OSCILLATIONS OF FUNCTIONAL DIFFERENTIAL EQUATIONS
GENERATED BY ADVANCED ARGUMENTS

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Introduction. We consider the functional differential equation

\[ L_n x(t) + \delta f(t, x[g_1(t)], \ldots, x[g_m(t)]) = 0, \tag{E, \delta} \]

where \( n \geq 2, \delta = \pm 1, L_0 x(t) = x(t), L_k x(t) = \left(1/a_k(t)\right)(L_{k-1} x(t)), k = 1, 2, \ldots, n, a_n = 1, \)
\( (= d/dt), a_i : R_+ \to R_+ \setminus \{0\}, i = 1, 2, \ldots, n - 1, g_i : R_+ \to R_+ = [0, \infty), i = 1, 2, \ldots, m, \)
\( f : R_+ \times R^m \to R = (-\infty, \infty) \) are continuous and \( g_i(t) \geq t \) on \( R_+ \) for \( i = 1, 2, \ldots, m. \)

We always assume

\[ \int_{-\infty}^{\infty} a_i(s) ds = \infty, \quad i = 1, 2, \ldots, n - 1. \tag{1} \]

We also introduce the following conditions:

\((C_1)\) There exist a continuous function \( q : R_+ \to R_+ \), and nonnegative numbers \( \lambda_i, i = 1, 2, \ldots, m \) with \( \sum_{i=1}^{m} \lambda_i = 1 \) such that

\[ f(t, x_1, \ldots, x_m) \operatorname{sgn} x_1 \geq q(t) \prod_{i=1}^{m} |x_i|^{\lambda_i} \]

for \( t \in R_+ \) and \( x_1 x_i > 0, i = 1, 2, \ldots, m. \)

\((C_2)\) There exist continuous functions \( q_i : R_+ \to R_+ \), \( i = 1, 2, \ldots, m \) such that

\[ f(t, x_1, \ldots, x_m) \operatorname{sgn} x_1 \geq \sum_{i=1}^{m} q_i(t)|x_i| \]

for \( t \in R_+ \) and \( xx_i > 0, i = 1, 2, \ldots, m. \)

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The domain of $L_n$, $D(L_n)$ is defined to be the set of all functions $x : [T_x, \infty) \to \mathbb{R}$ such that $L_jx(t), j = 0, 1, \ldots, n$ exist and are continuous on $[T_x, \infty)$. Our attention is restricted to those solutions $x \in D(L_n)$ of equation $(E, \delta)$ which satisfy $\sup\{|x(t)| : t > T\} > 0$ for any $T > T_x$. We make the standing hypothesis that equation $(E, \delta)$ is called oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory. Equation $(E, \delta)$ is said to be oscillatory if all of its solutions are oscillatory. Equation $(E, \delta)$ is said to be almost oscillatory if:

(i) for $\delta = 1$ and $n$ even, every solution of equation $(E, 1)$ is oscillatory.

(ii) for $\delta = 1$ and $n$ odd, every solution $x(t)$ of equation $(E, 1)$ is either oscillatory or

$L_jx(t) \to 0$ monotonically as $t \to \infty$, $j = 0, 1, \ldots, n - 1$;

(iii) for $\delta = -1$ and $n$ even, every solution $x(t)$ of equation $(E, -1)$ is oscillatory, $L_jx(t) \to 0$ monotonically as $t \to \infty$, $j = 0, 1, \ldots, n - 1$ or $|L_jx(t)| \to \infty$ monotonically as $t \to \infty$, $j = 0, 1, \ldots, n - 1$;

(iv) for $\delta = -1$ and $n$ odd, every solution $x(t)$ of equation $(E, -1)$ is either oscillatory or

$|L_jx(t)| \to \infty$ monotonically as $t \to \infty$, $j = 0, 1, \ldots, n - 1$.

The main purpose of this paper is to establish some criteria for equation $(E, \delta)$ to be oscillatory or almost oscillatory. The behavioural properties obtained for equation $(E, \delta)$ are generated by the advanced arguments $g_i (i = 1, 2, \ldots, m)$ and are not valid for the corresponding ordinary differential equations.

The oscillatory and asymptotic behaviour of solutions of differential equations with deviating arguments has been the subject of many recent investigations. Of particular importance, however, has been the study of oscillations which are caused by the deviating arguments and which do not appear in the corresponding ordinary differential equations. For more detail see ([1], [3], [6], [8] and [9]).

Most of the literature on the oscillation of advanced differential equations has been concerned with equations of the form

$$x^{(n)}(t) + q(t)x[g(t)] = 0, \quad n \text{ is even},$$

$(E*)$

where $q, g : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and $g(t) \geq t$.

Four of the more important such conditions which guarantee that all solutions of equation $(E*)$ oscillate on $\mathbb{R}_+$ are the following:

(I) Koplatadze and Chanturia [4, Theorem 8.5]

$$\liminf_{t \to \infty} t^n q(t) > \max\{-\beta(\beta - 1) \ldots (\beta - n + 1) : 0 \leq \beta \leq n - 1\};$$

(II) Kusano [5, Theorem 3]

$$\limsup_{t \to \infty} t \int_t^\infty s^{n-2} q(s) \, ds > (n - 1)!,$$

or

$$\liminf_{t \to \infty} t \int_t^\infty s^{n-2} q(s) \, ds > \frac{(n - 1)!}{4};$$

(III) Oláh [6, Theorems 1 and 4]

$$\limsup_{t \to \infty} \int_t^{g(t)} (s - t)^{n-1} q(s) \, ds > (n - 1)!,$$
or

\[ \min \left\{ \limsup_{t \to \infty} \int_t^\infty \frac{(s-t)s^{n-2}}{n-1} q(s) \, ds, \limsup_{t \to \infty} \int_t^\infty (u-s)^{n-2} q(u) \, du \right\} > (n-2)! ; \]

(IV) Webrowski [9, Corollary 1]

\[ \min \left\{ \liminf_{t \to \infty} \int_t^\infty s^{n-2} \int_s^\infty q(u) \, du \, ds, \liminf_{t \to \infty} \int_t^\infty (u-s)^{n-2} q(u) \, du \, ds \right\} > \frac{(n-1)!}{e} . \]

Webrowski’s result is actually more general than that stated above and also it is obtained for more general equations of type \((E, \delta)\) with \(L_nx = x^{(n)}\), and all \(a_i(t) \equiv 1\) for \(t > 0\) and \(i = 1, 2, \ldots, n-1\).

