 indicates that $a_n^{(2m+1)}$ is given by the n-th order polynomial of m, e.g.,

$$a_0^{(2m+1)} = 1,$$

$$a_1^{(2m+1)} = m,$$

$$a_2^{(2m+1)} = m\,(m+1),$$

$$a_3^{(2m+1)} = \frac{1}{6}\,m\,(m+1)\,(8\,m+7),$$

$$a_4^{(2m+1)} = \frac{1}{6}\,m\,(m+1)\,(14\,m^2+22\,m+9),$$

$$a_5^{(2m+1)} = \frac{1}{30}\,m\,(m+1)\,(8\,m+7)\,(19\,m^2+25\,m+9),$$

$$\begin{aligned}a_6^{(2m+1)} &= \frac{1}{180}\,m\,(m+1)\big(2360\,m^4+6544\,m^3\\&\qquad +6841\,m^2+3209\,m+576\big),\\ a_7^{(2m+1)} &= \frac{1}{2520}\,m\,(m+1)\big(99136\,m^5+330440\,m^4\\&\qquad +440960\,m^3+294775\,m^2+98919\,m+13410\big).\end{aligned}$$

We should note that the series $a_n^{(3)}$ coincides with the upper bound of the number of linearly independent Vassiliev invariants of degree n [Stoimenow 98].

Remark 3.15. It would be interesting to construct the explicit form of Kashaev's invariant for the arbitrary torus knot $\mathcal{K} = \mathrm{Trs}(m,p)$, and to study an asymptotic expansion as a q-series based on Equation (3–4).

Remark 3.16. The Rogers-Ramanujan identities are the following set of equations:

$$S_0(q) = \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q)_n} = \frac{1}{(q,q^4;q^5)_\infty}, \tag{3–44a}$$

$$S_1(q) = \sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q)_n} = \frac{1}{(q^2,q^3;q^5)_\infty}. \tag{3–44b}$$

With these functions, we have the modular property,

$$\begin{pmatrix} c_0(-\frac{1}{\tau}) \\ c_1(-\frac{1}{\tau}) \end{pmatrix} = \frac{2}{\sqrt{5}} \begin{pmatrix} \sin\left(\frac{2\pi}{5}\right) & \sin\left(\frac{\pi}{5}\right) \\ \sin\left(\frac{\pi}{5}\right) & -\sin\left(\frac{2\pi}{5}\right) \end{pmatrix} \begin{pmatrix} c_0(\tau) \\ c_1(\tau) \end{pmatrix}, \tag{3–45}$$

where we have set $q = \exp(2\pi \mathrm{i}\tau)$, and

$$c_0(q) = q^{1/40} S_0(q), \qquad c_1(q) = q^{9/40} S_1(q).$$

Conjecture 3.10 for $m = 2$ is an identity for a half differential of $S_1(q)$. We expect that there should exist an identity for $S_0(q)$. For this purpose, we define another series $\tilde{T}_n^{(5,2)}$:

$$\frac{\mathrm{sh}(4x)}{\mathrm{ch}(5x)} = 2\sum_{n=0}^{\infty}(-1)^n \frac{\tilde{T}_n^{(5,2)}}{(2n+1)!} x^{2n+1} = \sum_{n=0}^{\infty} \tilde{\chi}_{20}(n)\,\mathrm{e}^{-nx}. \tag{3–46}$$

Here, we have

$n \mod 20$	1	9	11	19	others
$\tilde{\chi}_{20}(n)$	1	-1	-1	1	0

and some of the $\tilde{T}_n^{(5,2)}$, $\tilde{T}_n^{(5,2)} = \frac{1}{2}(-1)^{n+1} L(-2n-1, \tilde{\chi}_{20})$ are as follows:

n	0	1	2	3	4	5
$\tilde{T}_n^{(5,2)}$	2	118	23762	10104358	7370639522	8214744720598

The Jacobi triple identity gives

$$(q^2, q^3, q^5; q^5)_\infty = (q)_\infty \cdot S_0(q) = \sum_{n=0}^{\infty} \tilde{\chi}_{20}(n)\, q^{\frac{1}{40}(n^2-1)}. \tag{3–47}$$

Conjecture 3.17. *We define*

$$\tilde{F}^{(5,2)}(q) = \sum_{\substack{a,b=0 \\ (a,b)\neq(0,0)}}^{\infty} q^{-ab}\,(q)_{a+b-1}. \tag{3–48}$$

Then, we have

$$\tilde{F}^{(5,2)}(\mathrm{e}^{-t}) = \mathrm{e}^{t/40} \sum_{n=0}^{\infty} \frac{\tilde{T}_n^{(5,2)}}{n!} \left(\frac{t}{40}\right)^n. \tag{3–49}$$

