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Simple Currents and Extensions of Vertex Operator Algebras

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Abstract: We consider how a vertex operator algebra can be extended to an abelian intertwining algebra by a family of weak twisted modules which are *simple currents* associated with semisimple weight one primary vectors. In the case that the extension is again a vertex operator algebra, the rationality of the extended algebra is discussed. These results are applied to affine Kac–Moody algebras in order to construct all the simple currents explicitly (except for E_8) and to get various extensions of the vertex operator algebras associated with integrable representations.

1. Introduction

Introduced in [B] and [FLM], *vertex operator algebras* are essentially *chiral algebras* as formulated in [BPZ] and [MoS], and provide a powerful algebraic tool for studying the general structure of conformal field theory. For a vertex operator algebra V, one wishes to adjoin certain simple V-modules to get a larger algebraic structure so that certain data such as fusion rules and braiding matrices are naturally incorporated. The introduction of the notions of *generalized vertex (operator) algebra* and *abelian intertwining algebra* in [DL] was made in this spirit. A similar notion called vertex operator *para-algebra* was independently introduced and studied in [FFR] with different motivations. Also see [M].

In this paper, we study how a vertex operator algebra can be extended to an abelian intertwining algebra by a family of weak twisted modules which are *simple currents* associated with semisimple weight one primary vectors. In the case that the extension is again a vertex operator algebra, we discuss the rationality of the extended algebra. Applying these results to affine Kac–Moody algebras we construct all the simple currents explicitly (except for E_8) and get various extensions of the vertex operator algebras associated with integrable representations.

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Many of the ideas discussed here are natural continuations of ideas discussed in [DL] and [Li4]. Recently, it is shown in [Hua] that the extension of the moonshine module vertex operator algebra $V^\natural$ [FLM] by $\mathbb{Z}_2$-twisted module forms an abelian intertwining algebra, where such an extension is called a nonmeromorphic extension.

Let G be a torsion group of automorphisms of V and $g \in G$. A G-simple current for a vertex operator algebra is an irreducible weak g-twisted module which gives a bijection between the equivalence classes of irreducible weak h-twisted modules and the equivalence classes of irreducible weak gh-twisted modules under the "tensor product" if $h \in G$ commutes with g. The theory of simple currents originated in the papers [FG] and [SY]. A *simple* vertex operator algebra V is always a G-simple current for any G. It is natural to expect that any simple current can be *deformed* from V by introducing a new action. Via this principle, a class of simple currents are constructed in the present paper. Let $U(V[1])$ be the universal enveloping algebra of the Lie algebra $V[1] = V \otimes \mathbb{C}[t,t^{-1}]/D(V \otimes \mathbb{C}[t,t^{-1}])$ defined in [Li3] (for more detail see Sect. 2) and let $\Delta(z) \in U(V[1])\{z\}$ satisfy conditions (2.13)–(2.16) below. Then for any weak h-twisted V-module $(M, Y_M(\cdot\,,z))$, we show that $(\tilde{M}, Y_{\tilde{M}}(\cdot\,,z)) := (M, Y_M(\Delta(z)\cdot\,,z))$ is a weak gh-twisted V-module where h is any automorphism of V of finite order which commutes with g and is not necessarily in G. If $\Delta(z) \in U(V^G[1])$ (V^G is the space of G-invariants of V) is invertible and $h \in G$, then $\tilde{M}$ is isomorphic to a tensor product module of M with $\tilde{V}$ and thus $\tilde{V}$ is a G-simple current. There is a simple way to construct such $\Delta(z)$ associated to any weight one primary vector α of V whose component operator $\alpha(0)$ is semisimple on V. The corresponding automorphism of V is given by $e^{2\pi i\alpha(0)}$. These results have been obtained in [Li4] in the case that $g = 1$.

This paper is organized as follows: In Sect. 2 we recall the definitions of weak twisted modules from [D2] and [FFR] and of intertwining operators among weak twisted modules from [FHL] and [X] using the language of formal variables. We present a notion of tensor product of two weak twisted modules in terms of universal mapping properties and relate the fusion rules (which are the dimensions of the space of intertwining operators of certain types) to the dimensions of certain spaces of homomorphisms of weak twisted modules. For certain elements $\Delta(z)$ in $U(V[1])\{z\}$ associated to an automorphism g of V of finite order we show how a weak h-twisted module can be deformed to a weak gh-twisted module if the automorphism h commutes with g. We discuss the relations among this deformation, the intertwining operators and tensor product. It turns out that the deformation of V is always a simple current. Such $\Delta(z)$ are constructed for semisimple primary vectors of V of weight one. These results are interpreted in the theory of vertex operator algebras associated with even lattices at the end of this section.

Section 3 deals with the extension of a simple vertex operator algebra by a family of simple currents constructed from semisimple primary vectors of weight one. We begin with a finite dimensional subspace H of V_1 which contains all semisimple vectors, and a lattice L contained in H such that the component operators $\alpha(0)$ ($\alpha \in L$) have only rational eigenvalues on V. We show that the direct sum U of all the deformations of V by using $\Delta(z)$ associated with $\alpha \in L$ is a generalized vertex algebra in the sense of [DL]. Then there is an even sublattice L_0 such that the corresponding deformation of V is isomorphic to V. We then prove that the quotient $\bar{U}$ of U modulo the isomorphic relations has a structure of abelian intertwining algebra. This section is technically complicated, involving the abelian cohomology of abelian groups introduced by Eilenberg–MacLane. We refer the reader to [DL] for the definition of abelian intertwining algebras and related terminology.

In Sect. 4 we discuss the rationality of a simple vertex operator algebra $V = \bigoplus_{g\in G} V^g$ which is graded by a finite abelian group G satisfying certain conditions. Such vertex operator algebras arise naturally in Sect. 3 and in [DL]. Under a mild assumption we show how the rationality of V^0 implies the rationality of V. In particular, if G consists of automorphisms constructed from semisimple primary weight one vectors, the assumption always holds. In Sect. 5 we apply the results obtained in the previous sections to affine algebras. After discussing simple currents for the vertex operator algebras associated to integrable representations and which are related to the work of [FG], we obtain various extensions of these vertex operator algebras by simple currents. Another result in this section is about the representations of these algebras. Under the assumption that certain elements in the Heisenberg subalgebra of a given affine algebra act nilpotently on a weak module for the vertex operator algebra we show the complete reducibility of this weak module[3]. In particular, this shows that any irreducible weak module is a standard module.

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2. Simple Currents and Twisted modules

In this section we first recall the definitions of twisted modules (cf. [D2 and FFR]) and intertwining operators among twisted modules (cf. [FHL and X]). We then discuss how a weak module for a vertex operator algebra V can be deformed to a twisted module by using certain elements in the vector space of formal power series with coefficients in the universal enveloping algebra $U(V[1])$ of $V[1]$ which is defined below. In general, we exhibit the deformation of intertwining operators among weak V-modules to intertwining operators among weak twisted modules. We also apply these results to vertex operator algebras associated with even positive-definite lattices. The ideas and techniques used in this section derive from those in [Li4].

Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra (cf. [B, FHL and FLM]) and let g be an automorphism of V of finite order T. Then V is a direct sum of eigenspaces of g:

$$V = \bigoplus_{r\in\mathbb{Z}/T\mathbb{Z}} V^r ,$$

where $V^r = \{v \in V | gv = e^{2\pi i r/T} v\}$. (We abuse notation and use $r \in \{0, 1, \ldots, T-1\}$ to denote both an integer and the corresponding residue class.) Following [D2 and FFR], a weak g-twisted *module* M for V is a vector space equipped with a linear map

$$V \to (\operatorname{End} M)\{z\} ,$$

$$v \mapsto Y_M(v, z) = \sum_{n\in\mathbb{Q}} v_n z^{-n-1} \quad (v_n \in \operatorname{End} M)$$

(where for any vector space W, we define $W\{z\}$ to be the vector space of W-valued formal series in z, with arbitrary complex powers of z) satisfying the following

[3] This assumption has been removed recently in [DLM1].

conditions for $u, v \in V, w \in M$, and $r \in \mathbb{Z}/T\mathbb{Z}$:

$$Y_M(v,z) = \sum_{n \in \frac{r}{T}+\mathbb{Z}} v_n z^{-n-1} \quad \text{for } v \in V^r \; ; \tag{2.1}$$

$$v_l w = 0 \quad \text{for } l \in \mathbb{Q} \text{ sufficiently large} \; ; \tag{2.2}$$

$$Y_M(\mathbf{1},z) = 1 \; ; \tag{2.3}$$

$$\begin{aligned} z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right) Y_M(u,z_1)Y_M(v,z_2) - z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right) Y_M(v,z_2)Y_M(u,z_1) \\ = z_2^{-1}\left(\frac{z_1-z_0}{z_2}\right)^{-r/T} \delta\left(\frac{z_1-z_0}{z_2}\right) Y_M(Y(u,z_0)v,z_2) \end{aligned} \tag{2.4}$$

if $u \in V^r$;

$$[L(m),L(n)] = (m-n)L(m+n) + \frac{1}{12}(m^3-m)\delta_{m+n,0}(\operatorname{rank} V)$$

for $m, n \in \mathbb{Z}$, where

$$L(n) = \omega_{n+1} \quad \text{for } n \in \mathbb{Z}, \text{ i.e., } Y_M(\omega,z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2} \; ;$$

$$\frac{d}{dz} Y_M(v,z) = Y_M(L(-1)v,z) \, . \tag{2.5}$$

This completes the definition. We denote this module by (M, Y_M) (or briefly by M).

