Quadratic Minima and Modular Forms

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CONTENTS

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We give upper bounds on the size of the gap between the constant term and the next nonzero Fourier coefficient of an entire modular form of given weight for $\Gamma_0(2)$. Numerical evidence indicates that a sharper bound holds for the weights $h \equiv 2 \pmod 4$. We derive upper bounds for the minimum positive integer represented by level-two even positive-definite quadratic forms. Our data suggest that, for certain meromorphic modular forms and $p = 2, 3$, the p-order of the constant term is related to the base-p expansion of the order of the pole at infinity.

1. INTRODUCTION

Carl Ludwig Siegel [1969] showed that the constant terms of certain level-one negative-weight modular forms T_h are nonvanishing ("Satz 2"), and that this implies an upper bound on the least positive exponent of a nonzero Fourier coefficient for any level-one entire modular form of weight h with a nonzero constant term. Theta functions fall into this category. Their Fourier coefficients code up representation numbers of quadratic forms. Consequently, for certain h, Siegel's result gives an upper bound on the least positive integer represented by a positive-definite even unimodular quadratic form in $n = 2h$ variables. This bound is sharper than Minkowski's for large n. (Mallows, Odlyzko and Sloane have improved Siegel's bound in [Mallows et al. 1975].)

John Hsia, in a private communication to Glenn Stevens, suggested that Siegel's approach is workable for higher-level forms. Following this hint, we constructed an analogue of T_h for $\Gamma_0(2)$, which we denote by $T_{2,h}$. To prove Satz 2, Siegel controlled the sign of the Fourier coefficients in the principal part of T_h. Following Siegel, we find upper bounds for the first positive exponent of a nonzero Fourier coefficient occuring in the expansion at infinity of an entire modular form with a nonzero constant term for $\Gamma_0(2)$. The whole Siegel argument carries over for weights $h \equiv 0 \pmod 4$. It is not clear that Siegel's method forces the nonvanishing of the $T_{2,h}$ constant terms when $h \equiv 2 \pmod 4$.

In the latter case, we took two approaches. We used a simple trick to derive a bound on the size of the gap after a nonzero constant term in the case $h \equiv 2 \pmod 4$ from our $h \equiv 0 \pmod 4$ result, avoiding the issue of the nonvanishing of the constant term of $T_{2,h}$, but at the cost of a weaker estimate. Also (at the suggestion of Glenn Stevens), we searched for congruences that would imply the nonvanishing of the constant term of $T_{2,h}$. We found numerical evidence that certain congruences dictate the 2- and 3-orders, not only of the constant terms of the $T_{2,h}$, but of a wider class of meromorphic modular forms of level $N \leq 3$. These congruences imply the nonvanishing of the constant term of $T_{2,h}$ for $h \equiv 2 \pmod 4$, but not for $h \equiv 0 \pmod 4$.

Denote the vector space of entire modular forms of weight h for $\Gamma_0(2)$ as $M(2,h)$. In Section 2, we prove that the second nonzero Fourier coefficient of an element of $M(2,h)$ with nonzero constant term must have exponent at most $\dim M(2,h)$ if $h \equiv 0 \pmod 4$, or at most $2\dim M(2,h)$ if $h \equiv 2 \pmod 4$. (We will see that, in fact, $\dim M(2,h) = 1 + \lfloor \frac{h}{4} \rfloor$.)

In Section 3, we describe the numerical experiments that indicate the nonvanishing of the constant terms of $T_{2,h}$. Specifically, the experiments suggest that if a meromorphic modular form for $\Gamma_0(N)$, where $1 \leq N \leq 3$, with a normalized integral Fourier expansion at infinity can be written as a quotient of two monomials in Eisenstein series, then for $p = 2, 3$, the p-order of the constant term is determined by the weight and the base-p expansion of the pole-order. (We are aware of several papers in which base-p expansions come up in analytical contexts, including discussions of the poles of coefficients of Bernoulli polynomials: [Kimura 1988; Adelberg 1992a; 1992b; 1996].)

In Section 4, we prove some of the congruences. In Section 5, we apply the results of Section 2 to the problem of level-two quadratic minima. We state some conjectures in Section 6.

The calculation of Fourier coefficients was usually done by formal manipulation of power series. When we could decompose a form into an infinite product (for example, the form Δ^{-s}), we applied the recursive relation of [Apostol 1976, Theorem 14.8], which is reproduced in Section 2.

2. BOUNDS FOR GAPS IN THE FOURIER EXPANSIONS OF ENTIRE MODULAR FORMS

Section 2A is introductory. We define several modular forms, some of which we will not need until Section 3. In Section 2B, we compute the Fourier expansions of some higher-level Eisenstein series. In Section 2C, we estimate the first positive exponent of a nonzero Fourier coefficient in the expansion of an entire modular form for $\Gamma_0(2)$ with a nonzero constant term.

2A. Some Modular Objects

This section is a tour of the objects mentioned in the article. The main building blocks are Eisenstein series with known divisors and computable Fourier expansions.

As usual, we denote by $\Gamma_0(N)$ the congruence subgroup

$$\Gamma_0(N) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z}) : c \equiv 0 \pmod N \right\}$$

and by $\Gamma(N)$ the subgroup

$$\left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z}) : \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \pmod N \right\}.$$

By $M(N,h)$ we denote the vector space of entire modular forms of one variable in the upper half-plane $\mathfrak{H}$ of weight h for $\Gamma_0(N)$ ("level N") and trivial character. We have an inclusion lattice satisfying

$$M(L,h) \subset M(N,h) \quad \text{if and only if} \quad L \mid N.$$

More particularly, any entire modular form for the group $\mathrm{SL}(2,\mathbb{Z})$ is also one for $\Gamma_0(2)$. The *conductor* of f is the least natural number N such that $f \in M(N,h)$. The dimension of $M(N,h)$ is denoted by $r(N,h)$, or r_h, or by r. We have the following formulas for positive even h. If $h \not\equiv 2 \pmod{12}$, then

$$r(1,h) = \left\lfloor \frac{h}{12} \right\rfloor + 1.$$

If $h \equiv 2 \pmod{12}$, then

$$r(1,h) = \left\lfloor \frac{h}{12} \right\rfloor.$$

For any positive even h,

$$r(2,h) = \left\lfloor \frac{h}{4} \right\rfloor + 1.$$

(The level-one formulas are standard; see [Serre 1973], for example. The level-two formula can be derived by similar methods.)

The subspace of cusp forms in $M(N,h)$ is denoted by $S(N,h)$. We use standard notation for divisor sums:

$$\sigma_\alpha(n) = \sum_{0<d\,|\,n} d^\alpha.$$

For complex z satisfying $\operatorname{Im}(z) > 0$, let $q = q(z) = e^{2\pi i z}$. For positive even $h \neq 2$, we denote the level-one, weight-h Eisenstein series with Fourier expansion at infinity

$$1 + \alpha_h \sum_{n=1}^{\infty} \sigma_{h-1}(n)\, q^n$$

by G_h or $G_h(z)$, where the numbers α_h are given as follows. (For $h > 0$, we follow [Serre 1973]; his E_k are our G_{2k}.) The Bernoulli numbers B_k are defined by the expansion

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty} (-1)^{k+1} B_k \frac{x^{2k}}{(2k)!}.$$

We set $\gamma_k = (-1)^k\, 4k/B_k$, and $\alpha_h = \gamma_{h/2}$ for $h > 0$, while $\alpha_0 = 0$. We have $\alpha_2 = -24$, $\alpha_4 = 240$, $\alpha_6 = -504$, $\alpha_8 = 480$, $\alpha_{10} = -264$, and $\alpha_{12} = \frac{65520}{691}$. The value of α_2 is included because, even though G_2 is not a modular form, we will mention it in some of the observations. We write Δ for the weight-12, level-one cusp form with Fourier series

$$\Delta = \sum_{n=1}^{\infty} \tau(n)\, q^n$$

and product expansion

$$\Delta = q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

Here, τ is the Ramanujan function. We denote the Klein modular invariant G_4^3/Δ by j, as usual. If $(N-1)\,|\,24$, we essentially follow Apostol's notation [1990], writing

$$\varphi_N(z) = \Delta(Nz)/\Delta(z),$$

$\alpha = 1/(N-1)$, and $\Phi_N = \varphi_N^\alpha$. The Φ_N are univalent meromorphic modular functions for $\Gamma_0(N)$.

We define some weight-24, level-one cusp forms as follows. For positive integers n, d, put

$$S_{n,d} = \Delta \left(\frac{n}{d} G_4^3 + \left(1 - \frac{n}{d}\right) G_6^2 \right).$$

We introduce the level-one functions T_h, which are elements in the construction of Siegel described in Section 5A. They are defined by the relation

$$T_h = G_{12r-h+2}\, \Delta^{-r}.$$

Here $N = 1$, so for even $h > 2$, if $h \equiv 2 \pmod{12}$, then $12r - h + 2 = 0$, and otherwise $12r - h + 2 = 14 - (h \bmod 12)$, where $a \bmod b = a - b\left\lfloor \frac{a}{b} \right\rfloor$, the least nonnegative integer A such that $A \equiv a \pmod{b}$. All poles of T_h lie at infinity, and it has weight $2 - h$.

