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Higher Order Terms in the Melvin–Morton Expansion of the Colored Jones Polynomial

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Abstract: We formulate a conjecture about the structure of "upper lines" in the expansion of the colored Jones polynomial of a knot in powers of $(q-1)$. The Melvin–Morton conjecture states that the bottom line in this expansion is equal to the inverse Alexander polynomial of the knot. We conjecture that the upper lines are rational functions whose denominators are powers of the Alexander polynomial. We prove this conjecture for torus knots and give experimental evidence that it is also true for other types of knots.

1. Introduction

Ever since the discovery of the Jones polynomial, its relation to the objects of the classical topology, i.e. the fundamental group of a knot, remained somewhat of a mystery. An apparent similarity between the skein relations for the Jones and Alexander polynomials did not lead to a better understanding of this relation. Therefore the discovery by P. Melvin and H. Morton [9] of the inverse Alexander polynomial inside the $(\check{q}-1)$ expansion of the colored Jones polynomial was a very interesting development.

Let $\mathscr{K}$ be a knot in S^3. We denote by $J_\alpha(\mathscr{K};K)$ its colored Jones polynomial normalized in such a way that it is multiplicative under a disconnected sum and

$$J_\alpha(\text{unknot};K) = \frac{\sin(\frac{\pi}{K}\alpha)}{\sin(\frac{\pi}{K})} = \frac{\check{q}^{\frac{\alpha}{2}} - \check{q}^{-\frac{\alpha}{2}}}{\check{q}^{\frac{1}{2}} - \check{q}^{-\frac{1}{2}}}, \quad \check{q} = e^{\frac{2\pi i}{K}}. \tag{1.1}$$

Another popular normalization for the Jones polynomial is

$$V_\alpha(\mathscr{K};K) = \frac{J_\alpha(\mathscr{K};K)}{J_\alpha(\text{unknot};K)}, \quad V_\alpha \in \mathbb{Z}[\check{q},\check{q}^{-1}]. \tag{1.2}$$

For a fixed value of color α we can expand the Jones polynomial $V_\alpha(\mathscr{K};K)$ in Taylor series in powers of

$$h = \check{q} - 1, \tag{1.3}$$

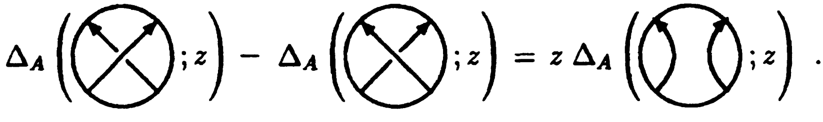

Fig. 1. A surgery link for producing a torus knot from an unknot.

or, equivalently, in powers of

$$\frac{1}{K} = \frac{1}{2\pi i} \log(1+h)\,. \tag{1.4}$$

The coefficients of this expansion are polynomials of finite degree in α:

$$V_\alpha(\mathcal{K};K) = \sum_{m,n\geq 0} D_{m,n}(\mathcal{K})\alpha^{2m}h^n, \quad D_{m,n}(\mathcal{K}) \in \mathbb{Q}\,. \tag{1.5}$$

The coefficients $D_{m,n}(\mathcal{K})$ are rational invariants of the knot $\mathcal{K}$. D. Bar-Natan [2] and J. Birman, X-S. Lin [4] showed that $D_{m,n}(\mathcal{K})$ were Vassiliev invariants of order n.

The following theorem was conjectured by P. Melvin and H. Morton [9] and later proved by D. Bar-Natan and S. Garoufalidis [3] (for a simple path integral proof see [12]).

Theorem 1.1. *Let $\mathcal{K}$ be a knot in S^3. Then the coefficients $D_{m,n}(\mathcal{K})$ of the expansion* (1.5) *satisfy the following two properties*:

$$D_{m,n}(\mathcal{K}) = 0 \quad \textit{for } m > \tfrac{n}{2}\,, \tag{1.6}$$

$$\sum_{m\geq 0} D_{m,2m}(\mathcal{K})a^{2m} = \frac{1}{\Delta_A(\mathcal{K}; e^{i\pi a} - e^{-i\pi a})}\,, \tag{1.7}$$

here $\Delta_A(\mathcal{K};z)$ is the Alexander–Conway polynomial satisfying the skein relation of Fig. 1 *and normalized in such a way that*

$$\Delta_A(\text{unknot}; z) = 1\,. \tag{1.8}$$

The bound on the powers of α in the expansion (1.5) allows us to rearrange it in "lines"

$$V_\alpha(\mathcal{K};K) = \sum_{n\geq 0} h^n \sum_{m\geq 0} D_{m,n+2m}(\alpha h)^{2m}\,. \tag{1.9}$$

From the quantum field theory point of view, the n^{th} line $\sum_{m\geq 0} D_{m,(n-1)+2m}(\alpha h)^{2m}$ is related to the n-loop contribution in the calculation of $V_\alpha(\mathcal{K};K)$ as a Chern–Simons path integral over the $SU(2)$ connections in the knot complement (see [12] for details).

The Taylor expansions (1.5), (1.9) and their link analogs played a key role in defining the "perturbative" invariants of rational homology spheres (see a review in [15] and references therein) and in establishing their relation [14] to Ohtsuki's invariants [11]. The latter application prompted us to look for "integrality" properties of the coefficients $D_{m,n}(\mathcal{K})$. We conjecture that this integrality can be exposed by an appropriate choice of expansion parameters in the series (1.5) and (1.9).

Let us introduce a new variable

$$z = \check{q}^{\frac{\alpha}{2}} - \check{q}^{-\frac{\alpha}{2}} . \tag{1.10}$$

We can express αh as a power series in z and h by expanding the r.h.s. of the equation

$$\alpha h = 2 \log \left(\sqrt{1 + \left(\frac{z}{2}\right)^2} + \frac{z}{2} \right) \frac{h}{\log(1+h)} = z + \mathcal{O}(z^3, h) . \tag{1.11}$$

After putting this expression in place of (αh) in Eq. (1.9) and assembling the powers of h and z we get a new expansion of the Jones polynomial

$$V_\alpha(\mathcal{K}; K) = \sum_{n=0}^{\infty} V^{(n)}(\mathcal{K}; z) h^n , \tag{1.12}$$

$$V^{(n)}(\mathcal{K}; z) = \sum_{m=0}^{\infty} d_m^{(n)}(\mathcal{K}) z^{2m} . \tag{1.13}$$

The form of the substitution (1.11) suggests immediately that the bottom line in the expansion (1.9) does not change:

$$\sum_{m=0}^{\infty} D_{m,n+2m}(\alpha h)^{2m} = V^{(0)}(\mathcal{K}; z) + \mathcal{O}(h) . \tag{1.14}$$

As a result, the second part of the Melvin–Morton conjecture (1.7) takes the form

$$V^{(0)}(\mathcal{K}; z) = \frac{1}{\Delta_A(\mathcal{K}; z)} \tag{1.15}$$

in new variables z, h. Since $\Delta_A(\mathcal{K}; z) \in \mathbb{Z}[z^2]$ and $\Delta_A(\mathcal{K}; 0) = 1$, it follows from Eq. (1.15) that

$$d_m^{(0)} \in \mathbb{Z} . \tag{1.16}$$

We conjecture that the upper lines $V^{(n)}(\mathcal{K}; z)$ satisfy the properties similar to those of (1.15) and (1.16).