Remark 2.1. If a weak g-twisted V-module M further is a $\mathbb{C}$-graded vector space:

$$M = \coprod_{\lambda \in \mathbb{C}} M_\lambda \, ,$$

such that for each $\lambda \in \mathbb{C}$,

$$\dim M_\lambda < \infty$$

$$M_{\lambda + \frac{n}{T}} = 0$$

for $n \in \mathbb{Z}$ sufficiently small, and

$$L(0)w = nw = (\mathrm{wt}\, w)w \quad \text{for } w \in M_n \; (n \in \mathbb{C}) \, ,$$

we call M a g-twisted V-module.

A weak 1-twisted V-module M is called a *weak V-module* and a 1-twisted V-module is called a *V-module*. A *g-homomorphism* f from a weak g-twisted V-module M to another weak g-twisted V-module W is a linear map $f : M \to W$ such that

$$f Y_M(u,z) = Y_W(u,z) f$$

for all $u \in V$. We denote the space of all g-homomorphisms from M to W by $\mathrm{Hom}_g(M,W)$. A *g-isomorphism* is a bijective g-homomorphism.

Next we shall define intertwining operators among weak g_k-twisted modules (M_k, Y_{M_k}) for $k = 1,2,3$, where g_k are commuting automorphisms of order T_k (cf. [X]). In this case V decomposes into the direct sum of common eigenspaces

$$V = \bigoplus_{j_1,j_2} V^{(j_1,j_2)} \, ,$$

where

$$V^{(j_1,j_2)} = \{v \in V | g_k v = e^{2\pi i j_k/T_k}, k = 1,2\} .$$

An *intertwining operator of type* $\left[{M_3 \atop M_1\ M_2}\right]$ associated with the given data is a linear map

$$\begin{aligned} M_1 &\to (\mathrm{Hom}(M_2, M_3))\{z\} , \\ w &\mapsto \mathcal{Y}(w,z) = \sum_{n\in\mathbb{C}} w_n z^{-n-1} \end{aligned} \tag{2.6}$$

such that for $w^i \in M_i$ $(i = 1,2)$, fixed $c \in \mathbb{C}$ and $n \in \mathbb{Q}$ sufficiently large

$$w^1_{c+n} w^2 = 0 ; \tag{2.7}$$

the following (generalized) Jacobi identity holds on M_2: for $u \in V^{(j_1,j_2)}$ and $w \in M_1$,

$$\begin{aligned} & z_0^{-1}\left(\frac{z_1 - z_2}{z_0}\right)^{j_1/T_1} \delta\left(\frac{z_1 - z_2}{z_0}\right) Y_{M_3}(u,z_1)\mathcal{Y}(w,z_2) \\ & - z_0^{-1}\left(\frac{z_2 - z_1}{z_0}\right)^{j_1/T_1} \delta\left(\frac{z_2 - z_1}{-z_0}\right)\mathcal{Y}(w,z_2)Y_{M_2}(u,z_1) \\ & = z_2^{-1}\left(\frac{z_1 - z_0}{z_2}\right)^{-j_2/T_2} \delta\left(\frac{z_1 - z_0}{z_2}\right)\mathcal{Y}(Y_{M_1}(u,z_0)w,z_2) \end{aligned} \tag{2.8}$$

and

$$\frac{d}{dz}\mathcal{Y}(w,z) = \mathcal{Y}(L(-1)w,z) , \tag{2.9}$$

where $L(-1)$ is the operator acting on M_1.

The intertwining operators of type $\left[{M_3 \atop M_1\ M_2}\right]$ associated with prescribed data clearly form a vector space, which we denote by $\mathcal{V}^{M_3}_{M_1 M_2}$. We set

$$N^{M_3}_{M_1 M_2} = \dim \mathcal{V}^{M_3}_{M_1 M_2} . \tag{2.10}$$

These numbers are called the *fusion rules* associated with the algebra, modules and auxiliary data. It is easy to observe that if $N^{M_3}_{M_1 M_2} > 0$ then $g_3 = g_1 g_2$ (see [X]). Thus we shall assume this relation in the following discussion.

The fusion rules have certain symmetry properties. Our next goal is to show that $N^{M_3}_{M_2 M_1} = N^{M_3}_{M_1 M_2}$. Let $\mathcal{Y}$ be an intertwining operator of type $\left[{M_3 \atop M_1\ M_2}\right]$. We define a linear map

$$\begin{aligned} M_2 &\to (\mathrm{Hom}(M_1, M_3))\{z\} \\ w &\mapsto \mathcal{Y}^{\pm}(w,z) = \sum_{n\in\mathbb{C}} w_n z^{-n-1} \end{aligned} \tag{2.11}$$

by the skew-symmetry

$$\mathcal{Y}^{\pm}(w^2,z)w^1 = e^{zL(-1)}\mathcal{Y}(w^1, e^{\pm\pi i}z)w^2 \tag{2.12}$$

for $w^i \in M_i$, where $Y(w^1, e^{\pm\pi i}z) = \sum_{n\in\mathbb{C}} w^1_n e^{\pm(-n-1)\pi i} z^{-n-1}$.

Lemma 2.2. (1) *The operator* $\mathscr{Y}^{\pm}(\,\cdot\,,z)$ *is an intertwining operator of type* $\left[{M_3 \atop M_2\ M_1}\right]$.

(2) *Each of the two maps* $\mathscr{Y} \mapsto \mathscr{Y}^{\pm}$ *is a linear isomorphism from* $\mathscr{V}_{M_1M_2}^{M_3}$ *to* $\mathscr{V}_{M_2M_1}^{M_3}$. *In particular* $N_{M_2M_1}^{M_3} = N_{M_1M_2}^{M_3}$.

Proof. (1) The relation (2.7) is clear. In order to prove (2.9) for $\mathscr{Y}^{\pm}(w^2,z)$ we first observe from (2.8) that

$$[L(-1),\mathscr{Y}(w^1,z)] = \mathscr{Y}(L(-1)w^1,z) = \frac{d}{dz}\mathscr{Y}(w^1,z)\,,$$

$$e^{L(-1)z_0}\mathscr{Y}(L(-1)w^1,z)e^{-L(-1)z_0} = \mathscr{Y}(L(-1)w^1,z+z_0)\,.$$

Thus we have the following calculation:

$$\begin{aligned}\frac{d}{dz}\mathscr{Y}^{\pm}(w^2,z)w^1 &= \frac{d}{dz}e^{zL(-1)}\mathscr{Y}(w^1,e^{\pm i\pi}z)w^2\\ &= e^{zL(-1)}L(-1)\mathscr{Y}(w^1,e^{\pm i\pi}z)w^2 - e^{zL(-1)}\mathscr{Y}(L(-1)w^1,e^{\pm i\pi}z)w^2\\ &= e^{zL(-1)}\mathscr{Y}(w^1,e^{\pm i\pi}z)L(-1)w^2\\ &= \mathscr{Y}^{\pm}(L(-1)w^2,z)w^1\,.\end{aligned}$$

The proof of Proposition 2.2.2 of [G] provides (as a special case) a proof of the Jacobi identity (2.8) for $Y_{M_i}(u,z_1)$ and $\mathscr{Y}^{\pm}(w^2,z_2)$.

(2) An easy verification shows that $(\mathscr{Y}^{+})^{-} = \mathscr{Y}$ for any $\mathscr{Y} \in \mathscr{V}_{M_1M_2}^{M_3}$. The isomorphism between $\mathscr{V}_{M_1M_2}^{M_3}$ and $\mathscr{V}_{M_2M_1}^{M_3}$ is now clear. □

Next we formulate the notion of tensor product $M_1 \boxtimes M_2$ of M_1 and M_2: it is a weak g_3-twisted V-module defined by a universal mapping property. See [Li3 and HL1–HL2] for the definition of tensor product of (ordinary) weak modules.

Definition 2.3. *A tensor product for the ordered pair* (M_1,M_2) *is a pair* $(M,F(\,\cdot\,,z))$ *consisting of a weak* g_3*-twisted* V*-module* M *and an intertwining operator* $F(\,\cdot\,,z)$ *of type* $\left({M \atop M_1\ M_2}\right)$ *such that the following universal property holds: for any weak* g_3*-twisted* V*-module* W *and any intertwining operator* $I(\,\cdot\,,z)$ *of type* $\left({W \atop M_1\ M_2}\right)$, *there exists a unique* V*-homomorphism* ψ *from* M *to* W *such that* $I(\,\cdot\,,z) = \psi\circ F(\,\cdot\,,z)$. *(Here* ψ *extends canonically to a linear map from* $M\{z\}$ *to* $W\{z\}$.*)*

The following proposition is a direct consequence of the definition and Lemma 2.2.

Proposition 2.4. *Let* $(M,F(\,\cdot\,,z))$ *be a tensor product of* M_1 *and* M_2.

(1) *For any weak* g_3*-twisted* V*-module* M_3, *the space* $\mathrm{Hom}_{g_3}(M,M_3)$ *is linearly isomorphic to* $\mathscr{V}_{M_1M_2}^{M_3}$.

(2) *The pair* $(M,F^{\pm}(\,\cdot\,,z))$ *is a tensor product of* M_2 *and* M_1. *In particular, the tensor product is commutative up to isomorphism.*

(3) *If* $g_1 = 1$ *and* $M_1 = V$ *then* M *is isomorphic to* M_2. *That is,* $V \boxtimes M_2 = M_2$.