We describe some level-two and level-three objects using three special divisor sums:

$$\sigma^{\mathrm{odd}}(n) = \sum_{\substack{0<d\,|\,n \\ d \text{ odd}}} d, \qquad \sigma_k^{\mathrm{alt}}(n) = \sum_{0<d\,|\,n} (-1)^d d^k,$$

and

$$\sigma_{N,k}^{*}(n) = \sum_{\substack{0<d\,|\,n \\ N \nmid (n/d)}} d^k.$$

Let $E_{\gamma,2}$ denote the unique normalized form in the one-dimensional space $M(2,2)$ (here "normalized" means the leading coefficient in the Fourier expansion of the form is a 1). The Fourier series is

$$E_{\gamma,2} = 1 + 24 \sum_{n=1}^{\infty} \sigma^{\text{odd}}(n)\, q^{n}. \tag{2–1}$$

$E_{\gamma,2}$ has a $\frac{1}{2}$-order zero at points of $\mathfrak{H}$ that are $\Gamma_0(2)$-equivalent to $-\frac{1}{2}+\frac{1}{2}i = \gamma$ (say). The vector space $M(2,4)$ is spanned by two forms $E_{0,4}$ and $E_{\infty,4}$, which vanish with order one at the $\Gamma_0(2)$-inequivalent zero and infinity cusps, respectively. They have Fourier expansions

$$E_{0,4} = 1 + 16 \sum_{n=1}^{\infty} \sigma_3^{\text{alt}}(n)\, q^{n} \tag{2–2}$$

and

$$E_{\infty,4} = \sum_{n=1}^{\infty} \sigma_{2,3}^{*}(n)\, q^{n}. \tag{2–3}$$

More generally, for $N = 2, 3$ and even $k > 2$, there is an Eisenstein series $E_{N,\infty,k}$ in $M(N,k)$ that vanishes at the infinity cusp, but not at cusps $\Gamma_0(N)$-equivalent to zero. (This exhausts the possibilities.) It has the Fourier expansion

$$E_{N,\infty,k} = \sum_{n=1}^{\infty} \sigma_{N,k-1}^{*}(n)\, q^{n}. \tag{2–4}$$

(With this notation, $E_{\infty,4} = E_{2,\infty,4}$.) We write

$$\Delta_2 = E_{0,4} E_{\infty,4}.$$

The singleton family $\{\Delta_2\}$ is a basis for the space $S(2,8)$.

We construct a level-two analogue of j (distinct from φ_2^{-1}, which also plays this role):

$$j_2 = E_{\gamma,2}^2 E_{\infty,4}^{-1}.$$

The function j_2 is analogous to j because it is modular (weight zero) for $\Gamma_0(2)$, holomorphic on the upper half-plane, has a simple pole at infinity, generates the field of $\Gamma_0(2)$-modular functions, and defines a bijection of a $\Gamma_0(2)$ fundamental set with $\mathbb{C}$. We show all this in Section 2C.

Finally, we introduce analogues of the T_h. They are used in our extension of Siegel's construction to level two. For $r = r(2,h)$, if $h \equiv 0 \pmod 4$, we set

$$T_{2,h} = E_{\gamma,2} E_{0,4} E_{\infty,4}^{-r}.$$

but if $h \equiv 2 \pmod 4$, we set

$$T_{2,h} = E_{\gamma,2}^2 E_{0,4} E_{\infty,4}^{-1-r}.$$

2B. The Fourier Expansions of the Higher Level Eisenstein Series

We will prove equations (2–1) and (for $N = 2$) (2–4); equation (2–3) follows immediately. Our tools are results in [Schoeneberg 1974]. The case $N = 3$ of (2–4) can be proved the same way we handle $N = 2$. This method also will give (2–2), but the calculations are longer. Equation (2–2) can also be proved in the following way. For a nonzero modular form in $M(2,h)$, the number of zeros in a fundamental region is exactly $h/4$ [Schoeneberg 1974, Theorem 8, p. 114]. We check that the exponent of the first nonzero Fourier coefficent , if any, in the expansion of $G_4 - E_{0,4} - 256E_{\infty,4}$ exceeds $h/4 = 1$. This exponent counts the number of zeros at $i\infty$. Hence

$$G_4 = E_{0,4} + 256 E_{\infty,4}. \tag{2–5}$$

We deduce (2–2) from (2–3) and (2–5).

The modular form $\mathsf{E}_{\gamma,2}$. Let ζ be the Riemann zeta function. Following Schoeneberg, let $G_2^*(z)$ be defined for $z \in \mathfrak{H}$ by

$$G_2^*(z) = 2\zeta(2) + 2 \sum_{m\geq 1} \sum_{n\in\mathbb{Z}} (mz+n)^{-2}.$$

Then, by [Schoeneberg 1974, p. 63, equation (16)],

$$G_2^*(z) = \frac{\pi^2}{3} - 8\pi^2 \sum_{n\geq 1} \sigma(n)\, e^{2\pi i z n}.$$

(Here σ is the usual sum of divisors.) For integers $N \geq 2$, let

$$E(z, N) = NG_2^*(Nz) - G_2^*(z).$$

We have the Fourier expansion

$$E(z,N) = \frac{N-1}{3}\pi^2 + 8\pi^2 \sum_{n \geq 1} \Bigg(\sum_{\substack{d \mid n,\ d>0 \\ d \not\equiv 0 \pmod N}} d \Bigg) e^{2\pi i n z}. \tag{2–6}$$

(We remark that there is a mistake in [1974, p. 177], where the preceding formula is printed with a minus sign before the term starting with $8\pi^2$.) The modular form $E(z, N)$ belongs to $M(2, N)$; see [Schoeneberg 1974, pp. 177–178]. We get (2–1) by setting $N = 2$ in (2–6) and noting that $r(2, 2) = 1$.

Higher-weight Eisenstein series. Let $\hat{\mathbb{C}}$ be the Riemann sphere. Let N and k be integers with $N \geq 1, k \geq 3$. Let $\boldsymbol{m} = \binom{m_1}{m_2}$ and $\boldsymbol{a} = \binom{a_1}{a_2}$ be matrices with entries in $\mathbb{Z}$. Schoeneberg defines the *inhomogenous Eisenstein series* $G_{N,k,\boldsymbol{a}} : \mathfrak{H} \to \hat{\mathbb{C}}$ as

$$G_{N,k,\boldsymbol{a}}(z) = \sum_{\substack{\boldsymbol{m} \equiv \boldsymbol{a} \pmod N \\ \boldsymbol{m} \neq \boldsymbol{0}}} (m_1 z + m_2)^{-k}.$$

If $N \geq 1$ and $k \geq 3$, then $G_{N,k,\boldsymbol{a}}$ has weight k for $\Gamma(N)$ [Schoeneberg 1974, p. 155, Theorem 1]. We put

$$\xi(t, N, k) = \sum_{\substack{dt \equiv 1 \pmod N \\ d>0}} \frac{\mu(d)}{d^k}.$$

Here μ is the Möbius function. We should note that Schoeneberg uses the symbol G^* in more than one way (differentiated by the subscripts) as we persist in following his notation. He introduces *reduced Eisenstein series* $G^*_{N,k,\boldsymbol{a}}$ for vectors $\boldsymbol{a}$ satisfying $\gcd(a_1, a_2, N) = 1$, requiring that

$$G^*_{N,k,\boldsymbol{a}} = \sum_{t \bmod N} \xi(t, N, k) G_{N,k,t\boldsymbol{a}}. \tag{2–7}$$

(This is equation (9), p. 159 of [Schoeneberg 1974], not his original definition.) Schoeneberg introduces series indexed by level-N congruence subgroups Γ_1 of $\mathrm{SL}(2, \mathbb{Z})$ as follows. Let μ_1 be the (finite) subgroup index $[\Gamma_1 : \Gamma(N)]$, and let one coset decomposition of Γ_1 be

$$\Gamma_1 = \bigcup_{\nu=1}^{\mu_1} \Gamma(N) A_\nu. \tag{2–8}$$

Then for $\gcd(a_1, a_2) = 1$ he defines $G^*_{\Gamma_1,k,\boldsymbol{a}}$ as

$$G^*_{\Gamma_1,k,\boldsymbol{a}} = \sum_{\nu=1}^{\mu_1} G^*_{N,k,({}^tA_\nu)\boldsymbol{a}}. \tag{2–9}$$

Remark 2.1. Schoeneberg [1974, pp. 161–162], shows that $G^*_{\Gamma_1,k,\boldsymbol{a}}$ is an entire weight-k level-N modular form for Γ_1. He shows also (p. 163) that, up to a multiplicative constant, there is only one $G^*_{\Gamma_1,k,\boldsymbol{a}}$ differing from 0 at exactly those cusps that have the form

$$V\left(-\frac{a_2}{a_1}\right), \quad \text{with } V \in \Gamma_1.$$

In view of (2–7)–(2–9), to calculate the Fourier expansion of $G^*_{\Gamma_1,k,\boldsymbol{a}}$ it is sufficient to know the Fourier expansions of the $G_{N,k,\boldsymbol{a}}$. They are as follows [Schoeneberg 1974, p. 157]. We write $\zeta_N = e^{2\pi i/N}$, $\delta(x) = 1$ if $x \in \mathbb{Z}$, $\delta(x) = 0$ otherwise. Then we may write

$$G_{N,k,\boldsymbol{a}}(z) = \sum_{\nu \geq 0} \alpha_\nu(N, k, \boldsymbol{a}) e^{2\pi i z \nu/N}, \tag{2–10}$$

where

$$\alpha_0(N, k, \boldsymbol{a}) = \delta\left(\frac{a_1}{N}\right) \sum_{\substack{m_2 \equiv a_2 \pmod N \\ \boldsymbol{m} \neq \boldsymbol{0}}} m_2^{-k}, \tag{2–11}$$

and, for $\nu \geq 1$,

$$\alpha_\nu(N, k, \boldsymbol{a}) = \frac{(-2\pi i)^k}{N^k (k-1)!} \times \sum_{\substack{m \mid \nu \\ (\nu/m) \equiv a_1 \pmod N}} m^{k-1} \operatorname{sgn} m \, \zeta_N^{a_2 m}. \tag{2–12}$$

The modular form $\mathsf{E}_{2,\infty,k}$. Set $\boldsymbol{u} = \binom{1}{0}$ and let ω be the leading Fourier coefficient in the expansion of $G^*_{\Gamma_0(N),k,\boldsymbol{u}}$, where $N = 2$ or 3. By Remark 2.1, $G^*_{\Gamma_0(N),k,\boldsymbol{u}}(i\infty) = 0$, so

$$E_{N,\infty,k} = \frac{1}{\omega}\, G^*_{\Gamma_0(N),k,\boldsymbol{u}} \tag{2–13}$$

is a normalized modular form in $M(N,k)$ that vanishes at infinity but not at zero.

