Two Short Presentations for Lyons′ Sporadic Simple Group

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CONTENTS

Keywords: Lyons' group, defining set of relations, constructing presentations, Double Coset Cannon Algorithm, group theoretic algorithms

Using an algorithm developed by the author, two new presentations for R. Lyons' sporadic simple group Ly are established, which contain fewer relations and are shorter than previously known ones.

1. INTRODUCTION

In [Gebhardt 2000] I introduced an algorithm, called the Double Coset Cannon Algorithm (DCCA), producing a rather short defining set of relations for a group G with respect to a given generating set of G. Given some subgroup $H < G$, the algorithm essentially uses the structure of the H-H-double cosets in G in combination with a test for redundancy of relators described in [Cannon 1973] and extends a defining set of relations for H to a defining set of relations for G. For its application, the permutation representation of the generators of G on the cosets of the subgroup H (i.e., the Cayley graph of G with respect to H and the generating set in question) is needed, and a base and a strong generating set for this permutation representation have to be known.

In this paper, we apply the algorithm to two permutation representations of R. Lyons' sporadic simple group Ly [Lyons 1972] and obtain two new presentations for this group, which involve the same generating sets as two previously known presentations published by H. Gollan [1998] and C. Sims [1973; Havas and Sims 1999], respectively, but contain fewer relations and have a smaller total length.

In the sequel, we denote by $\langle X \mid R\rangle$ the finitely presented group with set of generators X and set of relations R, where X is a finite set and $R \subset F(X)$ is a finite subset of the free group generated by X. For a subset X of any group G, $\langle X\rangle$ denotes the subgroup of G generated by X.

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2. SKETCH OF THE ALGORITHM

In this section we give a concise sketch of the used algorithm. For a detailed discussion and proofs see [Gebhardt 2000] or [Gebhardt 1999].

Assume $G = \langle H, t\rangle$ and let X_H be a generating set of H with defining set of relations R_H. Let Γ be the Cayley graph of G with respect to H and the generating set $X = X_H \cup \{t\}$ of G. Denote by $\Gamma|_H$ the graph obtained from Γ by removing all edges labelled with t, let $\Gamma_1, \dots, \Gamma_s$ be the connected components of $\Gamma|_H$ and for $i = 1, \dots, s$ fix a vertex v_i of Γ_i. Without loss of generality we can assume $v_1 = H$ and $v_2 = Ht^{-1}$. Fixing a maximal tree $\mathcal{T}$ of Γ, which contains maximal trees of all connected components of $\Gamma|_H$, one can (using the notion of *fundamental circuits*) associate to every edge ε of Γ a *fundamental relator* $\rho(\varepsilon)$ and moreover to every edge κ of some connected component Γ_i of $\Gamma|_H$ an element $g_{v_i}(\kappa) \in \mathrm{Stab}_H(v_i)$.

The following results are the basis for Cannon's algorithm [1973].

Theorem 2.1. *The set of all fundamental relators extends R_H to a defining set of relations of G with respect to the generating set X.*

Theorem 2.2. *Let $R \supseteq R_H$ be a set of relations holding in G and assume that in Γ edges are coloured such that the fundamental relator to every coloured edge is derivable from the relations in R. Let ρ be a relator derivable from R, which induces at some vertex of Γ a path containing exactly one uncoloured edge κ (counted with multiplicities).*

Then the fundamental relator $\rho(\kappa)$ is derivable from R.

The next two theorems are established in [Gebhardt 2000].

Theorem 2.3. *Let $\kappa_1, \dots, \kappa_r$ be edges of some connected component Γ_i of $\Gamma|_H$ and let $R \supseteq R_H$ be a set of relations holding in G, from which $\rho(\kappa_1), \dots, \rho(\kappa_r)$ are derivable. Let κ be an edge of Γ_i.*

Then $\rho(\kappa)$ is derivable from R if

$$g_{v_i}(\kappa) \in \langle g_{v_i}(\kappa_1), \dots, g_{v_i}(\kappa_r)\rangle.$$

Theorem 2.4. *Let v be a vertex of Γ, $k \in H \cap H^{t^{-1}}$ and let Γ_i and Γ_j be the connected components of $\Gamma|_H$ containing the vertices v and v^t, respectively. Let $R \supseteq R_H$ be a set of relations holding in G, from which the fundamental relators belonging to all edges of Γ_2, Γ_i and Γ_j are derivable.*

Then the fundamental relators belonging to the edges labelled with t starting at the vertices v and v^k, respectively, are equivalent modulo R.

