# Matrix generators for the Harada–Norton group

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**CONTENTS**



We show how to construct $133 \times 133$ matrices over $GF(5)$ generating the Harada–Norton group. We also obtain generators for its automorphism group. For many purposes this permits much faster calculations in the group than the alternative of permutations on 1,140,000 points. More importantly, it reduces storage requirements by a factor of around 500.

## 1. INTRODUCTION

The Harada–Norton group HN is one of the 26 sporadic simple groups. It has order

$$273{,}030{,}912{,}000{,}000 = 2^{14}.3^6.5^6.7.11.19.$$

It was first studied in [Harada 1976; Norton 1975]. The latter gives implicitly (but not explicitly) a construction of both the 133-dimensional real representation and the permutation representation on 1,140,000 points. Our goal is to construct a matrix representation of HN that can be used to carry out explicit calculations within the group: explicit matrix generators are available from the authors by electronic mail. Our paper provides a good illustration of the computational methods available for the construction of matrix groups.

We describe two constructions, the first closely following the method outlined in [Parker and Wilson 1990], and the second using a modified procedure that leads to a more complicated sequence of smaller machine computations.

From the point of view of efficient computer calculation, a small matrix representation over a small field seems the most desirable. The smallest representation of HN has degree 132 and is written over $GF(4)$. This representation is constructed in [Wilson 1993] as a byproduct of the construction of the Baby Monster. On the other hand, the smallest


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representation of HN: 2 is of degree 133 over GF(5), so it will also be useful to construct this representation in characteristic 5, which is in a sense the "natural" characteristic for this group. (In fact the smallest representation of HN in any other characteristic also has degree 133. This is easily seen by restricting putative modular characters to various subgroups, using [Jansen et al.].)

In Sections 2 and 3 we describe two constructions of a 133-dimensional matrix representation of HN. Both are based on the idea of amalgamating known subgroups of HN inside $\mathrm{GL}_{133}(5)$:

$$\begin{tikzcd}
& \mathrm{GL}_{133}(5) \arrow[d, dash] \arrow[ddl, dash, "f_1"'] \arrow[ddr, dash, "f_2"] & \\
& \mathrm{HN} \arrow[dl, dash] \arrow[dr, dash] & \\
H \arrow[dr, dash] & & K \arrow[dl, dash] \\
& L &
\end{tikzcd}$$

A general method for carrying out such computations is described in [Parker and Wilson 1990]; it consists of the following steps:

1. Select subgroups $H$ and $K$ of HN, and let $L = H \cap K$.
2. Construct representations $f_1 : H \to \mathrm{GL}_{133}(5)$ and $f_2 : K \to \mathrm{GL}_{133}(5)$.
3. Find generators for $f_1(L)$ inside $f_1(H)$ and $f_2(L)$ inside $f_2(K)$.
4. Conjugate $f_1$ to an equivalent representation $f_1'$ with $f_1'(L) = f_2(L)$.
5. Conjugate $f_1'$ to a representation $f_1''$ while keeping $f_1''(L) = f_1'(L)$, until $\langle f_1''(H), f_2(K)\rangle \cong \mathrm{HN}$.

This plan allows for considerable flexibility in the choice of $H$ and $K$; our two constructions show how different choices of these subgroups can lead to very different implementations of the basic plan. Other similar constructions described in the literature include O'Nan's group [Ryba 1988a; Jansen and Wilson 1994], Thompson's group [Linton 1989], the Baby Monster [Wilson 1993], Held's group [Ryba 1988b], and the double cover of the Higman–Sims group [Suleiman and Wilson 1992]. Our notation follows the *Atlas* [Conway et al. 1985].

In both constructions we perform our computer calculations with the collection of matrix manipulation programs known as the Meat-axe [Parker 1984]. In addition to well-known procedures for matrix multiplication, nullspace calculation, etc., the Meat-axe contains a pair of specialized procedures: CH, which "chops" the matrix representation defined by a pair of matrices into its irreducible constituents, and SB, which calculates a canonical basis for an absolutely irreducible representation. The complexity of the Meat-axe procedures depends on the degree $n$ of the representations that we work with: most of the procedures require space proportional to $n^2$ and time proportional to $n^3$.