Proposition 2.2. *The Fourier expansion at infinity of* $E_{2,\infty,k}$ *is*

$$E_{2,\infty,k} = \sum_{n=1}^{\infty} \sigma^*_{2,k-1}(n)\, q^n.$$

Proof. We choose $\Gamma_1 = \Gamma_0(2)$ and specialize (2–7)–(2–12) to this setting. The coset decomposition (2–8) is determined as follows. For $\Gamma_1 = \Gamma_0(2)$ we have $\mu_1 = 2$ because $[\Gamma_0(N) : \Gamma(N)] = N\varphi(N)$, where φ is Euler's function [Schoeneberg 1974, p. 79]. The matrices $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ are inequivalent modulo $\Gamma(2)$, so we have

$$\Gamma_0(2) = \Gamma(2)\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) \cup \Gamma(2)\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right). \tag{2–14}$$

We omit the remainder of the routine calculation. □

The product expansion of $\mathsf{E}_{\infty,4}$.

Proposition 2.3. *The modular form* $E_{\infty,4} \in M(2,4)$ *has the following product decomposition in the variable* $q = \exp(2\pi i z)$:

$$E_{\infty,4}(z) = q \prod_{0<n\in 2\mathbb{Z}} (1-q^n)^8 \prod_{0<n\in \boldsymbol{Z}\setminus 2\mathbb{Z}} (1-q^n)^{-8}. \tag{2–15}$$

Proof. We begin by showing that, for $\operatorname{Im}(z) > 0$,

$$E_{\infty,4}(z) = \eta(2z)^{16}\eta(z)^{-8}. \tag{2–16}$$

For now, denote $\eta(2z)^{16}\eta(z)^{-8}$ by $F(z)$. The function F is holomorphic on $\mathfrak{H}$ because η is nonvanishing on $\mathfrak{H}$. F has the product expansion

$$F(z) = q \prod_{0<n\in 2\mathbb{Z}} (1-q^n)^8 \prod_{0<n\in \mathbb{Z}\setminus 2\mathbb{Z}} (1-q^n)^{-8}. \tag{2–17}$$

This follows from the product expansion of η. It shows that F has a simple zero at infinity. The number of zeros in a $\Gamma_0(2)$ fundamental set in $\mathfrak{H}$ for a level-2, weight-4 modular form is one. If we showed that F has weight 4 for $\Gamma_0(2)$, it would follow that the divisors of $E_{\infty,4}$ and F are both $1 \cdot i\,\infty$. The expansion (2–17) shows that the Fourier series of F is monic. So is that of $E_{\infty,4}$. Thus F and $E_{\infty,4}$ would be monic modular forms with the same weight, level and divisor, hence identical. So, we only need to check the weight-4 modularity of F on a set of generators for $\Gamma_0(2)$. One such set is $\{T, V\}$, where $T = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $V = \left(\begin{smallmatrix}-1&-1\\2&1\end{smallmatrix}\right)$ [Apostol 1990, Theorem 4.3].

We calculate $F(T(z))/F(z)$ using the identity

$$\eta(z+b) = e^{\pi i b/12}\eta(z). \tag{2–18}$$

We have

$$\begin{aligned}\frac{F(T(z))}{F(z)} &= \frac{F(z+1)}{F(z)} = \frac{\eta^{16}\big(2(z+1)\big)}{\eta^8(z+1)}\,\frac{\eta^8(z)}{\eta^{16}(2z)}\\ &= \frac{e^{\pi i 2/12}\eta(2z)^{16}}{e^{\pi i/12}\eta(z)^8}\,\frac{\eta^8(z)}{\eta^{16}(2z)} = 1 = (0\,z+1)^4,\end{aligned}$$

which is what we needed.

To check modularity for V, we use Dedekind's functional equation. This implies that

$$\eta(V(z)) = (-i-2iz)^{1/2}\eta(z). \tag{2–19}$$

Equation (2–18) and Dedekind's equation also imply that

$$\begin{aligned}\eta(2V(z)) &= \eta\Big(\frac{-1}{2z+1} - 1\Big) = e^{-\pi i/12}\eta\Big(\frac{-1}{2z+1}\Big)\\ &= e^{-\pi i/12}(-i(2z+1))^{1/2}\eta(2z+1)\\ &= e^{-\pi i/12}(-i(2z+1))^{1/2}e^{\pi i/12}\eta(2z)\\ &= (-i-2iz)^{1/2}\eta(2z).\end{aligned} \tag{2–20}$$

By (2–19) and (2–20),

$$\begin{aligned}F(V(z)) &= \eta^{16}(2V(z))\eta^{-8}(V(z))\\ &= \frac{\big((-i-2z)^{1/2}\eta(2z)\big)^{16}}{\big((-i-2z)^{1/2}\eta(z)\big)^{8}}\\ &= (2z+1)^4 F(z).\end{aligned}$$

This verifies the weight-4 modularity for V and completes the proof. □

2C. Bounds for Gaps in Fourier Expansions of Level Two Entire Forms

At the step that Siegel called Satz 2, his argument and our extension of it depend on separate, fortuitous sign properties of particular modular forms. These lucky accidents probably bear further study.

Siegel's argument at level one. Denote the coefficient of q^n in the Fourier expansion of f at infinity by $c_n[f]$. Suppose that $f \in M(1,h)$ and $c_0[f] \neq 0$. Siegel showed that $c_n[f] \neq 0$ for some positive $n \leq \dim M(1,h) = r$ (say). We sketch his argument. Siegel sets

$$W = W(f) = (G_{h-12r+12})^{-1}\Delta^{1-r} f.$$

$W(f)$ turns out to be a polynomial in j.

The normalized meromorphic form T_h has a Fourier series of the form

$$T_h = C_{h,-r}q^{-r} + \cdots + C_{h,0} + \cdots,$$

with $C_{h,-r} = 1$. Siegel proves his Satz 1, $c_0[T_h f] = 0$, by showing that

$$T_h f = (2\pi i)^{-1} W(f)\,\frac{dj}{dz}.$$

(Since the right member of this equation is the derivative of a polynomial in j, the constant term of its Fourier series is zero.)

Siegel then proves his Satz 2, $C_{h,0} \neq 0$. To illustrate his approach, we present his argument specialized to weights $h \equiv 0 \pmod{12}$. Siegel employs the operator

$$\frac{d}{d\log q},$$

which we will abbreviate as D. At level one, for weights $h \equiv 0 \pmod{12}$, we have

$$T_h = -\Delta^{1-r} Dj.$$

Also, $j\Delta = G_4^3$. So

$$\begin{aligned} -T_h &= \Delta^{1-r}Dj = D(\Delta^{1-r}j) - jD(\Delta^{1-r}) \\ &= D(\Delta^{1-r}j) - j(1-r)\Delta^{-r}D(\Delta) \\ &= D(\Delta^{1-r}j) + (r-1)j\Delta^{-r}\Big(-\frac{1}{r}\Delta^{1+r}D(\Delta^{-r})\Big) \\ &= D(\Delta^{1-r}j) + \frac{1-r}{r}\,j\Delta D(\Delta^{-r}) \\ &= D(\Delta^{1-r}j) + \frac{1-r}{r}\,G_4^3 D(\Delta^{-r}). \end{aligned}$$

The term $D(\Delta^{1-r}j)$ is the derivative of a Fourier series, so it contributes nothing to the constant term of T_h. The Fourier series of G_4^3 has positive coefficients. The Fourier coefficients in the principal part of $D(\Delta^{-r})$ are negative, and it has no constant term, so the constant term of $G_4^3 D(\Delta^{-r})$ is negative. For $r > 1$ (the nontrivial case) it follows that $C_{h,0} < 0$.

Siegel completes his argument as follows. Let the Fourier expansion of f be

$$f = A_0 + A_1 q + A_2 q^2 + \cdots,$$

$A_0 \neq 0$. Then by Satz 1,

$$0 = c_0[T_h f] = C_{h,0}A_0 + \cdots + C_{h,-r}A_r.$$

By hypothesis, $A_0 \neq 0$, and by Satz 2,

$$A_0 = -(C_{h,0})^{-1}(C_{h,-1}A_1 + \cdots + C_{h,-r}A_r).$$

It follows that one of the A_n, for $n = 1, \ldots, r$, is nonzero.

Function theory at level two. We collect some familiar or easily verified facts. The point at infinity is denoted by $i\infty$ and the extended upper half-plane by $\mathfrak{H}^*$. The set of equivalence classes modulo $\Gamma_0(2)$ in $\mathfrak{H}^*$ we write as $\mathfrak{H}^*/\Gamma_0(2)$. This set has the structure of a genus-zero Riemann surface [Schoeneberg 1974, pp. 91–93, 103]. A set of representatives for $\mathfrak{H}^*/\Gamma_0(2)$ is called a *fundamental set* for $\Gamma_0(2)$, and a set F in $\mathfrak{H}^*$ containing a fundamental set, such that distinct $\Gamma_0(2)$-equivalent points in F must lie on its boundary, is called a *fundamental region* for

$\Gamma_0(2)$. Let S and T be the linear fractional transformations $S : z \mapsto -1/z$ and $T : z \mapsto z + 1$. Let

$$R = \left\{ z \in \mathfrak{H} : |z| > 1, \, |\operatorname{Re} z| < \tfrac{1}{2} \right\}.$$

Let V be the closure of $R \cup S(R) \cup ST(R)$ in the usual topology on $\mathbb{Z}$, and put $F_2 = V \cup \{i\infty\}$. Then F_2 is a fundamental region for $\Gamma_0(2)$. It has two $\Gamma_0(2)$-inequivalent cusps: zero and $i\infty$. The only noncusp in F_2 fixed by a map in $\Gamma_0(2)$ is $\gamma = -\frac{1}{2} + \frac{1}{2}i$.