Let V be a vertex operator algebra and g an automorphism of V of order T. We recall from [DLM2] the Lie algebra

$$V[g] = \bigoplus_{i=0}^{T-1} \left(V^i \otimes t^{i/T}\mathbb{C}[t,t^{-1}]/D\left(\bigoplus_{i=0}^{T-1} (V^i \otimes t^{i/T}\mathbb{C}[t,t^{-1}]) \right)\right)$$

associated with V and g, with bracket

$$[u(m), v(n)] = \sum_{i=0}^{\infty} \binom{m}{i} (u_i v)(m+n-i-1)\,,$$

where $D = L(-1) \otimes 1 + \frac{d}{dt} \otimes 1$ and $u(m)$ is the image of $u \otimes t^m$ in $V[g]$. Then $V^0[1]$ is a Lie subalgebra of both $V[1]$ and $V[g]$, and $V[g]$ acts on any weak g-twisted V-module.

Let $\Delta(z) \in U(V[1])\{z\}$ satisfy the following conditions:

$$z^{\frac{j}{T}} \Delta(z) a \in V[z, z^{-1}] \quad \text{for } a \in V^j\,; \tag{2.13}$$

$$\Delta(z)\mathbf{1} = \mathbf{1}\,; \tag{2.14}$$

$$[L(-1), \Delta(z)] = -\frac{d}{dz}\Delta(z)\,; \tag{2.15}$$

$$Y(\Delta(z_2 + z_0)a, z_0)\Delta(z_2) = \Delta(z_2)Y(a, z_0) \quad \text{for any } a \in V\,. \tag{2.16}$$

Let $G(V,g)$ be the set of all $\Delta(z)$ satisfying the conditions (2.13)–(2.16). Define $G^0(V,g)$ to be those $\Delta(z)$ in $G(V,g)$ which are invertible. The following lemma is obvious:

Lemma 2.5. (1) *Let $\Delta_1(z), \Delta_2(z) \in G(V,g)$. Then $\Delta_1(z)\Delta_2(z) \in G(V,g^2)$. In particular $G(V,1)$ is a semigroup with $\mathrm{id}_V \in G(V,1)$.*

(2) *If $\Delta(z) \in G(V,g)$ has an inverse $\Delta^{-1}(z) \in U(V[1])\{z\}$. Then $\Delta^{-1}(z) \in G(V,g^{-1})$.*

From now on we fix two commutative automorphisms g and h of V of order S and T respectively. The following result generalizes the corresponding results in [Li4] with $g = h = 1$.

Lemma 2.6. *Let $(M, Y_M(\,\cdot\,,z))$ be a weak h-twisted V-module and $\Delta(z) \in G(V,g)$. Set $\tilde{M} = M$ and $Y_{\tilde{M}}(\,\cdot\,,z) = Y_M(\Delta(z)\,\cdot\,,z)$. Then $(\tilde{M}, Y_{\tilde{M}}(\,\cdot\,,z))$ is a weak gh-twisted V-module.*

Proof. First, (2.14) implies that $\tilde{Y}_{\tilde{M}}(\mathbf{1},z) = \mathrm{Id}_{\tilde{M}}$. Second, it is easy to see that (2.15) implies that $\tilde{Y}_{\tilde{M}}(L(-1)a,z) = \frac{d}{dz}\tilde{Y}_{\tilde{M}}(a,z)$ for any $a \in V$. Let $a \in V^{(i,j)}$ and $b \in V$. Then we have

$$\begin{aligned}
&z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_M(\Delta(z_1)a, z_1)Y_M(\Delta(z_2)b, z_2)\\
&\quad - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y_M(\Delta(z_2)b, z_2)Y_M(\Delta(z_1)a, z_1)\\
&= z_2^{-1}\left(\frac{z_1 - z_0}{z_2}\right)^{\frac{-i}{S}} \delta\left(\frac{z_1 - z_0}{z_2}\right) Y_M(Y(\Delta(z_1)a, z_0)\Delta(z_2)b, z_2) =
\end{aligned}$$

$$= z_2^{-1}\left(\frac{z_1-z_0}{z_2}\right)^{\frac{-i}{S}}\delta\left(\frac{z_1-z_0}{z_2}\right)z_1^{-\frac{j}{T}}Y_M(Y((z_2+z_0)^{\frac{j}{T}}\Delta(z_2+z_0)a,z_0)\Delta(z_2)b,z_2)$$

$$= z_2^{-1}\left(\frac{z_1-z_0}{z_2}\right)^{\frac{-i}{S}}\delta\left(\frac{z_1-z_0}{z_2}\right)\left(\frac{z_2+z_0}{z_1}\right)^{\frac{j}{T}}Y_M(Y(\Delta(z_2+z_0)a,z_0)\Delta(z_2)b,z_2)$$

$$= z_2^{-1}\left(\frac{z_1-z_0}{z_2}\right)^{\frac{-i}{S}-\frac{j}{T}}\delta\left(\frac{z_1-z_0}{z_2}\right)Y_M(\Delta(z_2)Y(a,z_0)b,z_2)\,. \tag{2.17}$$

Thus $(\tilde{M}, Y_{\tilde{M}}(\,\cdot\,,z))$ is a weak gh-twisted V-module. □

We shall use the notation V^h for the h-fixed-point vertex operator subalgebra of V for an automorphism h.

Lemma 2.7. (1) *Assume that g commutes with each g_i. Let $\Delta(z)\in G(V^{g_1},g)$, let M_i $(i=1,2,3)$ be weak g_i-twisted V-modules and $I(\,\cdot\,,z)$ an intertwining operator of type $\binom{M_3}{M_1M_2}$. Then $\tilde{I}(\,\cdot\,,z)=I(\Delta(z)\,\cdot\,,z)$ is an intertwining operator of type $\binom{\tilde{M}_3}{M_1\tilde{M}_2}$.*

(2) *Let $\Delta(z)\in G(V,g)$ and let ψ be a h-homomorphism from W to M. Then ψ is a gh-homomorphism from $\tilde{W}$ to $\tilde{M}$.*

(3) *Let $\Delta(z)\in G(V^g,1)$ be such that $(V,Y(\Delta(z)\,\cdot\,,z)$ is isomorphic to the adjoint module $(V,Y(\,\cdot\,,z))$. Then there is a nonzero homomorphism of weak g-twisted modules from $(M,Y_M(\,\cdot\,,z))$ to $(M,Y_M(\Delta(z)\,\cdot\,,z))$ for any weak g-twisted V-module $(M,Y_M(\,\cdot\,,z))$. Moreover, if $\Delta(z)\in G^0(V^g,1)$, any such homomorphism is an isomorphism.*

Proof. The proof of (1) is similar to that of Lemma 2.6 and we omit details. (2) is a special case of (1) with $M_1=V, M_2=W$ and $M_3=M$. It remains to show (3). Clearly $Y_M(\,\cdot\,,z)$ is an intertwining operator of type $\binom{M}{VM}$. By Lemma 2.2, there is a nonzero intertwining operator of type $\binom{M}{MV}$. Now by (1) there is a nonzero intertwining operator of type $\binom{\tilde{M}}{M\tilde{V}}$, which yields a nonzero intertwining operator of type $\binom{\tilde{M}}{MV}$ by hypothesis. Consequently there is a nonzero intertwining operator $I(\,\cdot\,,z)$ of type $\binom{\tilde{M}}{VM}$ by Lemma 2.2 once more. Now $I(1,z)$ is the desired nonzero homomorphism. The other assertions are clear. □

Proposition 2.8. *Let $(W,F(\,\cdot\,,z))$ be a tensor product of a weak g_1-twisted module M_1 and a weak g_2-twisted module M_2 and assume that g commutes with each g_i. Then if $\Delta(z)\in G^0(V^{g_1},g),(\tilde{W},\tilde{F}(\,\cdot\,,z))$ is a tensor product of the pair $(M_1,\tilde{M}_2)$.*

Proof. First by Lemma 2.7 (1), we have an intertwining operator $\tilde{F}(\,\cdot\,,z)=F(\Delta(z)\,\cdot\,,z)$ of type $\binom{\tilde{W}}{M_1\tilde{M}_2}$. Let M be a weak gg_1g_2-twisted V-module and let $I(\,\cdot\,,z)$ be any intertwining operator of type $\binom{M}{M_1\tilde{M}_2}$. Then $I(\Delta(z)^{-1}\,\cdot\,,z)$ is an intertwining operator of type $\binom{\hat{M}}{M_1M_2}$, where $(\hat{M},Y_{\hat{M}}(\,\cdot\,,z))=(M,Y_M(\Delta(z)^{-1}\,\cdot\,,z))$. By the universal property of $(W,F(\,\cdot\,,z))$, there is a unique g_1g_2-homomorphism ψ from W to $\hat{M}$ such that $\hat{I}(\,\cdot\,,z)=\psi\circ\hat{F}(\,\cdot\,,z)$. By Lemma 2.7 (2), ψ is a g_1g_2-homomorphism from W to M. Since $\Delta(z)u$ only involves finitely many terms, we have: $I(\,\cdot\,,z)=\psi\circ F(\,\cdot\,,z)$. It is easy to check the uniqueness, so that the proof is complete. □

Corollary 2.9. *Let M be a weak h-twisted V-module and let $\Delta(z) \in G^0(V^h, g)$. Then $\tilde{M}$ is isomorphic to the tensor product module of M with $\tilde{V}$.*

Proof. It is easy to observe that $(M, F(\,\cdot\,, z))$ is a tensor product of M and V where $F(\,\cdot\,, z)$ is the transpose intertwining operator of $Y_M(\,\cdot\,, z)$ (also see Proposition 2.4 (3)). By Proposition 2.8, $(\tilde{M}, \tilde{F}(\,\cdot\,, z))$ is a tensor product of M and $\tilde{V}$. □

Remark 2.10. In the situation of Corollary 2.9 with $g = h = 1$, $\tilde{M}$ is defined as the tensor product of M with $\tilde{V}$ in the physics literature (cf. [MaS]). Corollary 2.9 asserts that this formulation coincides with our axiomatic notion of tensor product in this special case.

