# Large Deviations and the Distribution of Pre-images of Rational Maps

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**Abstract:** In this article we prove a large deviation result for the pre-images of a point in the Julia set of a rational mapping of the Riemann sphere. As a corollary, we deduce a convergence result for certain weighted averages of orbital measures, generalizing a result of Lyubich.

## 0. Introduction

Let $\hat{\mathbb{C}}$ denote the Riemann sphere and let $T : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of degree $d \geq 2$, say. Every point has $d$ pre-images (counted according to their multiplicities). There is a well-known result of Lyubich which shows that for a point $x \in \mathbb{J}$ in the Julia set an evenly distributed weight on the set of $d^n$ pre-images

$$S_n(x) = \{y \in \hat{\mathbb{C}} : T^n y = x\}$$

converges (in the weak* topology) to a measure $\mu_0$ as $n \to +\infty$ [4, 8]. The measure $\mu_0$ is precisely the unique measure of maximal entropy for the map $T$ [2, 5].

Since $T : \mathbb{J} \to \mathbb{J}$ is a continuous map on a compact metric space we can define the pressure of a continuous function $f : \mathbb{J} \to \mathbb{R}$ by

$$P(f) = \sup \left\{ h(v) + \int f \, dv : v \text{ is a } T\text{-invariant probability} \right\},$$

where $h(v)$ denotes the entropy of $T$ with respect to $v$. An equilibrium state for $f$ is a $T$-invariant probability $\mu$ realising this supremum.

Let $\mathscr{M}$ denote the set of all probability measures on $\mathbb{J}$. We shall show the following stronger "large deviation" result on the pre-images of a point $x \in \mathbb{J}$.

**Theorem 1.** *Let* $f : \mathbb{J} \to \mathbb{R}$ *be a Hölder continuous function such that* $P(f) > \sup f$ *and let* $\mu$ *be the unique equilibrium state for* $f$. *Let* $x \in \mathbb{J}$. *Then for any weak\* open neighbourhood* $\mathscr{U} \subset \mathscr{M}$ *of* $\mu$ *we have that the weighted proportion of*

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*the measures*

$$\mu_{y,n} = \frac{1}{n}(\delta_y + \delta_{Ty} + \cdots + \delta_{T^{n-1}y}) \notin \mathscr{U}$$

*tends to zero exponentially fast, in the sense that there exists $C > 0$ and $0 < \eta < 1$ such that*

$$\frac{\sum_{\substack{T^n y=x \\ \mu_{y,n} \notin \mathscr{U}}} e^{f^n(y)}}{\sum_{T^n y=x} e^{f^n(y)}} \leqq C\eta^n, \quad \text{for } n \geqq 0,$$

*where we denote $f^n(y) = f(y) + f(Ty) + \cdots + f(T^{n-1}y)$.*

*Remark.* The condition $P(f) > \sup f$ was first introduced by Urbanski in [9]. In the present article, as well as being required as a hypothesis for Lemmas 2 and 3, it is also used to give the lower bound required in establishing Lemma 6.

In the special case that $f = 0$ then $\mu$ becomes the measure of maximal entropy $\mu_0$ for $T : \mathbb{J} \to \mathbb{J}$. The theorem then reduces to the following.

**Corollary 1.** *Let $x \in \mathbb{J}$. For any weak\* open neighborhood $\mathscr{U} \subset \mathscr{M}$ of $\mu_0$ the proportion of the points $y \in S_n(x)$ such that*

$$\mu_{y,n} = \frac{1}{n}(\delta_y + \delta_{Ty} + \cdots + \delta_{T^{n-1}y}) \notin \mathscr{U}$$

*tends to zero exponentially fast, i.e., there exists $C > 0$ and $0 < \eta < 1$ such that*

$$\frac{1}{d^n} \#\{y \in S_n(x) : \mu_{y,n} \notin \mathscr{U}\} \leqq C\eta^n, \quad \text{for } n \geqq 0.$$

A second corollary to the theorem is given by the following convergence result.

**Corollary 2.** *Let $x \in \mathbb{J}$. Let $f : \mathbb{J} \to \mathbb{R}$ be a Hölder continuous function such that $P(f) > \sup f$ and let $\mu$ be the unique equilibrium state for $f$. Then the averages*

$$\frac{\sum_{T^n y=x} e^{f^n(y)} \mu_{y,n}}{\sum_{T^n y=x} e^{f^n(y)}}$$

*converges to $\mu$ in the weak\* topology as $n \to +\infty$.*

These two corollaries provide two different generalizations of the following well-known result of Lyubich.