Modular forms for $\Gamma_0(2)$ are not functions on $\mathfrak{H}^*/\Gamma_0(2)$, but the orders of their zeros and poles are well-defined. We write $\operatorname{ord}_z(f)$ for the order of a zero or pole of a modular form f at z. (This notation supresses the dependence on the subgroup Γ_1 in $\mathrm{SL}(2,\mathbb{Z})$ for which f is modular.) In nontrivial cases (that is, cases of even weight), $\operatorname{ord}_z(f)$ at a point z fixed by an element of $\Gamma_0(2)$ lies in $\frac{1}{2}\mathbb{Z}$, $\frac{1}{3}\mathbb{Z}$, or $\mathbb{Z}$, depending upon whether z is $\mathrm{SL}(2,\mathbb{Z})$-equivalent to i, to $\rho = e^{2\pi i/3}$, or otherwise. (The fixed point γ is $\mathrm{SL}(2,\mathbb{Z})$-equivalent to i.) If f and g are meromorphic modular forms for a subgroup Γ_1 of finite index in $\mathrm{SL}(2,\mathbb{Z})$, then

$$\operatorname{ord}_z(f) + \operatorname{ord}_z(g) = \operatorname{ord}_z(fg).$$

The number of zeros in a fundamental set of a nonzero function in $M(2,h)$ is $\frac{h}{4}$. To represent the divisor of a modular form for $\Gamma_0(2)$, we choose a fundamental set V_2 and write a formal sum

$$\operatorname{div}(f) = \sum_{\alpha \in V_2} \operatorname{ord}_\alpha(f)[\alpha].$$

If f and g are meromorphic modular forms for $\Gamma_0(2)$ of equal weight such that $\operatorname{div}(f) = \operatorname{div}(g)$, then $f = \lambda g$ for some constant λ. We recall that $\dim M(N,h)$ is denoted as $r(N,h)$ and that the subspace of cusp forms in $M(N,h)$ is denoted as $S(N,h)$.

Proposition 2.4. *If h is an even nonnegative number, then $r(2,h) = \left\lfloor \frac{h}{4} \right\rfloor + 1$.*

Sektch of the proof. First we note that multiplication by $\Delta_2 = E_{0,4}E_{\infty,4} \in S(2,8)$ is a vector space isomorphism between $M(2,h)$ and $S(2,h+8)$. Under the usual definitions (see, for example, [Ogg 1969, p. III-5]), evaluation of a modular form at a cusp is a linear functional. Therefore the map

$$\xi : M(2,h) \to \mathbb{Z} \times \mathbb{Z}$$

given by $\xi(f) = (f(0), f(i\infty))$ is linear, with kernel $S(2,h)$. For $h \geq 4$, let $h = 4n + 2m$, where $m = 0$ or 1. Since $E_{\infty,4}(0) \neq 0$, $E_{0,4}(i\infty) \neq 0$, $E_{\infty,4}(i\infty) = 0$, $E_{0,4}(0) = 0$, and $E_{\gamma,2}$ vanishes at neither cusp, the values of $\xi(aE^m_{\gamma,2}E^n_{\infty,4} + bE^m_{\gamma,2}E^n_{0,4})$ cover $\mathbb{Z} \times \mathbb{Z}$ as a, b range over $\mathbb{Z}$. Thus, ξ is surjective. Hence $\dim M(2,h) = 2 + \dim S(2,h)$. This fact allows an induction argument. One checks the initial cases by hand. For example, a form in $M(2,h)$ has precisely one zero (with order $\frac{1}{2}$) at a point $\Gamma_0(2)$-equivalent to γ in a fundamental set. This fixes the divisor, so $r(2,2) = 1$. □

Next, we show that $j_2 = E^2_{\gamma,2}E^{-1}_{\infty,4}$ has properties analogous to those of j.

Proposition 2.5. *The function j_2 is a modular function (weight-zero modular form) for $\Gamma_0(2)$. It is holomorphic on $\mathfrak{H}$ with a simple pole at infinity. It defines a bijection of $\mathfrak{H}/\Gamma_0(2)$ onto $\mathbb{Z}$ by passage to the quotient.*

Proof. The first two claims are obvious. To establish the last claim, let $f_\lambda = E^2_{\gamma,2} - \lambda E_{\infty,4}$ for $\lambda \in \mathbb{Z}$. Then $f_\lambda \in M(2,4)$. The sum of its zero orders in a fundamental set is 1. If f_λ has multiple zeros in a fundamental set, there must be exactly two of them at the equivalence class of γ, or exactly three at that of ρ. □

Proposition 2.6. *If f is meromorphic on $\mathfrak{H}^*$, the following statements are equivalent:*

(i) *f is a modular function for $\Gamma_0(2)$.*
(ii) *f is a quotient of two modular forms for $\Gamma_0(2)$ of equal weight.*
(iii) *f is a rational function of j_2.*

Proof. Clearly (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i). For $z \in \mathfrak{H}^*$, let $[z]$ be the equivalence class of z in $\mathfrak{H}/\Gamma_0(2)$. By an abuse of the notation, we may take f as in (i) as a function from $\mathfrak{H}^*/\Gamma_0(2)$ to $\hat{\mathbb{C}}$. The function j_2, also regarded in this fashion, is invertible. Let $\tilde{f}:\hat{\mathbb{C}}\to\hat{\mathbb{C}}$ satisfy $\tilde{f} = f \circ j_2^{-1}$. Then $\tilde{f}$ is meromorphic on $\hat{\mathbb{C}}$, so it is rational. If $z\in\hat{\mathbb{C}}$, let $u = j_2^{-1}(z) \in \mathfrak{H}^*/\Gamma_0(2)$. Then $f(u) = f(j_2^{-1}(z)) = \tilde{f}(z) = \tilde{f}(j_2(u))$. Thus f is a rational function in j_2. □

Next, we differentiate j_2.

Proposition 2.7. *For* $z \in \mathfrak{H}$,

$$\frac{d}{dz}\, j_2(z) = -2\pi i E_{\gamma,2}(z) E_{0,4}(z) E_{\infty,4}(z)^{-1}.$$

Proof. It follows from the functional equation that the derivative of a modular function (weight-zero modular form) has weight two. Therefore, both expressions represent weight-two meromorphic modular forms for $\Gamma_2(0)$. The only poles of either function lie at infinity. On each side, the principal part of the Fourier expansion at infinity consists only of the term $-2\pi i q^{-1}$. Therefore the form

$$\frac{d}{dz}\, j_2(z) + 2\pi i E_{\gamma,2}(z) E_{0,4}(z) E_{\infty,4}(z)^{-1}$$

is holomorphic, weight two. We find that it is zero in the same way that we established equation (2–5). □

Extension of Siegel's argument to level two. We introduce an analogue of Siegel's W map. For $h \equiv 0 \pmod 4$ and $f \in M(2,h)$, let

$$W_2(f) = f E_{\infty,4}^{-h/4}.$$

For $h \equiv 2 \pmod 4$, let

$$W_2(f) = f E_{\gamma,2} E_{\infty,4}^{-(h+2)/4}.$$

Proposition 2.8. *If h is positive, the restriction of W_2 to $M(2,h)$ is a vector space isomorphism onto the space of polynomials in j_2 of degree less than $r = r(2,h)$ ($h \equiv 0 \pmod 4$) or of degree between* 1 *and r inclusive ($h \equiv 2 \pmod 4$).*

Proof. Suppose $h \equiv 0 \pmod 4$ and $f \in M(2,h)$. In view of Proposition 2.4,

$$W_2(f) = f E_{\infty,4}^{1-r}.$$

For $d = 0, 1, \ldots, r-1$, the products $j_2^d E_{\infty,4}^{r-1}$ belong to $M(2,h)$. We have

$$W_2(j_2^d E_{\infty,4}^{r-1}) = j_2^d.$$

Let Q be the subspace of $M(2,h)$ generated by the modular forms $j_2^d E_{\infty,4}^{r-1}$, for $d = 0, 1, \ldots, r-1$, and let R be the space of polynomials in j_2 of degree at most $r-1$. W_2 carries Q isomorphically onto R. Therefore, $\dim Q = r$. Hence $Q = M(2,h)$. This proves the first claim.