Conjecture 1.1 (weak). *All coefficients $d_m^{(n)}$ in the expansion of the colored Jones polynomial $V_\alpha(\mathcal{K}; K)$ of a knot $\mathcal{K} \in S^3$ in powers of $z = \check{q}^{\frac{\alpha}{2}} - \check{q}^{-\frac{\alpha}{2}}$ and $h = \check{q} - 1$ are integer:*

$$d_m^{(n)} \in \mathbb{Z}, \quad m, n \geqq 0 . \tag{1.17}$$

Conjecture 1.2 (strong). *A line $V^{(n)}(\mathcal{K}; z)$ in the series (1.12) is a rational function of z:*

$$V^{(n)}(\mathcal{K}; z) = \frac{P^{(n)}(\mathcal{K}; z)}{\Delta_A^{2n+1}(\mathcal{K}; z)}, \quad P^{(n)}(\mathcal{K}; z) \in \mathbb{Z}[z^2] . \tag{1.18}$$

In other words, the expansion of the r.h.s. *of Eq.* (1.18) *in powers of z produces the series in the* r.h.s. *of Eq.* (1.13).

The weak conjecture can be derived from the strong one in the same way as we derived (1.16) from Eq. (1.15).

It may happen that for some knots the polynomials $P^{(n)}(\mathcal{K};z)$ defined by Eq. (1.18) would be divisible by powers of $\Delta_A(\mathcal{K};z)$. This means that for those knots a smaller power of $\Delta_A(\mathcal{K};z)$ could be placed in the denominator. Amphicheiral knots seem to present an example of such behavior. These are the knots which are isotopic to their mirror image.

If $\mathcal{K}'$ is the mirror image of $\mathcal{K}$, then

$$V_\alpha(\mathcal{K}';K) = V_\alpha(\mathcal{K};-K). \tag{1.19}$$

This symmetry is not easily seen in the coefficients of expansion (1.12) because it transforms h into $-\frac{h}{1+h}$ rather than into $-h$. Therefore it is natural to try another expansion parameter

$$\tilde{h} = \check{q}^{\frac{1}{2}} - \check{q}^{-\frac{1}{2}} = (1+h)^{\frac{1}{2}} - (1+h)^{-\frac{1}{2}} \tag{1.20}$$

instead of h in Eq. (1.12):

$$V_\alpha(\mathcal{K};K) = \sum_{n=0}^{\infty} \tilde{V}^{(n)}(\mathcal{K};z)\tilde{h}^n, \tag{1.21}$$

$$\tilde{V}^{(n)}(\mathcal{K};z) = \sum_{m=0}^{\infty} \tilde{d}_m^{(n)}(\mathcal{K})z^{2m}. \tag{1.22}$$

The symmetry (1.19) converts $\tilde{h}$ into $-\tilde{h}$ and z into $-z$, so for amphicheiral knots

$$\tilde{V}^{(2n+1)}(\mathcal{K};z) = 0, \quad n \geqq 0. \tag{1.23}$$

Since the relation between h and $\tilde{h}$ involves fractional powers, it does not follow from the weak Conjecture 1.1 that the coefficients $\tilde{d}_m^{(n)}$ would also be integer for any knot. In fact, our numerical estimates show that some of the first coefficients for the knot 6_1 are fractional. However, for the amphicheiral knots only the even powers of $\tilde{h}$ participate in the expansion (1.21) due to Eq. (1.23). Since the expansion

$$\tilde{h}^2 = \sum_{n=2}^{\infty} (-1)^n h^n \tag{1.24}$$

contains only integer coefficients (i.e. $(-1)^n$), then the weak Conjecture 1.1 implies that for amphicheiral knots, $\tilde{d}_m^{(n)}$ should also be integer:

Corollary 1.1. *For an amphicheiral knot* $\mathcal{K}$ *in addition to* (1.17),

$$\tilde{d}_m^{(n)} \in \mathbb{Z}. \tag{1.25}$$

Our experimental data also suggests (see Sect. 3 for details) that the following enhancement of the strong Conjecture 1.2 is true:

Conjecture 1.3. *For an amphicheiral knot* $\mathcal{K}$ *a line* $\tilde{V}^{(2n)}(\mathcal{K};z)$ *in the series* (1.21) *is a rational function*

$$\tilde{V}^{(2n)}(\mathcal{K};z) = \frac{\tilde{P}^{(2n)}(\mathcal{K};z)}{\Delta_A^{3n+1}(\mathcal{K};z)}, \quad \tilde{P}^{(2n)}(\mathcal{K};z) \in \mathbb{Z}[z^2]. \tag{1.26}$$

Note that $3n+1 \leqq 2(2n)+1$, $2(2n)+1$ being the power required by Eq. (1.18), so the amphicheiral knots require a smaller power of the Alexander polynomial in denominators.

The integrality of the coefficients of the polynomials $P^{(n)}(\mathcal{K};z)$ gives us a hope that similarly to the denominator $\Delta_A^{2n+1}(\mathcal{K};z)$, they may also have a direct interpretation in the framework of classical topology.

The weak Conjecture 1.1 has a "practical" application: we will use it in [16] in order to derive a p-adic convergence of the series of perturbative invariants to the total Witten–Reshetikhin–Turaev invariant of rational homology spheres constructed by rational surgeries on a knot in S^3.

In Sect. 2 we derive the strong Conjecture 1.2 for torus knots. In Sect. 3 we present experimental evidence that our conjectures are also true for other types of knots. In Sect. 4 we speculate about the possible explanation for the power of the Alexander polynomial in denominators of Eqs. (1.18) and Eq. (1.26).