On the other hand, the behavioural properties of equations of type \((E, \delta)\) have been vigorously investigated in the literature and as recent contributions to this study we refer to the papers of Grace and Lalli ([1] and [2]), Kitamura [3] and Philos [7] and the references cited therein.

In most of the results in [2], [3] and [7] and also other related works, the advanced equations have been treated as ordinary equations and hence the results fail to describe the oscillatory character of equations of the form

\[ \left( \frac{1}{t} \dot{x}(t) \right) + \frac{c}{t^3} x^\left(\frac{3}{2}\right) = 0, \quad c > 0 \text{ and } t > 0. \]

Also, the above state criteria (I) - (IV) are not applicable to the above equation.

In the present paper, we will establish some oscillation results for equation \((E, \delta)\) with advanced arguments which extend and unify the results of Webrowski [9]. We also mention here that the results of this paper are presented in a form which is essentially new and are independent of the analogous known ones for advanced differential equations see [1], [3] and [7]. The obtained results when specialized to equation \((E^*)\) are also independent of those given in [4], [5] and [6] which are mentioned above.

2. Main results. We begin by formulating some preparatory results which are needed for our results. For any continuous functions \(p_i : [t_0, \infty) \to R, i = 1, 2, \ldots, \) we define

\[ I_0 = 1 \]

\[ I_i(t, s; p_1, \ldots, p_i) = \int_t^s p_i(u) I_{i-1}(u, s; p_{i-1}, \ldots, p_1) \, du, \quad i = 1, 2, \ldots. \]

It is easy to verify that for \(i = 1, 2, \ldots, n-1\)

\[ I_i(t, s; p_1, \ldots, p_i) = (-1)^i I_i(s, t; p_i, \ldots, p_1) \quad \text{and} \quad I_i(t, s; p_1, \ldots, p_i) = \int_s^t p_i(u) I_{i-1}(t, u; p_1, \ldots, p_{i-1}) \, du. \]

The following three lemmas will be needed in the proofs of our results.

Lemma 1. If \(x \in D(L_n)\), then for \(t, s \in [t_0, \infty)\) and \(0 \leq i < k \leq n\)

\[ L_i x(t) = \sum_{j=1}^{k-1} I_{j-i}(t, s; a_{i+1}, \ldots, a_j) L_j x(s) \]

\[ + \int_s^t I_{k-i-1}(t, u; a_{i+1}, \ldots, a_{k-1}) a_k(u) L_k x(u) \, du . \]
\[ L_i x(t) = \sum_{j=1}^{k-1} (-1)^{j-i} I_{j-i}(s, t; a_j, \ldots, a_{i+1}) L_j x(s) \]
\[ + (-1)^{k-i} \int_t^s I_{k-i-1}(u, t; a_{k-i-1}, \ldots, a_{i+1}) a_k(u) L_k x(u) \, du. \]

This Lemma is a generalization of Taylor's formula with remainder encountered in calculus. The proof is immediate.

**Lemma 2.** Suppose condition (1) holds. If \( x \in D(L_n) \) and is of constant sign and not identically zero for all large \( t \), then there exist a \( t_x \geq t_0 \) and an integer \( \ell, \ 0 \leq \ell \leq n \) with \( n + \ell \) even for \( x(t)L_n x(t) \) nonnegative or \( n + \ell \) odd for \( x(t)L_n x(t) \) nonpositive and such that for every \( t \geq t_x \)
\[ \ell > 0 \ \text{implies} \ \ x(t)L_k x(t) > 0, \ (k = 0, 1, \ldots, \ell) \]
and
\[ \ell \leq n - 1 \ \text{implies} \ (-1)^{\ell-k} x(t)L_k x(t) > 0, \ (k = \ell, \ell + 1, \ldots, n). \]

This lemma generalizes a well-known lemma of Kiguradze and can be proved similarly.

**Lemma 3.** Consider the integro-differential inequality with advanced arguments
\[ \dot{y}(t) \text{ sgn} y(t) \geq \int_t^{\infty} Q(t, s) \prod_{i=1}^m [y[g_i(s)]]^{\alpha_i} \, ds, \]
where \( Q : R^2_+ \rightarrow R_+ \) and \( g_i : R_+ \rightarrow R_+ \) are continuous functions with \( g_i(t) \geq t \) on \( R_+ \), \( i = 1, 2, \ldots, m, \alpha_i \) are nonnegative numbers with \( \alpha_1 + \alpha_2 + \cdots + \alpha_m = 1 \). If
\[ \sum_{i=1}^m \alpha_i \liminf_{t \rightarrow \infty} \int_t^{\infty} Q(s, u) \, du \, ds > \frac{1}{e} \]
then every solution of inequality (2) is oscillatory.

For a proof see ([8],[9]). It will be convenient to make use of the following notations in the remainder of this paper. For every \( T \geq t_0 \) and all \( t \geq s \geq T \) we let
\[ \alpha_i[t, s] = I_i(t, s; a_1, \ldots, a_i), \quad i = 1, 2, \ldots, n - 1; \]
\[ \beta_i[t, s] = I_{n-i-1}(t, s; a_{n-i-1}, \ldots, a_{i+1}), \quad i = 1, 2, \ldots, n - 1; \]
\[ \gamma_i[t, s] = a_1(t) I_{i-1}(t, s; a_1, \ldots, a_i), \quad i = 2, 3, \ldots, n - 1, \]
and
\[ \gamma_1[t, T] = a_1(t); \]
\[ R_i[t, T] = \int_T^t a_i(s) \, ds \quad i = 1, 2, \ldots, n - 1; \]
\[ \rho_i[t, T] = \int_T^t \alpha_i-2[t, s] a_i-1(s) R_i[s, T] \, ds \quad i = 2, 3, \ldots, n - 1, \]
and
\[ \rho_1[t, T] = R_1[t, T]. \]

