Isomorphism Classes and Derived Series of Certain Almost-Free Groups

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CONTENTS

Baumslag defined a family of groups that are of interest because they closely resemble free groups, yet are not free. It was known that each group in this family has the same lower central series of quotients and the same first two terms in the derived series of quotients as does the free group F on two generators.

We have verified that there are different isomorphism types among the groups in the family, and that the third terms in the derived series of quotients are often distinct from that of F. Our basic technique is to count the number of homomorphisms from the groups of interest to a target group.

1. INTRODUCTION

While studying groups that resemble free groups, Baumslag [1967] defined the family

$$G_{ij} = \langle a, b, c : a = c^{-i} a^{-1} c^{i} a c^{-j} b^{-1} c^{j} b \rangle,$$

where $i, j \in \mathbb{Z}$. We describe a computer-assisted investigation of these groups, whose main result is:

Theorem. *There are several distinct isomorphism types among the G_{ij}. Furthermore, for some pairs (i, j), the derived series of quotients of G_{ij} differs from that of the free group F on two generators.*

Therefore some groups that strongly resemble free groups are in fact not free. To motivate our work, we recall earlier results about the G_{ij}.

As usual, given subgroups H, K of a group G, we define $[H, K]$ as the group generated by commutators $[h, k]$, for $h \in H$ and $k \in K$. We let

$$G_1 = G,\ G_2 = [G, G],\ \dots,\ G_i = [G_{i-1}, G_{i-1}],\ \dots$$

be the derived series of G, and

$$\gamma_1 G = G,\ \gamma_2 G = [G, G],\ \dots,\ \gamma_i G = [G, \gamma_i G],\ \dots$$

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the lower central series of G. We will usually consider the two series G/G_n and $G/\gamma_n G$, rather than the classical derived and central series; we refer to them as the *derived series of quotients* and the *central series of quotients*.

A fundamental and classic question is: How well do these towers capture the identity of a group? Can a group, for example, have the same towers as a free group and yet not be free?

It is easy to see that each G_{ij} modulo its commutator subgroup is the free Abelian group on two generators, $\langle a, b, c : a = 1, b^{-1}c^{-1}bc = 1\rangle$. Thus G_{ij} and the free group on two generators are alike in that

$$G_{ij}/[G_{ij}, G_{ij}] \cong F/[F, F].$$

In fact the resemblance goes much deeper. Baumslag showed that $G_{ij}/\gamma_n G_{ij} \cong F/\gamma_n F$ for every n. Moreover,

$$\bigcap_{n=1}^{\infty} \gamma_n G_{ij} = \bigcap_{n=1}^{\infty} \gamma_n F = 1.$$

Are the G_{ij} free? By a result of Whitehead [Magnus et al. 1976], G_{ij} is free if and only if the single defining relator

$$a^{-1}c^{-i}a^{-1}c^{i}ac^{-j}b^{-1}c^{j}b \qquad (1.1)$$

of G_{ij} is part of some free basis of the free group on three generators. Baumslag showed that this is impossible, so none of the G_{ij} is free.

Since they are not free, the next question is: Are the G_{ij} all distinct? Baumslag showed that they all have the same lower central series of quotients. For the derived series of quotients, hand computation shows that the first two entries are the same for all G_{ij}, and isomorphic to those of the free group F. Beyond that the computation gets very difficult; we shall return to this question later.

2. STRATEGY AND RESULTS

To show that two groups G and G' are distinct, one may try to show that they map differently to some third group—in other words, that the homomorphism sets $\mathrm{Hom}(G, H)$ and $\mathrm{Hom}(G', H)$ are distinct for some group H. For instance, one can try to choose H so the two sets of homomorphism are finite, and count the number of homomorphisms. This approach has been used several times before: see, for example, [Havas and Kovacs 1984; Holt and Rees 1990].

A nilpotent target group H will fail to discriminate between G_{ij} and G_{kl}, because all the G_{ij} have the same lower central series of quotients, and therefore the same homomorphisms into any nilpotent group. Having no theoretically attractive choice for the target, we simply choose reasonably small finite groups. Matrix groups over cyclic groups $\mathbb{Z}/n$ are an obvious choice, since they are usually not nilpotent, and matrices are easily representable on the computer. We use $\mathrm{SL}(2, \mathbb{Z}/n)$.

To count the homomorphisms we use the following well-known fact. Suppose $G = \langle x_1, \dots, x_n : w_1, \dots, w_r\rangle$ is a finitely presented group, H is any group, and $h_1, \dots, h_n$ are elements of H. A necessary and sufficient condition for there to be a homomorphism $G \to H$ taking x_i to h_i is that the elements of H obtained from the relators $w_1, \dots, w_r$ by replacing each x_i with h_i all be trivial. It is obvious that if such a homomorphism exists it is unique.