2. The Jones Polynomial of Torus Knots

We denote a type (p,q) torus knot in S^3 as $\mathcal{K}_{p,q}$. An expression for its colored Jones polynomial was derived in [6] within the framework of the quantum Chern–Simons theory. The $(\check{q}-1)$ expansion of the polynomial was studied in [10] and [1]. We derived the explicit expansion (1.9) for torus knots in Eq. (A.4) of Appendix of [12]. In our notations there

$$Z_\alpha(S^3,\mathcal{K}_{m,n};K-2)=\sqrt{\frac{2}{K}}\sin\left(\frac{\pi}{K}\alpha\right)V_\alpha(\mathcal{K}_{m,n};K)\,. \tag{2.1}$$

We will rederive the formula of [12] in a slightly different way that will allow us to prove the conjectures of the previous section.

Lemma 2.1. *The expansion* (1.12) *for a torus knot* $\mathcal{K}_{p,q}$ *comes from the formula*

$$\begin{aligned} V_\alpha(\mathcal{K}_{p,q};K) &= \frac{1}{z}\frac{e^{\frac{i\pi}{4}\operatorname{sign}(pq)}}{\sqrt{2K|pq|}}e^{\frac{i\pi}{2K}\left(pq-\frac{p}{q}-\frac{q}{p}\right)}\int\limits_{\substack{-\infty\\ [\beta=0]}}^{+\infty} d\beta\, e^{-\frac{i\pi}{2K}\frac{\beta^2}{pq}}\frac{z_\beta}{\Delta_A(\mathcal{K}_{p,q};z_\beta)} \\ &= \frac{1}{z}(1+h)^{\frac{1}{4}\left(pq-\frac{p}{q}-\frac{q}{p}\right)}\sum_{m=0}^{\infty}\frac{(2m)!}{(m!)^2}\left(\frac{\log(1+h)}{4pq}\right)^m\left(\frac{Kpq}{i\pi}\right)^{2m} \\ &\quad\times\frac{\partial^{(2m)}}{\partial\beta^{2m}}\frac{z_\beta}{\Delta_A(\mathcal{K}_{p,q};z_\beta)}\bigg|_{\beta=0}\,, \end{aligned} \tag{2.2}$$

here

$$z_\beta=\check{q}^{\frac{1}{2}\left(\alpha+\frac{\beta}{pq}\right)}-\check{q}^{-\frac{1}{2}\left(\alpha+\frac{\beta}{pq}\right)}\,, \tag{2.3}$$

$\Delta_A(\mathcal{K}_{p,q};z)$ *is the Alexander polynomial of* $\mathcal{K}_{p,q}$:

$$\Delta_A(\mathcal{K}_{p,q};t-t^{-1})=\frac{(t^{pq}-t^{-pq})(t-t^{-1})}{(t^p-t^{-p})(t^q-t^{-q})}\,, \tag{2.4}$$

and the symbol $\int_{-\infty\,[\beta=0]}^{+\infty}$ *means that we have to take only the contribution of the stationary phase point* $\beta=0$ *to the integral of Eq.* (2.2).

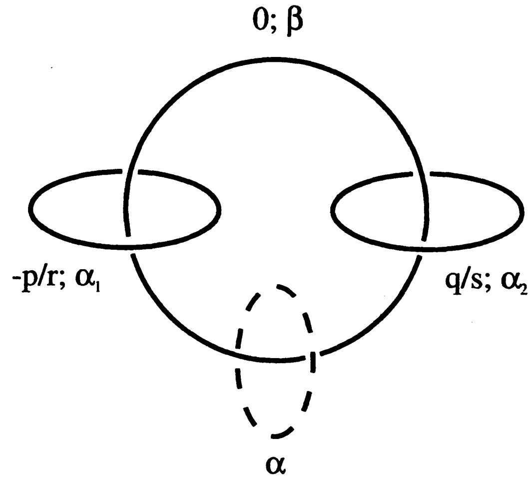

Fig. 2. A surgery link for producing a torus knot from an unknot.

Note that each derivative ∂_β extracts a factor of $\frac{i\pi}{Kpq}$ from z_β. These factors cancel the prefactor $(\frac{Kpq}{i\pi})^{2m}$, so that Eq. (2.2) presents an expansion in powers of

$$\log(1+h) = h - \sum_{n=2}^{\infty}(-1)^n \frac{h^n}{n} \,. \tag{2.5}$$

Proof of Lemma 2.1. If $\mathcal{K}_{p,q}$ is a torus knot, then the numbers $p, q \in \mathbb{Z}$ are coprime. Therefore we can choose the numbers $r, s \in \mathbb{Z}$ in such a way that

$$ps - qr = 1 \,. \tag{2.6}$$

It is not hard to see that the surgeries on the 3-component solid link in S^3 of Fig. 2 (with framings $(-\frac{p}{r}, \frac{q}{s}, 0)$) produce again S^3. However the dashed unknot of Fig. 2 becomes a torus knot $\mathcal{K}_{p,q}$ in the new S^3. Therefore its colored Jones polynomial can be calculated by a Reshetikhin–Turaev surgery formula applied to the link of Fig. 2. The colored Jones polynomial of that link is

$$J_{\alpha_1,\alpha_2,\alpha,\beta} = \frac{\sin(\frac{\pi}{K}\alpha_1\beta)\sin(\frac{\pi}{K}\alpha_2\beta)\sin(\frac{\pi}{K}\alpha\beta)}{\sin^2(\frac{\pi}{K}\beta)\sin(\frac{\pi}{K})} \,. \tag{2.7}$$

The "quantum factor" for a rational surgery with the framing $\frac{p}{q}$ was worked out by L. Jeffrey [7]:

$$\tilde{U}^{(p,q)}_{\alpha 1} = i\frac{\operatorname{sign}(q)}{\sqrt{2K|q|}} e^{-\frac{i\pi}{4}\Phi(U^{(p,q)})} \sum_{n=0}^{q-1} \sum_{\mu=\pm 1} \mu$$

$$\times \exp\left(\frac{i\pi}{2K}\frac{1}{q}(p\alpha^2 - 2\alpha(2Kn+\mu) + s(2Kn+\mu)^2)\right), \tag{2.8}$$

here $\Phi(U^{(p,q)})$ is the Rademacher function:

$$\Phi\begin{bmatrix} p & r \\ q & s \end{bmatrix} = \frac{p+s}{q} - 12s(p,q), \tag{2.9}$$

and $s(p,q)$ is the Dedekind sum:

$$s(p,q) = \frac{1}{4q}\sum_{j=1}^{q-1} \cot\left(\pi\frac{j}{q}\right)\cot\left(\pi\frac{pj}{q}\right). \tag{2.10}$$