Now let $h \equiv 2 \pmod 4$. Then

$$W_2(f) = f E_{\gamma,2} E_{\infty,4}^{-r}.$$

For $d = 0, 1, \ldots, r-1$, the products $j_2^d E_{\gamma,2} E_{\infty,4}^{r-1}$ belong to $M(2,h)$. We have

$$W_2(j_2^d E_{\gamma,2} E_{\infty,4}^{r-1}) = j_2^{d+1}.$$

W_2 carries $E_{\gamma,2}Q$ isomorphically onto $j_2 R$. Therefore, $\dim E_{\gamma,2}Q = r$. Hence $E_{\gamma,2}Q = M(2,h)$. □

Proposition 2.9. *For even nonnegative h and $f \in M(2,h)$, the constant term in the Fourier expansion at infinity of $fT_{2,h}$ is zero.*

Proof. Suppose $h \equiv 0 \pmod 4$. Then

$$\begin{aligned} W_2(f)\frac{d}{dz} j_2 &= -f E_{\infty,4}^{1-r} 2\pi i E_{\gamma,2} E_{0,4} E_{\infty,4}^{-1} \\ &= -2\pi i f T_{2,h}. \end{aligned}$$

If $h \equiv 2 \pmod 4$, we get the same result by a similar calculation. Thus, $fT_{2,h}$ is the derivative of a polynomial in j_2, so it can be expressed in a neighborhood of infinity as the derivative with respect to z of a power series in the variable $q = \exp(2\pi i z)$. This derivative is a power series in q with vanishing constant term. □

Proposition 2.10. *For positive $h \equiv 0 \pmod 4$, the constant term in the Fourier expansion at infinity of $T_{2,h}$ is nonzero.*

Proof. Let $u = 2\pi i z = \log q$. We retain the notation D for the operator d/du, which has the property that $D(q^n) = nq^n$. Let $m_2 = j_2 - 64$. Arguing as in the proof of (2–5), we see that $E_{\gamma,2}^2 = E_{0,4}+64E_{\infty,4}$, so $m_2 = E_{0,4}E_{\infty,4}^{-1}$. Thus

$$\frac{d}{dz}m_2 = \frac{d}{dz}j_2 = -2\pi i E_{\gamma,2}E_{0,4}E_{\infty,4}^{-1},$$

so that $D(m_2) = -E_{\gamma,2}E_{0,4}E_{\infty,4}^{-1}$. It follows that

$$T_{2,h} = -E_{\infty,4}^{1-r}D(m_2).$$

Hence

$$\begin{aligned}
&E_{\infty,4}^{1-r}D(m_2)\\
&= D(E_{\infty,4}^{1-r}m_2) - m_2 D(E_{\infty,4}^{1-r})\\
&= D(E_{\infty,4}^{1-r}m_2) - m_2(1-r)E_{\infty,4}^{-r}D(E_{\infty,4})\\
&= D(E_{\infty,4}^{1-r}m_2) + (r-1)m_2E_{\infty,4}^{-r}\Bigl(-\frac{1}{r}E_{\infty,4}^{1+r}D(E_{\infty,4}^{-r})\Bigr)\\
&= D(E_{\infty,4}^{1-r}m_2) + \frac{1-r}{r}m_2E_{\infty,4}D(E_{\infty,4}^{-r})\\
&= D(E_{\infty,4}^{1-r}m_2) + \frac{1-r}{r}E_{0,4}D(E_{\infty,4}^{-r}).
\end{aligned}$$

The term $D(E_{\infty,4}^{1-r}m_2)$ makes no contribution to the constant term. Therefore the constant term of $T_{2,h}$ is the same as that of $\frac{r-1}{r}E_{0,4}D(E_{\infty,4}^{-r})$. We now examine the principal part of $D(E_{\infty,4}^{-r})$.

An absolutely convergent monic power series can be written as an infinite product. The technique was used by Euler to prove the Pentagonal Number Theorem. It has been codified as follows [Apostol 1976, Theorem 14.8]:

Lemma 2.11. *For a given set A and a given arithmetical function f, the numbers $p_{A,f}(n)$ defined by the equation*

$$\prod_{n\in A}(1-x^n)^{-f(n)/n} = 1 + \sum_{n=1}^{\infty} p_{A,f}(n)x^n$$

satisfy the recursion formula

$$np_{A,f}(n) = \sum_{k=1}^{n} f_A(k)p_{A,f}(n-k),$$

where $p_{A,f}(0) = 1$ and

$$f_A(k) = \sum_{\substack{d\,|\,k\\ d\in A}} f(d).$$

Proposition 2.3 and this lemma imply that, for fixed s,

$$E_{\infty,4}^{-s} = q^{-s}\sum_{n=0}^{\infty} R(n)q^n, \tag{2–21}$$

where $R(0) = 1$ and $n > 0$ implies that

$$R(n) = \frac{8s}{n}\sum_{a=1}^{n}\sigma_1^{\mathrm{alt}}(a)R(n-a). \tag{2–22}$$

Because $\sigma_1^{\mathrm{alt}}(a)$ alternates sign, the alternation of the sign of $R(n)$ follows by an easy induction argument from (2–22). To be specific, $R(n) = U_n(-1)^n$ for some $U_n > 0$. Thus we may write

$$\begin{aligned}
E_{\infty,4}^{-r} &= U_0(-1)^0q^{-r} + U_1(-1)^1q^{1-r}\\
&\quad + \cdots + U_{r-1}(-1)^{r-1}q^{-1} + U_r(-1)^r + \cdots;
\end{aligned}$$

hence $D(E_{\infty,4}^{-r})$ equals

$$\begin{aligned}
&-rU_0(-1)^0q^{-r} + (1-r)U_1(-1)^1q^{1-r}\\
&\quad + \cdots + (-1)U_{r-1}(-1)^{r-1}q^{-1} + 0 + \cdots,
\end{aligned}$$

which in turn equals

$$\begin{aligned}
&V_r(-1)^1q^{-r} + V_{r-1}(-1)^2q^{1-r}\\
&\quad + \cdots + V_1(-1)^rq^{-1} + 0 + \cdots
\end{aligned}$$

for positive V_n.

On the other hand, the Fourier coefficient of q^n, for $n \geq 0$, in the expansion of $E_{0,4}$ is $W_n(-1)^n$ for positive W_n, by (2–2). Thus the constant term of $E_{0,4}D(E_{\infty,4}^{-r})$ is

$$\sum_{n=1}^{r} V_n(-1)^{r+1-n}W_n(-1)^n = (-1)^{r+1}\sum_{n=1}^{r} V_nW_n,$$

and that of $T_{2,h}$ is the number

$$\frac{r-1}{r}(-1)^{r+1}\sum_{n=1}^{r} V_nW_n.$$

For weights $h \geq 4$, $r > 1$. □

The signs of the Fourier coefficients are not as cooperative in the case $h \equiv 2 \pmod 4$, and so far we do not have a result corresponding to Proposition 2.10 in this situation.

Theorem 2.12. *Suppose $f \in M(2,h)$ with Fourier expansion at infinity*

$$f(z) = \sum_{n=0}^{\infty} A_n q^n, \quad \text{with } A_0 \neq 0.$$

If $h \equiv 0 \pmod 4$, then some $A_n \neq 0$, for $1 \leq n \leq r(2,h)$. If $h \equiv 2 \pmod 4$, then some $A_n \neq 0$, for $1 \leq n \leq 2r(2,h)$.

Proof. First suppose that $h \equiv 0 \pmod 4$. The argument tracks Siegel's in the level-one case. We still denote the coefficient of q^n in the Fourier expansion of f at infinity as $c_n[f]$. The normalized meromorphic form $T_{2,h}$ has a Fourier series of the form

$$T_{2,h} = C_{h,-r}q^{-r} + \cdots + C_{h,0} + \cdots,$$

with $C_{h,-r} = 1$. By Proposition 2.9,

$$0 = c_0[T_{2,h}f] = C_{h,0}A_0 + \cdots + C_{h,-r}A_r.$$

By hypothesis, $A_0 \neq 0$. By Proposition 2.10, $C_{h,0}$ is nonzero, so

$$A_0 = -(C_{h,0})^{-1}(C_{h,-1}A_1 + \cdots + C_{h,-r}A_r).$$

It follows that one of the A_n is nonzero.

Now suppose $h \equiv 2 \pmod 4$, say $h = 4k+2$, and $f \in M(2,h)$. For some monic q-series F and some nonzero constant C_t, $f = 1 + C_t\, q^t F$. Let $g = f^2 \in M(2,2h)$. Then

$$g = 1 + 2C_t\, q^t F + C_t^2 q^{2t} F^2.$$

Since $2h \equiv 0 \pmod 4$, we have $t \leq r(2,2h) = 1 + \lfloor (2h/4) \rfloor = 1 + \lfloor (8k+4)/4 \rfloor = 2k+2$. On the other hand, $r(2,h) = r(2,4k+2) = 1 + \lfloor (4k+2)/4 \rfloor = 1+k$. $\square$

The only obstacle to obtaining the bound $r+1$ instead of $2r$ in the second case is the lack of a version of Proposition 2.10 for weights $h \equiv 2 \pmod 4$. In Section 3, we present experimental evidence for, among other things, an extended Proposition 2.10.

While it is possible that the level-two result extends to the other levels N at which $\Gamma_0(N)$ has genus zero, namely $N = 1, \ldots, 10, 12, 13, 16, 25$, Glenn Stevens has raised the question whether, because of the absence of an analogue for j, higher genus is an obstruction to this sort of argument.

3. OBSERVATIONS

The divisor of a meromorphic modular form f, normalized so that the leading Fourier coefficient is 1, determines the Fourier expansions of f , because the divisor determines f. This suggests the problem of finding effective rules governing the map from divisors to Fourier series. Some results in this direction are known. For example, Fourier expansions of Eisenstein series with prescribed behavior at the cusps are stated in [Schoeneberg 1974].

Here we study rules by which the divisor governs congruences for the Fourier expansion. The theory of congruences among holomorphic modular forms is significant in number theory, so it is natural to scrutinize any new congruences among modular forms. Regularities among the constant terms suggest an empirical basis for such a theory in the meromorphic setting.

In Sections 3A and 3B, we discuss three rules (for conductors $N = 1, 2, 3$) governing the constant term of the Fourier expansion at infinity. We describe numerical evidence for congruences obeyed by certain meromorphic modular forms. The congruences relate geometric and arithmetic data: the divisor, and the 2-order or 3-order of the constant terms. This connection is expressed in terms of the weight and the sum of the digits in the base two or base three expansion of the pole order.

These rules are described for modular forms of level $N \leq 3$. They do not apply to all the objects we surveyed, and we don't know how to sort the deviant from nondeviant forms, except by inspection. The deviations are systematic in the sense that the constant terms at a given level still obey simple rules. We can also manufacture linear combinations of nondeviant forms that depart from the congruence rules in a stronger sense: the 2-order and the 3-order of the constant terms are arbitrary. This means that the constant terms of some of the deviant forms are controlled by invariants of the divisor other than the weight and the order of the pole at infinity.