We let
\[ \sigma(t) = \min\{g_1(t), g_2(t), \ldots, g_m(t)\}, \quad t \geq T, \]
and
\[ \tau(t) = \max\{g_1(t), g_2(t), \ldots, g_m(t)\}, \quad t \geq T. \]
Theorem 1. Let conditions (1) and \((C_1)\) hold. A sufficient condition for equation \((E, \delta)\) to be almost oscillatory is that:

(i) when \(\delta = 1\) and \(n\) is even,
\[
\sum_{i=1}^{m} \lambda_i \liminf_{t \to \infty} \int_t^{g_i(t)} \gamma_{\ell}[s, T] \int_s^{\infty} \beta_{\ell}[u, s] q(u) \, du \, ds > \frac{1}{e} \tag{4;\ell}
\]
for all large \(T\), \((\ell = 1, 3, \ldots, n - 1)\) holds;

(ii) when \(\delta = 1\) and \(n\) is odd, condition \((4;\ell)\) \((\ell = 2, 4, \ldots, n - 1)\) and
\[
\int_{-\infty}^{\infty} \beta_0[s, T] q(s) \, ds = \infty, \quad \text{for all large } T, \text{ holds;} \tag{5}
\]

(iii) when \(\delta = -1\) and \(n\) is even, conditions \((4;\ell)\) \((\ell = 2, 4, \ldots, n - 2)\), \((5)\) and
\[
\int_{-\infty}^{\infty} q(s) \prod_{i=1}^{n} (\alpha_{n-1}[g_i(s), T])^{\lambda_i} \, ds = \infty, \quad \text{for all large } T, \text{ holds;} \tag{6}
\]

(iv) when \(\delta = -1\) and \(n\) is odd, conditions \((4;\ell)\) \((\ell = 1, 3, \ldots, n - 2)\) and \((6)\) hold.

**Proof:** Suppose that equation \((E, \delta)\) has a nonoscillatory solution \(x(t) \neq 0\) for \(t \geq t_0 \geq 0\). Then from equation \((E, \delta)\), condition \((C_1)\) and Lemma 2 it follows that there exist a \(t_1 \geq t_0\) and an integer \(\ell \in \{0, 1, \ldots, n\}\) with \(n + \ell\) odd if \(\delta = 1\) or \(n + \ell\) even if \(\delta = -1\) such that
\[
\begin{cases}
x(t)L_k x(t) > 0, \quad \text{for } t \geq t_1, & (k = 1, 2, \ldots), \\
(-1)^{\ell-k} x(t)L_k x(t) > 0, \quad \text{for } t \geq t_1, & (k = \ell, \ell + 1, \ldots, n).
\end{cases} \tag{7}
\]
Suppose \(\ell \in \{2, 3, \ldots, n - 1\}\). Then, from Lemma 1 (ii) we obtain
\[
L_\ell x(t) = \sum_{j=\ell}^{n-1} (-1)^{j+\ell} I_{-\ell}(s, t; a_j, \ldots, a_{\ell+1}) L_j x(s)
\]
\[+ (-1)^{n-\ell} \int_t^{s} I_{n-\ell-1}(u, t; a_{n-1}, \ldots, a_{\ell+1}) L_n x(u) \, du \quad \text{for } s \geq t \geq t_1.
\]
Using \((7)\) and letting \(s \to \infty\) we have
\[
|L_\ell x(t)| \geq \int_t^{\infty} I_{n-\ell-1}(u, t; a_{n-1}, \ldots, a_{\ell+1}) q(u) \prod_{i=1}^{m} |x[g_i(u)]|^{\lambda_i} \, du
\]
\[= \int_t^{\infty} \beta_\ell[u, \ell] q(u) \prod_{i=1}^{m} |x[g_i(u)]|^{\lambda_i} \, du.
\]
Again, from Lemma 1 (i) we get
\[
\dot{x}(t) = a_1(t) \sum_{j=1}^{\ell-1} I_{j-1}(t, t_1; a_2, \ldots, a_j) L_j x(t_1)
\]
\[+ a_1(t) \int_{t_1}^{t} I_{\ell-2}(t, s; a_2, \ldots, a_{\ell-1}) a_\ell(s) L_\ell x(s) \, ds, \quad t \geq t_1.
\]
Using (7) and the fact that the function $L_\ell x(t)$ is nonincreasing for $t \geq t_1$ we obtain
\begin{equation}
|x(t)| \geq a_1(t) \int_{t_1}^{t} I_{\ell-2}(t, s; a_2, \ldots, a_{\ell-1})a_\ell(s) \, ds \cdot |L_\ell x(t)|
\end{equation}
\begin{equation}
= a_1(t)I_{\ell-1}(t, t_1; a_2, \ldots, a_\ell)|L_\ell x(t)|
= \gamma_\ell [t, t_1]|L_\ell x(t)| \quad \text{for } t \geq t_1.
\end{equation}
Combining (8) and (9), we get
\begin{equation}
|x(t)| \geq \gamma_\ell [t, t_1] \int_{t}^{\infty} \beta_\ell [u, t] \prod_{i=1}^{m} |x[g_i(u)]|^\lambda_i \, du, \quad t \geq t_1.
\end{equation}
Inequality (10), in view of condition (4;£) and Lemma 3, has only oscillatory solutions, a contradiction to the fact that $|x(t)| > 0$ for $t \geq t_1$.

Next, suppose $\ell = 1$. This is the case when $\delta = 1$ and $n$ is even or when $\delta = -1$ and $n$ is odd. Now, we apply Lemma 1 (ii) and obtain
\begin{equation}
\dot{x}(t) = a_1(t)L_1 x(t)
\end{equation}
\begin{equation}
= a_1(t) \sum_{j=1}^{n-1} (-1)^j I_{j-1}(s, t; a_j, \ldots, a_2)L_j x(s)
+ (-1)^{n-1}a_1(t) \int_{t}^{s} I_{n-2}(u, s; a_{n-1}, \ldots, a_2)L_n x(u) \, du.
\end{equation}
Thus,
\begin{equation}
|\dot{x}(t)| \geq a_1(t) \int_{t}^{\infty} I_{n-2}(u, t; a_{n-1}, \ldots, a_2)q(u) \prod_{i=1}^{m} |x[g_i(u)]|^\lambda_i \, du
\end{equation}
\begin{equation}
= a_1(t) \int_{t}^{\infty} \beta_1 [u, t] q(u) \prod_{i=1}^{m} |x[g_i(u)]|^\lambda_i \, du, \quad t \geq t_1.
\end{equation}
From (4;1) and Lemma 3, it follows that $x(t)$ must be oscillatory. But this contradicts the fact that $x(t) \neq 0$ for $t \geq t_0$.