Conjecture 3.18. *The q-series $\tilde{F}^{(5,2)}(q)$ with $q \to \omega \equiv \mathrm{e}^{2\pi\mathrm{i}/N}$ has an asymptotic expansion in $N \to \infty$ as*

$$\begin{aligned} \tilde{F}^{(5,2)}(\omega) &= \sum_{\substack{a,b=0 \\ 1\leq a+b\leq N}}^{N} \omega^{-ab}\,(\omega)_{a+b-1} \\ &\simeq \frac{2}{\sqrt{5}} N^{\frac{3}{2}}\,\mathrm{e}^{\frac{\pi \mathrm{i}}{4} - \frac{\pi\mathrm{i}}{20N}} \Big(2\sin\big(\frac{2\pi}{5}\big)\,\mathrm{e}^{-\frac{N\pi\mathrm{i}}{20}} \\ &\quad + \sin\big(\frac{\pi}{5}\big)\,\mathrm{e}^{-\frac{9N\pi\mathrm{i}}{20}}\Big) + \mathrm{e}^{-\frac{\pi\mathrm{i}}{20N}} \sum_{n=0}^{\infty} \frac{\tilde{T}_n^{(5,2)}}{n!} \left(\frac{\pi}{20\,\mathrm{i}\,N}\right)^n. \end{aligned} \tag{3–50}$$

With the above conjecture and Equation (3–15b), the transformation property (3–17) should be reformulated as a variant of Equation (3–45); we set

$$\Phi^{(5)}(\alpha) = \mathrm{e}^{\frac{9}{20}\pi\mathrm{i}\alpha} F^{(5,2)}(\mathrm{e}^{2\pi\mathrm{i}\alpha}), \tag{3–51}$$

$$\Psi^{(5)}(\alpha) = \mathrm{e}^{\frac{1}{20}\pi\mathrm{i}\alpha} \tilde{F}^{(5,2)}(\mathrm{e}^{2\pi\mathrm{i}\alpha}). \tag{3–52}$$

Using the fact that we have

$$\Phi^{(5)}(0) = 1, \qquad \Psi^{(5)}(0) = 2,$$

and a recursion relation,

$$\Phi^{(5)}(\alpha+1) = \mathrm{e}^{\frac{9}{20}\pi\mathrm{i}} \cdot \Phi^{(5)}(\alpha),$$

$$\Psi^{(5)}(\alpha+1) = \mathrm{e}^{\frac{1}{20}\pi\mathrm{i}} \cdot \Psi^{(5)}(\alpha),$$

we get for $n \in \mathbb{Z}$

$$\Phi^{(5)}(n) = \mathrm{e}^{\frac{9}{20}\pi i n}, \qquad \Psi^{(5)}(n) = 2\,\mathrm{e}^{\frac{1}{20}\pi i n}.$$

As a result, we find that the functions $\Psi^{(5)}$ and $\Phi^{(5)}$ can be regarded as a set of "nearly" modular functions [Zagier 01] satisfying

$$\begin{aligned} &\begin{pmatrix} \Psi^{(5)}(\frac{1}{N}) \\ \Phi^{(5)}(\frac{1}{N}) \end{pmatrix} \\ &+ (-\mathrm{i}\,N)^{\frac{3}{2}} \cdot \frac{2}{\sqrt{5}} \begin{pmatrix} \sin\left(\frac{2\pi}{5}\right) & \sin\left(\frac{\pi}{5}\right) \\ \sin\left(\frac{\pi}{5}\right) & -\sin\left(\frac{2\pi}{5}\right) \end{pmatrix} \begin{pmatrix} \Psi^{(5)}(-N) \\ \Phi^{(5)}(-N) \end{pmatrix} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} \tilde{T}_n^{(5,2)} \\ T_n^{(5,2)} \end{pmatrix} \left(\frac{\pi}{20\,\mathrm{i}\,N}\right)^n. \end{aligned} \tag{3–53}$$

Note that the transformation matrix coincides with that of Equation (3–45).

Remark 3.19. As a generalization of the previous remark to the case $m > 2$, we define a formal q-series

$\tilde{F}^{(2m+1,2;a)}(q)$ for $a = 0, 1, \ldots, m-2$ by

$$\tilde{F}^{(2m+1,2;a)}(q) = \sum_{\substack{0 \le c_1 < \infty \\ 0 \le c_{2m-2} \le \cdots \le c_2 < \infty \\ 0 < c_1 + c_2}} (-1)^{\sum_{j=3}^{2m-2} c_j} q^{-c_1\, c_2 + \frac{1}{2} \sum_{j=3}^{2m-2} c_j\,(c_j+1) - \sum_{j=1}^{a} c_{2j+2}} \frac{(q)_{c_1+c_2-1}}{\prod_{j=2}^{2m-3} (q)_{c_j - c_{j+1}}}. \quad (3\text{–}54)$$

Conjecture 3.20. *We have*

$$\tilde{F}^{(2m+1,2;a)}(\mathrm{e}^{-t}) = \mathrm{e}^{\frac{(2m-2a-3)^2}{8(2m+1)}\, t} \sum_{n=0}^{\infty} \frac{\tilde{T}_n^{(2m+1,2;a)}}{n!} \left(\frac{t}{2^3\,(2\,m+1)} \right)^n, \quad (3\text{–}55)$$

where $a = 0, 1, \ldots, m-2$. We have used the T-series

$$\frac{\mathrm{sh}\big((2\,a+4)\,x\big)}{\mathrm{ch}\big((2\,m+1)\,x\big)} = \sum_{n=0}^{\infty} \tilde{\chi}_{8m+4}^{(a)}(n)\, \mathrm{e}^{-nx} = 2 \sum_{n=0}^{\infty} (-1)^n \frac{\tilde{T}_n^{(2m+1,2;a)}}{(2\,n+1)!}\, x^{2n+1}, \quad (3\text{–}56)$$

and the periodic function is defined by

$n \mod (8\,m+4)$	$\tilde{\chi}_{8m+4}^{(a)}(n)$
$2\,m - 2\,a - 3$	1
$2\,m + 2\,a + 5$	-1
$6\,m - 2\,a - 1$	-1
$6\,m + 2\,a + 7$	1
others	0