By defining a map $\mathrm{wt} : F(X) \to \mathbb{Z}^{|X|}$ in the obvious way and considering the images of R and of the possible extensions of R by a single fundamental relator under wt, some heuristics for adding new fundamental relators to R can be established [Gebhardt 2000].

From these results, one obtains the following algorithm:

Algorithm (DCCA). Let X_H be a generating set for $H < G$ with defining set of relations R_H and let $X = X_H \cup \{t\}$ be a generating set for G.

The following algorithm extends R_H to a defining set R of relations for G with respect to the generating set X. (See next page for the procedure DCCAColour.)

[initialization]
construct Γ, $\Gamma|_H$, Γ_i, v_i $(i = 1, \dots, s)$, and $\mathcal{T}$ as above
colour the edges of $\mathcal{T}$ and the edges of Γ_1
$R := R_H$
$S_i := \langle 1\rangle$ $(i = 2, \dots, s)$

[add one relator belonging to an edge with label t]
find edge ε with minimal $\left|\mathbb{Z}^{|X|} : \langle \mathrm{wt}\,(R \cup \{\rho(\varepsilon)\})\rangle_{\mathbb{Z}}\right|$
$R := R \cup \{\rho(\varepsilon)\}$
DCCAColour( )

[Step 1: treat edges of $\Gamma|_H$]
for $i = 2, \dots, s$
  while $|S_i| < \left|\mathrm{Stab}_H(v_i)\right|$
    find edge ε of Γ_i such that $g_{v_i}(\varepsilon) \notin S_i$
    $R := R \cup \{\rho(\varepsilon)\}$
    DCCAColour( )
  end while
end for

[Step 2: treat remaining edges of Γ]
while there exists an uncoloured edge
  find uncolored edge ε with minimal $\left|\mathbb{Z}^{|X|} : \langle \mathrm{wt}\,(R \cup \{\rho(\varepsilon)\})\rangle_{\mathbb{Z}}\right|$
  $R := R \cup \{\rho(\varepsilon)\}$
  DCCAColour( )
end while

```
procedure DCCAColour( )
repeat
  quit := true
  for all vertices v of Γ
    for all ρ ∈ R
      if the path induced by ρ at v contains exactly one uncoloured edge ε
        colour ε
        quit := false
        if ε has label t
          if the requirements of Theorem 2.4 are satisfied
            colour the edges with label t starting at all vertices in w^{H∩H^{t^{-1}}}, where w is the starting
          end if                                                                                   vertex of ε
        else
          find i ∈ {2, ..., s} such that ε is an edge of Γ_i
          S_i := ⟨S_i, g_{v_i}(ε)⟩
          colour all edges ε' of Γ_i satisfying g_{v_i}(ε') ∈ S_i
          if |S_i| = |Stab_H(v_i)|
            [treatment of Γ_i has just been completed]
            check for new possibilities of applying Theorem 2.4
          end if
        end if
      end if
    end for
  end for
until quit = true
```

The procedure DCCAColour.

3. PRESENTATIONS OBTAINED USING THE SUBGROUP $G_2(5)$

3A. The Presentation by C. Sims

In the course of his original proof of existence and uniqueness of Ly, C. Sims in [Sims 1973; Havas and Sims 1999] gave a presentation of Ly on 5 generators a, b, c, d, z plus 34 auxiliary generators which contains 86 relations. After substituting the auxiliary generators, there remains a set R_G^{Sims} of 52 nontrivial relations involving the generators a, b, c, d, z with a total length of 1297. From this presentation, a permutation representation of G on 8 835 156 digits can be obtained via coset enumeration with respect to the subgroup generated by a, b, c and d [Havas and Sims 1999].

According to [Havas and Sims 1999], we can write $\langle a, b, c, d, z \mid \rangle \xrightarrow{\pi} \langle a, b, c, d, z \mid R_G^{\text{Sims}} \rangle =: G \simeq \text{Ly}$ and

$$\pi\left(\langle a, b, c, d \mid \rangle\right) =: H_1 \simeq G_2(5).$$

3B. A New Presentation via $G_2(5)$

Restricting the permutation representation of degree 8 835 156 to the smallest H_1-orbit, one obtains a permutation representation of H_1 of degree 19 530. A base and a strong generating set for this permutation representation can easily be computed using MAGMA [Bosma et al. 1997]. Since H_1 is a point stabilizer in the permutation representation of degree 8 835 156 of G, this leads to a base and a strong generating set for this permutation representation of G.