Since

$$\sum_{1\leqq\alpha\leqq K-1} \sin\left(\frac{\pi}{K}\alpha\beta\right)\tilde{U}_{\alpha 1}^{(p,q)} = \sqrt{\frac{K}{2}}\tilde{U}_{\beta 1}^{(-q,p)}, \tag{2.11}$$

we conclude that

$$J_\alpha(\mathscr{K}_{p,q};K) = \frac{e^{-\frac{i\pi}{2K}pq(\alpha^2-1)}}{\sqrt{\frac{2}{K}}\sin(\frac{\pi}{K})}\sum_{1\leqq\beta\leqq K-1}\frac{\sin(\frac{\pi}{K}\alpha\beta)}{\sin(\frac{\pi}{K}\beta)}\tilde{U}_{\beta 1}^{(r,p)}\tilde{U}_{\beta 1}^{(-s,q)}. \tag{2.12}$$

We did not include the manifold framing correction because the surgery produces S^3 in the canonical framing. However we had to include the knot framing correction factor $e^{-\frac{i\pi}{2K}pq(\alpha^2-1)}$ because the torus knot is produced with the framing pq.

By substituting Eq. (2.8) in Eq. (2.12) and using the relation

$$\Phi\begin{bmatrix} r & -s \\ p & -q \end{bmatrix} + \Phi\begin{bmatrix} -s & r \\ q & -p \end{bmatrix} = -3\,\mathrm{sign}(pq), \tag{2.13}$$

we arrive at the following formula for the colored Jones polynomial:

$$\begin{aligned} J_\alpha(\mathscr{K}_{p,q};K) = {} & \frac{e^{\frac{i\pi}{4}\mathrm{sign}(pq)}}{4\sqrt{2K|pq|}}\frac{e^{-\frac{i\pi}{2K}pq(\alpha^2-1)}}{\sin(\frac{\pi}{K})}\sum_{n_1=0}^{p-1}\sum_{n_2=0}^{q-1}\sum_{\mu_{1,2,3}=\pm 1}\mu_1\mu_2\mu_3 \\ & \times \sum_{1\leqq\beta\leqq K-1}\frac{1}{\sin(\frac{\pi}{K}\beta)}\exp\left[-\frac{i\pi}{2K}\left(\frac{\beta^2}{pq}\right.\right. \\ & +2\beta\left(\frac{2Kn_1+\mu_1}{p}+\frac{2Kn_2+\mu_2}{q}+\mu_3\alpha\right) \\ & \left.\left.+\frac{q}{p}(2Kn_1+\mu_1)^2+\frac{p}{q}(2Kn_2+\mu_2)^2\right)\right]. \end{aligned} \tag{2.14}$$

The sum over β in this formula is completely similar to the sums in Eqs. (2.8), (2.9) of [13] if we substitute there $g=0$, $n=3$, $m_1=n_1$, $m_2=n_2$, $K\frac{m_3}{p_3}=\mu_3\alpha$. We will not present the analysis of the large K asymptotics of the formula (2.14) since it is exactly the same as the one in Sect. 3 of [13]. We will rather use the final result expressed in Eqs. (3.21), (3.23) and Eq. (3.44) of Proposition 3.1 in that paper. Namely, if

$$\alpha < \frac{K}{|pq|}, \tag{2.15}$$

then all the "irreducible" contributions (3.45) of [13] cancel out (note, by the way, that there are no irreducible flat connections in the knot complement whose holonomy along the knot meridian is equal to $\exp(\frac{i\pi}{K}\alpha\sigma_3)$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ for α satisfying (2.15)). The only survivor is the "reducible" contribution

$$J_\alpha(\mathcal{K}_{p,q};K) = \frac{e^{\frac{i\pi}{4}\mathrm{sign}(pq)}}{8\sqrt{2K|pq|}} \frac{e^{-\frac{i\pi}{2K}pq(\alpha^2-1)}}{\sin(\frac{\pi}{K})} \sum_{\mu_{1,2,3}=\pm 1} \mu_1\mu_2\mu_3 \int_{\substack{-\infty \\ [\beta=-pq\mu_3\alpha]}}^{+\infty} \frac{d\beta}{\sin(\frac{\pi}{K}\beta)}$$
$$\times \exp\left[-\frac{i\pi}{2K}\left(\frac{\beta^2}{pq} + 2\beta\left(\frac{\mu_1}{p} + \frac{\mu_2}{q} + \mu_3\alpha\right) + \frac{q}{p} + \frac{p}{q}\right)\right]$$
$$= \frac{e^{\frac{i\pi}{4}\mathrm{sign}(pq)}}{\sqrt{2K|pq|}} \frac{e^{\frac{i\pi}{2K}(pq-\frac{p}{q}-\frac{q}{p})}}{2i\sin(\frac{\pi}{K})}$$
$$\times \int_{\substack{-\infty \\ [\beta=0]}}^{+\infty} d\beta\, e^{-\frac{i\pi}{2K}\frac{\beta^2}{pq}} \frac{(e^{i\pi p\frac{\alpha_\beta}{K}} - e^{-i\pi p\frac{\alpha_\beta}{K}})(e^{i\pi q\frac{\alpha_\beta}{K}} - e^{-i\pi q\frac{\alpha_\beta}{K}})}{(e^{i\pi pq\frac{\alpha_\beta}{K}} - e^{-i\pi pq\frac{\alpha_\beta}{K}})}, \tag{2.16}$$

here we used a notation

$$\alpha_\beta = \alpha + \frac{\beta}{pq}. \tag{2.17}$$

The symbol $\int_{-\infty\,[\beta=0]}^{+\infty}$ means that we have to take only the contribution of the stationary phase point $\beta = 0$ to the integral of Eq. (2.16). In other words, we have to expand the preexponential factor in Taylor series in β around $\beta = 0$ and integrate each monomial with the gaussian factor $e^{-\frac{i\pi}{2K}\frac{\beta^2}{pq}}$ term by term with the help of the formula

$$\int_{-\infty}^{+\infty} e^{-\frac{i\pi}{2K}\frac{\beta^2}{pq}} \beta^{2m}\, d\beta = \sqrt{2K|pq|}e^{-\frac{i\pi}{4}\mathrm{sign}(pq)} \frac{(2m)!}{m!}\left(\frac{Kpq}{2\pi i}\right)^m. \tag{2.18}$$