In our surveys, a meromorphic modular form f that obeys the congruences always has a normalized rational Fourier expansion and a pole at infinity. The T_h and $T_{2,h}$ were the first examples. We looked for other instances of this behavior and found it exhibited by some standard objects. We then conducted a more or less systematic survey of similar objects.

We describe two sets of data. The first survey suggests rules regarding the 2-order or 3-order of constant terms of a family of level-N objects, with $1 \leq N \leq 3$. The second survey looks at negative powers of the functions $E_{N,\infty,k}$, with $N = 2, 3$. These examples form families of their own, and within these families the behavior of the constant term is again predictable.

Congruences for constant terms seem to have implications for the whole Fourier expansion of related meromorphic forms. In Section 3C, we report observations on the Fourier expansion of j that support this idea.

3A. Observations on the Constant Terms: First Survey

We list several thousand forms obeying rules governing their constant terms. Let $d_b(n)$ be the sum of the digits in the base-b expansion of the positive integer n and $c_n[f]$ be the coefficient c_n in the Fourier series

$$f = \sum_n c_n\, q^n.$$

Let p be prime. If an integer n can be factored as $n = p^a m$, with $(p, m) = 1$, we write

$$\mathrm{ord}_p(n) = a.$$

In addition we write $\mathrm{ord}_p(0) = \infty$. If a rational number x can be written $\frac{n}{d}$ as a quotient of integers, we set $\mathrm{ord}_p(x) = \mathrm{ord}_p(n) - \mathrm{ord}_p(d)$.

We write C_2 for the set of level-two meromorphic modular forms f of any weight with rational Fourier expansion at infinity, leading coefficient 1, and a pole at infinity of order $s = s(f) > 0$ such that

$$\mathrm{ord}_2(c_0[f]) = 3d_2(s)\,.$$

The set of level-three meromorphic modular forms f of any weight with rational Fourier expansion at infinity, leading coefficient 1, and a pole at infinity of order $s = s(f) > 0$ such that

$$\mathrm{ord}_3(c_0[f]) = d_3(s)$$

will be denoted C_3.

For a function f with a pole of order $s = s(f)$ at infinity, let $\beta = d_2(s)$ and $\gamma = d_3(s)$. Membership in C_2 is a congruence relation, since

$$\mathrm{ord}_2(n) = a \iff n \equiv 2^a \pmod{2^{a+1}},$$

but membership in C_3 means a choice of two congruences:

$$\mathrm{ord}_3(n) = a \iff n \equiv \pm 3^a \pmod{3^{a+1}}.$$

We define two subsets of C_3, the members of which make this choice systematically:

$$D_3 = \bigl\{ f \in C_3 : c_0[f] \equiv (-1)^s 3^\gamma \pmod{3^{\gamma+1}} \bigr\}$$

and

$$E_3 = \bigl\{ f \in C_3 : c_0[f] \equiv 3^\gamma \pmod{3^{\gamma+1}} \bigr\}\,.$$

If f is a meromorphic modular form, let $w = w(f)$ be the weight of f. As above, let $s = s(f)$ be the order of the pole of f at infinity. Finally, we will write $L = L(f)$ for the largest digit in the base-3 expansion of $s(f)$.

In this survey, the constant terms of the meromorphic forms we studied have three modes of behavior, depending upon the conductor.

(1) The meromorphic forms f for $\mathrm{SL}(2, \mathbb{Z})$ (conductor one forms) obey the following rule.

(a) If $w \equiv 0 \pmod 4$, then $f \in C_2$.
(b) If $w \equiv 2 \pmod 4$, then $2^{4\beta} \mid c_0[f]$.
(c) If $w \equiv 0 \pmod 3$, then $f \in D_3$.
(d) If $w \equiv 1 \pmod 3$ and $L = 1$, then $f \in E_3$.
(e) If $w \equiv 1 \pmod 3$ and $L = 2$, then $3^{\gamma+1} \mid c_0[f]$.
(f) If $w \equiv 2 \pmod 3$, then $3^{\gamma+1} \mid c_0[f]$.

(2) Forms with conductor 2 obey (a)–(b), but not, in general, (c)–(f).
(3) Forms with conductor 3 obey (c)–(f), but not, in general (a)–(b).

Conductor one. What follows is a list of objects obeying rule (1) above. (The function G_2 isn't modular in the ordinary sense, but we assigned it weight 2 to see what would happen.)

Δ^{-a}	for $1 \le a \le 140$
j^a	for $1 \le a \le 50$
$j\Delta^{-a}$	for $1 \le a \le 100$
$j^a\Delta^{-b}$	for $1 \le a, b \le 50$
$G_6^a\Delta^{-b}$	for $1 \le a, b \le 50$

(If we set $a = 1$, these are the functions T_h, with $h \equiv 8 \pmod{12}$, $8 \le h \le 596$.)

$G_4^aG_6^b\Delta^{-c}$	for $1 \le a, c \le 50,\ 0 \le b \le 11$
$G_{10}^a\Delta^{-b}$	for $1 \le a, b \le 50$
$G_{14}^a\Delta^{-b}$	for $1 \le a, b \le 50$
$G_{2a}\Delta^{-b}$	for $1 \le a \le 7,\ 1 \le b \le 140$

(If we set $a = 2$, these are the functions T_h, with $h \equiv 10 \pmod{12}$, $10 \le h \le 1678$, and if we set $a = 4$, they are the functions T_h, with $h \equiv 6 \pmod{12}$, $6 \le h \le 1674$.)

$G_{2a}\Delta^{-b}$	for $8 \le a \le 24,\ 1 \le b \le 50$,
$G_{2a}^{-1}\Delta^{-b}$	for $1 \le a \le 18,\ 1 \le b \le 50$,
$S_{n,d}^{-a}$	for $1 \le a \le 50,\ 1 \le d \le 4,\ 1 \le n \le d$,
$G_{10}^aS_{1,2}^{-b}$	for $1 \le a, b \le 50$,
$G_{14}^aS_{1,2}^{-b}$	for $1 \le a, b \le 50$,
$G_4^aG_6^bS_{1,2}^{-c}$	for $1 \le a, c \le 50,\ 1 \le b \le 5$.

In an earlier survey, we found that $\Delta^{-s} \in C_2 \cap C_3$ for $1 \le s \le 3525$. We also found that $j^s \in C_2 \cap C_3$ for $1 \le s \le 200$, and that $j^k\Delta^{-m} \in C_2 \cap C_3$ for $1 \le k, m \le 100$. The computing power we exploited at the time (with Roger Frye's assistance) was not available when we were conducting the experiments described here, so we do not have data on membership in D_3 for the additional functions.

Conductor two. This is a list of objects obeying rule (2) above. The first two items are the first few functions $T_{2,h}$. Evidently, rule (2) does not force the vanishing of the constant terms of the functions $E_{\gamma,2}E_{0,4}E_{\infty,4}^{-a} = T_{2,h}$ for $h \equiv 0 \pmod 4$, but would imply a level-2 Satz 2 for $h \equiv 2 \pmod 4$ if it held for all $E_{\gamma,2}^2E_{0,4}E_{\infty,4}^{-a}$, a positive.

$E_{\gamma,2}E_{0,4}E_{\infty,4}^{-a}$	for $1 \le a \le 100$
$E_{\gamma,2}^2E_{0,4}E_{\infty,4}^{-a}$	for $1 \le a \le 100$
j_2^a	for $1 \le a \le 100$
φ_2^{-a}	for $1 \le a \le 100$
$G_{2a}E_{\infty,4}^{-b}$	for $0 \le a \le 24,\ 1 \le b \le 50$
$G_{2a}^{-1}E_{\infty,4}^{-b}$	for $1 \le a \le 11,\ 1 \le b \le 50$
$G_4^aE_{\infty,4}^{-b}$	for $1 \le a, b \le 50$
$G_6^aE_{\infty,4}^{-b}$	for $1 \le a, b \le 50$
$G_4^aG_6E_{\infty,4}^{-b}$	for $1 \le a, b \le 50$
$G_{10}^aE_{\infty,4}^{-b}$	for $1 \le a, b \le 50$
$E_{\gamma,2}^aE_{\infty,4}^{-b}$	for $1 \le a, b \le 50$
$E_{\gamma,2}^a\Delta^{-b}$	for $1 \le a, b \le 50$
$E_{0,4}^aE_{\infty,4}^{-b}$	for $1 \le a, b \le 50$
Δ_2^{-a}	for $1 \le a \le 100$

Conductor three. This is a brief list of objects obeying rule (3) above. It should be noted that φ_3^{-1} has a double pole at infinity. More objects obeying (3) are listed in the next section.

φ_3^{-a}	for $1 \le a \le 100$
$G_{10}^{-a}\varphi_3^{-b}$	for $1 \le a, b \le 50$
Φ_3^{-a}	for $1 \le a \le 50$
$G_{2a}\Phi_3^{-b}$	for $1 \le a \le 24,\ 1 \le b \le 50$
$G_4^a\Phi_3^{-b}$	for $1 \le a, b \le 50$
$G_{10}^a\Phi_3^{-b}$	for $1 \le a, b \le 50$

3B. Second Survey, with Deviations from Rules (1)–(3)

Given a pair of objects of the same conductor, pole order and weight obeying rules (1)–(3), one can find a linear combination that violates the rules. For example, let $r(N,k) > 1$ and let f, g be distinct normalized forms in $M(N,k)$, for $1 \le N \le 3$. Let s be a positive integer. Then $\varphi = f\Delta^{-s}$ and $\gamma = g\Delta^{-s}$ are normalized, and they have equal pole order, weight and conductor. Suppose they are subject to one of the above rules that dictates for $p = 2$ or 3 that $\operatorname{ord}_p(c_0[\varphi]) = \operatorname{ord}_p(c_0[\gamma]) = \varepsilon$ (say). Further, let the constant terms of φ and γ be $p^\varepsilon(a/b)$ and $p^\varepsilon(c/d)$ (with none of a, b, c, d divisible by p). Let $x = (p^\sigma - ad)/(bc - ad)$ for an arbitrary number $\sigma \ne 0$. Then the meromorphic modular form $\zeta = (1-x)\varphi + x\gamma$ is also normalized with the same weight and pole order as φ and γ. It may have lower conductor if $N \ne 1$, but whichever rule dictated the values of $\operatorname{ord}_p(c_0[\varphi])$ and $\operatorname{ord}_p(c_0[\gamma])$ is also part of rule (1). Yet it fails, because $c_0[\zeta] = p^{\varepsilon+\sigma}/bd$. This shows that features of the divisor other than the weight and the order of the pole at infinity influence the arithmetic of the constant term.