Our first construction was carried out by the second author, using an implementation of the Meat-axe system written by Richard Parker of Perihelion Software Ltd., on the Birmingham University Computer Centre's IBM 3090. This version of the Meat-axe allows for computations with representations of degrees up to around 1000: we were able to treat HN as a "small" group and we followed the steps of [Parker and Wilson 1990] very closely. The largest computation that we used was a chop on a 462-dimensional representation. In this case the application of Step 5 involved testing 250 possibilities for $f_1''$.

Our second construction was carried out by the first author, using a smaller implementation of the Meat-axe on a Sun workstation. This version of the Meat-axe allows for matrices of dimensions up to about 200: the size limitation forced us to treat HN as a "large" group and we modified the steps of [Parker and Wilson 1990] accordingly. The largest individual step of this computation consisted of the calculation of the nullspace of a matrix of size $134 \times 168$; we finished the construction by locating the group HN as one of 12 possible matrix groups. As a preliminary, we needed some local analysis of HN; this is given in Sections 3.1 and 3.2.

## 2. THE FIRST CONSTRUCTION OF HN IN $\mathrm{GL}_{133}(5)$

The general strategy is to start with $A_{12}$, find a subgroup $M_{12}$, and adjoin an outer automorphism of $M_{12}$. Since $A_{12}$ is a maximal subgroup of HN [Norton and Wilson 1986], the group can be generated in this way. The degree-133 character of HN restricts to $A_{12}$ mod 5 as $1 + 43 + 89$ [Jansen et al.], and since the representation is self-dual it is a direct sum. Restricting further to $M_{12}$, the 43 becomes a uniserial module with factors 16, 11a, 16*, while the 89 breaks up as $1 \oplus 45 \oplus [43']$, where $[43']$ denotes a uniserial module with factors 16*, 11b, 16. (The structure of these modules was determined using the Meat-axe: see below.) The outer automorphism of $M_{12}$ now fuses the two indecomposables [43] and $[43']$, and acts as the regular representation of $C_2$ on the fixed space of $M_{12}$. The situation is illustrated as follows:

$$\begin{tikzcd} & \mathrm{HN} \arrow[dl, dash] \arrow[dr, dash] & \\ 1 \oplus 43 \oplus 89 \quad A_{12} & & M_{12}{:}2 \quad 1 \oplus 1' \oplus [86] \oplus 45 \\ & M_{12} \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

$$1 \oplus [43] \oplus 1 \oplus [43'] \oplus 45$$

### 2.1. Making the Representation of $A_{12}$

All three required irreducibles are constituents of the permutation representation of degree 462, on the cosets of $A_{12} \cap (S_6 \wr S_2)$. This can easily be made from the deleted permutation representation on 12 points, by using the Meat-axe program VP (vector permute) to find the images of a vector fixed by the given subgroup. This permutation representation on 462 points can then be made into a matrix representation and chopped up with the Meat-axe in the usual way [Parker 1984]. Finally we make the direct sum of the three representations.

### 2.2. Finding a Subgroup $M_{12}$ in $A_{12}$

A fixed-point-free element of order three and an involution of cycle type $2^4 1^4$ have a reasonable probability of generating $M_{12}$, so a random search will quickly produce words giving a subgroup $M_{12}$. If we do this in such a way that the product of our two generators has order 11, we have what we shall call "standard generators" for $M_{12}$. More precisely, there are up to conjugacy just two such pairs of generators, and these are interchanged by the outer automorphism. In one case the product is in class 11A, in the other, 11B.

### 2.3. Putting the Representation into Canonical Form

Using our "standard generators" for $M_{12}$ we need to define a "standard basis" for the representation, and write all our matrices with respect to this basis. Of course, the representation of $M_{12}$ is not completely reducible, so this is not as easy as it might otherwise be. One problem is that the 43-dimensional indecomposable summands are not well-defined: there are five of each.

The usual way of using the Meat-axe program SB (standard base) is to choose an element $f$ of the group algebra that has nullity 1 in its action on one of the irreducibles, and nullity 0 on all the others. Spinning up a null vector of $f$ in its action on the whole space then gives a standard basis for an invariant subspace on which the group acts in the specified manner. In the present instance we can generalise slightly, by taking a group algebra element $f$ having nullity 1 on 16 and nullity 0 on everything else. Then $f$ has nullity 2 on the full 133-dimensional representation, and five of the six 1-spaces in this nullspace spin up to 43-dimensional invariant subspaces. We can take any one of these to be our standard indecomposable summand, and SB will produce a standard base for the given 43-space. Similarly, we obtain standard bases for the other summands, and concatenate them in order, to get a standard base for the whole space. (In fact we chose our two 43-spaces to be the ones contained in simple $A_{12}$-submodules. However, this choice does not seem to be better than any other.)