Conjecture 3.21. *In the case where q is the N-th root of unity, we have*

$$\begin{aligned} \tilde{F}^{(2m+1,2;a)}(\omega) &\simeq \frac{2}{\sqrt{2\,m+1}}\, N^{\frac{3}{2}}\, \mathrm{e}^{\frac{\pi \mathrm{i}}{4} - \frac{\pi \mathrm{i}}{N}\, \frac{(2m-2a-3)^2}{4(2m+1)}} \\ &\times \sum_{k=0}^{m-1} (-1)^k\,(m-k)\, \sin\left((a+2)\, \frac{2\,k+1}{2\,m+1}\, \pi \right)\, \mathrm{e}^{-N\pi \mathrm{i}\, \frac{(2k+1)^2}{4(2m+1)}} \\ &+ \mathrm{e}^{-\frac{\pi \mathrm{i}}{N}\, \frac{(2m-2a-3)^2}{4(2m+1)}} \sum_{n=0}^{\infty} \frac{\tilde{T}_n^{(2m+1,2;a)}}{n!} \left(\frac{\pi}{4\,(2\,m+1)\,N\,\mathrm{i}} \right)^n. \end{aligned} \quad (3\text{–}57)$$

We define functions $\Psi_a^{(2m+1)}(\alpha)$ for $a = 0, 1, \ldots, m-2$ by

$$\Psi_a^{(2m+1)}(\alpha) = \mathrm{e}^{\frac{(2m-2a-3)^2}{4(2m+1)}\pi \mathrm{i} \alpha}\, \tilde{F}^{(2m+1,2;a)}(\mathrm{e}^{2\pi \mathrm{i} \alpha}), \quad (3\text{–}58)$$

and with Equation (3–16) introduce a vector $\mathbf{\Phi}^{(2m+1)}(\alpha)$,

$$\mathbf{\Phi}^{(2m+1)}(\alpha) = \begin{pmatrix} \Psi_{m-2}^{(2m+1)}(\alpha) \\ \vdots \\ \Psi_0^{(2m+1)}(\alpha) \\ \Phi^{(2m+1)}(\alpha) \end{pmatrix}. \quad (3\text{–}59)$$

The above conjecture indicates a nearly modular property of weight 1/2,

$$\mathbf{\Phi}^{(2m+1)}(\frac{1}{N}) + (-\mathrm{i}\,N)^{\frac{3}{2}}\, \mathbf{M}^{(2m+1)}\, \mathbf{\Phi}^{(2m+1)}(-N) = \sum_{n=0}^{\infty} \frac{\boldsymbol{T}_n^{(2m+1)}}{n!} \left(\frac{\pi}{4\,(2m+1)\,\mathrm{i}\,N} \right)^n, \quad (3\text{–}60)$$

where $\mathbf{M}^{(2m+1)}$ is an $m \times m$ matrix with an entry

$$\begin{aligned} \Big(\mathbf{M}^{(2m+1)}\Big)_{1 \le i,j \le m} &= (-1)^{j-1}\, \frac{2}{\sqrt{2\,m+1}}\, \sin\left(\frac{(m-i+1)\,(2\,j-1)}{2\,m+1}\, \pi \right) \\ &= \frac{2}{\sqrt{2\,m+1}}\, \cos\left(\frac{(2\,i-1)\,(2\,j-1)}{2\,(2\,m+1)}\, \pi \right), \end{aligned}$$

and

$$\boldsymbol{T}_n^{(2m+1)} = \begin{pmatrix} \tilde{T}_n^{(2m+1,2;m-2)} \\ \vdots \\ \tilde{T}_n^{(2m+1,2;1)} \\ \tilde{T}_n^{(2m+1,2;0)} \\ T_n^{(2m+1,2)} \end{pmatrix}.$$

We define an a-series as an expansion of $\tilde{F}^{(2m+1,2;a)}(q)$ with $q \to 1 - x$:

$$\tilde{F}^{(2m+1,2;a)}(1-x) = \sum_{n=0}^{\infty} \tilde{a}_n^{(2m+1;a)}\, x^n. \quad (3\text{–}61)$$

Using a b-series defined by

$$\tilde{F}^{(2m+1,2;a)}(\mathrm{e}^{-t}) = \sum_{n=0}^{\infty} \frac{\tilde{b}_n^{(2m+1;a)}}{n!}\, t^n, \quad (3\text{–}62)$$