Let $\ell = 0$. Then $\delta = 1$ and $n$ is odd or $\delta = -1$ and $n$ is even. It follows from (7) that
\begin{equation}
(-1)^i x(t)L_i x(t) > 0, \quad \text{for } t \geq t_1, \quad (i = 0, 1, \ldots, n).
\end{equation}
Since $x(t)\dot{x}(t) < 0$ for $t \geq t_1$, $|x(t)| \to c \geq 0$ as $t \to \infty$. If $c > 0$ there exists a $t_2 \geq t_1$ so that
\begin{equation}
|x[g_i(t)]| \geq \frac{c}{2} \quad \text{for } t \geq t_2, \quad (i = 1, 2, \ldots, m).
\end{equation}
By Lemma 1 (ii) we get
\begin{equation}
x(t_2) = \sum_{j=0}^{n-1} (-1)^j I_j(s, t_2; a_j, \ldots, a_1)L_j x(s)
+ (-1)^n \int_{t_2}^{s} I_{n-1}(u, t_2; a_{n-1}, \ldots, a_1)L_n x(u) \, du.
\end{equation}
Using (11), we have

$$|x(t_2)| \geq \int_{t_2}^{s} q(u)\beta_0[u,t_2] \prod_{i=1}^{m} |x[g_i(u)]|^\lambda_i \, du$$

$$\geq \prod_{i=1}^{m} \left( \frac{C}{2} \right)^{\lambda_i} \int_{t_2}^{s} q(u)\beta_0[u,t_2] \, du$$

As $s \to \infty$ we have $|x(t_2)| = \infty$, a contradiction. Thus, $c = 0$.

Finally, suppose $\ell = n$. Then $\delta = -1$ and $n$ is either odd or even. From (7) we obtain

$$x(t)L_ix(t) > 0 \quad \text{for} \quad t \geq t_1, \quad (i = 0, 1, \ldots, n). \quad (12)$$

On the other hand, by l'Hôpital's rule,

$$\lim_{t \to \infty} \frac{|x(t)|}{\alpha_{n-1}[t, t_1]} = \lim_{t \to \infty} |L_{n-1}x(t)| > 0.$$ 

Since $g_i(t) \to \infty$ as $t \to \infty$, $i = 1, 2, \ldots, m$, there exist a constant $c > 0$ and a $t_2 \geq t_1$ so that

$$|x[g_i(t)]| \geq c\alpha_{n-1}[g_i(t), t_1] \quad \text{for} \quad t \geq t_2 \quad (i = 1, 2, \ldots, m). \quad (13)$$

Integrating equation $(E, -1)$ from $t_2$ to $t$ and using (13) we obtain

$$|L_{n-1}x(t)| \geq L_{n-1}|x(t_2)| + \int_{t_2}^{t} q(s) \prod_{i=1}^{m} |x[g_i(s)]|^\lambda_i \, ds$$

$$\geq \prod_{i=1}^{m} c^{\lambda_i} \int_{t_2}^{t} q(s) \prod_{i=1}^{m} (\alpha_{n-1}[g_i(s), t_1])^{\lambda_i} \, ds \to \infty \quad \text{as} \quad t \to \infty.$$

Thus,

$$\lim_{t \to \infty} |L_{n-1}x(t)| = \infty$$

and consequently, $|L_{i}x(t)| \to \infty$ monotonically as $t \to \infty$, $i = 0, 1, \ldots, n-1$. This completes the proof.

The following examples are illustrative:

**Example 1.** Consider the third order advanced equation

$$\left( \frac{1}{t} \ddot{x}(t) \right) + \frac{c}{t^4} \left( x \left[ \frac{3}{2} t \right] \right)^{2/3} (x[2t])^{1/3} = 0, \quad t \geq 1, \quad (E_1)$$

where $c$ is a positive constant. Since we have

$$a_1(t) = 1, \quad a_2(t) = t, \quad q(t) = \frac{c}{t^4}, \quad g_1(t) = \frac{3}{2} t, \quad g_2(t) = 2t, \quad \lambda_1 = \frac{2}{3} \quad \text{and} \quad \lambda_2 = \frac{1}{3},$$

it is easily checked that

$$\beta_0[t, 1] \sim \frac{t^3}{6} (t \to \infty),$$

$$\beta_2[t, s] = 1,$$

$$\gamma_2[t, s] = a_1(t) \int_{s}^{t} a_2(u) \, du = \frac{1}{2} (t^2 - s^2), \quad t \geq s \geq 1,$$


and

\[ \frac{1}{3} \liminf_{t \to \infty} \int_{t}^{\frac{3}{2}t} \frac{1}{2} (s^2 - 1) \int_{s}^{\infty} \frac{c}{u^4} \, du \, ds + \frac{2}{3} \liminf_{t \to \infty} \int_{t}^{2t} \frac{1}{2} (s^2 - 1) \int_{s}^{\infty} \frac{c}{u^4} \, du \, ds = \frac{c}{18} \ln 6. \]

Thus, if \( c > \frac{18}{\ln 6} \approx 3.67 \), all the conditions of Theorem 1 (ii) are satisfied and hence every solution \( x(t) \) of equation \((E_1)\) is either oscillatory or \( L_i x(t) \to 0 \) as \( t \to \infty \) monotonically, \( i = 0, 1, 2 \).

It is interesting to note that:

(i) When \( c = 8 \sqrt{6} > 3.67 \), equation \((E_1)\) has the nonoscillatory solution \( x(t) = \frac{1}{t} \to 0 \) as \( t \to \infty \).

(ii) When \( c = \frac{3 \sqrt{6}}{16} < 3.67 \), equation \((E_1)\) has the nonoscillatory solution \( x(t) = \sqrt{t} \not\to 0 \) as \( t \to \infty \).

Accordingly, we believe that condition \((4; \ell)\) is sharp.