**Lyubich's Theorem** ([4, 8, 2]). *Let $x \in \mathbb{J}$. Then the averages*

$$\frac{1}{d^n} \sum_{T^n y=x} \mu_{y,n}$$

*converge to $\mu_0$ in the weak\* topology as $n \to +\infty$.*

*Remark.* Strictly speaking, Lyubich established that the averages $\frac{1}{d^n}\sum_{T^n y=x}\mu_{y,n}$ converge to a non-atomic $T$-invariant probability measure supported on $\mathbb{J}$. This measure was subsequently shown to maximise the entropy in [2] and it was shown that there is a unique measure of maximal entropy in [5]. Whereas Lyubich's

Theorem can be established using normality of sequences of functions and the Montel–Carathéodory theorem [8], these stronger results (Corollaries 1 and 2) seem to require a different argument.

Previous applications of large deviation ideas to rational maps include the result of Lopes dealing with almost everywhere convergence of Birkhoff averages [3]. (Lopes restricted himself to the case of hyperbolic Julia sets but, as Przytycki observed [7] this is unnecessary provided we assume $P(f) > \sup f$.)

## 1. Some Properties of Rational Maps

In this section we shall recall some of the basic properties of rational maps which we shall need later. Let $T : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of degree $d \geqq 2$, i.e.

$$T(z) = \frac{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0}{b_m z^m + b_{m-1} z^{m-1} + \cdots + b_1 z + b_0},$$

where $a_n, \ldots, a_0, b_m, \ldots, b_0 \in \mathbb{C}$ (with $a_n, b_m \neq 0$) and $d = \max\{n, m\} \geqq 2$.

Counted according to multiplicity, every point $x \in \hat{\mathbb{C}}$ will have $d$-pre-images. If we consider the $n^{\text{th}}$ iterate $T^n : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ then the set of pre-images $S_n(x) = \{y : T^n y = x\}$ of a point $x \in \hat{\mathbb{C}}$ will have cardinality $d^n$.

**Definition.** *The **Julia set** $\mathbb{J} \subset \mathbb{C}$ is defined to be the closure of the set of all periodic points $T^n x = x$ for which $|(T^n)'(x)| > 1$.*

Clearly $\mathbb{J}$ is a closed $T$-invariant set and we shall be interested in the restriction $T : \mathbb{J} \to \mathbb{J}$ of the map $T$. We shall write $C^0(\mathbb{J})$ for the space of real valued continuous functions on $\mathbb{J}$. We denote by $\mathcal{M}_T$ the space of all $T$-invariant probability measures on $\mathbb{J}$.

**Definition.** *For any continuous function $f : \mathbb{J} \to \mathbb{R}$ we can define the **pressure** by*

$$P(f) = \sup\{h(v) + \int f \, dv : v \in \mathcal{M}_T\},$$

*where $h(v)$ denotes the entropy of $T$ with respect to the measure $v$.*

We let $h(T) = P(0)$ denote the *topological entropy* of $T : \mathbb{J} \to \mathbb{J}$.
The following results about entropy will be useful to us.

**Lemma 1** [2, 4].

  (i) $h(T) = \log d$.
  (ii) *The map $v \to h(v)$ is upper semi-continuous in the weak$^*$ topology.*
  (iii) *There is a unique measure of maximal entropy $\mu_0$ for $T : \mathbb{J} \to \mathbb{J}$.*

Statement (iii) of Lemma 1 has the following generalization.

**Lemma 2** (*Denker and Urbanski* [1]). *If $f : \mathbb{J} \to \mathbb{R}$ is a Hölder continuous function such that $P(f) > \sup f$ then $f$ has a unique equilibrium state $\mu$.*

The following lemma gives us some information about the relationship between pressure and the pre-images of a point.

**Lemma 3** (*Przytycki* [6]). *Let* $x \in \mathbb{J}$. *Let* $f$ *be a Hölder continuous function on the Julia set* $\mathbb{J}$ *such that* $P(f) > \sup f$ *and let* $g$ *be a continuous function on* $\mathbb{J}$. *Then*

(i)

$$\lim_{n \to +\infty} \frac{1}{n} \log \sum_{T^n y = x} e^{f^n(y)} = P(f) \, ;$$

(ii)

$$\limsup_{n \to +\infty} \frac{1}{n} \log \sum_{T^n y = x} e^{f^n(y) + g^n(y)} \leqq P(f + g) \, .$$

Finally, the following lemma gives us an alternative characterisation of the entropy.