m	n	0	1	2	3	4	5	6	7
2	$a_n^{(5)}$	1	2	6	23	109	621	4149	31851
	$\tilde{a}_n^{(5;0)}$	2	3	9	35	168	966	6496	50103
3	$a_n^{(7)}$	1	3	12	62	402	3162	29308	312975
	$\tilde{a}_n^{(7;0)}$	2	5	20	105	690	5478	51102	548244
	$\tilde{a}_n^{(7;1)}$	3	6	24	127	840	6699	62689	674091
4	$a_n^{(9)}$	1	4	20	130	1070	10738	127316	1741705
	$\tilde{a}_n^{(9;0)}$	2	7	35	231	1925	19481	232309	3191199
	$\tilde{a}_n^{(9;1)}$	3	9	45	300	2520	25641	306915	4227525
	$\tilde{a}_n^{(9;2)}$	4	10	50	335	2825	28821	345618	4767048
5	$a_n^{(11)}$	1	5	30	235	2345	28623	413441	6896695
	$\tilde{a}_n^{(11;0)}$	2	9	54	429	4329	53235	772863	12939498
	$\tilde{a}_n^{(11;1)}$	3	12	72	578	5880	72702	1059436	17785437
	$\tilde{a}_n^{(11;2)}$	4	14	84	679	6944	86163	1258684	21168134
	$\tilde{a}_n^{(11;3)}$	5	15	90	730	7485	93039	1360788	22905630
6	$a_n^{(13)}$	1	6	42	385	4515	64911	1105573	21759966
	$\tilde{a}_n^{(13;0)}$	2	11	77	715	8470	122584	2097326	41414087
	$\tilde{a}_n^{(13;1)}$	3	15	105	985	11760	171084	2937544	58154346
	$\tilde{a}_n^{(13;2)}$	4	18	126	1191	14301	208845	3595347	71312841
	$\tilde{a}_n^{(13;3)}$	5	20	140	1330	16030	234682	4047162	80376063
	$\tilde{a}_n^{(13;4)}$	6	21	147	1400	16905	247800	4277077	84995664
7	$a_n^{(15)}$	1	7	56	588	7924	131124	2572640	58354762
	$\tilde{a}_n^{(15;0)}$	2	13	104	1105	15028	250172	4928300	112114184
	$\tilde{a}_n^{(15;1)}$	3	18	144	1545	21168	354105	6998985	159603426
	$\tilde{a}_n^{(15;2)}$	4	22	176	1903	26224	440363	8726795	199383701
	$\tilde{a}_n^{(15;3)}$	5	25	200	2175	30100	506880	10064600	230275675
	$\tilde{a}_n^{(15;4)}$	6	27	216	2358	32724	552096	10976580	251378289
	$\tilde{a}_n^{(15;5)}$	7	28	224	2450	34048	574966	11438630	262082935

TABLE 4.

we obtain a relationship between the T-series and the a-series as a result of Equation (3–42):

$$\tilde{b}_n^{(2m+1;a)} = \left(\frac{(2\,m - 2\,a - 3)^2}{8\,(2\,m+1)} \right)^n \times \sum_{k=0}^{n} \binom{n}{k} \frac{\tilde{T}_k^{(2m+1,2;a)}}{(2\,m-2\,a-3)^{2k}}, \tag{3–63}$$

$$\tilde{a}_n^{(2m+1;a)} = \frac{1}{n!} \sum_{k=1}^{n} s(n,k)\, \tilde{b}_k^{(2m+1;a)}. \tag{3–64}$$

A table of these series is given in Table 4. For convention, we have also included the a-series defined by Equation (3–38).

4. HYPERBOLIC KNOTS

In the previous section, we analytically studied an asymptotic behavior of Kashaev's invariant of the torus knot. Here, we consider numerically an asymptotic formula for the invariant of the hyperbolic knots and links: knots up to 6-crossing, the Whitehead link, and Borromean rings. Our conjecture based on both analytic results for the torus knot (Corollary 3.9) and numerical results for hyperbolic knots is summarized as follows.

Conjecture 4.1. *Kashaev's invariant behaves in a large N limit as*

$$\log\left|\langle\mathcal{K}\rangle_N\right| \sim v_3 \cdot \left\|S^3 \setminus \mathcal{K}\right\| \cdot \frac{N}{2\pi} + \frac{3}{2}\,\#(\mathcal{K}) \cdot \log N + O(N^0), \tag{4–1}$$

where $\#(\mathcal{K})$ is the number of prime factors of a knot as a connected-sum of prime knots.

Numerical computation is performed with the help of PARI/GP [PARI 00]. We compute Kashaev's invariant $\langle\mathcal{K}\rangle_N$ for the hyperbolic knot $\mathcal{K}$ numerically (see Figures 2–8). We plot $\Re\big(\frac{2\pi}{N}\log\langle\mathcal{K}\rangle_N\big)$ as a function of N, and numerical data is given as $\bullet$ in those figures. The solid line denotes a result of the least-squares method with a trial function,

$$\begin{aligned} v_{\mathcal{K}}(N) &= \frac{2\pi}{N}\log\langle\mathcal{K}\rangle_N \\ &= c_1(\mathcal{K}) + c_2(\mathcal{K}) \cdot \frac{2\pi}{N}\log N + \frac{c_3(\mathcal{K})}{N} + \frac{c_4(\mathcal{K})}{N^2}. \end{aligned} \tag{4–2}$$

This trial function is motivated from an analytic result (3–4) of the torus knot.