Definition 2.11. *Let V be a vertex operator algebra and G a torsion group of automorphisms of V. We denote the set of equivalence classes of irreducible weak h-twisted modules by* $\mathrm{Irr}_h(V)$ *for $h \in G$. For convenience we write* $\mathrm{Irr}(V) = \mathrm{Irr}_1(V)$. *An irreducible weak g-twisted V-module M for $g \in G$ is called a G-***simple current** *if the tensor functor "$M \boxtimes \cdot$" is a bijection from* $\mathrm{Irr}_h(V)$ *to* $\mathrm{Irr}_{gh}(V)$ *for any $h \in G$ which commutes with g. A 1-simple current ($G = 1$) is called a simple current.*

Clearly, a G-simple current $M \in \mathrm{Irr}(V)$ acts on $\mathrm{Irr}_h(V)$ for $h \in G$ as a permutation via the tensor product $M \boxtimes \cdot$ for any h.

Proposition 2.12. *For any $\Delta(z) \in G^0(V^G, g), (V, Y(\Delta(z)\,\cdot\,, z))$ is a G-simple current if V is a simple vertex operator algebra.*

Proof. By Corollary 2.9 for any weak h-twisted module $M, \tilde{M} = (M, Y_M(\Delta(z)\,\cdot\,, z)$ is isomorphic to the tensor product of M with $(V, Y(\Delta(z)\,\cdot\,, z))$, and M is isomorphic to the tensor product of $\tilde{M}$ with $(V, Y(\Delta(z)^{-1}\,\cdot\,, z))$. Thus if M is irreducible so is $\tilde{M}, \tilde{V} \mapsto \tilde{V} \boxtimes M$ being a bijection from $\mathrm{Irr}_h(V)$ to $\mathrm{Irr}_{gh}(V)$. □

Conjecture 2.13. *Let V be a vertex operator algebra and G a group of automorphisms of finite order of V. Define $\bar{V}$ to be the direct sum of all G-simple currents of V. Then $\bar{V}$ is an abelian intertwining algebra in the precise sense of* [DL].

In the next section we will prove this conjecture in some special cases.

Let V be a vertex operator algebra and let $\alpha \in V$ satisfying the following conditions:

$$L(n)\alpha = \delta_{n,0}\alpha, \quad \alpha(n)\alpha = \delta_{n,1}\gamma\mathbf{1} \quad \text{for any } n \in \mathbb{Z}_+\,, \tag{2.18}$$

where γ is a fixed complex number. Notice that α in (2.18) is a primary vector of weight one. Furthermore, we assume that $\alpha(0)$ acts semisimply on V with rational eigenvalues. It is clear that $e^{2\pi i\alpha(0)}$ is an automorphism of V. If the denominators of all eigenvalues of $\alpha(0)$ are bounded, then $e^{2\pi i\alpha(0)}$ is of finite order. Define

$$\Delta(\alpha, z) = z^{\alpha(0)}\exp\left(\sum_{k=1}^{\infty}\frac{\alpha(k)}{-k}(-z)^{-k}\right) \in U(V^{e^{2\pi i\alpha(0)}}[1])\{z\}\,. \tag{2.19}$$

Recall the following proposition from [Li2].

Proposition 2.14. *Let τ be a finite-order automorphism of V such that $\tau\alpha = \alpha$ and let $(M, Y_M(\,\cdot\,, z))$ be any τ-twisted V-module. Then $(M, Y(\Delta(\alpha, z)\,\cdot\,, z))$ is a $\sigma_\alpha\tau$-twisted weak V-module, where $\sigma_\alpha = e^{-2\pi i\alpha(0)}$.*

The main part of the proof of Proposition 2.14 in [Li2] is to establish that $\Delta(\alpha,z)$ satisfies the condition (2.16).

We end this section by discussing an example, namely vertex operator algebras associated with even positive-definite lattices. Let L be a positive definite rational lattice with form $\langle\, ,\rangle$, let L_0 be an even sublattice of L and let $V = V_{L_0}$ be the vertex operator algebra constructed in [B and FLM]. For any $\beta \in L$, $V_{\beta+L_0}$ is a twisted V_{L_0}-module for the inner automorphism $\sigma_\beta = e^{-2\beta(0)\pi i}$ (see [DM and Le]). It is easy to see that $V_{\beta+L_0}$ is isomorphic to the adjoint module V_{L_0} if and only if $\beta \in L_0$.

Proposition 2.15. *Let $\beta \in L$. Then as a σ_β-twisted module, $(V_{L_0}, Y(\Delta(\beta,z)\,\cdot\,,z))$ is isomorphic to $V_{L_0+\beta}$.*

Proof. First, since $\Delta(\beta,z)$ is invertible and V_{L_0} is a simple vertex operator algebra, it follows from Proposition 2.12 that $(V_{L_0}, Y(\Delta(\beta,z)\,\cdot\,,z))$ is an irreducible weak σ_β-twisted V_{L_0}-module. For any $\alpha \in H = \mathbb{C}\otimes_{\mathbb{Z}} L_0$, we have:

$$\Delta(\beta,z)\alpha = \Delta(\beta,z)\alpha(-1)\mathbf{1} = \alpha + z^{-1}\langle\beta,\alpha\rangle\,. \tag{2.20}$$

Let ψ be the algebra automorphism of $U(V_{L_0}[1])$ such that $\psi(Y(a,z)) = Y(\Delta(\beta,z)\, a,z)$ for $a \in V_{L_0}$. Then we have:

$$\psi(\alpha(n)) = \alpha(n) + \delta_{n,0}\langle\beta,\alpha\rangle \quad \text{for } \alpha \in H, n \in \mathbb{Z}\,. \tag{2.21}$$

Then the action of $\psi(H(0))$ on V_{L_0} is also semisimple with $L_0+\beta$ as the set of H-weights and V_{L_0} is still a completely reducible module for the Heisenberg algebra $\psi(\tilde{H})$. It follows from the classification result [D1] that $(V_{L_0}, Y(\Delta(\beta,z)\,\cdot\,,z))$ is isomorphic to $V_{L_0+\beta}$. $\square$

Let P be the dual lattice of L. Then $V_{\beta+L_0}$ is a V_{L_0}-module if $\beta \in P$. It is proved in [D1] that there is a 1–1 correspondence between the equivalence classes of irreducible modules for V_{L_0} and the cosets of P/L_0. More specifically, V_P is the direct sum of all inequivalent irreducible V_{L_0}-modules:

$$V_P = V_{L_0+\beta_1} \oplus \dots \oplus V_{L_0+\beta_k}\,, \tag{2.22}$$

where $k = |P/L_0|$. For the vertex operator algebra V_{L_0}, intertwining operators are explicitly constructed and fusion rules are calculated in [DL]. Moreover, it is established in [DL] that V_P is an abelian intertwining algebra.

For any fixed $\beta \in L$, we consider all the irreducible σ_β-twisted V_{L_0}-modules. It was essentially proved [D1] that any irreducible σ_β-twisted V_{L_0}-module is isomorphic to $V_{\beta+\beta_i+L_0}$ for some $1 \leqq i \leqq k$. It follows from Proposition 2.15 that all irreducible σ_β-twisted V_{L_0}-modules can be obtained as $(V_{L_0}, Y(\Delta(\gamma,z)\,\cdot\,,z))$ for $\gamma \in L$, so that by Proposition 2.12 all irreducible σ_β-twisted V_{L_0}-modules are simple currents.

3. Abelian Intertwining Algebras

In this section we consider extensions of a vertex operator algebra V, whose weight one subspaces is nonzero, by incorporating certain twisted modules. The corresponding twist elements are automorphisms $e^{2\pi i h(0)}$, where $h \in V_1$ is a primary vector and $h(0)$ acts semisimply on V. (There should be no confusion between h in Sect. 2 – an automorphism of V, and h in this section – an element in V_1.) We prove that such an extension is a generalized vertex operator algebra and that the quotient space modulo isomorphic relations is an abelian intertwining algebra in the sense of Dong and Lepowsky [DL]. We refer the reader to [DL] for the definitions of generalized vertex operator algebras and of abelian intertwining algebras. In this section we assume that V is a simple vertex operator algebra.

Let $h \in V$ satisfy condition (2.18), that is, $L(n)h = \delta_{n,0}h$ and $h(n)h = \delta_{n,1}\gamma\mathbf{1}$ for some fixed complex number γ. For any $s \in \mathbb{Q}$, set

$$E^{\pm}(sh,z) = \exp\left(\sum_{k=1}^{\infty} \frac{sh(\pm k)}{k} z^{\mp k}\right) . \tag{3.1}$$

Then we have:

$$E^{+}(sh,z_1)E^{-}(th,z_2) = \left(1-\frac{z_2}{z_1}\right)^{-\gamma st} E^{-}(th,z_2)E^{+}(sh,z_1) \tag{3.2}$$

for $s,t \in \mathbb{Q}$ (cf. formula (4.3.1) of [FLM]).