### 2.4. A Second Set of Generators for $M_{12}$

We now must find, as words in our standard generators for $M_{12}$, a second set of standard generators, conjugate to the first by an outer automorphism of $M_{12}$. A repeat of the random search will produce candidates for these. To show that our new generators are not conjugate to the old ones inside $M_{12}$, it suffices to show that from a representation (such as 11a) not invariant under $\operatorname{Aut} M_{12}$ we obtain an inequivalent representation.

### 2.5. A Second Standard Base

What we are looking for is a suitable involution conjugating our first set of generators for $M_{12}$ to the second. First we use the same procedure as above to produce a standard base for the space. This time however we have to consider all five possibilities for each 43-dimensional indecomposable. Thus there are 25 possibilities in all.

Now each of these 25 matrices will conjugate our first pair of generators for $M_{12}$ to the second. Moreover we can multiply this by any matrix centralizing $M_{12}$. Thus we obtain a large number of possible conjugating matrices. However, there is no point in conjugating by something centralizing $A_{12}$, either before or after conjugating by our standard base matrix. This means that in fact we need only consider elements which act as involutions of $\mathrm{GL}_2(5)$ on the fixed 2-space of $M_{12}$, and trivially elsewhere. Moreover, these only need to be considered modulo scalars, making ten cases for each standard base matrix, or 250 in all.

Of these 250 cases, 249 give rise to groups containing elements of order greater than 40. Thus they cannot be HN. On the other hand, HN does have a representation which can be constructed in this way, so the one remaining case must generate HN.

## 3. THE SECOND CONSTRUCTION OF HN IN $\mathrm{GL}_{133}(5)$

We now give an alternative computer construction of HN as an explicit subgroup of $\mathrm{GL}_{133}(5)$. Our plan is to amalgamate representations of subgroups $U \cong U_3(8){:}\,3$ and $T \cong 2^3.2^2.2^6{:}\,(F_{21} \times 3)$ over an intersection $B = U \cap T \cong 2^{3+6}{:}\,(F_{21} \times 3)$ to produce a representation of HN:

$$\begin{tikzcd}[column sep=small] & \mathrm{HN} \arrow[dl, dash] \arrow[dr, dash] & \\ U_3(8){:}3 \cong U & & T = \langle B, x\rangle \cong 2^3.2^2.2^6{:}(F_{21}\times 3) \\ & B \cong 2^{3+6}{:}(F_{21}\times 3) \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

The biggest obstacle that we encounter is the computation of 133-dimensional representations of the subgroups $U$ and $T$. We remark that there are two isomorphism types of split extension $U_3(8){:}\,3$. We shall reserve the notation for the group $U_3(8){:}\,3_1$ [Conway et al. 1985, p. 66]; this particular extension arises as a subgroup in both HN and $E_7(5)$ (the latter embedding is exhibited in [Griess and Ryba 1994]). A 56-dimensional representation of $U_3(8){:}\,3$ was computed in [Griess and Ryba 1994]; in Section 3.3 we make use of the fact that this representation of $U$ extends to the simple group $E_7(5)$ to obtain a 133-dimensional representation of $U$.

The group $T$ is solvable, but it is not conveniently described in the literature: we devote Sections 3.1 and 3.2 to a theoretical study of $T$ in order to facilitate its construction. We show that $T$ can be constructed by adjoining a particular element $x$ to the readily accessible maximal subgroup $B$ of $U$. The results of Sections 3.1–3.3 are summarized in Proposition 3.3, which can be regarded as a link between the work on subgroups and the construction of HN itself. The remaining sections are devoted to a computer construction of a matrix representation of the element $x$ (and thus of $\mathrm{HN} \cong \langle U, x\rangle$).