We also give some computations which support numerical results of $c_1(\mathcal{K})$. Although there is no mathematically rigorous proof of the asymptotics of each invariant, it is known [Kashaev 95] that a semirigorous proof works well to obtain an asymptotic limit of Kashaev's invariant. In the limit $N \to \infty$, we may replace the ω-series with the dilogarithm function,

$$\begin{aligned} \frac{2\pi\,\mathrm{i}}{N}\log(\omega)_n &= \frac{2\pi\,\mathrm{i}}{N}\sum_{j=1}^{n}\log(1-\exp(2\pi\,\mathrm{i}\,j/N)) \\ &\sim \int_0^x \mathrm{d}t\,\log(1-\mathrm{e}^t) \\ &= \frac{\pi^2}{6} - \mathrm{Li}_2(\mathrm{e}^x). \end{aligned}$$

Thus, we formally obtain a *potential* from Kashaev's invariant $\langle\mathcal{K}\rangle_N$ by the following steps (we set $\frac{2\pi\mathrm{i}}{N}a_i = \log x_i$):

$$\begin{aligned} \omega^{a_i a_j} &\to \exp\left(-\frac{\mathrm{i}\,N}{2\pi}\log x_i \log x_j\right), \\ (\omega)_{a_i} &\to \exp\left(\frac{\mathrm{i}\,N}{2\pi}\left(\mathrm{Li}_2(x_i) - \frac{\pi^2}{6}\right)\right), \\ (\omega)^*_{a_i} &\to \exp\left(\frac{\mathrm{i}\,N}{2\pi}\left(-\mathrm{Li}_2({x_i}^{-1}) + \frac{\pi^2}{6}\right)\right). \end{aligned} \tag{4–3}$$

This computation is essentially the same as that of the central charge from the character [Richmond and Szekeres 81]. The invariant may be represented by the integral of the potential $V_{\mathcal{K}}(\boldsymbol{x})$,

$$\langle\mathcal{K}\rangle_N \sim \iiint \prod_i \mathrm{d}x_i\,\exp\left(\frac{\mathrm{i}\,N}{2\pi}V_{\mathcal{K}}(\boldsymbol{x})\right). \tag{4–4}$$

In the large N limit, we apply a stationary phase approximation, and obtain a saddle point $\boldsymbol{x}_0$ as a solution of the set of equations,

$$\left.\frac{\partial}{\partial x_i}V_{\mathcal{K}}(\boldsymbol{x})\right|_{\boldsymbol{x}=\boldsymbol{x}_0} = 0. \tag{4–5}$$

With this solution, we may obtain

$$\lim_{N\to\infty}\frac{2\pi}{N}\log\langle\mathcal{K}\rangle_N = \mathrm{i}\,V_{\mathcal{K}}(\boldsymbol{x}_0), \tag{4–6}$$

whose real part is expected to coincide with the hyperbolic volume (Conjecture 2.3).

In the following, for several hyperbolic knots and links we give a list of numerical data $c_1(\mathcal{K}), \ldots, c_4(\mathcal{K})$, potential $V_{\mathcal{K}}(\boldsymbol{x})$, and a saddle point $\boldsymbol{x}_0$ of the potential. We will see that

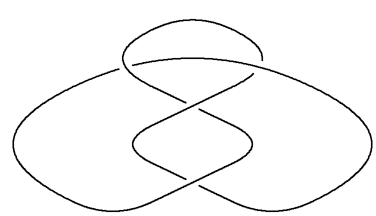
$$c_1(\mathcal{K}) = \Re\big(\mathrm{i}\,V_{\mathcal{K}}(\boldsymbol{x}_0)\big) = \mathrm{Vol}(S^3 \setminus \mathcal{K}), \tag{4–7}$$

$$c_2(\mathcal{K}) = \frac{3}{2}, \tag{4–8}$$

$$c_3(\mathcal{K}) < 0, \tag{4–9}$$

which supports Equation (4–1) (Conjecture 4.1) for $\mathcal{K}$ a prime knot. Note that Equation (3–21) proves this conjecture for $\mathcal{K} = \mathrm{Trs}(m, p)$.

Figure-Eight Knot 4_1.



$$\langle 4_1\rangle_N = \sum_{a=0}^{N-1}\left|(\omega)_a\right|^2. \tag{4–10}$$

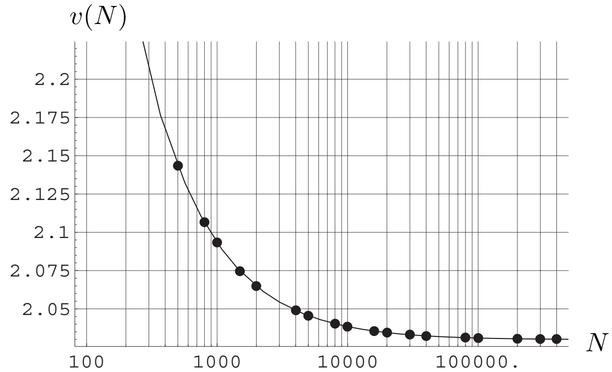


FIGURE 2. Figure-eight knot.

- Numerical result (Figure 2):

$$
\begin{aligned}
\mathrm{Vol}(S^3 \setminus 4_1) &= 2\,D(\mathrm{e}^{\pi \mathrm{i}/3}) = 2.029883212819307...\\
c_1 &= 2.029883193056962 \pm 7.77 \times 10^{-9}\\
c_2 &= 1.50002685 \pm 2.42 \times 10^{-6}\\
c_3 &= -1.7269321 \pm 0.000095\\
c_4 &= 3.575981 \pm 0.0027.
\end{aligned}
$$

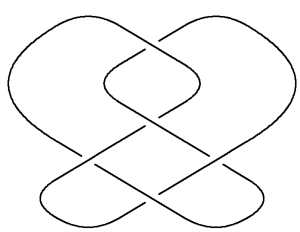
- Potential and saddle point:

$$V_{4_1}(x) = \mathrm{Li}_2(x) - \mathrm{Li}_2(x^{-1}), \tag{4–11}$$

$$x_0 = \exp(-\pi\,\mathrm{i}/3).$$

Note that asymptotic behavior of this ω-series is proved rigorously (see, e.g., [Murakami 00]).