Let us recall some elementary results from [Li4].

Lemma 3.1. *Let $h \in V_1$ such that* (2.18) *holds. Then for any $a \in V$,*

$$e^{z(L(1)-h(1))}e^{-zL(1)} = \exp\left(\sum_{k=1}^{\infty} \frac{h(k)}{k}(-z)^k\right) , \tag{3.3}$$

$$e^{z(L(-1)+h(-1))}e^{-zL(-1)} = \exp\left(\sum_{k=1}^{\infty} \frac{h(-k)}{k} z^k\right) , \tag{3.4}$$

$$\begin{aligned} Y(E^{-}(h,z_1)a,z_2) &= E^{-}(h,z_1+z_2)E^{-}(-h,z_2) \\ &\quad \cdot Y(a,z_2)z_2^{-h(0)}E^{+}(h,z_2)(z_2+z_1)^{h(0)}E^{+}(-h,z_2+z_1)\,, \end{aligned} \tag{3.5}$$

$$E^{-}(h,z_1)Y(a,z_2)E^{-}(-h,z_1) = Y(\Delta(-h,z_2-z_1)\Delta(h,z_2)a,z_2)\,. \quad \square \tag{3.6}$$

Let H be a finite-dimensional subspace of V_1 satisfying the following conditions:

$$L(n)h = \delta_{n,0}h,\ h(n)h' = \langle h,h'\rangle\delta_{n,1}\mathbf{1} \quad \text{for } n \in \mathbb{Z}_{+}, h,h' \in H\,, \tag{3.7}$$

where $\langle \cdot , \cdot \rangle$ is assumed to be a nondegenerate symmetric bilinear form on H. Then we may identify H with its dual H^*. We also assume that for any $h \in H, h(0)$ acts semisimply on V. Then

$$V = \bigoplus_{\alpha\in H} V^{(0,\alpha)}, \quad \text{where } V^{(0,\alpha)} = \{u \in V | h(0)u = \langle \alpha,h\rangle u \text{ for } h \in H\}\,. \tag{3.8}$$

Let L be a lattice in H such that for each $\alpha \in L, \alpha(0)$ has rational eigenvalues on V. *From now on, we assume that there is a positive integer T such that the eigenvalues of $T\alpha(0)$ on V are integers.* One can show that this holds if V is finitely generated. Let G be the group of automorphisms of V generated by $e^{2\pi i\alpha(0)}$ for $\alpha \in L$. Then G is an abelian torsion group. Note that $\Delta(\alpha,z) \in G^0(V^G,\sigma_\alpha)$ and $(V, Y(\Delta(\alpha,z)\cdot,z))$ is a G-simple current by Proposition 2.12 where $\sigma_\alpha = e^{-2\pi i\alpha(0)}$.

For any $\alpha \in L$ and for any weak V-module $(M, Y_M(\cdot,z))$, we have a weak σ_α-twisted module

$$(M^{(\alpha)}, Y_\alpha(\cdot,z)) = (M, Y_M(\Delta(\alpha,z)\cdot,z))\,.$$

This yields a linear isomorphism ψ_α from $M^{(\alpha)}$ onto M such that

$$\psi_\alpha(Y_\alpha(a,z)u) = Y(\Delta(\alpha,z)a,z)\psi_\alpha(u) \quad \text{for } a \in V, u \in M^{(\alpha)} . \tag{3.9}$$

For $\alpha = 0$, we may choose $\psi_0 = \mathrm{id}_M$.

Set $U = \bigoplus_{\alpha \in L} V^{(\alpha)}$. For $\alpha, \beta \in L$, we have a linear isomorphism $\psi_{\beta-\alpha}^{-1}\psi_\beta$ from $V^{(\beta)}$ onto $V^{(\beta-\alpha)}$ satisfying the following condition:

$$\psi_{\beta-\alpha}^{-1}\psi_\beta(Y_\beta(a,z)u) = Y_{\beta-\alpha}(\Delta(\alpha,z)a,z)\psi_{\beta-\alpha}^{-1}\psi_\beta(u) \quad \text{for } a \in V, u \in V^{(\beta)} . \tag{3.10}$$

Then we may extend ψ_α to an automorphism of U such that

$$\psi_\alpha(Y_\beta(a,z)u) = Y_{\beta+\alpha}(\Delta(\alpha,z)a,z)\psi_\alpha(u) \quad \text{for } a \in V, u \in V^{(\alpha)} . \tag{3.11}$$

Then it is easy to see that $\psi_{\alpha+\beta} = \psi_\alpha\psi_\beta$ for any $\alpha, \beta \in L$. In other words, ψ gives rise to a representation of L on U.

By a simple calculation we get:

$$\Delta(\alpha,z)\beta = \beta + z^{-1}\langle\alpha,\beta\rangle\mathbf{1}, \quad \Delta(\alpha,z)\omega = \omega + z^{-1}\alpha + z^{-2}\frac{\langle\alpha,\alpha\rangle}{2}\mathbf{1} . \tag{3.12}$$

Lemma 3.2. *For any $\alpha \in L, h \in H$, we have*

$$\psi_\alpha h(n) = h(n)\psi_\alpha + \delta_{n,0}\langle\alpha,h\rangle \quad \textit{for } n \in \mathbb{Z} , \tag{3.13}$$

$$\psi_\alpha\Delta(h,z) = z^{\langle\alpha,h\rangle}\Delta(h,z)\psi_\alpha , \tag{3.14}$$

$$\psi_\alpha e^{zL(-1)}\psi_{-\alpha}e^{-zL(-1)} = E^-(\alpha,z) . \tag{3.15}$$

Proof. By definition, we get $\Delta(h,z)\alpha = \alpha + z^{-1}\langle\alpha,h\rangle\mathbf{1}$. Then (3.13) is clear and (3.14) follows from (3.13). By (3.11) and (3.12) we get:

$$\psi_\alpha L(-1) = (L(-1) + \alpha(-1))\psi_\alpha . \tag{3.16}$$

Thus $\psi_\alpha e^{zL(-1)}\psi_\alpha^{-1} = e^{z(L-1)+\alpha(-1))}$. Then (3.15) easily follows from Lemma 3.1. □

For any $\alpha \in L, h \in H$, we define:

$$V^{(\alpha,h)} = \{u \in V^{(\alpha)} | h'(0)u = \langle h', h+\alpha\rangle u \text{ for } h' \in H\} . \tag{3.17}$$

Then by Lemma 3.2 we have:

$$V^{(\alpha)} = \bigoplus_{h \in H} V^{(\alpha,h)}, \psi_\alpha V^{(\alpha,h)} = V^{(0,h)} \quad \text{for } h \in H . \tag{3.18}$$

Let $P = \{\lambda \in H | V^{(0,\lambda)} \neq 0\}$. As V is simple it is easy to prove that P is a subgroup of H (cf. [LX]). Let $A = L \times P$ be the product group. We define:

$$\eta((\alpha_1,\lambda_1),(\alpha_2,\lambda_2)) = -\langle\alpha_1,\alpha_2\rangle - \langle\alpha_1,\lambda_2\rangle - \langle\alpha_2,\lambda_1\rangle \in \frac{1}{T}\mathbb{Z}/2\mathbb{Z} , \tag{3.19}$$

$$C((\alpha_1,\lambda_1),(\alpha_2,\lambda_2)) = e^{(\langle\alpha_1,\lambda_2\rangle - \langle\alpha_2,\lambda_1\rangle)\pi i} \in \mathbb{C}^* \tag{3.20}$$

for any $(\alpha_i, \lambda_i) \in A, i = 1, 2$. Then $\eta(\cdot, \cdot)$ and $C(\cdot, \cdot)$ satisfy the following conditions:

$$\eta(a,b) = \eta(b,a), \quad \eta(a+b,c) = \eta(a,c) + \eta(b,c)\,, \tag{3.21}$$

$$C(a,a) = 1, \quad C(a,b) = C(b,a)^{-1}, \quad C(a+b,c) = C(a,c)C(b,c) \tag{3.22}$$

for $a, b, c \in G$. $C(\cdot, \cdot)$ will be our commutator map later.

Definition 3.3. *For $u \in V^{(\alpha)}, v \in V^{(\beta)}, \alpha, \beta \in L$, we define $Y_\alpha(u,z)v \in V^{(\alpha+\beta)}\{z\}$ as follows:*

$$Y_\alpha(u,z)v = \psi_{-\alpha-\beta}E^-(\alpha,z)Y(\psi_\alpha \Delta(\beta,z)u,z)\Delta(\alpha,-z)\psi_\beta(v)\,. \tag{3.23}$$

Set $U = \bigoplus_{\alpha \in L} V^{(\alpha)}$. Then this defines a map $Y(\cdot, z)$ from U to $(\mathrm{End}(U))\{z\}$ via $Y(u,z) = Y_\alpha(u,z)$ for $u \in V^{(\alpha)}$. Notice that for any $u \in V^{(\alpha,h_1)}, v \in V^{(\beta,h_2)}$, $Y(u,z)v \in V^{(\alpha+\beta,h_1+h_2)}\{z\}$.

Proposition 3.4. *The following $L(-1)$-derivative property holds:*

$$Y(L(-1)u,z)v = \frac{d}{dz}Y(u,z)v \tag{3.24}$$

for any $u, v \in U$.