**Lemma 4.**

(i) *If* $v \in \mathcal{M}_T$ *then*

$$h(v) = \inf \{ P(g) - \int g \, dv \colon g \in C^0(\mathbb{J}) \} \, .$$

(ii) *If* $v \in \mathcal{M} - \mathcal{M}_T$ *then*

$$0 \geqq \inf \{ P(g) - \int g \, dv \colon g \in C^0(\mathbb{J}) \} \, .$$

*Proof.* In fact, these two results hold for any continuous mapping $T$ of a compact metric space for which $h(T) < +\infty$ and, in the case of (i), the map $v \to h(v)$ is upper semi-continuous (cf. [10, pp. 221–222]).

## 2. Proof of Theorem 1

In this section we will give the proof of Theorem 1, using the results from the previous section. As before $f : \mathbb{J} \to \mathbb{R}$ is a Hölder continuous function satisfying $P(f) > \sup f$. We can define a map $Q : C^0(\mathbb{J}) \to \mathbb{R}$ by $Q(g) = P(f + g) - P(f)$. For $v \in \mathcal{M}$, we then denote the *Legendre transform* of $Q(g)$ by

$$I(v) = \sup_{g \in C^0(\mathbb{J})} \left( \int g \, dv - Q(g) \right).$$

Given any weak* closed (and hence compact) subset $\mathcal{K} \subset \mathcal{M}$ we define $\rho = \rho_{\mathcal{K}} := \inf_{v \in \mathcal{K}} I(v)$.

Our proof will be based upon the following estimate.

**Lemma 5.**

$$\limsup_{n \to +\infty} \frac{1}{n} \log \left( \frac{\sum_{\substack{T^n y = x \\ \mu_{y, n} \in \mathcal{K}}} e^{f^n(y)}}{\sum_{T^n y = x} e^{f^n(y)}} \right) \leqq -\rho \, . \tag{1}$$

*Proof.* Fix a choice of $\varepsilon > 0$. From the definition of $\rho$, for every $v \in \mathcal{K}$, there exists $g \in C^0(\mathbb{J})$ such that

$$\int g \, dv - Q(g) > \rho - \varepsilon \, .$$

Thus we have that

$$\mathcal{K} \subset \bigcup_{g \in C^0(\mathbb{J})} \left\{ v \in \mathcal{M} : \int g \, dv - Q(g) > \rho - \varepsilon \right\}$$

and by weak$^*$ compactness we can choose a finite subcover

$$\mathcal{K} \subset \bigcup_{i=1}^{k} \left\{ v \in \mathcal{M} : \int g_i \, dv - Q(g_i) > \rho - \varepsilon \right\}.$$

Therefore we have the inequality

$$\sum_{\substack{T^n y = x \\ \mu_{y,n} \in \mathcal{K}}} e^{f^n(y)} \leqq \sum_{i=1}^{k} \left( \sum_{\substack{T^n y = x \\ \frac{1}{n} g_i^n(y) - Q(g_i) > \rho - \varepsilon}} e^{f^n(y)} \right)$$

$$\leqq \sum_{i=1}^{k} e^{-n(Q(g_i) + (\rho - \varepsilon))} \left( \sum_{T^n y = x} e^{f^n(y) + g_i^n(y)} \right).$$

Taking limits we get that

$$\limsup_{n \to +\infty} \frac{1}{n} \log \left( \frac{\sum_{\substack{T^n y = x \\ \mu_{y,n} \in \mathcal{K}}} e^{f^n(y)}}{\sum_{T^n y = x} e^{f^n(y)}} \right)$$

$$\leqq \sup_{1 \leqq i \leqq k} \left\{ -Q(g_i) - \rho + \varepsilon + \limsup_{n \to +\infty} \frac{1}{n} \log \left( \sum_{T^n y = x} e^{f^n(y) + g_i^n(y)} \right) \right.$$

$$\left. - \liminf_{n \to +\infty} \frac{1}{n} \log \left( \sum_{T^n y = x} e^{f^n(y)} \right) \right\}$$

$$\leqq \sup_{1 \leqq i \leqq k} \{ -Q(g_i) - \rho + \varepsilon + P(f + g_i) - P(f) \} = -\rho + \varepsilon,$$

where the second inequality uses Lemma 3. Since $\varepsilon > 0$ can be chosen arbitrarily small this completes the proof of the lemma.