5_2 Knot.



$$\langle 5_2 \rangle_N = \sum_{0 \le a \le b \le N-1} \frac{\left((\omega)_b\right)^2}{(\omega)_a^{\,*}} \omega^{-(b+1)a}. \tag{4–12}$$

- Numerical result (Figure 3):

$$
\begin{aligned}
\mathrm{Vol}(S^3 \setminus 5_2) &= 2.828122088330783...\\
c_1 &= 2.8281219744 \pm 1.5571 \times 10^{-8}\\
c_2 &= 1.5000269858 \pm 2.01 \times 10^{-6}\\
c_3 &= -2.648116951 \pm 0.0000732\\
c_4 &= 4.22788 \pm 0.00169.
\end{aligned}
$$

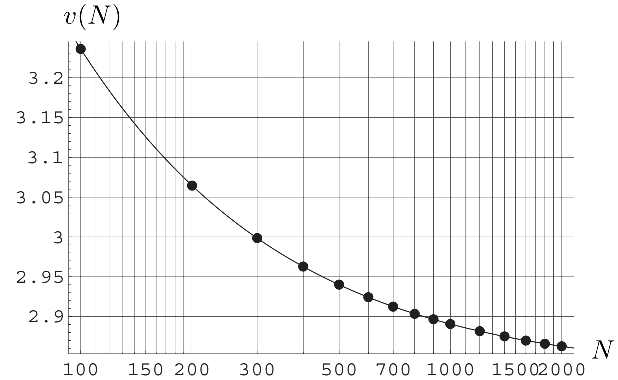


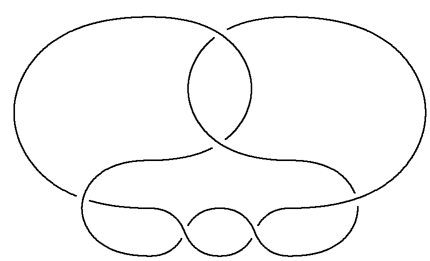
FIGURE 3. Knot 5_2.

- Potential and saddle point:

$$V_{5_2}(x,y) = 2\,\mathrm{Li}_2(y) + \mathrm{Li}_2(x^{-1}) + \log x \log y - \frac{\pi^2}{2}, \tag{4–13}$$

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0.122561 + 0.744862\,\mathrm{i} \\ 0.337641 - 0.56228\,\mathrm{i} \end{pmatrix}.$$

6_1 Knot.



$$\langle 6_1 \rangle_N = \sum_{\substack{a,b,c=0 \\ a+b \le c}}^{N-1} \frac{\left|(\omega)_c\right|^2}{(\omega)_a\,(\omega)_b^{\,*}} \omega^{(c-a-b)(c-a+1)}. \tag{4–14}$$

- Numerical result (Figure 4):

$$
\begin{aligned}
\mathrm{Vol}(S^3 \setminus 6_1) &= 3.16396322888...\\
c_1 &= 3.1639628602 \pm 3.04 \times 10^{-8}\\
c_2 &= 1.5000356 \pm 1.88 \times 10^{-6}\\
c_3 &= -4.0343627 \pm 0.0000611\\
c_4 &= 3.971777 \pm 0.000970.
\end{aligned}
$$

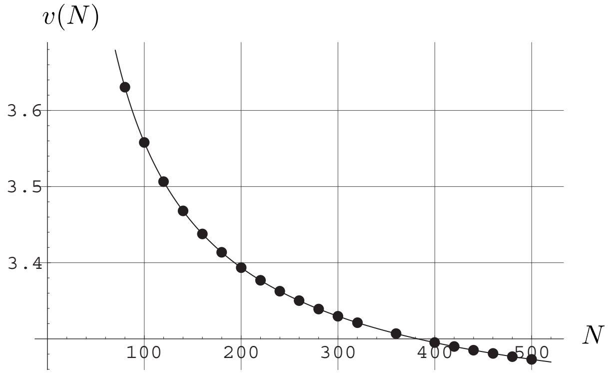


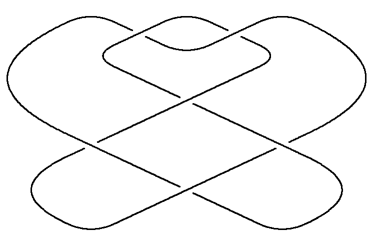
FIGURE 4. Knot 6_1.