Proof. Let $\alpha, \beta \in L$ and let $u \in V^{(\alpha)}, v \in V^{(\beta)}$. Then

$$\begin{aligned} Y(L(-1)u,z)v &= \psi_{-\alpha-\beta}E^-(\alpha,z)Y(\psi_\alpha \Delta(\beta,z)L(-1)u,z)\Delta(\alpha,-z)\psi_\beta(v) \\ &= \psi_{-\alpha-\beta}E^-(\alpha,z)Y(\psi_\alpha[\Delta(\beta,z),L(-1)]u,z)\Delta(\alpha,-z)\psi_\beta(v) \\ &\quad + \psi_{-\alpha-\beta}E^-(\alpha,z)Y([\psi_\alpha,L(-1)]\Delta(\beta,z)u,z)\Delta(\alpha,-z)\psi_\beta(v) \\ &\quad + \psi_{-\alpha-\beta}E^-(\alpha,z)Y(L(-1)\psi_\alpha\Delta(\beta,z)u,z)\Delta(\alpha,-z)\psi_\beta(v)\,. \end{aligned} \tag{3.25}$$

Note that (3.16) is equivalent to

$$[\psi_\alpha, L(-1)] = \alpha(-1)\psi_\alpha\,.$$

From (2.8) with $u = \alpha(-1)\cdot\mathbf{1}$, which is σ_h-invariant for any $h \in H$, we have

$$\begin{aligned} Y(\alpha(-1)w,z) &= \sum_{i=0}^{\infty} \binom{-1}{i} ((-z)^i\alpha(-1-i)Y(w,z) + z^{-1-i}Y(w,z)\alpha(i)) \\ &= \sum_{i=0}^{\infty} (z^i\alpha(-1-i)Y(w,z) + z^{-1-i}Y(w,z)\alpha(i)) \\ &= \alpha(z)^-Y(w,z) + Y(w,z)\alpha(z)^+\,, \end{aligned}$$

where

$$\alpha(z)^- = \sum_{n=0}^{\infty}\alpha(-n-1)z^n, \alpha(z)^+ = \sum_{n=1}^{\infty}\alpha(n)z^{-n-1}\,.$$

Thus

$$\begin{aligned}
&\psi_{-\alpha-\beta}E^-(\alpha,z)Y([\psi_\alpha,L(-1)]\Delta(\beta,z)u,z)\Delta(\alpha,-z)\psi_\beta(v)\\
&\quad=\psi_{-\alpha-\beta}E^-(\alpha,z)\alpha(z)^-Y(\psi_\alpha\Delta(\beta,z)u,z)\Delta(\alpha,-z)\psi_\beta(v)\\
&\qquad+\psi_{-\alpha-\beta}E^-(\alpha,z)Y(\psi_\alpha\Delta(\beta,z)u,z)\alpha(z)^+\Delta(\alpha,-z)\psi_\beta(v)\\
&\quad=\psi_{-\alpha-\beta}\left(\frac{d}{dz}E^-(\alpha,z)\right)Y(\psi_\alpha\Delta(\beta,z)u,z)\Delta(\alpha,-z)\psi_\beta(v)\\
&\qquad+\psi_{-\alpha-\beta}E^-(\alpha,z)Y\left(\psi_\alpha\left(\frac{d}{dz}\Delta(\beta,z)u,z\right)\right)\Delta(\alpha,-z)\psi_\beta(v)\,.
\end{aligned}$$

Now from (2.15) and Proposition 2.15 we obtain

$$Y(L(-1)u,z)v=\frac{d}{dz}Y(u,z). \qquad \square$$

Theorem 3.5. *For any* $u\in V^{(\alpha,h_1)}, v\in V^{(\beta,h_2)}, w\in V^{(\gamma,h_3)}, \alpha,\beta,\gamma\in L, h_1,h_2,h_3\in P$, *we have the following generalized Jacobi identity*:

$$\begin{aligned}
&z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)\left(\frac{z_1-z_2}{z_0}\right)^{\eta((\alpha,h_1),(\beta,h_2))}Y(u,z_1)Y(v,z_2)w\\
&\quad-C((\alpha,h_1),(\beta,h_2))z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)\left(\frac{z_2-z_1}{z_0}\right)^{\eta((\alpha,h_1),(\beta,h_2))}Y(v,z_2)Y(u,z_1)w\\
&=z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)\left(\frac{z_2+z_0}{z_1}\right)^{\eta((\alpha,h_1),(\gamma,h_3))}Y(Y(u,z_0)v,z_2)w\,. \qquad (3.26)
\end{aligned}$$

Moreover, $(U,\mathbf{1},\omega,Y,T,A,\eta(\cdot,\cdot),C(\cdot,\cdot))$ *is a generalized vertex algebra in the sense of* [*DL*].

Proof. By (3.21), (3.22) and Proposition 3.4 all the axioms in the definition of a generalized vertex algebra (Chapter 9 of [DL]) hold except the Jacobi identity. By Definition 3.3 and Lemma 3.1 together with the relation (3.14) we have:

$$\begin{aligned}
&Y(u,z_1)Y(v,z_2)w\\
&\quad=z_1^{\langle\alpha,\beta+\gamma\rangle}\psi_{-\alpha-\beta-\gamma}E^-(\alpha,z_1)Y(\Delta(\beta+\gamma,z_1)\psi_\alpha u,z_1)\Delta(\alpha,-z_1)\psi_{\beta+\gamma}Y(v,z_2)w\\
&\quad=z_1^{\langle\alpha,\beta+\gamma\rangle}z_2^{\langle\beta,\gamma\rangle}\psi_{-\alpha-\beta}E^-(\alpha,z_1)Y(\Delta(\beta+\gamma,z_1)\psi_\alpha u,z_1)\Delta(\alpha,-z_1)\\
&\qquad\cdot E^-(\beta,z_2)Y(\Delta(\gamma,z_2)\psi_\beta v,z_2)\Delta(\beta,-z_2)\psi_\gamma w\\
&\quad=\left(1-\frac{z_2}{z_1}\right)^{\langle\alpha,\beta\rangle}z_1^{\langle\alpha,\beta+\gamma\rangle}z_2^{\langle\beta,\gamma\rangle}\psi_{-\alpha-\beta-\gamma}\\
&\qquad\cdot E^-(\alpha,z_1)E^-(\beta,z_2)Y(\Delta(\beta,z_1-z_2)\Delta(\gamma,z_1)\psi_\alpha u,z_1)\\
&\qquad\cdot Y(\Delta(\alpha,-z_1+z_2)\Delta(\gamma,z_2)\psi_\beta v,z_2)\Delta(\alpha,-z_1)\Delta(\beta,-z_2)\psi_\gamma w
\end{aligned}$$

$$= (z_1 - z_2)^{\langle\alpha,\beta\rangle} z_1^{\langle\alpha,\gamma\rangle} z_2^{\langle\beta,\gamma\rangle} \psi_{-\alpha-\beta-\gamma}$$
$$\cdot E^-(\alpha,z_1)E^-(\beta,z_2)Y(\Delta(\beta,z_1-z_2)\Delta(\gamma,z_1)\psi_\alpha u,z_1)$$
$$\cdot Y(\Delta(\alpha,-z_1+z_2)\Delta(\gamma,z_2)\psi_\beta v,z_2)\Delta(\alpha,-z_1)\Delta(\beta,-z_2)\psi_\gamma w\,. \qquad (3.27)$$

Symmetrically,

$$Y(v,z_2)Y(u,z_1)w$$
$$= (z_2 - z_1)^{\langle\alpha,\beta\rangle} z_1^{\langle\alpha,\gamma\rangle} z_2^{\langle\beta,\gamma\rangle} \psi_{-\alpha-\beta-\gamma}$$
$$\cdot E^-(\alpha,z_1)E^-(\beta,z_2)Y(\Delta(\alpha,z_2-z_1)\Delta(\gamma,z_2)\psi_\beta v,z_2)$$
$$\cdot Y(\Delta(\beta,-z_2+z_1)\Delta(\gamma,z_1)\psi_\alpha u,z_1)\Delta(\alpha,-z_1)\Delta(\beta,-z_2)\psi_\gamma w\,. \qquad (3.28)$$

Since

$$\beta(0)\Delta(\gamma,z_1)\psi_\alpha u = \Delta(\gamma,z_1)\beta(0)\psi_\alpha u = \langle\beta,h_1\rangle\Delta(\gamma,z_1)\psi_\alpha u\,,$$

we see that $(z_1-z_2)^{-\langle\beta,h_1\rangle}\Delta(\beta,z_1-z_2)\Delta(\gamma,z_1)\psi_\alpha u$ involves only integral powers of (z_1-z_2). Similarly, $(z_1-z_2)^{-\langle\alpha,h_2\rangle}\Delta(\alpha,-z_1+z_2)\Delta(\gamma,z_2)\psi_\beta v$ involves only integral powers of (z_1-z_2). Using properties of δ-functions (cf. [FLM]) we see that