### Example 2.

Consider the second order advanced equation

\[ (\frac{1}{t} \dot{x}(t)) + \frac{c}{t^3} x[\frac{3}{2}t] = 0, \quad t \geq 1, \tag{E_2} \]

where \( c \) is a positive constant. Here, we take

\[ a_1(t) = t, \quad q(t) = \frac{c}{t^3}, \quad g_1(t) = \frac{3}{2} \quad \text{and} \quad \lambda_1 = 1. \]

Now it is easy to calculate that

\[ \beta_1[t, s] = 1 \]

\[ \gamma_1[t, 1] = 1 \]

and

\[ \liminf_{t \to \infty} \int_{t}^{\frac{3}{2}t} s \int_{s}^{\infty} \frac{c}{u^4} \, du \, ds = \frac{c}{2} \ln 3 \]

All the conditions of Theorem 1(i) are satisfied if \( c > \frac{2}{c \ln \frac{3}{2}} \approx 1.84 \), and hence equation \((E_2)\) is oscillatory.

Next, if we take \( c = \frac{\sqrt{6}}{4} < 1.84 \), equation \((E_2)\) has the nonoscillatory solution \( x(t) = \sqrt{t} \).

### Remark.

As we already mentioned in the introduction, many oscillation criteria for functional differential equations with deviating arguments that appeared in the literature have treated the advanced equations as ordinary equations, e.g. the results in [2], [3] and [7] fail to apply to the advanced equation

\[ \ddot{x}(t) + \frac{1}{4t^2} x[6t] = 0, \quad t \geq 1, \tag{E_3} \]

since in these papers, equation \((E_3)\) is considered as the nonoscillatory ordinary equation

\[ \ddot{x}(t) + \frac{1}{4t^2} x(t) = 0, \quad t \geq 1. \tag{E_4} \]

However, equation \((E_3)\) generates oscillations.

We note that the results in [4]-[6] are also not applicable to the equation \((E_3)\).
Theorem 2. Let conditions (1) and \((C_2)\) hold. Equation \((E, \delta)\) is almost oscillatory if:

(i) for \(\delta = 1\) and \(n\) even,
\[
\liminf_{t \to \infty} \int_t^\sigma \int_s^\infty \gamma(t) \beta(u, s) \sum_{i=1}^m q_i(u) \, du \, ds > \frac{1}{e},
\]
for all large \(T\), \((\ell = 1, 3, \ldots, n-1)\);

(ii) for \(\delta = 1\) and \(n\) odd, condition \((14; \ell)\) \((\ell = 2, 4, \ldots, n-1)\) and
\[
\int_{-\infty}^{\infty} \beta_0(s, T) \sum_{i=1}^m q_i(s) \, ds = \infty \quad \text{for all large } T,
\]
hold;

(iii) for \(\delta = -1\) and \(n\) even, conditions \((14; \ell)\) \((\ell = 2, 4, \ldots, n-2)\), \((15)\) and
\[
\int_{-\infty}^{\infty} \sum_{i=1}^m q_i(s) \alpha_{n-1}(g_i(s), T) \, ds = \infty \quad \text{for all large } T,
\]
hold;

(iv) for \(\delta = -1\) and \(n\) odd, conditions \((14; \ell)\) \((\ell = 1, 3, \ldots, n-2)\) and \((16)\) hold.

Proof: Suppose that equation \((E, \delta)\) has a nonoscillatory solution \(x(t) \neq 0\) for \(t \geq t_0 \geq 0\). As in the proof of Theorem 1 we obtain (7).

Let \(\ell \in \{1, 2, \ldots, n-1\}\). Then, from (7) we have that
\[
|x(t)| \quad \text{is increasing for } \quad t \geq t_1.
\]
Thus,
\[
|x[g_i(t)]| \geq |x[\sigma(t)]| \quad \text{for } t \geq t_1, \quad (i = 1, 2, \ldots, m).
\]
From equation \((E, \delta)\), \((C_2)\) and (17) we obtain
\[
-L_n x(t) \, \text{sgn} \, x(t) \geq |x[\sigma(t)]| \sum_{i=1}^m q_i(t), \quad t \geq t_1.
\]
Proceeding as in the proof of Theorem 1, we prove that the case \(\ell \in \{1, 2, \ldots, n-1\}\) is impossible.

The cases when \(\ell = 0\) and \(\ell = n\) can be treated similarly as in the proof for those cases in Theorem 1.

Theorem 3. Let conditions (1) and \((C_2)\) hold. A sufficient condition for equation \((E, \delta)\) to be almost oscillatory is that:

(i) When \(\delta = 1\) and \(n\) is even,
\[
\liminf_{t \to \infty} \int_t^{r(t)} a(t) \int_v^{\infty} \frac{\beta(t, u)}{R(t)} \sum_{i=1}^m q_i(u) \, du \, dv > \frac{1}{e},
\]
for all large \(T\), \((\ell = 1, 3, \ldots, n-1)\) holds;

(ii) when \(\delta = 1\) and \(n\) is odd, conditions \((18; \ell)\) \((\ell = 2, 4, \ldots, n-1)\) and \((15)\) hold;

(iii) when \(\delta = -1\) and \(n\) is even, condition \((18; \ell)\) \((\ell = 2, 4, \ldots, n-2)\), \((15)\) and \((16)\) hold;

(iv) when \(\delta = -1\) and \(n\) is odd, conditions \((18; \ell)\) \((\ell = 1, 3, \ldots, n-2)\) and \((16)\) hold.
Proof: Assume that equation \((E, \delta)\) possesses a nonoscillatory solution \(x(t) \neq 0\) for \(t \geq t_0\). As in the proof of Theorem 1 we obtain (7).