We next want to show that if $\mathcal{K}$ does not contain $\mu$ then $\rho > 0$. This will follow from the next lemma.

**Lemma 6.**

    (i) *If $v \neq \mu$ then $I(v) > 0$.*
    (ii) *The map $v \to I(v)$ is lower semi-continuous on $\mathcal{M}_T$ and $I$ is bounded away from $0$ on $\mathcal{M} - \mathcal{M}_T$.*

*Proof.* For part (i) we have that

$$
\begin{aligned}
I(v) &= \sup_{g \in C^0(\mathbb{J})} \left( \int g\,dv - P(f+g) + P(f) \right) \\
&= \sup_{g \in C^0(\mathbb{J})} \left( \int (g-f)\,dv - P(g) + P(f) \right) \\
&= \sup_{g \in C^0(\mathbb{J})} \left( \int g\,dv - P(g) \right) + P(f) - \int f\,dv \\
&= - \inf_{g \in C^0(\mathbb{J})} \left( P(g) - \int g\,dv \right) + P(f) - \int f\,dv \,.
\end{aligned}
$$

If $v \in \mathscr{M}_T$ then, by part (i) of Lemma 4, this is equal to $-h(v) + P(f) - \int f\,dv$, and, by the uniqueness of the equilibrium state $\mu$, $-h(v) + P(f) - \int f\,dv > 0$. On the other hand, if $v \in \mathscr{M} - \mathscr{M}_T$, then

$$
\inf_{g \in C^0(\mathbb{J})} \left( P(g) - \int g\,dv \right) < 0
$$

by part (ii) of Lemma 4 and so

$$
I(v) > 0 + P(f) - \int f\,dv \geqq P(f) - \sup f > 0 \,.
$$

For the proof of (ii) we first notice that $I(v) = -h(v) + P(f) - \int f\,dv$. We then complete the proof with the lower bound in the proof of (i) above. This completes the proof of the lemma.

Since $\mathscr{K}$ is compact, we can conclude that if $\mu \notin \mathscr{K}$ then $\rho > 0$. Theorem 1 now follows by setting $\mathscr{K} = \mathscr{M} - \mathscr{U}$.

*Proof of Corollary 2.* We shall show that for any $g \in C^0(\mathbb{J})$ we have that

$$
\frac{1}{\Sigma(f,n)} \sum_{T^n y = x} e^{f^n(y)} \frac{g^n(y)}{n} \to \int g\,d\mu, \quad \text{as } n \to +\infty \,,
$$

where $\Sigma(f,n) = \sum_{T^n y = x} e^{f^n(y)}$.

Given $\varepsilon > 0$, define an open neighbourhood $\mathscr{U}$ of $\mu$ by

$$
\mathscr{U} = \left\{ v \in \mathscr{M} : \left| \int g\,dv - \int g\,d\mu \right| < \varepsilon \right\} \,.
$$

Then we may write

$$
\begin{aligned}
&\frac{1}{\Sigma(f,n)} \sum_{T^n y = x} e^{f^n(y)} \frac{g^n(y)}{n} \\
&= \frac{1}{\Sigma(f,n)} \sum_{\substack{T^n y = x \\ \mu_{y,n} \in \mathscr{U}}} e^{f^n(y)} \frac{g^n(y)}{n} + \frac{1}{\Sigma(f,n)} \sum_{\substack{T^n y = x \\ \mu_{y,n} \notin \mathscr{U}}} e^{f^n(y)} \frac{g^n(y)}{n} \\
&= \frac{1}{\Sigma(f,n)} \sum_{\substack{T^n y = x \\ \mu_{y,n} \in \mathscr{U}}} e^{f^n(y)} \left\{ \int g\,d\mu + E_n(y) \right\} + O(\eta^n) \,,
\end{aligned}
$$

where $|E_n(y)| < \varepsilon$ and $0 < \eta < 1$ is given by Theorem 1.

Thus we conclude (by adding appropriate constants to $g$ if necessary) that

$$\limsup_{n \to +\infty} \frac{1}{\Sigma(f,n)} \sum_{T^n y = x} e^{f^n(y)} \frac{g^n(y)}{n} \leqq \int g d\mu + \varepsilon$$

and

$$\liminf_{n \to +\infty} \frac{1}{\Sigma(f,n)} \sum_{T^n y = x} e^{f^n(y)} \frac{g^n(y)}{n} \geqq \int g d\mu - \varepsilon .$$

Since $\varepsilon > 0$ is arbitrary, the result is proved.

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