$$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y(u,z_1)Y(v,z_2)w$$
$$= z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)(z_1-z_2)^{\langle\alpha,\beta\rangle} z_1^{\langle\alpha,\gamma\rangle} z_2^{\langle\beta,\gamma\rangle}\psi_{-\alpha-\beta-\gamma}$$
$$\cdot E^-(\alpha,z_1)E^-(\beta,z_2)\left(\frac{z_1-z_2}{z_0}\right)^{\langle\beta,h_1\rangle} Y(\Delta(\beta,z_0)\Delta(\gamma,z_1)\psi_\alpha u,z_1)$$
$$\cdot \left(\frac{z_1-z_2}{z_0}\right)^{\langle\alpha,h_2\rangle} Y(\Delta(\alpha,-z_0)\Delta(\gamma,z_2)\psi_\beta v,z_2)\Delta(\alpha,-z_1)\Delta(\beta,-z_2)\psi_\gamma w$$
$$= z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)\left(\frac{z_1-z_2}{z_0}\right)^{\langle\alpha,\beta\rangle+\langle\alpha,h_2\rangle+\langle\beta,h_1\rangle} z_1^{\langle\alpha,\gamma\rangle} z_2^{\langle\beta,\gamma\rangle} z_0^{\langle\alpha,\beta\rangle}\psi_{-\alpha-\beta-\gamma}$$
$$\cdot E^-(\alpha,z_1)E^-(\beta,z_2)Y(\Delta(\beta,z_0)\Delta(\gamma,z_1)\psi_\alpha u,z_1)$$
$$\cdot Y(\Delta(\alpha,-z_0)\Delta(\gamma,z_2)\psi_\beta v,z_2)\Delta(\alpha,-z_1)\Delta(\beta,-z_2)\psi_\gamma w\,, \qquad (3.29)$$

and that

$$z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y(v,z_2)Y(u,z_1)w$$
$$= z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)(z_2-z_1)^{\langle\alpha,\beta\rangle} z_1^{\langle\alpha,\gamma\rangle} z_2^{\langle\beta,\gamma\rangle}\psi_{-\alpha-\beta-\gamma}$$
$$\cdot E^-(\alpha,z_1)E^-(\beta,z_2)Y(\Delta(\alpha,z_2-z_1)\Delta(\gamma,z_2)\psi_\beta v,z_2)$$
$$\cdot Y(\Delta(\beta,-z_2+z_1)\Delta(\gamma,z_1)\psi_\alpha u,z_1)\Delta(\alpha,-z_1)\Delta(\beta,-z_2)\psi_\gamma w$$

$$= z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)(z_2-z_1)^{\langle\alpha,\beta\rangle}z_1^{\langle\alpha,\gamma\rangle}z_2^{\langle\beta,\gamma\rangle}\psi_{-\alpha-\beta-\gamma}E^-(\alpha,z_1)E^-(\beta,z_2)$$

$$\cdot\left(\frac{z_2-z_1}{-z_0}\right)^{\langle\alpha,h_2\rangle}Y(\Delta(\alpha,-z_0)\Delta(\gamma,z_2)\psi_\beta v,z_2)$$

$$\cdot\left(\frac{-z_2+z_1}{z_0}\right)^{\langle\beta,h_1\rangle}Y(\Delta(\beta,z_0)\Delta(\gamma,z_1)\psi_\alpha u,z_1)\Delta(\alpha,-z_1)\Delta(\beta,-z_2)\psi_\gamma w$$

$$= e^{(\langle\beta,h_1\rangle-\langle\alpha,h_2\rangle)\pi i}z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)\left(\frac{z_2-z_1}{z_0}\right)^{\langle\alpha,\beta\rangle+\langle\alpha,h_2\rangle+\langle\beta,h_1\rangle}$$

$$\cdot z_1^{\langle\alpha,\gamma\rangle}z_2^{\langle\beta,\gamma\rangle}z_0^{\langle\alpha,\beta\rangle}\psi_{-\alpha-\beta-\gamma}$$

$$\cdot E^-(\alpha,z_1)E^-(\beta,z_2)Y(\Delta(\alpha,-z_0)\Delta(\gamma,z_2)\psi_\beta v,z_2)$$

$$\cdot Y(\Delta(\beta,z_0)\Delta(\gamma,z_1)\psi_\alpha u,z_1)\Delta(\alpha,-z_1)\Delta(\beta,-z_2)\psi_\gamma w\,. \tag{3.30}$$

Thus

$$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)\left(\frac{z_1-z_2}{z_0}\right)^{\eta((\alpha,h_1),(\beta,h_2))}Y(u,z_1)Y(v,z_2)w$$

$$-C((\alpha,h_1),(\beta,h_2))z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)\left(\frac{z_2-z_1}{z_0}\right)^{\eta((\alpha,h_1),(\beta,h_2))}Y(v,z_2)Y(u,z_1)w$$

$$= z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)z_1^{\langle\alpha,\gamma\rangle}z_2^{\langle\beta,\gamma\rangle}z_0^{-\langle\alpha,\beta\rangle}\psi_{-\alpha-\beta-\gamma}E^-(\alpha,z_1)E^-(\beta,z_2)$$

$$\cdot Y(Y(\Delta(\beta,z_0)\Delta(\gamma,z_1)\psi_\alpha(u),z_0)\Delta(\alpha,-z_0)\Delta(\gamma,z_2)\psi_\beta(v),z_2)$$

$$\cdot \Delta(\alpha,-z_1)\Delta(\beta,-z_2)\psi_\gamma(w)\,. \tag{3.31}$$

On the other hand,

$$Y(Y(u,z_0)v,z_2)w$$

$$= E^-(\alpha+\beta,z_2)\psi_{-\alpha-\beta-\gamma}Y(\psi_{\alpha+\beta}\Delta(\gamma,z_2)Y(u,z_0)v,z_2)\Delta(\alpha+\beta,-z_2)\psi_\gamma w$$

$$= E^-(\alpha+\beta,z_2)\psi_{-\alpha-\beta-\gamma}Y(\psi_{\alpha+\beta}\Delta(\gamma,z_2)E^-(\alpha,z_0)\psi_{-\alpha-\beta}$$

$$\cdot Y(\psi_\alpha\Delta(\beta,z_0)u,z_0)\Delta(\alpha,-z_0)\psi_\beta v,z_2)\Delta(\alpha+\beta,-z_2)\psi_\gamma w$$

$$= z_2^{\langle\alpha+\beta,\gamma\rangle}z_0^{\langle\alpha,\beta\rangle}E^-(\alpha+\beta,z_2)\psi_{-\alpha-\beta-\gamma}Y(\Delta(\gamma,z_2)E^-(\alpha,z_0)$$

$$\cdot Y(\Delta(\beta,z_0)\psi_\alpha u,z_0)\Delta(\alpha,-z_0)\psi_\beta v,z_2)\Delta(\alpha+\beta,-z_2)\psi_\gamma w$$

$$= \left(1+\frac{z_0}{z_2}\right)^{\langle\alpha,\gamma\rangle}z_2^{\langle\alpha+\beta,\gamma\rangle}z_0^{\langle\alpha,\beta\rangle}E^-(\alpha+\beta,z_2)\psi_{-\alpha-\beta-\gamma}Y(E^-(\alpha,z_0)\Delta(\gamma,z_2)$$

$$\cdot Y(\Delta(\beta,z_0)\psi_\alpha u,z_0)\Delta(\alpha,-z_0)\psi_\beta v,z_2)\Delta(\alpha+\beta,-z_2)\psi_\gamma w$$

$$= (z_2+z_0)^{\langle\alpha,\gamma\rangle} z_2^{\langle\beta,\gamma\rangle} z_0^{\langle\alpha,\beta\rangle} E^-(\alpha+\beta,z_2)E^-(\alpha,z_0+z_2)E^-(-\alpha,z_2)\psi_{-\alpha-\beta-\gamma}$$
$$\cdot\ Y(\Delta(\gamma,z_2)Y(\Delta(\beta,z_0)\psi_\alpha u,z_0)\Delta(\alpha,-z_0)\psi_\beta v,z_2)$$
$$\cdot\ z_2^{-\alpha(0)}(z_2+z_0)^{\alpha(0)}E^+(\alpha,z_2)E^+(-\alpha,z_2+z_0)\Delta(\alpha+\beta,-z_2)\psi_\gamma w$$
$$= (z_2+z_0)^{\langle\alpha,\gamma\rangle} z_2^{\langle\beta,\gamma\rangle} z_0^{\langle\alpha,\beta\rangle} E^-(\beta,z_2)E^-(\alpha,z_0+z_2)\psi_{-\alpha-\beta-\gamma}$$
$$\cdot\ Y(\Delta(\gamma,z_2)Y(\Delta(\beta,z_0)\psi_\alpha u,z_0)\Delta(\alpha,-z_0)\psi_\beta v,z_2)$$
$$\cdot\ z_2^{-\langle\alpha,h_3\rangle}(z_2+z_0)^{\langle\alpha,h_3\rangle}E^+(-\beta,z_2)E^+(-\alpha,z_2+z_0)(-z_2)^{\langle\alpha+\beta,h_3\rangle}\psi_\gamma w$$
$$= (z_2+z_0)^{\langle\alpha,\gamma\rangle} z_2^{\langle\beta,\gamma\rangle} z_0^{\langle\alpha,\beta\rangle} E^-(\beta,z_2)E^-(\alpha,z_0+z_2)\psi_{-\alpha-\beta-\gamma}$$
$$\cdot\ Y(Y(\Delta(\gamma,z_2+z_0)\Delta(\beta,z_0)\psi_\alpha u,z_0)\Delta(\gamma,z_2)\Delta(\alpha,-z_0)\psi_\beta v,z_2)$$
$$\cdot\ z_2^{-\langle\alpha,h_3\rangle}(z_2+z_0)^{\langle\alpha,h_3\rangle}E^+(-\beta,z_2)E^+(-\alpha,z_2+z_0)(-z_2)^{\langle\alpha+\beta,h_3\rangle}\psi_\gamma w\,. \tag{3.32}$$