- Potential and saddle point:

$$V_{6_1}(x,y,z) = \mathrm{Li}_2(z) - \mathrm{Li}_2(z^{-1}) - \mathrm{Li}_2(x) + \mathrm{Li}_2(y^{-1}) - \log\left(\frac{z}{x\,y}\right)\log(z/x) + 2\,\pi\,\mathrm{i}\log(x/z), \tag{4–15}$$

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} 0.17385 + 1.06907\,\mathrm{i} \\ 0.322042 + 0.15778\,\mathrm{i} \\ 0.278726 - 0.48342\,\mathrm{i} \end{pmatrix}.$$

6_2 Knot.



$$\langle 6_2 \rangle_N = \sum_{\substack{a,b,c=0 \\ a \le b \\ 0 \le a+c \le N-1}}^{N-1} \omega^{-a(b+c+1)} \left(\frac{(\omega)_b}{\left|(\omega)_a\right|}\right)^2 \frac{(\omega)_{a+c}}{(\omega)_{b-a}}. \tag{4–16}$$

- Numerical result (Figure 5):

$$\begin{aligned} \mathrm{Vol}(S^3 \setminus 6_2) &= 4.40083251... \\ c_1 &= 4.400828513 \pm 2.97 \times 10^{-7} \\ c_2 &= 1.500213389 \pm 9.83 \times 10^{-6} \\ c_3 &= -4.685095 \pm 0.00028 \\ c_4 &= 6.02178 \pm 0.00266. \end{aligned}$$

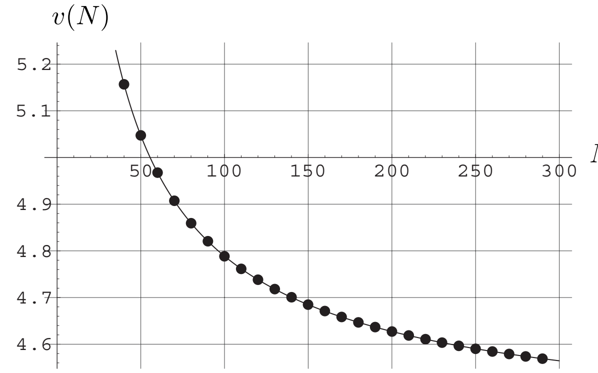


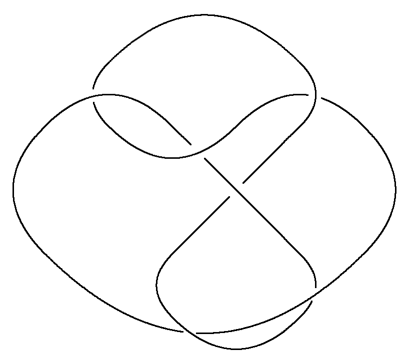
FIGURE 5. Knot 6_2.

- Potential and saddle point:

$$V_{6_2}(x,y,z) = 2\,\mathrm{Li}_2(y) + \mathrm{Li}_2(x\,z) - \mathrm{Li}_2(x) + \mathrm{Li}_2(x^{-1}) - \mathrm{Li}_2(y/x) + \log(x)\,\log(y\,z) - \frac{\pi^2}{3}, \tag{4–17}$$

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} 0.09043267 + 1.60288\,\mathrm{i} \\ -0.232705 - 1.09381\,\mathrm{i} \\ -0.964913 - 0.621896\,\mathrm{i} \end{pmatrix}.$$

6_3 Knot.



$$\langle 6_3 \rangle_N = \sum_{\substack{a,b,c=0 \\ a+b+c \le N-1}}^{N-1} \left|\frac{(\omega)_{a+b+c}}{(\omega)_b\,(\omega)_c}\right|^2 (\omega)^*_{a+b}\,(\omega)_{a+c}\,\omega^{(a+1)(b-c)}. \tag{4–18}$$

- Numerical result (Figure 6):

$$\begin{aligned} \mathrm{Vol}(S^3 \setminus 6_3) &= 5.69302109... \\ c_1 &= 5.69289987 \pm 0.0000124 \\ c_2 &= 1.50411 \pm 0.00026 \\ c_3 &= -5.6162 \pm 0.0066 \\ c_4 &= 10.315 \pm 0.0397. \end{aligned}$$

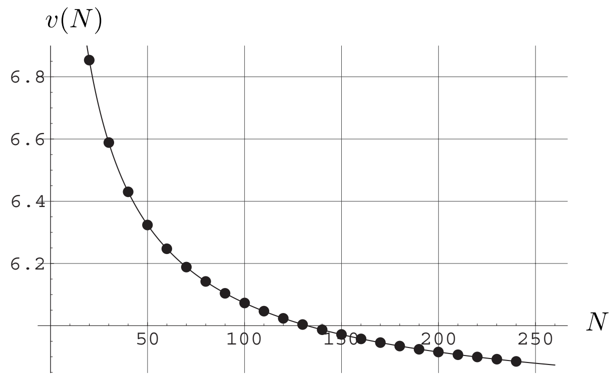


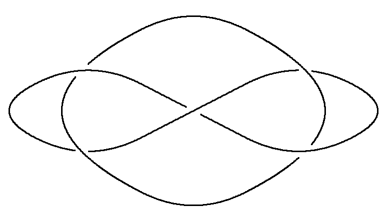
FIGURE 6. 6_3 Knot.