The result will be precisely Eq. (A.4) of [12]. We present that formula in a slightly different form. We use the formula (2.4) for the Alexander polynomial of the torus knot and by introducing a notation

$$z_\beta = \check{q}^{\frac{\alpha_\beta}{2}} - \check{q}^{\frac{-\alpha_\beta}{2}} \tag{2.19}$$

we arrive at Eq. (2.2). □

The formula (2.2) proves half of the strong Conjecture 1.2, namely,

Lemma 2. *For a torus knot $\mathcal{K}_{p,q}$,*

$$V^{(n)}(\mathcal{K}_{p,q};z) = \frac{P^{(n)}(\mathcal{K}_{p,q};z)}{\Delta_A^{2n+1}(\mathcal{K}_{p,q};z)}, \quad P^{(n)}(\mathcal{K}_{p,q};z) \in \mathbb{Q}[z^2]. \tag{2.20}$$

Proof of Lemma 2.2. For a smooth function $f(z_\beta)$

$$\left(\frac{Kpq}{i\pi}\right)^2 \frac{\partial^2}{\partial\beta^2} f(z_\beta) = z_\beta f'(z_\beta) + (z_\beta^2+4)f''(z_\beta)\,, \tag{2.21}$$

therefore

$$\left(\frac{Kpq}{i\pi}\right)^{2m} \frac{\partial^{(2m)}}{(\partial\beta)^{2m}} \frac{z_\beta}{\Delta_A(\mathscr{K}_{p,q};z_\beta)}\Bigg|_{\beta=0} = \frac{z\tilde{P}_m(z)}{\Delta_A^{2m+1}(\mathscr{K}_{p,q};z)}, \quad \tilde{P}_m(z)\in \mathbf{Z}[z^2]\,. \tag{2.22}$$

The numerator of the r.h.s. of Eq. (2.22) is proportional to z because in view of Eq. (2.21), the l.h.s. of this equation is an odd function of z. Finally, Eq. (2.2) demonstrates that a line $V^{(n)}(\mathscr{K}_{p,q};z)$ is a linear combination of the functions (2.22) for $m \leqq n$. □

The fractions in the expansion of $(1+h)^{\frac{1}{4}(pq-\frac{p}{q}-\frac{q}{p})}$ and $\frac{\log(1+h)}{4pq}$ in powers of h separate us from the complete proof of the strong Conjecture 1.2. We prove the next lemma in order to eliminate half of these fractions.

Lemma 3. *For a torus knot $\mathscr{K}_{p,q}$,*

$$P^{(n)}(\mathscr{K}_{p,q};z)\in \mathbf{Z}\left[z^2,\frac{1}{p}\right]\,. \tag{2.23}$$

Proof of Lemma 2.3. We are going to absorb the factor

$$(e^{i\pi p\frac{\alpha\beta}{K}} - e^{-i\pi p\frac{\alpha\beta}{K}})$$

together with some other factors of the integrand in Eq. (2.16) inside the gaussian factor $e^{-\frac{i\pi}{2K}\frac{\beta^2}{pq}}$ by completing the square. We achieve this by introducing a new variable β' such that

$$\beta = \beta' + \mu p - q - pq\,. \tag{2.24}$$

Now the integral of Eq. (2.16) can be rewritten as

$$V_\alpha(\mathscr{K}_{p,q};K) = \frac{1}{z}\frac{e^{\frac{i\pi}{4}\operatorname{sign}(pq)}}{\sqrt{2K|pq|}}\sum_{\mu=\pm1} \check{q}^{\frac{1}{2}(p+1)(q-\mu)}\check{q}^{\frac{\alpha}{2}(\mu p-q-pq)}$$

$$\times \int_{\substack{-\infty\\ [\beta'=0]}}^{+\infty} d\beta'\, \check{q}^{-\frac{1}{4}\frac{\beta'^2}{pq}} \frac{1-\check{q}^{q\alpha+\frac{\beta}{p}}}{1-\check{q}^{pq\alpha+\beta}}\,. \tag{2.25}$$

We kept β in the last factor of the integrand in this equation meaning that it is a function (2.24) of β'.

The last factor of the integrand in Eq. (2.25) can be presented as a geometric series

$$\frac{1-\check{q}^{q\alpha+\frac{\beta}{p}}}{1-\check{q}^{pq\alpha+\beta}} = \lim_{x\to1^-}\sum_{n\in(\mathbf{Z}_+)-(\mathbf{Z}_++\frac{1}{p})} (\check{q}^{pq\alpha+\beta})^n x^n\,, \tag{2.26}$$

here we used a notation

$$\sum_{n\in(\mathbb{Z}_+)-(\mathbb{Z}_++\frac{1}{p})} f(n) = \sum_{n=0}^{\infty} f(n) - \sum_{n=0}^{\infty} f\left(n+\frac{1}{p}\right). \tag{2.27}$$

Since

$$\frac{e^{\frac{i\pi}{4}\operatorname{sign}(pq)}}{\sqrt{2K|pq|}} \int_{-\infty}^{+\infty} d\beta'\, \check{q}^{-\frac{1}{4}\frac{\beta'^2}{pq}} \check{q}^{n\beta'} = \check{q}^{-pqn^2}, \tag{2.28}$$

Eq. (2.25) becomes

$$V_\alpha(\mathcal{K}_{p,q};K) = \frac{1}{z}\sum_{\mu=\pm1} \mu(1+h)^{\frac{1}{2}(p+1)(q-\mu)}(\check{q}^{\frac{\alpha}{2}})^{\mu p-q-pq} \tag{2.29}$$

$$\times \lim_{x\to1^-} \sum_{n\in(\mathbb{Z}_+)-(\mathbb{Z}_++\frac{1}{p})} (\check{q}^{\frac{\alpha}{2}})^{2pqn}(1+h)^{n(\mu p-q-pq)}(1+h)^{-pqn^2}x^n. \tag{2.30}$$

Our immediate task is to expand this expression in powers of h with coefficients being hopefully rational functions of $\check{q}^{\frac{\alpha}{2}}$. Since p and q are coprime, at least one of them is odd. Therefore $\frac{1}{2}(p+1)(q-\mu)\in\mathbb{Z}$ and

$$(1+h)^{\frac{1}{2}(p+1)(q-\mu)} \in \mathbb{Z}[[h]]. \tag{2.31}$$

It remains to treat the sum over n. Consider an expansion

$$(1+h)^{-pqn^2} = \sum_{m=0}^{\infty}(-1)^m \frac{\prod_{l=0}^{m-1}(pqn^2+l)}{m!}h^m. \tag{2.32}$$