### 3.1. Properties of the Group B and its Subgroups

Let $H \cong \mathrm{HN}$, and let $H \geq U \geq B$ with $U \cong U_3(8){:}\,3$, $B \cong 2^{3+6}{:}\,(F_{21} \times 3)$ [Conway et al. 1985, pp. 66, 166]. Let

$$\begin{aligned} W &= O_2(B) \cong 2^{3+6}, \\ K &= Z(W) \cong 2^3, \\ \bar{W} &= W/K \cong 2^6. \end{aligned}$$

Let $F$ be a subgroup of $B$ isomorphic to $F_{21}$, and let $\alpha$ be a 3-element of $C_B(F)$.

The next proposition summarizes some structural information about the small solvable group $B$ and its subgroups. We computed these properties with the matrix represenation of $B$ given in [Griess and Ryba 1994, Sec. 3].

**Proposition 3.1.** (i) *$K$ is an absolutely irreducible $F$-module.*

(ii) *$\bar{W}$ is a direct sum of two isomorphic absolutely irreducible (three-dimensional) $F$-modules; thus $\bar{W}$ has exactly three proper $F$-submodules, $\bar{W}_1$, $\bar{W}_2$, $\bar{W}_3$, say.*

(iii) *Let $W_i$ be the preimage of $\bar{W}_i$ in $W$; then each $W_i$ is the direct product of three cyclic groups of order 4.*

(iv) *As $F$-modules, each $\bar{W}_i \cong K$; the isomorphism is given by the squaring map, $\bar{w} \mapsto w^2$.*

(v) *The element $\alpha$ cyclically permutes $W_1$, $W_2$ and $W_3$.*

### 3.2. The Subgroup T of HN

Let $H \cong \mathrm{HN}$, $U \cong U_3(8){:}3$, $B \cong 2^{3+6}{:}(F_{21} \times 3)$, $W \cong 2^{3+6}$, $K \cong 2^3$, $W_1 \cong W_2 \cong W_3 \cong 4^3$, $F \cong F_{21}$ and $\alpha$ be as in 3.1. Let $\sigma$ be a 7-element in $F$.

Let $N = N_H(K)$, so that [Norton and Wilson 1986, Sec. 3.1]

$$N \cong 2^3.2^2.2^6.(L_3(2) \times 3).$$

Let $T = \langle O_2(N), F, \alpha\rangle \cong 2^3.2^2.2^6{:}(F_{21} \times 3)$; then $T \cap U = B$ (since $B$ is maximal in $U$). Moreover, since $U$ is maximal in $H$, $H = \langle U, T\rangle$. Let $C$ be the unique normal subgroup of $N$ with structure $2^3.2^2.2^6.3$; then $C = C_N(K)$, since $N/C$ is the unique quotient of $N$ between $F_{21}$ and $L_3(2)$, and

$$F_{21} \cong F/(F \cap C_N(K)) \subseteq N/C_N(K) \subseteq \mathrm{Aut}\, K \cong L_3(2).$$

Let $X$ be any normal subgroup of $N$ which contains $K$ and has structure $2^3.2^2$. Let $\bar{X} = X/K \cong 2^2$. We claim that $|C_X(\sigma)| = 4$ (because $|C_X(\sigma)| \equiv |X| \equiv 4 \pmod 7$, and $|C_X(\sigma)| \le 4$ since $\sigma$ acts fixed point freely on $K$). Let $x$ be one of the three nonidentity elements of $C_X(\sigma)$; $x$ must be in class 2A of $H$ [Conway et al. 1985, p. 164].

**Lemma 3.2.** (i) $[X, W] \le K$.

(ii) *The element $x$ centralizes exactly one of $W_1$, $W_2$, and $W_3$; it inverts every element of the other pair of these groups.*

(iii) *The group $H$ can be generated from its subgroup $U$ by adjoining a 2A-involution that centralizes $\langle W_1, F\rangle$ and inverts all elements of $W_2$ and $W_3$.*

*Proof.* (i) Observe that $\bar{W}$ acts on $\bar{X}$ via the conjugation action of $W$ on $X$ (since $W$ and $X$ both centralize $K$). Thus $C_{\bar{W}}(\bar{X})$ is an $F$-invariant submodule of $\bar{W}$ (since $W$ and $X$ are $F$-invariant). But $\mathrm{Aut}\, \bar{X} \cong S_3$, so the co-dimension of $C_{\bar{W}}(\bar{X})$ in $\bar{W}$ is at most 1; hence $C_{\bar{W}}(\bar{X}) = \bar{W}$ (by 3.1(ii)). Therefore $[X, W] \le K$.