Suppose \(\ell \in \{2, 3, \ldots, n-1\}\). Then, from Lemma 1 (ii), (7) and condition \((C_2)\) we obtain

\[
|L_{\ell}x(t)| \geq \int_t^{\infty} \beta_{\ell}[u, t] \sum_{i=1}^{m} q_i(u)|x[g_i(u)]| du, \quad t \geq t_1.
\]  
(19)

Again, from Lemma 1 (i) and (7), we get

\[
|x[g_i(t)]| \geq \int_{t_1}^{g_i(t)} \alpha_{\ell-2}[g_i(t), s]a_{\ell-1}(s)|L_{\ell-1}x(s)| ds, \quad i = 1, 2, \ldots, m.
\]  
(20)

Using the fact that \(L_{\ell}x(t)\) is nonincreasing for \(t \geq t_1\) we have

\[
|L_{\ell-1}x(t)| \geq R_{\ell}[t, t_1]|L_{\ell}x(t)| \quad \text{for} \quad t \geq t_1,
\]
and that the function

\[
\frac{|L_{\ell-1}x(t)|}{R_{\ell}[t, t_1]} \quad \text{is nonincreasing for} \quad t \geq t_1.
\]  
(21)

Thus, inequality (20) becomes

\[
|x[g_i(t)]| \geq \frac{|L_{\ell-1}x[\tau(t)]|}{R_{\ell}[\tau(t), t_1]} \int_{t_1}^{g_i(t)} \alpha_{\ell-2}[g_i(t), s]a_{\ell-1}(s)R_{\ell}[s, t_1] ds
\]

\[
= \rho_{\ell}[g_i(t), t_1]|L_{\ell-1}x[\tau(t)]|, \quad (i = 1, 2, \ldots, m), \quad t \geq t_1.
\]  
(22)

Combining (19) and (22) we get

\[
|L_{\ell-1}x(t)| \geq a_{\ell}(t) \int_t^{\infty} \frac{\beta_{\ell}[u, t]}{R_{\ell}[\tau(u), t_1]} \left( \sum_{i=1}^{m} q_i(u)\rho_{\ell}[g_i(t), t_1] \right) |L_{\ell-1}x[\tau(u)]| du.
\]

From condition \((18; \ell)\) \((\ell = 2, 3, \ldots, n-1)\) and Lemma 3, it follows that \(L_{\ell-1}x(t)\) oscillates, which is a contradiction.

Next, suppose \(\ell = 1\). Then, \(n\) is even and \(\delta = 1\) or \(n\) is odd and \(\delta = -1\) and from Lemma 1 (ii) and (21), we obtain

\[
|\dot{x}(t)| \geq a_1(t) \int_t^{\infty} \frac{\beta_1[u, t]}{R_1[\tau(t), t_1]} \left( \sum_{i=1}^{m} q_i(u)\rho_1[g_i(u), t_1] \right) |x[\tau(u)]| du,
\]
and from condition \((18; 1)\), we have the desired contradiction.

A similar proof as in Theorems 1 and 2 covers the cases where \(\ell = 0\) and \(\ell = n\). This completes the proof.

Remark. It is noteworthy that condition \((14; \ell)\) of Theorem 2 and condition \((18; \ell)\) of Theorem 3 are independent. To demonstrate that we consider the following examples.

Example 3. Consider the second order advanced equation

\[
\left( \frac{1}{t} \ddot{x}(t) \right) + \frac{e}{(e+1)t^3} x[et] + \frac{1}{(e+1)t^3} x[e^2t] = 0, \quad t \geq 1. \quad (E_5)
\]
Since
\[ a_1(t) = t, \quad q_1(t) = \frac{e}{(e+1)t^3}, \quad q_2(t) = \frac{1}{(e+1)t^3}, \]
\[ g_1(t) = et \quad \text{and} \quad g_2(t) = e^2t, \]
we obtain
\[ \sigma(t) = et, \quad \tau(t) = e^2t, \]
and we can easily calculate that
\[ \beta_1[t, s] = 1, \]
\[ \gamma_1[t, s] = t, \]
\[ \rho_1[t, T] = \frac{t^2 - T^2}{2}, \quad T > 1 \]
and
\[ \liminf_{t \to \infty} \int_t^{et} \int_s^{\infty} \left( \frac{e}{(e+1)u^3} + \frac{1}{(e+1)u^3} \right) du \, ds = \frac{1}{2} > \frac{1}{e}, \]
i.e., condition (14;1) of Theorem 2 is satisfied and hence equation (E_5) is oscillatory.

Now, for \( T > 1 \),
\[ \liminf_{t \to \infty} \int_t^{e^2t} \int_s^{\infty} \frac{2}{e^4u^2 - T^2} \left[ \frac{e}{(e+1)u^3} \left( \frac{e^2u^2 - T^2}{2} \right) + \frac{1}{(e+1)u^3} \frac{e^4u^2 - T^2}{2} \right] du \, ds = \frac{1}{e} \]
And hence condition (18;1) of Theorem 3 is not satisfied.

**Example 4.** Now we consider the second order advanced equation
\[ \left( \frac{1}{t} \dot{x}(t) \right) + \frac{1}{3e^2t^2} [x[t+1] + x[t+2] + x[t+3]] = 0, \quad t \geq 1. \quad (E_6) \]

Here, we take
\[ a_1(t) = t, \quad q_i(t) = \frac{1}{3et^2}, \quad i = 1, 2, 3, \quad g_1(t) = t + 1, \quad g_2(t) = t + 2 \quad \text{and} \quad g_3(t) = t + 3. \]

It is easy to check that
\[ \sigma(t) = t + 1, \quad \tau(t) = t + 3, \]
\[ \beta_1[t, s] = 1, \]
\[ \gamma_1[t, s] = t, \quad T \geq 1, \]
\[ \rho_1[t, T] = \frac{t^2 - T^2}{2}, \quad T > 1 \]
and
\[ \liminf_{t \to \infty} \int_t^{t+3} \int_s^{\infty} \frac{1}{(u+3)^2 - T^2} \left[ \frac{(u+1)^2 - T^2}{3eu^2} + \frac{(u+2)^2 - T^2}{3eu^2} + \frac{(u+3)^2 - T^2}{3eu^2} \right] du \, ds \]
\[ = \frac{3}{e}; \]
i.e., condition (18;1) of Theorem 3 is satisfied. Thus, equation (E_6) is oscillatory by Theorem 3(i).

Next, we see that
\[
\liminf_{t \to \infty} \int_t^{t+1} \int_s^{\infty} \frac{1}{e u^2} du \, ds = \frac{1}{e},
\]
i.e., condition (14;1) of Theorem 2 is not satisfied.