Each term in the sum over m is a polynomial in n which takes integer values for $n\in\mathbb{Z}$. Therefore it can be presented as an integer linear combination of binomial polynomials $\binom{n}{k}=\frac{n!}{(n-k)!k!}$:

$$(-1)^m\frac{\prod_{l=0}^{m-1}(pqn^2+l)}{m!} = \sum_{k=0}^{2m} C_{m,k}(p,q)\binom{n}{k}, \quad C_{m,k}(p,q)\in\mathbb{Z}. \tag{2.33}$$

A binomial polynomial can be expressed as a derivative

$$\binom{n}{k} = \frac{1}{k!}\partial_\varepsilon^{(k)}(1+\varepsilon)^n\Big|_{\varepsilon=0}. \tag{2.34}$$

We combine Eqs. (2.32)–(2.34) together and substitute them back into Eq. (2.29). After summing up a geometric series

$$\lim_{x\to1^-} \sum_{n\in(\mathbb{Z}_+)-(\mathbb{Z}_++\frac{1}{p})} (\check{q}^{\frac{\alpha}{2}})^{2pqn}(1+h)^{n(\mu p-q-pq)}(1+\varepsilon)^n x^n = \frac{1-T(h,\varepsilon)}{1-T^p(h,\varepsilon)}, \tag{2.35}$$

$$T(h,\varepsilon) = (\check{q}^{\frac{\alpha}{2}})^{2q}(1+h)^{\mu-q-\frac{q}{p}}(1+\varepsilon)^{\frac{1}{p}}, \tag{2.36}$$

we get

$$V_\alpha(\mathscr{K}_{p,q};K) = \frac{1}{z}\sum_{\mu=\pm 1}\mu(1+h)^{\frac{1}{2}(p+1)(q-\mu)}(\check{q}^{\frac{\alpha}{2}})^{\mu p-q-pq}\sum_{m=0}^{\infty}h^m$$

$$\times \sum_{k=0}^{2m} C_{m,k}(p,q)\left(\frac{1}{k!}\partial_\varepsilon^{(k)}\right)\frac{1-T(h,\varepsilon)}{1-T^p(h,\varepsilon)}\bigg|_{\varepsilon=0}. \quad (2.37)$$

We see from this expression that a line $V^{(n)}(\mathscr{K}_{p,q};z)$ of the expansion (1.12) is a linear combination over $\mathbb{Z}[\check{q}^{\frac{\alpha}{2}},\check{q}^{-\frac{\alpha}{2}}]$ of coefficients at monomials

$$h^{n_1}\varepsilon^{n_2}, \quad n_1+\frac{n_2}{2}\leqq n \quad (2.38)$$

in the expansion of $\frac{1-T(h,\varepsilon)}{1-T^p(h,\varepsilon)}$. A coefficient at $h^{n_1}\varepsilon^{n_2}$ would have taken the form

$$\frac{P_{n_1,n_2}(\check{q}^{\frac{\alpha}{2}},\check{q}^{-\frac{\alpha}{2}})}{\left(\frac{1-(\check{q}^{\frac{\alpha}{2}})^{2pq}}{1-(\check{q}^{\frac{\alpha}{2}})^{2q}}\right)^{n_1+n_2+1}} \quad (2.39)$$

with some polynomial $P_{n_1,n_2}\in\mathbb{Z}[\check{q}^{\frac{\alpha}{2}},\check{q}^{-\frac{\alpha}{2}}]$ if not for the fractional powers in $(1+h)^{\frac{1}{p}}$ and $(1+\varepsilon)^{\frac{1}{p}}$. However a simple lemma

$$\frac{1}{n!}\prod_{l=0}^{n-1}\left(\frac{1}{p}+l\right)\in\mathbb{Z}\left[\frac{1}{p}\right] \quad (2.40)$$

guarantees that

$$P_{n_1,n_2}\in\mathbb{Z}\left[\check{q}^{\frac{\alpha}{2}},\check{q}^{-\frac{\alpha}{2}},\frac{1}{p}\right]. \quad (2.41)$$

Therefore we find that

$$V^{(n)}(\mathscr{K}_{p,q};z) = \frac{P_n(\check{q}^{\frac{\alpha}{2}})}{\left(\frac{1-(\check{q}^{\frac{\alpha}{2}})^{2pq}}{1-(\check{q}^{\frac{\alpha}{2}})^{2q}}\right)^{2n+1}}, \quad P_n(\check{q}^{\frac{\alpha}{2}})\in\mathbb{Z}\left[\check{q}^{\frac{\alpha}{2}},\check{q}^{-\frac{\alpha}{2}},\frac{1}{p}\right]. \quad (2.42)$$

By using Eq. (2.4) we can rewrite this as

$$V^{(n)}(\mathscr{K}_{p,q};z) = \frac{\tilde{P}_n(\check{q}^{\frac{\alpha}{2}})}{(1-(\check{q}^{\frac{\alpha}{2}})^{2p})^{2n+1}\Delta_A^{2n+1}(\mathscr{K}_{p,q};\check{q}^{\frac{\alpha}{2}}-\check{q}^{-\frac{\alpha}{2}})},$$

$$\tilde{P}_n(\check{q}^{\frac{\alpha}{2}})\in\mathbb{Z}\left[\check{q}^{\frac{\alpha}{2}},\check{q}^{-\frac{\alpha}{2}},\frac{1}{p}\right]. \quad (2.43)$$

Comparing this with Eq. (2.20) we conclude that $\tilde{P}_n(\check{q}^{\frac{\alpha}{2}})$ is divisible by $(1-(\check{q}^{\frac{\alpha}{2}})^{2p})^{2n+1}$ over $\mathbb{Q}[\check{q}^{\frac{\alpha}{2}},\check{q}^{-\frac{\alpha}{2}}]$. However since the process of division by $1-(\check{q}^{\frac{\alpha}{2}})^{2p}$ does not introduce any new fractions in the coefficients of the polynomials, the polynomial $P^{(n)}(\mathscr{K}_{p,q};z)$ of Eq. (2.20) can have only the divisors of p in denominators of its coefficients:

$$P^{(n)}(\mathcal{K}_{p,q};\check{q}^{\frac{\alpha}{2}}-\check{q}^{-\frac{\alpha}{2}})=\frac{\tilde{P}_n(\check{q}^{\frac{\alpha}{2}})}{(1-(\check{q}^{\frac{\alpha}{2}})^{2p})^{2n+1}}\in\mathbb{Z}\left[\check{q}^{\frac{\alpha}{2}},\check{q}^{-\frac{\alpha}{2}},\frac{1}{p}\right]. \tag{2.44}$$