(ii) The map $[x,\ ] : W \to K$ induces an $F$-invariant linear map from $\bar{W}$ to $K$, because

$$[x, w^f] = [x^f, w^f] = [x, w]^f.$$

In particular, for each $i$, $[x,\ ] : W_i \to K$ is either the trivial map or the squaring map. Observe that $[x,\ ]$ cannot be trivial on all three groups $W_i$ (since $C_H(x)/\langle x\rangle \cong \mathrm{HS}.2$ has no subgroup isomorphic to $W$ [Conway et al. 1985, p. 80]). Moreover, if $w_1 \in W_1$, then $w_2 = w_1^\alpha \in W_2$, and $w_3 = w_1 w_2 \in W_3$; thus

$$[x, w_1][x, w_2][x, w_3] = 1$$

(since $[x, W] \le K \le Z(\langle X, W\rangle)$). It follows that $[x,\ ]$ is trivial on exactly one of the three subgroups $W_i$, and it gives the squaring map on the other two subgroups.

(iii) We may assume that $[x, W_1] = [x^\alpha, W_2] = [x^{\alpha^2}, W_3] = 1$ (by replacing $x$ by a conjugate under $\langle\alpha\rangle$ if necessary). Hence,

$$\{1, x\} = C_{C_X(\sigma)}(W_1),$$

and therefore $F$ centralizes $x$ (since $F$ normalizes $W_1$, $X$ and $\langle\sigma\rangle$). The claim follows, since $U$ is maximal in $H$, and $x \notin U$. □

### 3.3. Representations of $U_3(8){:}\,3$

We begin by describing our computer construction of a 133-dimensional representation of $U_3(8){:}\,3$. We made use of the explicit embedding

$$U_3(8){:}\,3 \le 2E_7(5) \le \mathrm{GL}_{56}(5)$$

and of the corresponding matrix representations of the fundamental roots of an invariant Lie algebra $\mathcal{E}$ of type $E_7$ over $F_5$ (the relevant matrices for $U_3(8){:}\,3$ and $\mathcal{E}$ are displayed in Section 8 of [Griess and Ryba 1994]).

Conjugation by $U_3(8){:}\,3$ preserves $\mathcal{E}$ and therefore we can compute a 133-dimensional matrix representation of $U_3(8){:}\,3$ as follows:

**Calculation 3.3.** 1. Compute and store matrix representatives $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_{133}$ (in $\mathrm{GL}_{56}(5)$) for a Chevalley basis of $\mathcal{E}$.
2. Select a generating pair of $56 \times 56$ matrices $U_1$, $U_2$ for $U_3(8){:}\,3$.
3. Calculate each of the 266 matrices $\mathcal{E}_i^{U_k}$, and find the corresponding linear combinations $\mathcal{E}_i^{U_k} = \Sigma A_{ij}^k \mathcal{E}_j$.

Then the $133 \times 133$ matrices $A_{ij}^1$ and $A_{ij}^2$ generate a group $U \cong U_3(8){:}\,3$. In principle, Step 3 could be carried out by applying Gaussian elimination 266 times to various $134 \times 133^2$ matrices. In practice, this process can be enormously speeded up by the following standard sampling technique, which was also used in [Griess and Ryba 1994, Sec. 8].

**Calculation 3.3′.** 3a. Select a random $3 \times 56$ matrix, say $M$.
3b. Calculate the $3 \times 56$ matrices $F_i^k = M\mathcal{E}_i^{U_k}$ and $F_j = M\mathcal{E}_j$.
3c. Apply Gaussian elimination 266 times on $134 \times 168$ matrices to obtain the values of $A_{ij}^k$ as coefficients in: $F_i^k = \Sigma A_{ij}^k F_j$. (In the unlikely event that more than one such combination is ever found we go back to (3a) and make a more random choice for $M$. There must always be at least one such linear combination since $U_3(8){:}\,3$ preserves the Lie algebra $\mathcal{E}$.)

The matrix generators of $U_3(8){:}\,3$ given in [Griess and Ryba 1994] include matrix generators for the subgroups $B$, $F$, $W_1$, $W_2$, and $W_3$. We applied the two preceding calculations to these matrices to obtain explicit copies of these subgroups inside our matrix group $U = \langle A_{ij}^1, A_{ij}^2 \rangle$.