This fact led us to search for other deviants. We found systematic deviations from rules (1)–(3), but for these examples, the 2- and 3-orders of the constant terms were still determined by the weight and the order of the pole at infinity.

The following functions obey rule (2):

$$\begin{aligned}
&E^{-a}_{\infty,4} && \text{for } 1 \le a \le 51\\
&E^{-a}_{2,\infty,k} && \text{for } 1 \le a \le 51,\\
& && \quad 6 \le k \le 22,\ k \equiv 2 \pmod 4\\
&E^{-a}_{2,\infty,k} && \text{for } 2 \le a \le 50,\ a \text{ even},\\
& && \quad 8 \le k \le 24,\ k \equiv 0 \pmod 4
\end{aligned}$$

The following functions obey rule (3):

$$\begin{aligned}
&E^{-a}_{3,\infty,6} && \text{for } 1 \le a \le 98\\
&E^{-a}_{3,\infty,k} && \text{for } 3 \le a \le 48,\ a \equiv 0 \pmod 3,\\
& && \quad 12 \le k \le 24,\ k \equiv 0 \pmod 6\\
&E^{-a}_{3,\infty,k} && \text{for } 2 \le a \le 98,\ a \equiv 0 \text{ or } 2 \pmod 3,\\
& && \quad 8 \le k \le 20,\ k \equiv 2 \pmod 6\\
&E^{-a}_{3,\infty,k} && \text{for } 1 \le a \le 97,\ a \equiv 1 \pmod 3,\\
& && \quad 8 \le k \le 20,\ k \equiv 2 \pmod 6,\ L = 2\\
&E^{-a}_{3,\infty,k} && \text{for } 1 \le a \le 98,\\
& && \quad 4 \le k \le 22,\ k \equiv 4 \pmod 6
\end{aligned}$$

The following functions deviate from rule (2):

$$E^{-a}_{2,\infty,k}, \qquad \text{for } 1 \le a \le 51,\ a \text{ odd}, \quad 8 \le k \le 24,\ k \equiv 0 \pmod 4$$

Rule (2) predicts that $\operatorname{ord}_2\bigl(c_0[E^{-a}_{2,\infty,k}]\bigr) = 3d_2(a)$ in this situation. Instead the constant terms obey the rule

$$\operatorname{ord}_2(c) = 3d_2(a) + \operatorname{ord}_2(a+1) + k - 5. \tag{3–1}$$

The following functions deviate from rule (3):

$$E^{-a}_{3,\infty,k}, \qquad \text{for } 1 \le a \le 49,\ a \equiv 1 \pmod 3, \quad 12 \le k \le 24,\ k \equiv 0 \pmod 6$$

The weights of these functions are divisible by 3, so rule (3) predicts that

$$c_0[E^{-a}_{3,\infty,k}] \equiv (-1)^a 3^{d_3(a)} \pmod{3^{d_3(a)+1}}.$$

Instead,

$$c_0[E^{-a}_{3,\infty,k}] \equiv (-1)^{a+1} 3^{d_3(a)} \pmod{3^{d_3(a)+1}}. \tag{3–2}$$

The functions

$$E^{-a}_{3,\infty,k}, \qquad \text{for } 2 \le a \le 47, a \equiv 2 \pmod 3, \quad 12 \le k \le 24,\ k \equiv 0 \pmod 6$$

also depart from rule (3). In this situation, it is not true that $\operatorname{ord}_3(c_0[E^{-a}_{3,\infty,k}]) = d_3(a)$, as predicted by rule (3). Instead

$$\operatorname{ord}_3(c_0[E^{-a}_{3,\infty,k}]) = d_3(a) + \operatorname{ord}_3(a+1) = \delta \ \text{(say)}. \tag{3–3}$$

We have not yet understood how these functions choose between the congruences

$$c_0[E^{-a}_{3,\infty,k}] \equiv \pm 3^\delta \pmod{3^{\delta+1}},$$

except that our data indicate that it depends only on the value of a.

The last set of functions in this survey deviating from rule (3) is

$$E^{-a}_{3,\infty,k},\ \text{for}\ 1 \le a \le 94,\ a \equiv 1 \pmod 3,\ 8 \le k \le 20,\ k \equiv 2 \pmod 6,\ L = 1.$$

Here $w \equiv 1 \pmod 3$, so rule (3) predicts that

$$c_0[E^{-a}_{3,\infty,k}] \equiv 3^{d_3(a)} \pmod{3^{d_3(a)+1}}.$$

Actually for this set we have

$$c_0[E^{-a}_{3,\infty,k}] \equiv -3^{d_3(a)} \pmod{3^{d_3(a)+1}}. \tag{3–4}$$

3C. Divisibility Properties of the Fourier Coefficients of j, Δ and Their Reciprocals

We observed a pattern of connections between corresponding Fourier coefficients (not the constant terms) of $1/\Delta$ and j, and between corresponding Fourier coefficients (not the constant terms) of Δ and $1/j$. These experiments were motivated by the following considerations. Membership of f^s in C_2 or C_3 for integers s with $1 \le s \le B$, for some bound B, imposes conditions modulo powers of 2 or 3 on the Fourier coefficients of f with exponent at most $B-1$. It is easy to check, for example, that if f has a simple pole at infinity and $f^s \in C_2$ for $1 \le s \le 4$, then

$$\begin{aligned}
c_0[f] &\equiv 8 \pmod{16},\\
c_1[f] &\equiv 4 \pmod{8},\\
c_2[f] &\equiv 0 \pmod{128},\\
c_3[f] &\equiv 2 \pmod{4}.
\end{aligned}$$

It is possible to extend these calculations indefinitely. They suggest that there is a systematic relationship between the 2- and 3-orders of corresponding coefficients of any two functions satisfying the above requirements on f. This led us to compare these orders in the expansions of j and $1/\Delta$.

Denote $\operatorname{ord}_p(c_n[j]) - \operatorname{ord}_p(c_n[1/\Delta])$ by $\delta_{p,n}$. For $-1 \le n \le 2470$ $(n \ne 0)$, we found that

$$\begin{aligned}
n \equiv 0 \pmod 2 &\implies \delta_{2,n} = 3\operatorname{ord}_2(n) + 1,\\
n \equiv 0 \pmod 3 &\implies \delta_{3,n} = 2\operatorname{ord}_3(n),\\
n \equiv 1 \pmod 3 &\implies \delta_{3,n} = -1,
\end{aligned}$$

It is interesting to compare these rules with the congruences of Lehner (see [Lehner 1949] or [Apostol 1990, p. 91]), which are

$$\begin{aligned}
c_{2^\alpha n}[j] &\equiv 0 \pmod{2^{3\alpha+8}},\\
c_{3^\alpha n}[j] &\equiv 0 \pmod{3^{2\alpha+3}},\\
c_{5^\alpha n}[j] &\equiv 0 \pmod{5^{\alpha+1}},\\
c_{7^\alpha n}[j] &\equiv 0 \pmod{7^{\alpha}}.
\end{aligned}$$

There are tantalizing hints of similar relations. For example, if $5 \le n \le 2470$ with $n \equiv 0 \pmod 5$ and $n \ne 2245$, then $\delta_{5,n} = \operatorname{ord}_5(n)$; but $\delta_{5,2245} = 2$.

On general principles, we also compared Δ with $1/j$, and found that $\operatorname{ord}_p(c_n[1/j]) = \operatorname{ord}_p(c_n[\Delta])$ for $p = 2, 3$ and $1 \le n \le 4096$. This also holds for $p = 5$, if $n \not\equiv 3$ or $4 \pmod 5$ and $1 \le n \le 1225$. These observations have some independent interest, because $c_n[\Delta] = \tau(n)$, the Ramanujan tau function. For example, one may imagine a proof of Lehmer's conjecture that the Ramanujan tau function is nonvanishing, consisting of two parts: a proof of the above relations for all positive n, and a proof that $c_n[1/j]$ is nonvanishing.

4. CONGRUENCES

The following scenario plays out only when we are lucky. Given the power series of a modular form $f(x) = 1 + \sum_{n=1}^{\infty} a(n)x^n$, one uses Möbius inversion and Lemma 2.11 to find the first few factors in the product expansion. One then guesses the whole product expansion. The product expansion then is used to guess how to write the form as a monomial in Dedekind's η function, and this relation is proved with the analytic theory of modular forms. Then one derives the product expansion from that of η, and the recursion among the Fourier coefficients using Lemma 2.11. Finally, the recursion is used to prove a special case of rules (1)–(3) from page 268.

To illustrate, we will prove the first of the following theorems, which is an example of rule (2):

Theorem 4.1. *If* $s = 2^x$, *with* $x \in \mathbb{N}$, *then*

$$\mathrm{ord}_2\left(c_0[E_{\infty,4}^{-s}]\right) = 3.$$

Theorem 4.2. *If* $s = 2^x D$, *with* $x \in \mathbb{N}$ *and* $D = 1, 3$ *or* 5, *then* Δ^{-s} *lies in* C_2.