**Example 5.** Consider the third order advanced equation
\[
\left(\frac{1}{t} \ddot{x}(t)\right) + \frac{c_1}{t^4} \dot{x}\left[\frac{3}{2} t\right] + \frac{c_2}{t^4} x[2t] = 0, \quad t \geq 1, \quad (E_7)
\]
where \(c_1\) and \(c_2\) are positive constants. Since
\[
a_1(t) = 1, \quad a_2(t) = t, \quad q_1(t) = \frac{c_1}{t^4}, \quad q_2(t) = \frac{c_2}{t^4}, \quad g_1(t) = \frac{3}{2} t \quad \text{and} \quad g_2(t) = 2t,
\]
we obtain
\[
\sigma(t) = \frac{3}{2} t,
\]
and we can easily calculate that
\[
\beta_0[t, T] \approx \frac{t^3}{6} (t \to \infty), \quad T \geq 1,
\]
\[
\beta_2[t, s] = 1,
\]
\[
\gamma_2[t, s] = \frac{t^2 - s^2}{2},
\]
and
\[
\liminf_{t \to \infty} \int_t^{\frac{3}{2} t} \frac{s^2 - T^2}{2} \int_s^{\infty} \frac{c_1 + c_2}{u^4} du \, ds = \frac{c_1 + c_2}{6} \ln \frac{3}{2}.
\]
All the conditions of Theorem 2(ii) are satisfied if we take \(c_1 + c_2 > \frac{6}{e \ln \frac{3}{2}} \approx 5.52\), and hence every solution \(x(t)\) of equation \((E_7)\) is either oscillatory or \(L_i x(t) \to 0\) is monotonically as \(t \to \infty, \ i = 0, 1, 2\).

We note that:

(i) when \(c_1 = \frac{189}{16}\) and \(c_2 = \frac{1}{4}\), equation \((E_7)\) has the nonoscillatory solution \(x(t) = \frac{1}{t} \to 0\) as \(t \to \infty\) and satisfies the conclusions given above

(ii) when \(c_1 = \frac{2\sqrt{2} - 8}{12\sqrt{3}}\) and \(c_2 = \frac{1}{4}\), equation \((E_7)\) has the nonoscillatory solution \(x(t) = t^{\frac{3}{2}} \to \infty\) as \(t \to \infty\).

The following results are concerned with the oscillatory and asymptotic behaviour of equation \((E, -1)\).

In what follows we suppose that there exist continuous functions \(\xi_i : R_+ \to R_+, \ (i = 1, 2, \ldots, m)\) such that
\[
t \leq \xi_i(t) \leq g_i(t), \quad (i = 1, 2, \ldots, m), \quad t \geq 0.
\]
We let
\[
\xi(t) = \min\{\xi_1(t), \xi_2(t), \ldots, \xi_m(t)\}, \quad t \geq 0.
\]
Theorem 4. Let $n \geq 3$ and assume that conditions (1) and $(C_1)$ hold. Suppose also that for every $\ell \in \{1, 2, \ldots, n-1\}$ with $n + \ell$ even and for all large $T$, conditions (4; $\ell$) hold and
\[
\sum_{j=1}^{m} \lambda_j \liminf_{t \to \infty} \int_{t}^{\xi_j(t)} q(s) \prod_{i=1}^{m} (\alpha_{n-1}[g_i(s), \xi_i(s)])^{\lambda_i} ds > \frac{1}{\varepsilon}.
\] (23)

Then, for $n$ odd, equation $(E, -1)$ is oscillatory. If, in addition, condition (5) holds, then for $n$ even, every solution $x(t)$ of equation $(E, -1)$ is either oscillatory or $\lim_{t \to \infty} L_i x(t) = 0$ monotonically $(i = 0, 1, \ldots, n-1)$.

Theorem 5. Suppose that $n \geq 3$, and that conditions (1) and $(C_2)$ hold. In addition either conditions (14; $\ell$) or conditions (18; $\ell$) hold for $\ell \in \{1, 2, \ldots, n-1\}$ with $n + \ell$ even and all large $T$ and
\[
\liminf_{t \to \infty} \int_{t}^{\xi(t)} \sum_{i=1}^{m} q_i(s) \alpha_{n-1}[g_i(s), \xi_i(s)] ds > \frac{1}{\varepsilon}.
\] (24)

Then every solution of equation $(E, -1)$ is oscillatory provided that $n$ is odd.

Moreover, if condition (6) is satisfied and $n$ is even, then every solution $x(t)$ of equation $(E, -1)$ is oscillatory or $L_i x(t) \to 0$ monotonically as $t \to \infty$, $i = 0, 1, \ldots, n-1$.

Proof of Theorems 4 and 5: Suppose that equation $(E, -1)$ has a nonoscillatory solution $x(t) \neq 0$ for $t \geq t_0 \geq 0$. As in the proof of Theorem 1, we obtain (7). The cases where $\ell \in \{0, 1, \ldots, n-1\}$ can be treated exactly as in Theorems 1-3 and hence will be omitted. Now, let $\ell = n$. Then, from Lemma 1(i) we have
\[
x(t) = \sum_{j=0}^{n-1} I_j(t, s; a_1, \ldots, a_j) L_j x(s) + \int_{s}^{t} I_{n-1}(t, u; a_1, \ldots, a_{n-1}) L_n x(u) du.
\]

Using (7) for the case $\ell = n$, we get
\[
|x(t)| \geq \alpha_{n-1}[t, s]|L_{n-1}x(s)| \quad \text{for} \quad t \geq s \geq t_1.
\]

Which gives for $i = 1, 2, \ldots, m$
\[
|x[g_i(t)]| \geq \alpha_{n-1}[g_i(t), \xi_i(t)] |L_{n-1}x[\xi_i(t)]|, \quad \text{for} \quad t \geq t_1.
\] (25)

Therefore, from equation $(E, -1)$, $(C_1)$, $(C_2)$ and (25) we obtain for $t \geq t_1$
\[
|L_n x(t)| \geq q(t) \prod_{i=1}^{m} |x[g_i(t)]|^{\lambda_i}
\]
\[
\geq q(t) \prod_{i=1}^{m} (\alpha_{n-1}[g_i(t), \xi_i(t)])^{\lambda_i} (|L_{n-1}x[\xi_i(t)]|)^{\lambda_i}
\]
and
\[
|L_n x(t)| \geq \sum_{i=1}^{m} q_i(t) |x[g_i(t)]|
\]
\[
\geq \sum_{i=1}^{m} q_i(t) \alpha_{n-1}[g_i(t), \xi_i(t)] |L_{n-1}x[\xi_i(t)]|.
\]
From the above inequalities, in view of (23) and (24) and Theorem 3 and 4 in [8], it follows that \( L_{n-1}x(t) \) must be oscillatory, which is a contradiction.