We used the Meat-axe program CH on $U$ to prove that this matrix group is irreducible (on its natural 133-dimensional represenation). Note also that $U$ preserves the Killing form on $\mathcal{E}$, so it gives a self-dual irreducible 133-dimensional representation of $U_3(8){:}\,3$. The group $U_3(8){:}\,3$ has exactly three different self-dual 133-dimensional representations; but these representations are equivalent under another outer automorphism of $U_3(8)$. Now, observe that the 133-dimensional complex representations of HN restrict to self-dual irreducible complex representations of $U_3(8){:}\,3$ (see [Conway et al. 1985]). It follows that any self-dual irreducible 133-dimensional 5-modular representation of $U_3(8){:}\,3$ must extend to HN (because 5 does not divide $|U_3(8){:}\,3|$). We conclude that $U$ can be extended to HN inside $\mathrm{GL}_{133}(5)$. Together with Lemma 3.2 this gives:

**Proposition 3.3.** *There is a matrix $x$ in $\mathrm{GL}_{133}(5)$ with the following properties*:

(i) $\langle U, x\rangle \cong \mathrm{HN}$.
(ii) *$x$ has trace 1 as an element of $F_5$*.
(iii) $[x, W_1] = [x, F] = 1$.
(iv) *$wxw - x = 0$ for all $w \in W_2 \cup W_3$*.
(v) $\langle B, x\rangle = T$.

### 3.4. Calculation of Centralizer Algebras

We now investigate the collection of all matrices satisfying the condition (iii) of the previous proposition. This requires the calculation of a centralizer algebra, and, as a preliminary, an analysis of the restriction of the natural 133-dimensional $F_5U$-module to a subgroup

$$G = \langle W_1, F\rangle \cong 4^3{:}\,F_{21}.$$

The proof of the following result is a computation carried out with the Meat-axe program CH.

**Lemma 3.4.1.** *Let $V$ denote the natural 133-dimensional $F_5U$-module. Then the restriction $V_G$ has constituents*

$$1 \oplus 6 \oplus 7\mathrm{a}^2 \oplus 7\mathrm{b}^2 \oplus 7\mathrm{c}^2 \oplus 21\mathrm{a}^2 \oplus 21\mathrm{b}^2.$$

In this statement we use 1, 6, 7a, 7b, 7c, 21a and 21b to denote the different isomophism types of irreducible constituents of $V$; the superscripts denote multiplicities. The dimension of a constituent representation is given by its name. Thus, for instance, $V_G$ has two irreducible summands of type 7a; these summands are each 7-dimensional. We remark that the restriction $V_G$ actually has

$$6 = \frac{5^2-1}{5-1}$$

irreducible submodules of type 7a; our application of the Meat-axe has selected an arbitrary pair of these (and the other such submodules are contained in the linear span of our selected pair). The program CH shows that the 6-dimensional submodule of $V_G$ decomposes as a direct sum of two algebraically conjugate absolutely irreducible three-dimensional summands defined over $F_{25}$; the calculation also shows that the other irreducible constituents of $V_G$ are absolutely irreducible.

Although we established Lemma 3.4.1 by applying the Meat-axe it is also easy to obtain a non-computer proof by character theory. The advantage of using the Meat-axe is that we obtain actual lists of vectors giving bases of the irreducible constituents of $V_G$. Once we have determined bases for the constituents of $V_G$ it is useful to turn them into standard bases (by means of the Meat-axe program SB). (We must work over the field $F_{25}$ in order to obtain a standard basis for the six-dimensional constituent of $V_G$.)

The decomposition of $V_G$ leads to a decomposition of $\mathcal{C}$, the centralizer algebra, consisting of all $133\times 133$ matrices that commute with the matrices in $G$. For each isomorphism class, $n$ say, of irreducible summand of $V_G$, we let $\mathcal{C}_n$ denote the subalgebra of $\mathcal{C}$ consisting of matrices with row spaces in the sum of the subspaces of type $n$ in $V_G$.