Since, according to Lemma 2, $P^{(n)}(\mathcal{K}_{p,q};z)\in\mathbb{Q}[z^2]$, Eq. (2.44) proves the lemma. □

Proof of Conjecture 1.2. A proof similar to that of Lemma 3 would prove that

$$P^{(n)}(\mathcal{K}_{p,q};z)\in\mathbb{Z}\left[z^2,\frac{1}{q}\right].$$

Since p and q are coprime, $\mathbb{Z}[z^2,\frac{1}{p}]\cap\mathbb{Z}[z^2,\frac{1}{q}]=\mathbb{Z}[z^2]$ and this proves the strong Conjecture 1.2. □

We used the formula (2.2) to calculate the polynomials $P^{(n)}(z)$, $n=1,2,3$ for the simplest torus knots:

(2,3):
$$\begin{aligned}\Delta_A&=1+z^2\\P^{(1)}&=2z^2+z^4\\P^{(2)}&=1-3z^2-z^4\\P^{(3)}&=-3+13z^2-z^6,\end{aligned}\tag{2.45}$$

(2,5):
$$\begin{aligned}\Delta_A&=1+3z^2+z^4\\P^{(1)}&=10z^2+21z^4+12z^6+2z^8\\P^{(2)}&=3-19z^2-24z^4+58z^6+145z^8+128z^{10}\\&\quad+56z^{12}+12z^{14}+z^{16},\end{aligned}\tag{2.46}$$

(2,7):
$$\begin{aligned}\Delta_A&=1+6z^2+5z^4+z^6\\P^{(1)}&=28z^2+126z^4+180z^6+110z^8+30z^{10}+3z^{12}\\P^{(2)}&=6-66z^2-138z^4+1398z^6+7248z^8+15747z^{10}\\&\quad+19635z^{12}+15360z^{14}+7776z^{16}+2544z^{18}\\&\quad+519z^{20}+60z^{22}+3z^{24},\end{aligned}\tag{2.47}$$

(3,5):
$$\begin{aligned}\Delta_A&=1+8z^2+14z^4+7z^6+z^8\\P^{(1)}&=40z^2+314z^4+908z^6+1224z^8+846z^{10}\\&\quad+308z^{12}+56z^{14}+4z^{16}.\end{aligned}\tag{2.48}$$

3. Experimental Results

In this section we will present the results of numerical calculations of the coefficients $d_m^{(n)}(\mathcal{K})$ of Eqs. (1.12), (1.13) for some simple knots. L. Kauffman and S. Lins [8] presented conveniently normalized formulas for the Jones polynomial $V_\alpha(\mathcal{K};K)$ as an element of $\mathbb{Z}[\check{q}]$. Choose an integer number $N\geqq 0$. If we calculate the polynomial for $1\leqq\alpha\leqq N+1$, then after substituting $\check{q}=1+h$ and extracting the coefficients in front of the first powers h^n, $0\leqq n\leqq 2N$ we can determine all the coefficients $D_{m,n}(\mathcal{K})$, $0\leqq m\leqq N$, $0\leqq n\leqq 2N$ of Eq. (1.5) by solving the system of linear equations for every power of h (each equation in a particular system corresponds to a specific value of α). Finally we can recalculate the coefficients $D_{m,n}$ into the coefficients $d_m^{(n)}$, $0 \leqq 2N$, $0 \leqq m \leqq N - \frac{n}{2}$ of Eq. (1.13). The results are presented in Tables 1 and 2 for the knots 5_2 and 6_1 respectively (see e.g. [5] for the table of knots). As we see, all the coefficients in the tables are indeed integer in agreement with the weak Conjecture 1.1. It is easy to check that the coefficients in the top line are consistent with the claim (1.15) of the Melvin–Morton conjecture.

We checked the strong Conjecture 1.2 by using all available coefficients $d_m^{(n)}$ in order to calculate "approximate" polynomials $P_{\rm appr}^{(n)}$ by the formula

$$P_{\rm appr}^{(n)}(\mathcal{K};z) = \Delta_A^{2n+1}(\mathcal{K};z) \sum_{0 \leqq m \leqq N-\frac{n}{2}} d_m^{(n)} z^{2m} + \mathcal{O}(z^{2N-n+1})\,. \tag{3.1}$$

In other words, we used the highest values of N that our computer could handle, in order to find the exact values of the coefficients $D_{m,n}(\mathcal{K})$, $0 \leqq m \leqq N$, $0 \leqq n \leqq 2N$ and $d_m^{(n)}$, $0 \leqq 2N$, $0 \leqq m \leqq N - \frac{n}{2}$. Thus we determined the exact coefficients of the series

$$\Delta_A^{2n+1}(\mathcal{K};z) \sum_{m \geqq 0} d_m^{(n)} z^{2m} \tag{3.2}$$

(cf. Eqs. (1.13) and (1.18)) up to the power z^{2N-2}. This is indicated by the term $\mathcal{O}(z^{2N-n+1})$ in the r.h.s. of Eq. (3.1) and the corresponding terms in the following equations (one can deduce the values of N that were used in the actual calculations from the powers of z in these terms):

$$
\begin{aligned}
5_2: \quad & \Delta_A = 1 + 2z^2 \\
& P_{\rm appr}^{(1)} = -6z^2 - 5z^4 + \mathcal{O}(z^{18}) \\
& P_{\rm appr}^{(2)} = 2 - 7z^2 + 36z^4 + 54z^6 + 23z^8 + \mathcal{O}(z^{18}) \\
& P_{\rm appr}^{(3)} = 4 - 83z^2 + 140z^4 - 156z^6 - 467z^8 - 358z^{10} - 103z^{12} + \mathcal{O}(z^{16})\,,
\end{aligned} \tag{3.3}
$$

$$
\begin{aligned}
6_1: \quad & \Delta_A = 1 - 2z^2 \\
& P_{\rm appr}^{(1)} = 2z^2 - z^4 + \mathcal{O}(z^{20}) \\
& P_{\rm appr}^{(2)} = -2 + z^2 + 17z^4 - 10z^6 + 3z^8 + \mathcal{O}(z^{20}) \\
& P_{\rm appr}^{(3)} = -35z^2 + 35z^4 + 166z^6 - 113z^8 + 50z^{10} - 11z^{12} + \mathcal{O}(z^{18})\,.
\end{aligned} \tag{3.4}
$$