Moreover, we use the permutation representation of H_1 of degree 19 530 in order to verify that $H_2 := \pi\left(\langle a, b, c \mid \rangle\right) \simeq 5_+^{1+4} : \mathrm{GL}_2(5)$ and that the order of H_2 is $|H_2| = 1\,500\,000$, hence $|H_1 : H_2| = 3\,906$.

We start by choosing a defining set R_{H_2} of relations from the set R_G^{Sims}: Let

$$R_{H_3} := \left\{a^8, b^5, (ab)^4, [a^2, b], [a, b]^3\right\} \subset R_G^{\text{Sims}}$$

and

$$R_{H_2} := R_{H_3} \cup \{c^5,\, c^{a^2} = c^3,\\ c^{ba} = c^{a^2 b} cbcb^{-1},\, c^{b^2} = c^2 c^{b^{-1}} (c^b)^{-2}\} \subset R_G^{\mathrm{Sims}}.$$

We apply the coset enumeration programme ACE [Havas 1991] in order to compute the group order $|\langle a, b \,|\, R_{H_3}\rangle| = 480$. One also checks that the subgroup of $\langle a, b, c \,|\, R_{H_2}\rangle$ generated by a and b is of index 3125. This proves that

$$|\langle a, b, c \,|\, R_{H_2}\rangle| = 1\,500\,000;$$

hence, R_{H_2} is a defining set of relations for H_2 containing 9 relations with a total length of 80.

Then, by applying the DCCA twice (to $H_1 = \langle H_2, d\rangle$ with subgroup H_2 and to $G = \langle H_1, z\rangle$ with subgroup H_1), we construct sets R_{H_1} and R_G of relations, such that $R_{H_2} \cup R_{H_1}$ is a defining set of relations for $H_1 \simeq G_2(5)$ and $R_{H_2} \cup R_{H_1} \cup R_G$ is a defining set of relations for $G \simeq \mathrm{Ly}$.

For the first application of the DCCA, the permutation representation of H_1 corresponding to the action of H_1 on the cosets of H_2 is needed. This permutation representation of degree 3906, as well as a base and a strong generating set for it, can easily be constructed from the permutation representation of H_1 of degree 19530 using MAGMA.

There are 3 nontrivial H_2-H_2-double cosets in H_1. As can be expected from the results of [Gebhardt 2000], the R_{H_1} obtained consists of 7 relations:

$$\begin{aligned} R_{H_1} := \{&(a\bar{b}a)^d = a\bar{b}a^5, \quad (b^2\bar{a})^d = \bar{a}^2 b^2 \bar{a},\\ &(ba\bar{c}ba\bar{b}^2a)^{dcd} = \bar{a}b\bar{a}\bar{b}a\bar{b}^2ac\bar{b}cba\bar{c},\\ &(a^2\bar{c}ba\bar{c}b\bar{a}\bar{b})^{dcd} = \bar{a}^2\bar{b}\bar{a}(\bar{b}c)^2b^2,\\ &(b^2acba)^{dc\bar{a}bcd} = \bar{a}b\bar{a}\bar{b}\bar{a}\bar{b}^2c\bar{a}\bar{b}\bar{c}ba,\\ &(a\bar{c}^2b)^{dc\bar{a}bcd} = \bar{a}^4b^2\bar{c}\bar{b}a\bar{b}ca\bar{b},\\ &c\bar{a}\bar{c}a\bar{c}\bar{a}cad\bar{c}\bar{a}\bar{c}ac\bar{a}cdca\bar{c}\bar{a}\bar{c}acd\} \end{aligned}$$

Here and elsewhere we use the convention $\bar{a} := a^{-1}$, and so on. The total length of R_{H_1} is 160. The CPU time for this computation on an IBM RS/6000-590 was 12 seconds.

The data necessary for the second application of the DCCA have already been provided. There are 4 nontrivial H_1-H_1-double cosets in G. The DCCA thus produces a set R_G containing 9 relations:

$$\begin{aligned} R_G := \{&a^z = \bar{a}^3, \quad a^{zdz} = a^3,\\ &(\bar{c}\bar{d}cbab^2(cb)^2\bar{c}d)^{zdz} =\\ &\qquad \bar{c}bc^2a\bar{c}b\bar{d}ca\bar{c}b\bar{c}\bar{b}^2c\bar{a}c\bar{d}cb^2cda\bar{d}\bar{c}\bar{d}cb\bar{a}b,\\ &[z, \bar{d}\bar{c}ba\bar{b}\bar{d}c\bar{d}\bar{c}d\bar{c}\bar{b}a\bar{b}c\bar{d}cd] = bc\bar{b}\bar{c},\\ &a^{zd\bar{b}z} = \bar{a}\bar{d}\bar{c}b\bar{c}^2a\bar{b}c\bar{a}b^2c\bar{b}c\bar{a}\bar{c}d\bar{b}\bar{a},\\ &(\bar{c}\bar{a}\bar{d}\bar{c}ba\bar{b}a\bar{c}bca\bar{c}\bar{d}cdac\bar{b}abac)^{zd\bar{b}z} =\\ &\qquad \bar{a}c\bar{a}^3\bar{c}\bar{b}^2\bar{c}d\bar{c}a\bar{c}b^2c\bar{b}c\bar{a}\bar{c}d\bar{b}\bar{c}\bar{d}\bar{c}babdc\bar{a},\\ &(\bar{a}bdc\bar{a}\bar{b}ab\bar{a}\bar{c}\bar{b}ca)^{zdcdz} = \bar{c}a^3b\bar{c}\bar{b}\bar{a}c\bar{d}\bar{c}b\bar{a}\bar{b},\\ &(\bar{d}cb\bar{a}\bar{b})^{zdcdz} = ac\bar{a}ba\bar{c}bca\bar{c}\bar{b}\bar{a}\bar{b}ab\bar{d}ca\bar{c}\bar{b}\bar{c}a,\\ &a\bar{d}\bar{c}b\bar{a}^2\bar{d}\bar{c}\bar{b}^2\bar{c}d\bar{c}(a\bar{c})^2b\bar{a}c^2\bar{b}c\bar{d}cac\bar{b}a\bar{d}\bar{z}bz\bar{b}z\} \end{aligned}$$

The total length of R_G is 309. The CPU time for this computation on an IBM RS/6000-590 was 12.3 hours, and about 450 MB of memory were used.

The complete defining set $R_{H_2} \cup R_{H_1} \cup R_G$ of relations for G with respect to the generators a, b, c, d and z thus contains 25 relations of total length 549.

4. PRESENTATIONS OBTAINED USING THE SUBGROUP $\hat{3}$McL : 2

4A. The Presentation by H. Gollan

H. Gollan [1998] gave a proof for the existence and uniqueness of Lyons' group that is independent of Sims' proof. As a part of this proof, he also established a presentation for Ly. The starting point is a transitive permutation group

$$G = \pi(\langle a, b, c, d, e, f, t, y \,|\, \rangle)$$

with 8 generators on 9 606 125 digits, with a subgroup

$$H := \pi(\langle a, b, c, d, e, f, t \,|\, \rangle) \simeq \hat{3}\,\mathrm{McL} : 2$$

which he proves to be the full point stabilizer

$$\mathrm{Stab}_G(2252).$$

With this at hand he eventually proves that $G \simeq \mathrm{Ly}$.

According to [Conway et al. 1985] and [Gollan 1998], the set R_H shown at the top of the next page is a defining set of relations for $H \simeq \hat{3}\,\mathrm{McL} : 2$ with respect to the generators a, b, c, d, e, f, t, where a, b, c, d, e, f correspond to generators of $\hat{3}\,\mathrm{McL}$ and t to the outer automorphism of this group. The total length of R_H is 556.

From his so-called double coset trick which he used to prove the equality $H = \mathrm{Stab}_G(2252)$, Gollan

$$
\begin{aligned}
R_H := \bigl\{ & a^2,\ b^2,\ c^2,\ d^2,\ e^2,\ f^2,\ t^2,\ (ab)^3,\ (ac)^2,\ (ad)^2,\ (ae)^4,\\
& (af)^2,\ (bc)^5,\ (bd)^2,\ (be)^2,\ (bf)^2,\ (cd)^3,\ (ce)^3,\ (cf)^4,\ (de)^2,\\
& (df)^3,\ (ef)^6,\ a(cf)^2,\ b(ef)^3,\ (eab)^3,\ (bce)^5,\ (aecd)^4,\ (cef)^{21},\\
& (abc)^5,\ (abce)^{24},\ a^t \cdot abcdfcbecfdcba,\ b^t \cdot (bc)^2 dfeacbcfefdc,\\
& c^t \cdot r(bc)^2 r^2 (bc)^2 r^2 bcafdcbcefcbcdecdfe,\ d^t \cdot (bc)^4 abacdecbfcdecbac,\\
& e^t \cdot (bc)^3 r (bc)^2 rbcrafdeacbceaefdcfea,\ f^t \cdot bcr^2((bc)^2 r^2)^2 abfcbdcbeacbd \bigr\}
\end{aligned}
$$

Set R_H of relations for the presentations of Section 4, where $r := cdefdcbefedcfdecfd(cef)^{14}$.

as a by-product also obtained a set R_G^{Gollan} of 78 additional relations with a very large total length (far more than 4000), so that $R_H \cup R_G^{\mathrm{Gollan}}$ is a defining set of relations for $G \simeq \mathrm{Ly}$ with respect to the generators a, b, c, d, e, f, t, y.