**Corollary 3.4.** *As an algebra,*

$$\mathcal{C} = \mathcal{C}_1 \oplus \mathcal{C}_6 \oplus \mathcal{C}_{7\mathrm{a}} \oplus \mathcal{C}_{7\mathrm{b}} \oplus \mathcal{C}_{7\mathrm{c}} \oplus \mathcal{C}_{21\mathrm{a}} \oplus \mathcal{C}_{21\mathrm{b}}.$$

*Moreover, as $F_5$-algebras, $\mathcal{C}_1 \cong F_5$, $\mathcal{C}_6 \cong F_{25}$, and $\mathcal{C}_{7\mathrm{a}} \cong \mathcal{C}_{7\mathrm{b}} \cong \mathcal{C}_{7\mathrm{c}} \cong \mathcal{C}_{21\mathrm{a}} \cong \mathcal{C}_{21\mathrm{b}} \cong M_2(5)$, where $M_2(5)$ is the algebra of $2\times 2$ matrices over $F_5$.*

This follows immediately from Lemma 3.4.1 and Schur's Lemma (see [Alperin 1986, Sec. 2], for example). In particular, we see that $\dim \mathcal{C} = 23$. In order to compute matrices corresponding to particular elements of the centralizer algebra $\mathcal{C}$, we make use of the standard bases described above. For example, if $b_1, b_2, \ldots, b_7$ and $b'_1, b'_2, \ldots, b'_7$ are standard bases of our two summands of type 7a, the element of $\mathcal{C}_{7\mathrm{a}}$ that corresponds to $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ is the matrix of $\mathrm{GL}_{133}(5)$ that maps $b_i$ to $\alpha b_i + \beta b'_i$ and $b'_i$ to $\gamma b_i + \delta b'_i$, and is zero on the complementary submodules. A similar construction produces bases for the matrix algebras $\mathcal{C}_1$, $\mathcal{C}_{7\mathrm{b}}$, $\mathcal{C}_{7\mathrm{c}}$, $\mathcal{C}_{21\mathrm{a}}$, and $\mathcal{C}_{21\mathrm{b}}$. A slightly more complicated procedure yields a basis of $\mathcal{C}_6$; however, we shall only use elements of the one-dimensional subalgebra $\mathcal{C}_6^* \le \mathcal{C}_6$ with $\mathcal{C}_6^* \cong F_5$. As a basis element of $\mathcal{C}_6^*$ we take the $133\times 133$ matrix that acts as the identity on the 6-dimensional summand of the $F_5G$-module $V$, and acts as zero on the complementary submodule.

Although this is not strictly necessary for our later purposes, it is informative to carry out an exactly similar calculation for the larger group $W{:}F$, proving that:

**Lemma 3.4.2.** *The centralizer algebra of the 133-dimensional matrix group $W{:}F$ is 7-dimensional.*

This implies that there is a 7-parameter family of $133\times 133$ matrices satisfying conditions (iii) and (iv) of Proposition 3.3.

### 3.5. Computer Construction of a Representation of x

We are now in a position to compute all matrices $x$ that satisfy the conditions of Proposition 3.3. We begin by noting that $x$ belongs to the centralizer algebra $\mathcal{C}$ (see Corollary 3.4), and that the components of $x$ in our decomposition of $\mathcal{C}$ must all square to 1. We observe that both square roots of 1 in $\mathcal{C}_6 \cong F_{25}$ lie in the prime subfield $\mathcal{C}_6^* \cong F_5$; therefore $x$ must actually belong to the 22-dimensional subalgebra

$$\mathcal{C}^* = \mathcal{C}_1 \oplus \mathcal{C}_6^* \oplus \mathcal{C}_{7\mathrm{a}} \oplus \mathcal{C}_{7\mathrm{b}} \oplus \mathcal{C}_{7\mathrm{c}} \oplus \mathcal{C}_{21\mathrm{a}} \oplus \mathcal{C}_{21\mathrm{b}}$$

of $\mathcal{C}$. We shall work with an explicit basis (consisting of matrices) $c_1, c_2, \ldots, c_{22}$ for $\mathcal{C}^*$ (we described the construction of such a basis immediately after Corollary 3.4).

The elements of $\mathcal{C}^*$ that satisfy condition (iv) of Proposition 3.3 form a subspace $\mathcal{C}^{**}$. The following calculation provides an explicit basis of $\mathcal{C}^{**}$.