The proof of Theorem 4.2 is similar, and the reader can reproduce it by imitating part of the proof of Theorem 4.1 The proof of Theorem 4.2 is in fact simpler, because there is no need to derive the product expansions, but as D increases it becomes messy. It seems that this process can be continued, but we have no reason to believe that it will work for every odd D. We would be surprised if similar verifications of rule (1) could not also be written for $\mathrm{ord}_3\left(c_0\left[\Delta^{-s}\right]\right)$.

Proof of Theorem 4.1. The Fourier series of $E_{\infty,4}$ is monic integral, and therefore so are those of its integral powers. Thus the terms $R(n-a)$ on the right side of (2–22) are integral. If $s = 2^x$, then $0 < n < s$ implies that $\mathrm{ord}_2(n) < x$. So (2–22) implies that $R(n) \equiv 0 \pmod{16}$. Also, by (2–22),

$$R(s) = 8 \sum_{a=1}^{s} \sigma_1^{\mathrm{alt}}(a) R(s-a). \tag{4–1}$$

All the terms in the sum on the right side of (4–1), except the one corresponding to $a = s$, are congruent to zero modulo 16. Therefore,

$$\begin{aligned} R(s) &\equiv 8\sigma_1^{\mathrm{alt}}(s) R(0) \\ &\equiv 8(2^x + 2^{x-1} + \cdots + 2 - 1) \cdot 1 \\ &\equiv 8 \pmod{16}. \end{aligned}$$

Thus, $\mathrm{ord}_2(R(s)) = 3$. But $R(s) = c_0[E_{\infty,4}^{-s}]$. ◻

For the project of improving the bound in Theorem 2.12 in the case $h \equiv 2 \pmod 4$, the nonvanishing of the constant terms of the Fourier expansions of the $T_{2,h}$ forms is the key to our approach. We state some partial results in this direction for T-series of both levels. The arguments follow the approach used above and appear in [Brent 1994, Chapter 5].

Theorem 4.3. (i) *If* $h \equiv 8 \pmod{12}$ *and* $r(1,h) = 2^x$, *with* $x \geq 1$, *then* $c_0[T_h] \equiv 16 \pmod{32}$.
(ii) *If* $h \equiv 2 \pmod{12}$ *and* $r(1,h) = 2^x$, *with* $x \geq 1$, *then* $c_0[T_h] \equiv 8 \pmod{32}$.
(iii) *If* $h = 2^x - 6 > 0$, *then* $c_0[T_{2,h}] \equiv 8 \pmod{16}$.
(iv) *If* $h = 2^x - 4 > 0$, *then* $c_0[T_{2,h}] \equiv 16 \pmod{32}$.

5. APPLICATIONS TO THE THEORY OF QUADRATIC FORMS

5A. Quadratic Forms and Modular Forms

We tell how certain quadratic forms give rise to level-two modular forms. For even v, set $\boldsymbol{x} = {}^t(x_1, \ldots, x_v)$, so that $\boldsymbol{x}$ is a column vector. Let A be a $v \times v$ square symmetric matrix with integer entries, even entries on the diagonal, and positive eigenvalues. Then $Q_A(\boldsymbol{x}) = {}^t\boldsymbol{x}A\boldsymbol{x}$ is a homogenous second-degree polynomial in the x_i. We refer to Q_A as the even positive-definite quadratic form associated to A. If $\boldsymbol{x} \in \mathbb{Z}^v$, then $Q_A(\boldsymbol{x})$ is a nonnegative even number, and vanishes only if $\boldsymbol{x}$ is the zero vector. The level of Q_A is the smallest positive integer N such that NA^{-1} also has integer entries and even entries on the diagonal. Let $\#Q_A^{-1}(n)$ denote the cardinality of the inverse image in $\mathbb{Z}^v$ of an integer n under the quadratic form Q_A.

Proposition 5.1. *Suppose that* Q_A *is a level-two quadratic form. Then the function* $\Theta_A : \mathfrak{H} \to \mathbb{Z}$ *satisfying*

$$\Theta_A(z) = \sum_{n=0}^{\infty} \#Q_A^{-1}(2n) q^n$$

lies in $M(2, v/2)$.

Proof. We use machinery from [Miyake 1989]. Let $\chi : \mathbb{Z} \to \mathbb{Z}$ be a Dirichlet character mod N, and let $\alpha \in \Gamma_0(N)$ be the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. By abuse of notation we also let χ denote the character $\chi : \Gamma_0(N) \to \mathbb{Z}$ that acts by the map $\alpha \mapsto \chi(d)$. We have the stroke operator $f|_h$ defined by

$$(f|_h\, \alpha)(z) = (cz + d)^{-h} f(\alpha z), \quad \text{for } z \in \mathfrak{H}.$$

Denote by $M(h, \Gamma_0(N), \chi)$ the vector space of functions f holomorphic on $\mathfrak{H}^*$ such that $f|_h\alpha = \chi(\alpha)f$ for all $\alpha \in \Gamma_0(N)$. Thus $M(h, \Gamma_0(2), \chi)$ and $M(2, h)$ coincide for trivial χ. The space $M(h, \Gamma_0(N), \chi)$ is itself trivial if $\chi(-1) \neq (-1)^h$ [Miyake 1989, Lemma 4.3.2, p. 115] Thus the only nontrivial space $M(h, \Gamma_0(2), \chi)$ is $M(2, h)$.

Let $(n \mid m)$ be the Kronecker symbol. Let $A^{-1} = (b_{ij})$. We put

$$\psi_A(m) = ((-1)^{v/2} \det A \mid m)$$

and

$$\Delta_A = \sum_{1 \le i,j \le v} b_{ij} \frac{\partial^2}{\partial x_i \, \partial x_j}.$$

A spherical function of degree ν with respect to A is a complex homogenous polynomial $P(x_1, \ldots, x_v) = P(\boldsymbol{m})$ (say) of degree ν annihilated by Δ_A. For $z \in \mathfrak{H}$, let

$$\theta_{A,P}(z) = \sum_{\boldsymbol{m} \in \mathbb{Z}^v} P(\boldsymbol{m}) \exp\left(2\pi i \frac{Q_A(\boldsymbol{m})}{2} z\right).$$

Then $\theta_{A,P} \in M(v/2 + \nu, \, \Gamma_0(2), \, \psi_A)$ [Miyake 1989, eq. (3), p. 192]. Evidently,

$$\Theta_A = \theta_{A,1} \in M\left(v/2, \, \Gamma_0(2), \, \psi_A\right).$$

In particular, $M(v/2, \, \Gamma_0(2), \, \psi_A)$ is nontrivial, so it must be $M(2, \, v/2)$. □

Since $M(2, h)$ is nontrivial only for even h, it also follows that $4 \mid v$.

5B. Quadratic Minima

In this section we apply Theorem 2.12 to the problem of quadratic minima. It is possible to improve the result slightly by an application of Theorem 4.3 to the sparse family of weights $h \equiv 2 \pmod 4$ mentioned there. It would be substantially improved by a proof that the constant term of $T_{2,h}$ is nonzero for all $h \equiv 2 \pmod 4$, since this would improve Theorem 2.12.

Theorem 5.2. *If Q is an even positive-definite quadratic form of level two in v variables, with $8 \mid v$, then Q represents a positive integer $2n \le 2 + v/4$. If $v \equiv 4 \pmod 8$, then Q represents a positive integer $2n \le 2 + v/2$.*

Proof. Let A be the matrix associated to Q, so that $Q = Q_A$. Suppose $v = 8u$. Then $\Theta_A \in M(2, 4u)$ by Proposition 5.1. By Theorem 2.12, $\#Q_A^{-1}(2n) \neq 0$ for some n in the range $1 \le n \le r(2, 4u) = 1 + u$. That is, Q represents an integer $2n \le 2(1+u) = 2 + v/4$. On the other hand, suppose $v = 8u+4$. Then $\Theta_A \in M(2, 4u+2)$, and $\#Q_A^{-1}(2n) \neq 0$ for some n in the range $1 \le n \le 2r(2, 4u+2) = 2(1+u)$. Thus Q represents an integer $2n \le 4 + 4u = 2 + v/2$. □

6. CONCLUSION

We don't know how to frame natural descriptions of the families obeying the rules (1)–(3) from Section 3 (page 3A). We will only remark that some of our experiments indicate that the arithmetic of the constant terms comes from the modularity of the underlying functions, but not from the properties of formal power series as they relate to Ramanujan's congruences for the Ramanujan τ function. At the suggestion of Glenn Stevens, we formed nonmodular series obeying the Ramanujan congruences and checked the constant terms of their negative powers without turning up examples of rules (1)–(3). It seems to be the modularity of Δ, for example, but not in a direct way its obedience to the Ramanujan congruences, that causes it to obey rule (1).

On the basis of the observations reported in Section 3, we could make many narrow conjectures. Several seem to be worth stating.

Conjecture 6.1. (i) *The constant terms of the $T_{2,h}$ follow rule* (2) *on page* 268.

(ii) *The forms Δ^{-s} and j^s, for s a positive integer, follow rule* (1).

(iii) *The forms Δ and j^{-1} satisfy the relations between them stated in Section* 3C *for all integers $n \ge 1$, and the reciprocal forms Δ^{-1} and j satisfy the relations between them stated in Section* 3C *for all nonzero integers $n \ge -1$.*

Part (i) of this conjecture would have the following consequence:

Conjecture 6.2. *Suppose* $f \in M(2, h)$ *with Fourier expansion at infinity*

$$f(z) = \sum_{n=0}^{\infty} A_n q^n, \quad \text{with } A_0 \neq 0.$$

If $h \equiv 2 \pmod 4$, *then some* $A_n \neq 0$, *for* $1 \leq n \leq 1 + r(2, h)$.

This in turn would imply:

Conjecture 6.3. *If* Q *is a level-two even positive-definite quadratic form in* v *variables, where* $v \equiv 4 \pmod 8$, *then* Q *represents a positive integer* $2n \leq 3 + v/4$.

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