For illustration, we consider the following examples.

**Example 6.** Consider the third order advanced equation

\[
\left( \frac{1}{t} \dot{x}(t) \right)' - \frac{c}{t^4} x[2t] = 0, \quad t \geq 1, \quad (E_8)
\]

where \( C \) is a positive constant. Here, we take

\[
a_1(t) = 1, \quad a_2(t) = t, \quad q_1(t) = \frac{c}{t^4} \quad \text{and} \quad g_1(t) = 2t.
\]

It is easy to check that

\[
\alpha_2[t, s] = \frac{t^3}{6} - \frac{s^2 t}{2} + \frac{s^3}{3},
\]

\[
\beta_1[t, s] = \frac{t^2 - s^2}{2};
\]

\[
\gamma_1[t, T] = 1, \quad T \geq 1;
\]

and

\[
\liminf_{t \to \infty} \int_t^{2t} \int_s^{\infty} \frac{u^2 - s^2}{2t^4} \frac{c}{u^4} \, du \, ds = \frac{c}{3} \ln 2.
\]

Thus, all the conditions of Theorem 2(iv) are satisfied if \( c > \frac{3}{5} \approx 1.58 \), and hence every solution \( x(t) \) of equation \((E_8)\) is either oscillatory or \( |L_i x(t)| \to \infty \) as \( t \to \infty \) monotonically, \( i = 0, 1, 2 \).

We observe that if \( c = \frac{5}{8} \), equation \((E_8)\) has the nonoscillatory solution \( x(t) = \sqrt{t} \to \infty \), \( |L_1 x(t)| = \frac{1}{\sqrt{t}} \not\to \infty \) as \( t \to \infty \) and \( |L_2 x(t)| = \frac{1}{2\sqrt{t}} \to 0 \) as \( t \to \infty \).

Next, we choose \( \xi(t) = \frac{3}{2} t \). Then,

\[
\liminf_{t \to \infty} \int_t^{\frac{3}{2} t} \frac{c}{u^4} \alpha_2[2u, \frac{3}{2} u] \, du = \frac{5c}{24} \ln \frac{3}{2}.
\]

If \( c > \frac{24}{5e \ln \frac{3}{2}} \approx 4.42 \), then equation \((E_8)\) is oscillatory by Theorem 5.

**Example 7.** Consider the fourth order advanced equation

\[
\left( \frac{1}{t} \ddot{x}(t) \right)''' - \frac{c}{t^5} x[2t] = 0, \quad t \geq 1, \quad (E_9)
\]

where \( c \) is a positive constant. Since

\[
a_1(t) = 1, \quad a_2(t) = t, \quad a_3(t) = 1, \quad q_1(t) = \frac{c}{t^4} \quad \text{and} \quad g_1(t) = 2t.
\]

we obtain

\[
\alpha_3[t, T] = \beta_0[t, T] \sim \frac{t^4}{12} (t \to \infty), \quad T \geq 1,
\]

\[
\beta_2[t, s] = t - s,
\]

\[
\gamma_2[t, s] = \frac{t^2 - s^2}{2},
\]
and
\[
\liminf_{t \to \infty} \int_t^{2t} \frac{s^2 - t^2}{2} \int_s^\infty \frac{c(u - s)}{u^5} \, du \, ds = \frac{c}{24} \ln 2.
\]

All the conditions of Theorem 2(iii) are satisfied if we take \( c > \frac{24}{\pi \ln^2 2} \approx 12.6 \) and hence every solution \( x(t) \) of equation \((E_9)\) is oscillatory, \( L_i x(t) \to 0 \) monotonically as \( t \to \infty, i = 0, 1, 2, 3 \) or \( |L_j x(t)| \to \infty \) monotonically as \( t \to \infty, j = 0, 2, 3 \).

It is easy to check that:

(i) when \( c = \frac{45\sqrt{2}}{128} \), equation \((E_9)\) has the nonoscillatory solution \( x(t) = t^{\frac{3}{2}} \) with \( L_1 x(t) = \frac{3}{2} t^{\frac{3}{2}} \to \infty \) as \( t \to \infty \) and \( L_2 x(t) = \frac{15}{4} t^{-\frac{1}{2}} \) and \( L_3 x(t) = \frac{15}{8} t^{-\frac{3}{2}} \to 0 \) as \( t \to \infty \),

(ii) when \( c = 80 \), equation \((E_9)\) has the nonoscillatory solution \( x(t) = \frac{1}{t} \), which satisfies one of the above conclusions.

Next, we let \( \xi(t) = \frac{3}{2} t \). Then
\[
\alpha_3[t, s] = \frac{t^4}{12} - \frac{t^3 s}{6} + \frac{t s^3}{6} - \frac{s^4}{12},
\]
and
\[
\liminf_{t \to \infty} \int_t^{2t} \frac{c}{u^5} \alpha_3[2u, \frac{3}{2} u] \, du = \frac{7c}{192} \ln \frac{3}{2}.
\]

Thus, all the conditions of Theorem 5 are satisfied if \( c > \frac{192}{7c \ln \frac{3}{2}} \approx 25.23 \), and hence every solution of equation \((E_9)\) is either oscillatory or \( L_i x(t) \to 0 \) monotonically as \( t \to \infty, i = 0, 1, 2, 3 \).

We note that if \( c = 80 \), equation \((E_9)\) has the nonoscillatory solution \( x(t) = \frac{1}{t} \) which satisfies the above conclusion.

Remarks.

(1) The results of the present paper are new and general. Which are applicable to functional differential equations with advanced arguments.

(2) If \( a_i(t) = 1, i = 1, 2, \ldots, n - 1 \) then some of our results include those of Webrowski [9].

(3) The behavioural properties of equations given in Examples 1-7 are not covered by any of the results discussed in [1]-[9]. Also, are not discernible from previously know criteria.

(4) As we mentioned earlier, advanced arguments preserve oscillations, and in some cases, as in equation \((E_3)\), generate oscillations.

(5) In view of some examples we have provided, we believe that the conditions presented here are “sharp”.

REFERENCES