**Calculation 3.5.** 1. Pick a 4-element, $w$ say, of $W_2$. Calculate the matrices $c_i' = wc_iw - c_i$.
2. Use a sampling technique (similar to Calculation 3.3′) to find a basis for the space of vectors $\lambda_i$ with $\Sigma_i \lambda_i c_i' = 0$.
3. Let $\mathcal{C}^{**}$ be the subspace of $\mathcal{C}^*$ spanned by the corresponding combinations: $\Sigma_i \lambda_i c_i$.

We performed Calculation 3.5 with a particular choice of $w$ and obtained a seven-dimensional dimensional space $\mathcal{C}^{**}$. Lemma 3.4.2 shows that there is no point in trying to cut down to an even smaller candidate space for $x$ by using further choices for $w$. The seven-dimensional space $\mathcal{C}^{**}$ consists of matrices in $\mathcal{C}$ whose components correspond to the elements of $F_5$ and $M_2(5)$ given by $\alpha$, $\alpha$, $\left(\begin{smallmatrix} \beta & 2\gamma \\ 3\delta & \varepsilon \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} \zeta & \eta \\ \eta & \zeta \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 4\beta & \gamma \\ 2\delta & 4\varepsilon \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} \varepsilon & 3\delta \\ 2\gamma & \beta \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 4\varepsilon & 4\delta \\ 4\gamma & 4\beta \end{smallmatrix}\right)$, where $\alpha, \ldots, \eta$ are seven parameters from the field $F_5$. (We note that the actual $2 \times 2$ matrix components that we have presented depend on the particular choices of summands of Lemma 3.4.1. For example, $V_G$ has six subspaces of type 7a, and we selected two of these as summands. A different choice of a pair of summands would lead to a conjugate of the $2 \times 2$ matrix in the corresponding component of the centralizer algebra).

The trace condition, Proposition 3.3(ii), leads to an equation: $\alpha + 6\alpha + 7(\beta+\varepsilon) + 7(\zeta+\zeta) + 7(4\beta + 4\varepsilon) + 21(\varepsilon+\beta) + 21(4\varepsilon+4\beta) \equiv 1 \pmod 5$, and thus $2\alpha + 4\zeta \equiv 1 \pmod 5$. Combined with Proposition 3.3(i), this forces $\alpha = 1$, $\zeta = 1$ and $\eta = 0$. The order of $x$, now leaves us with just twelve possible sets of choices for the other four parameters. The corresponding twelve $133 \times 133$ matrices can be calculated, as linear combinations of $b_1, b_2, \ldots, b_{22}$. Eleven of the twelve possible matrices generate elements of order bigger than 40 when combined with elements of $U$. The remaining choice of $x$ must therefore extend the matrices of $U$ to a group isomorphic to HN (by Proposition 3.3).

## 4. EXTENDING HN TO HN: 2

The principle here is essentially the same as that used in Section 2 to extend $M_{12}$ to $M_{12}{:}\,2$, but this time the procedure is much more straightforward, since the representation is irreducible.

First we obtain "standard generators" $x$ and $y$ for HN. We chose $x \in 2\mathrm{A}$, $y \in 3\mathrm{B}$ with $xy \in 19\mathrm{A/B}$. Then we write our representation with respect to a "standard base" defined by $(x, y)$, as before. Then we find, as words in $x$ and $y$, a pair of generators $(x', y')$ that we believe to be automorphic to $(x, y)$. Using the Meat-axe program SB again we can verify this: $(x', y')$ is automorphic to $(x, y)$ if and only if the matrices representing $x'$ and $y'$ with respect to their standard basis are identical to those representing $x$ and $y$ with respect to the standard basis for $(x, y)$.

Further, the base change matrix conjugates $(x, y)$ to $(x', y')$. Thus by adjoining this matrix we obtain some subgroup of $4 \times \mathrm{HN}{:}\,2$, where the central 4 is represented by scalar matrices. Provided $(x, y)$ and $(x', y')$ are not conjugate in HN, we obtain generators for HN: 2 by multiplying our base-change matrix by a suitable scalar. Two of the possible scalars produce the two representations of HN: 2, while any other scalar gives a group $G$ containing elements of order 50, so $G$ cannot be HN: 2.

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## ELECTRONIC AVAILABILITY

Generators for the matrix representations described in this article can be obtained from the authors by e-mail (see addresses below).

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