- Potential and saddle point:

$$\begin{aligned} V_{6_3}(x,y,z) = {} & \mathrm{Li}_2(xyz) - \mathrm{Li}_2((xyz)^{-1}) - \mathrm{Li}_2(y) \\ & + \mathrm{Li}_2(y^{-1}) - \mathrm{Li}_2(z) + \mathrm{Li}_2(z^{-1}) \\ & - \mathrm{Li}_2((xy)^{-1}) + \mathrm{Li}_2(xz) - \log(x)\ \log(y/z), \end{aligned} \tag{4–19}$$

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} 0.204323 - 0.978904\,\mathrm{i} \\ 1.60838 + 0.558752\,\mathrm{i} \\ 0.554788 + 0.19273\,\mathrm{i} \end{pmatrix}.$$

Whitehead Link 5_1^2.



$$\langle 5_1^2 \rangle_N = \sum_{\substack{a,b,c=0 \\ b \leq a \\ a+c \leq N-1}}^{N-1} \frac{(\omega)_{a+c}^*\,(\omega)_a}{(\omega)_b\,(\omega)_c^*}\,\omega^{c(a-b)}, \tag{4–20}$$

- Numerical result (Figure 7):

$$\begin{aligned} \mathrm{Vol}(S^3 \setminus 5_1^2) &= 3.66386237... \\ c_1 &= 3.663960 \pm 0.000113 \\ c_2 &= 1.49978 \pm 0.00190 \\ c_3 &= -3.2729 \pm 0.0461 \\ c_4 &= 6.1846 \pm 0.2549. \end{aligned}$$

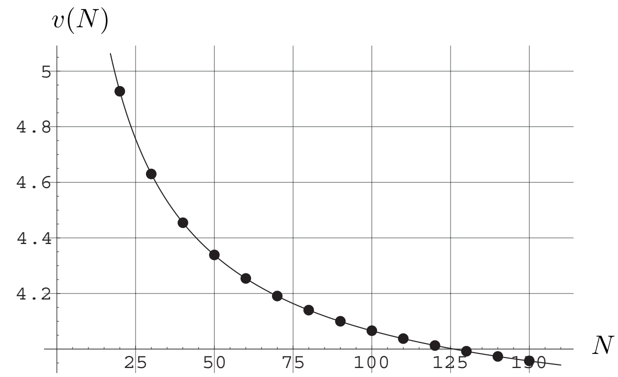


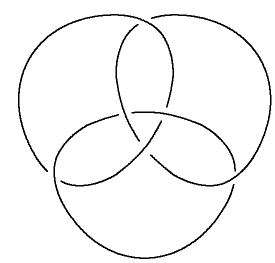
FIGURE 7. Whitehead link.

- Potential and saddle point:

$$\begin{aligned} V_{5_1^2}(x,y,z) = {} & -\mathrm{Li}_2((x\,z)^{-1}) + \mathrm{Li}_2(x) - \mathrm{Li}_2(y) \\ & + \mathrm{Li}_2(z^{-1}) - \log(z)\log(x/y), \end{aligned} \tag{4–21}$$

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} -\mathrm{i} \\ \mathrm{i} \\ \frac{1}{2}(1+\mathrm{i}) \end{pmatrix}.$$

Borromean Rings 6_2^3.



$$\begin{aligned} \langle 6_2^3 \rangle_N = & \sum_{\substack{a,b,c,d=0 \\ a \leq b \leq a+c \leq N-1 \\ b+d \leq N-1}} \left| \frac{(\omega)_{a+c}\,(\omega)_{b+d}}{(\omega)_d\,(\omega)_{a+c-b}} \right|^2 \\ & \times \frac{1}{(\omega)_a\,(\omega)_{b-a}^*}\,\omega^{(b+1)(c-d+a-b)}. \end{aligned} \tag{4–22}$$

- Numerical result (Figure 8):

$$\begin{aligned} \mathrm{Vol}(S^3 \setminus 6_2^3) &= 7.32772475... \\ c_1 &= 7.3276812 \pm 4.1 \times 10^{-6} \\ c_2 &= 1.50176 \pm 0.00011 \\ c_3 &= -8.76447 \pm 0.00296 \\ c_4 &= 11.116 \pm 0.025. \end{aligned}$$

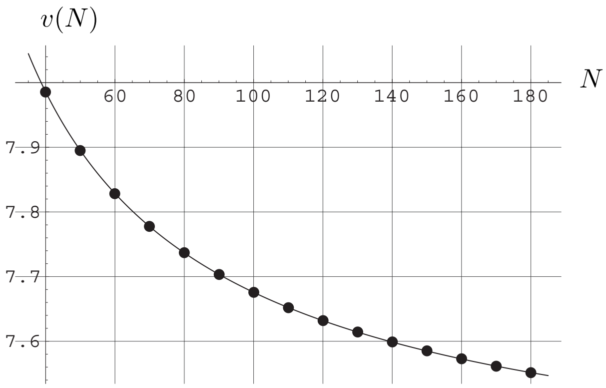


FIGURE 8. Borromean rings.

- Potential and saddle point:

$$V_{6^3_2}(x,y,z,w) = \mathrm{Li}_2(z) - \mathrm{Li}_2(z^{-1}) + \mathrm{Li}_2(y\,w) - \mathrm{Li}_2(\frac{1}{y\,w}) - \mathrm{Li}_2(w) + \mathrm{Li}_2(w^{-1}) - \mathrm{Li}_2(z/y) + \mathrm{Li}_2(y/z) - \mathrm{Li}_2(x) + \mathrm{Li}_2(x/y) - \log y \log(\frac{z}{y\,w}), \tag{4–23}$$

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ w_0 \end{pmatrix} = \begin{pmatrix} 0 \\ -\mathrm{i} \\ 1-\mathrm{i} \\ \frac{1+\mathrm{i}}{2} \end{pmatrix}.$$

Remark 4.2. There may exist q-series identities which arise from Kashaev's invariant for hyperbolic knots and links.

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