Table 1. The coefficients $d_m^{(n)}$ for the knot 5_2

m	0	1	2	3	4	5	6
$d_m^{(0)}$	1	−2	4	−8	16	−32	64
$d_m^{(1)}$	0	−6	31	−114	360	−1040	2832
$d_m^{(2)}$	2	−27	226	−1286	5843	−22974	81684
$d_m^{(3)}$	4	−139	1750	−14100	86613	−443388	1991453
$d_m^{(4)}$	19	−832	14664	−158554	1262646	−8145921	45047755
$d_m^{(5)}$	93	−5720	133890	−1866899	18679183	−148104718	988048870

Table 2. The coefficients $d_m^{(n)}$ for the knot 6_1

m	0	1	2	3	4	5	6
$d_m^{(0)}$	1	2	4	8	16	32	64
$d_m^{(1)}$	0	2	11	42	136	400	1104
$d_m^{(2)}$	−2	−19	−93	−340	−1037	−2754	−6428
$d_m^{(3)}$	0	−35	−455	−3264	−17389	−7720	−300255
$d_m^{(4)}$	15	328	2843	14830	50071	74117	−399260
$d_m^{(5)}$	13	1226	24996	274355	2107672	12766200	65058967

Table 3. The coefficients $d_m^{(2n)}$ for the knot 4_1

m	0	1	2	3	4	5	6
$d_m^{(0)}$	1	1	1	1	1	1	1
$d_m^{(2)}$	−1	−5	−14	−30	−55	−91	−140
$d_m^{(4)}$	4	48	266	996	2926	7280	16044
$d_m^{(6)}$	−35	−780	−7214	−41875	−180510	−631436	−1890680
$d_m^{(8)}$	543	19434	270472	2251006	13395371	62736271	245214729

The coefficients of the series (3.2) are "stable" up to z^{2N-2}: they do not change if one uses higher values of N.

As we see from these equations, the stable coefficients in the series (3.2) appear to be zero starting from some power of z. An assumption that higher stable coefficients (which could be determined by the use of bigger values of N in computer calculations) would also be zero, leads to the strong Conjecture 1.2 with the exact polynomials $P^{(n)}$ being equal to the r.h.s. of Eqs. (3.3), (3.4) with the terms $O(z^{2N-n+1})$ removed.

Tables 3 and 4 contain the lists of the coefficients $\tilde{d}_m^{(2n)}$ for the simplest amphicheiral knots 4_1 (the "figure 8" knot) and 8_3. All coefficients appear to be integer in agreement with Corollary 1.1. The approximate line polynomials $\tilde{P}_{\mathrm{appr}}^{(2n)}$ for the same amphicheiral knots were calculated by the formula

$$\tilde{P}_{\mathrm{appr}}^{(2n)}(\mathcal{K};z) = \Delta_A^{3n+1}(\mathcal{K};z) \sum_{m=0}^{N-n} \tilde{d}_m^{(2n)} z^{2m} + \mathcal{O}(z^{2N-2n+1}) . \tag{3.5}$$

As we see, they are also of a limited degree:

$$\begin{aligned} 4_1: \quad & \Delta_A = 1 - z^2 \\ & \tilde{P}_{\mathrm{appr}}^{(2)} = -1 - z^2 + \mathcal{O}(z^{20}) \\ & \tilde{P}_{\mathrm{appr}}^{(4)} = 4 + 20z^2 + 14z^4 + 2z^6 + \mathcal{O}(z^{18}) \\ & \tilde{P}_{\mathrm{appr}}^{(6)} = -35 - 430z^2 - 989z^4 - 635z^6 - 140z^8 - 11z^{10} + \mathcal{O}(z^{16}) . \end{aligned} \tag{3.6}$$

Table 4. The coefficients $d_m^{(2n)}$ for the knot 8_3

m	0	1	2	3	4	5	6
$d_m^{(0)}$	1	4	16	32	64	128	256
$d_m^{(2)}$	−4	−76	−821	−6868	−49504	−323456	−1970944
$d_m^{(4)}$	60	2746	58210	840696	9594881	93259044	806300400

8_3:

$$\Delta_A = 1 - 4z^2$$

$$\tilde{P}^{(2)}_{\mathrm{appr}} = -4 - 12z^2 + 11z^4 - 4z^6 + \mathcal{O}(z^{20}) \tag{3.7}$$

$$\tilde{P}^{(4)}_{\mathrm{appr}} = 60 + 1066z^2 + 1482z^4 + 928z^6 + 513z^8 - 248z^{10} + 80z^{12} + \mathcal{O}(z^{18})$$

This confirms Conjecture 1.3.

4. Discussion

Let us speculate briefly about the possible origin of the strong Conjecture 1.2 and Conjecture 1.3. We plan to present the expansion (1.9) of the colored Jones polynomial $V_\alpha(\mathcal{K};K)$ of a knot $\mathcal{K} \in S^3$ as a contribution of a particular stationary phase point into a certain finite dimensional integral (such a representation seems to appear naturally when one tries to derive the expansion (1.9) and strong Conjecture 1.2 from the universal R-matrix). The determinant of the quadratic form of second derivatives of the exponent in the rapidly oscillating exponential is equal to the Alexander polynomial of $\mathcal{K}$. Thus, in accordance with Eq. (1.15), the stationary phase point contribution is inversely proportional to the Alexander polynomial in the leading approximation in K^{-1}.

The subleading terms in the K^{-1} expansion of $V_\alpha(\mathcal{K};K)$ can be calculated by Feynman rules. In other words, the lines $V^{(n)}(\mathcal{K};z)$ of the expansion (1.12) will be related to closed $(n+1)$-loop graphs. The edges of the graphs represent the inverse matrix of the second derivative quadratic form. The valence of the vertices of the graphs matches the order of the higher order terms in the Taylor expansion of the rapidly oscillating exponent around the stationary phase point.

The matrix elements of the inverse quadratic form are inversely proportional to the determinant of that form which is equal to the Alexander polynomial. As a result, the highest order of the Alexander polynomial in the denominator of $V^{(n)}(\mathcal{K};z)$ will be equal to the maximum number of edges in a closed $(n+1)$-loop diagram (plus 1 coming from the leading approximation). For a fixed n, this number is determined by vertices with the smallest valence $v_{\min}$:

$$\#\text{edges} \leqq \frac{v_{\min}}{v_{\min} - 2}\,. \tag{4.1}$$

The exponent of the rapidly oscillating exponential turns out to be even. Hence, for a general knot, $v_{\min} = 4$, so that the r.h.s. of Eq. (4.1) is equal to $2n$. This leads to the power $2n + 1$ in the denominator of Eq. (1.18).