In our case, then, we can compute the number of homomorphisms from G_{ij} into $H = \mathrm{SL}(2, \mathbb{Z}/n)$ by taking all triples $(x, y, z) \in H \times H \times H$ and counting how many satisfy

$$x^{-1}z^{-i}x^{-1}z^{i}xz^{-j}y^{-1}z^{j}y = 1, \qquad (2.1)$$

the G_{ij} having a single defining relator (1.1). Liriano wrote a program to do this and ran it on a PC. With $n = 2$, all tested combinations of i and j yielded $36 = 6^2$ homomorphisms. With $n = 3$, all combinations yielded $576 = 24^2$. This was the "expected" number in each case, because if G_{ij} were really free on two generators, the generators' images could be assigned arbitrarily in any target group H, yielding $|H|^2$ homomorphisms.

(i,j)	$\lvert\mathrm{Hom}(G_{ij},H)\rvert$	(i,j)	$\lvert\mathrm{Hom}(G_{ij},H)\rvert$
$(1,2)$	20640	$(1,7)$	22560
$(1,3)$	11520	$(2,3)$	19200
$(1,4)$	20640	$(2,5)$	15840
$(1,5)$	17760	$(3,5)$	16320
$(1,6)$	9600	$(4,2)$	11040

TABLE 1. Number of homomorphisms from G_{ij} into $H = \mathrm{SL}(2,\mathbb{Z}/5)$, for various (i,j). Note that each entry is a multiple of $\lvert\mathrm{SL}(2,\mathbb{Z}/5)\rvert = 120$.

When $n = 5$ the results were more gratifying, as shown in Table 1. This proves the first statement in the Theorem: not all the G_{ij} are isomorphic.

The following observation by Lewis allows a significant improvement to the algorithm described. For $x \in H$, let $\mathrm{Sol}(x)$ be the set of $(y,z) \in H \times H$ such that the relation (2.1) is satisfied. If x_1 and x_2 are conjugate, $\mathrm{Sol}(x_1)$ and $\mathrm{Sol}(x_2)$ have same cardinality; indeed, any element of H conjugating x_1 and x_2 also conjugates $\mathrm{Sol}(x_1)$ and $\mathrm{Sol}(x_2)$. It follows immediately that

$$\begin{aligned}\lvert\mathrm{Hom}(G_{ij},H)\rvert &= \sum_{x\in H}\lvert\mathrm{Sol}(x)\rvert \\ &= \sum_{x\in \mathrm{Conj}}\lvert\mathrm{Sol}(x)\rvert\,\lvert\mathrm{conj}(x)\rvert,\end{aligned}$$

where Conj denotes a set of representatives of the conjugacy classes of H and $\mathrm{conj}(x)$ denotes the conjugacy class of x. Thus, instead of examining every triple $(x,y,z) \in H \times H \times H$, we need only look at triples $(x,y,z) \in \mathrm{Conj} \times H \times H$. Computing the conjugacy classes of H usually takes negligible time and can be done once and for all. In our case the running time is cut by a factor of $120/9 \cong 13$, since $\mathrm{SL}(2,\mathbb{Z}/n)$ has order 120 and nine conjugacy classes; each line of Table 1 now takes about six minutes to compute on a MacIIci.

Recall that the original motivation for studying the groups G_{ij} was their strong resemblance to a free group. We know that the lower central series of quotients of every G_{ij} is the same as that of the free group on two generators. What about the derived series of quotients? Baumslag showed that $G_{ij}/(G_{ij})_2$ and $G_{ij}/(G_{ij})_3$ are the same as for the free group, but could not prove anything for $G_{ij}/(G_{ij})_4$. To apply the computational technique we need a target group H with nontrivial H/H_4; we take $H = \mathrm{SL}(2,\mathbb{Z}/4)$. Now $\lvert\mathrm{SL}(2,\mathbb{Z}/4)\rvert = 48$, so the "expected" number of homomorphisms is $48^2 = 2304$. Lewis found that 2304 is indeed the predominant value, but that

$$\lvert\mathrm{Hom}(G_{ij},\mathrm{SL}(2,\mathbb{Z}/4))\rvert = 3072 = 48 \cdot 64$$

for $(i,j) = (1,3)$, $(1,7)$, $(3,5)$, $(5,3)$, $(5,7)$, and $(7,5)$. (Each of these results takes about a minute to compute on a MacIIci.) Only the values 2304 and 3072 have been observed. Robert W. Johnson later verified these results.

This establishes that the derived series of quotients of some of the G_{ij} differ from that of a free group, and proves the second part of the Theorem.

We conclude with two questions:

1. Are all the G_{ij} distinct?
2. Is the cardinality of $\mathrm{Hom}(G,\ \mathrm{SL}(2,\mathbb{Z}/n))$ always a multiple of the order of $\mathrm{SL}(2,\mathbb{Z}/n)$? If so, why?

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