4B. A New Presentation via $\hat{3}$McL : 2

Because of Gollan's result $H = \mathrm{Stab}_G(2252)$, the permutation representation of $G \simeq \mathrm{Ly}$ of degree 9 606 125 can be interpreted as the permutation representation of G on the cosets of

$$H \simeq \hat{3}\,\mathrm{McL} : 2$$

in G. From [Gollan 1998], a base and a strong generating set for this permutation representation are available, too.

Although the defining set R_H of relations for H is not minimal, it is used in the sequel, since the relations are very natural and since they nicely reflect the group structure of $\hat{3}\,\mathrm{McL} : 2$.

Thus, the data necessary for applying the DCCA are available. There are 4 nontrivial H-H-double cosets in G, and so we obtain a set R_G containing 9 relations with a total length of 828 (see sidebar below). The total CPU time on an IBM RS/6000-590 was approximately 33.9 hours and about 400 MB of memory were used.

The complete defining set $R_H \cup R_G$ thus contains 45 relations with a total length of 1384. Note, however, that no optimization was applied to the presentation of H, yet.

5. CONCLUSIONS

The computations described in this paper have been performed using the author's implementation of the DCCA which exists in the form of a C++ class library. Some information can be found in [Gebhardt 1999]; the source code is available from the author on request.

It has already been noted in [Havas and Sims 1999] that presentations of Ly are important to representation theory, in that proofs of the correctness of several matrix representations of Ly rely upon

$$
\begin{aligned}
R_G := \bigl\{ & (bcbdtcefectcbtb)^y = ebftefcbteacbtcbatbefdctbfteacbtcbaecdcbcfetftcbatbtdfdceb,\\
& (bcbdtcfecbcetcecb)^y = tatatbtbctfteacbtcbatbefdctefcbteacbtcbatbefdcfbtbfteacbtcbatabtabde,\\
& (abceacdtbcetefea)^{ydy} = btabctfteatbfteatbtabctbcaetftabctbcaetfbtbtabctbcaetfbadtdeatce,\\
& (abceacdtecetctebcb)^{ydy} = tbtabctbcaetftabctbcaetfbtbfcdfebtabctbcaetbcfetbatceatbcaetfbcdtetce,\\
& (atetadtbetabtbtatbcdfab)^{ycby} = tbtbtabctcdfebtabctbcaetbcefeteacbtcbacectbtbcbetaeabfecta,\\
& (atetadtbetacebcbtcbaet)^{ycby} = tbteacbtcbatbtbtabctbcaetfbtatdfecfdcbtcbatbefdcftabctbcaetftbetct,\\
& (tdtatadceatedtfecbae)^{yatby} = btabtbfteacbtcbatbtbtabctbcaetftabctbcaetftabctbcaetfbabectcfteacbtcbaeft,\\
& (tdtatadcebcbctabctb)^{yatby} = tbftetbfteacbtcbatfteacbtcbatfteacbtcbatbtbfcdfebtabctbcaetbcfetfbtcdfctct,\\
& tbtatbftdfefdtabctbacetftcbatftefcbteacbtcbatbefdcftabctbacdfetfbty^{-1}btay^{-1}tdtatadceyatcdabtatdcycby \bigr\}
\end{aligned}
$$

Set R_G of relations for the presentation in Section 4B.

checking that the generating matrices satisfy defining relations. Having in mind matrix representations of larger degree, the total length of such a presentation becomes important, since it essentially determines the number of matrix-matrix multiplications which have to be performed to verify the correctness of a representation.

In this respect, especially the new presentation obtained via $G_2(5)$ may be useful, since the number of relations is just half of the number of relations of Sims' original presentation. Its total length is smaller by a factor of more than 2.3.

For computations involving the maximal subgroup $\hat{3}\,\mathrm{McL} : 2$ of Ly, also the second presentation constructed here may be of use, since it reflects the structure of this subgroup quite nicely.

It should be noted, however, that due to the rather efficient elimination of redundant generators, it is fairly difficult to apply coset enumeration techniques to presentations obtained from the DCCA in general.

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