Hence

$$z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)Y(Y(u,z_0)v,z_2)w$$
$$= z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)(z_2+z_0)^{\langle\alpha,\gamma\rangle} z_2^{\langle\beta,\gamma\rangle} z_0^{\langle\alpha,\beta\rangle} E^-(\beta,z_2)E^-(\alpha,z_1)\psi_{-\alpha-\beta-\gamma}$$
$$\cdot\ Y(Y(\Delta(\gamma,z_2+z_0)\Delta(\beta,z_0)\psi_\alpha u,z_0)\Delta(\gamma,z_2)\Delta(\alpha,-z_0)\psi_\beta v,z_2)$$
$$\cdot\ z_2^{-\langle\alpha,h_3\rangle}(z_2+z_0)^{\langle\alpha,h_3\rangle}E^+(-\beta,z_2)E^+(-\alpha,z_1)(-z_2)^{\langle\alpha+\beta,h_3\rangle}\psi_\gamma w$$
$$= z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)(z_2+z_0)^{\langle\alpha,\gamma\rangle} z_2^{\langle\beta,\gamma\rangle} z_0^{\langle\alpha,\beta\rangle} E^-(\beta,z_2)E^-(\alpha,z_1)\psi_{-\alpha-\beta-\gamma}$$
$$\cdot\ \left(\frac{z_2+z_0}{z_1}\right)^{\langle\gamma,h_1\rangle} Y(Y(\Delta(\gamma,z_1)\Delta(\beta,z_0)\psi_\alpha u,z_0)\Delta(\gamma,z_2)\Delta(\alpha,-z_0)\psi_\beta v,z_2)$$
$$\cdot\ \left(\frac{z_2+z_0}{z_1}\right)^{\langle\alpha,h_3\rangle}\Delta(\alpha,-z_1)\Delta(\beta,-z_2)\psi_\gamma w$$
$$= z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)\left(\frac{z_2+z_0}{z_1}\right)^{-\eta((\alpha,h_1),(\gamma,h_3))} z_1^{\langle\alpha,\gamma\rangle} z_2^{\langle\beta,\gamma\rangle} z_0^{\langle\alpha,\beta\rangle} E^-(\beta,z_2)E^-(\alpha,z_1)$$
$$\cdot\ \psi_{-\alpha-\beta-\gamma}Y(Y(\Delta(\gamma,z_1)\Delta(\beta,z_0)\psi_\alpha u,z_0)\Delta(\gamma,z_2)\Delta(\alpha,-z_0)\psi_\beta v,z_2)$$
$$\cdot\ \Delta(\alpha,-z_1)\Delta(\beta,-z_2)\psi_\gamma w\,. \tag{3.33}$$

The generalized Jacobi identity (3.26) now follows immediately. □

Next we shall extend certain V-modules to modules for U considered as a generalized vertex algebra via Theorem 3.5. Let M be an irreducible V-module (with finite-dimensional homogeneous subspaces). Since $[L(0),h(0)]=0$ for any $h\in H$,

H preserves each homogeneous subspace of M so that there exist $0 \neq u \in M, \lambda \in H^*$ such that $h(0)u = \lambda(h)u$ for $h \in H$. Since H acts semisimply on V (by assumption) and u generates M by V (from the irreducibility of M), H also acts semisimply on M. For any $\lambda \in H^*$, we define

$$M^{(0,\lambda)} = \{u \in M | h(0)u = \lambda(h)u \text{ for } h \in H\}. \tag{3.34}$$

Set

$$P(M) = \{\lambda \in H^* | M^{(0,\lambda)} \neq 0\}. \tag{3.35}$$

Since M is irreducible, $P(M)$ is an irreducible $P(V)$-set. Thus $L \times P(M)$ is an irreducible $(L \times P(V))$-set. *Suppose that for any* $\alpha \in L, \alpha(0)$ *has rational eigenvalues on* M. Let $\lambda_0 \in P(M)$ be any H-weight of M. Then $P(M) = \lambda_0 + P(V)$. By using a basis of L, we see that there is a positive integer K such that $\langle \lambda_0, \alpha \rangle \in \frac{1}{K}\mathbb{Z}$ for any $\alpha \in L$. Therefore

$$\langle \lambda, \alpha \rangle \in \frac{1}{TK}\mathbb{Z} \quad \text{for any } \lambda \in P(M), \alpha \in L. \tag{3.36}$$

Using formula (3.19) we extend the definition of $\eta(\cdot,\cdot)$ to $(L \times P(M)) \times (L \times P(M))$ with values in $\frac{1}{TK}\mathbb{Z}$.

Recall that $(M^{(\alpha)}, Y_\alpha(\cdot, z))$ is a weak σ_α-twisted V-module for any $\alpha \in L$. Set $W = \bigoplus_{\alpha \in L} M^{(\alpha)}$. For $a \in V^{(\alpha)}, u \in M^{(\beta)}, \alpha, \beta \in L$, we define $Y_W(a,z)u \in M^{(\alpha+\beta)}\{z\}$ as follows:

$$Y_W(a,z)u = \psi_{-\alpha-\beta} E^-(\alpha,z) Y_M(\psi_\alpha \Delta(\beta,z)a, z)\Delta(\alpha,-z)\psi_\beta(u). \tag{3.37}$$

Then the same argument used in the proof of Theorem 3.5 shows that for any $a \in V^{(\alpha,h_1)}, b \in V^{(\beta,h_2)}, u \in M^{(\gamma,h_3)}$, where $\alpha, \beta, \gamma \in L, h_1, h_2 \in P, h_3 \in P(M)$, we have:

$$\begin{aligned} &z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)\left(\frac{z_1 - z_2}{z_0}\right)^{\eta((\alpha,h_1),(\beta,h_2))} Y_W(a,z_1)Y_W(b,z_2)u \\ &\quad - C((\alpha,h_1),(\beta,h_2))z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right)\left(\frac{z_2 - z_1}{z_0}\right)^{\eta((\alpha,h_1),(\beta,h_2))} Y_W(b,z_2)Y_W(a,z_1)u \\ &= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)\left(\frac{z_2 + z_0}{z_1}\right)^{\eta((\alpha,h_1),(\gamma,h_3))} Y_W(Y(a,z_0)b,z_2)u. \end{aligned} \tag{3.38}$$

Then we have:

Theorem 3.6. (W, Y_W) *is a module for the generalized vertex algebra* U *in the sense of* [DL].

In view of Proposition 2.15, it is possible that various $V^{(\alpha)}$ in U may be V-isomorphic to each other. Next we shall reduce U to a smaller space $\bar{U}$ such that the multiplicity of any σ_α-twisted V-module $V^{(\alpha)}(\alpha \in L)$ is one, and $\bar{U}$ is an abelian intertwining algebra in the sense of [DL] rather than a generalized vertex algebra.

Set

$$L_0 = \{\alpha \in L | \sigma_\alpha = \mathrm{id}_V,\ V^{(\alpha)} \simeq V\}. \tag{3.39}$$

Lemma 3.7. *Let $\alpha, \beta \in L$. Then $\sigma_\alpha = \sigma_\beta$ and $V^{(\alpha)}$ is isomorphic to $V^{(\beta)}$ as a weak σ_α-twisted V-module if and only if $\alpha - \beta \in L_0$.*

Proof. Suppose that $\sigma_\alpha = \sigma_\beta$ and $V^{(\alpha)}$ is isomorphic to $V^{(\beta)}$ as a weak σ_α-twisted V-module. Let ϕ be a V-isomorphism from $V^{(\alpha)}$ onto $V^{(\beta)}$. Then $\psi_\beta\phi\psi_{-\beta}$ is a linear isomorphism from $V^{(\alpha-\beta)}$ onto V such that

$$\psi_\beta\phi\psi_{-\beta}(Y(a,z)u) = Y(\Delta(\beta,z)\Delta(-\beta,z)a,z)\psi_\beta\phi\psi_{-\beta}(u) = Y(a,z)\psi_\beta\phi\psi_{-\beta}(u) \quad (3.40)$$

for any $a \in V, u \in V^{(\alpha-\beta)}$. Then by definition, $\alpha - \beta \in L_0$.

On the other hand, suppose that $\alpha - \beta \in L_0$ for some $\alpha, \beta \in L$. Since $\sigma_{\alpha-\beta} = \mathrm{id}_V$, we have $\sigma_\alpha = \sigma_\beta$. Let ψ be a V-isomorphism from $V^{(\alpha-\beta)}$ onto V. Then $\psi_{-\beta}\psi\,\psi_\beta$ is a linear isomorphism from $V^{(\alpha)}$ onto $V^{(\beta)}$ satisfying the condition:

$$\begin{aligned}\psi_\beta^{-1}\psi\,\psi_\beta(Y_\alpha(a,z)u) &= \psi_{-\beta}\psi Y(\Delta(\beta,z)a,z)\psi_\beta u = \psi_\beta Y(\Delta(\beta,z)a,z)\psi\,\psi_\beta u\\ &= Y_\beta(\Delta(-\beta,z)\Delta(\beta,z)a,z)\psi_{-\beta}\psi\,\psi_\beta u = Y_\beta(a,z)\psi_{-\beta}\psi\,\psi_\beta u \quad (3.41)\end{aligned}$$

for any $a \in V$. Then $V^{(\alpha)}$ is V-isomorphic to $V^{(\beta)}$. The proof is complete. □

For any $\alpha, \beta \in L_0$, by definition we have $V^{(\alpha)} \simeq V \simeq V^{(\beta)}$. From Lemma 3.7, $\alpha - \beta \in L_0$. Thus L_0 is a sublattice of L.

Lemma 3.8. *L_0 is an even sublattice such that $L_0 \subseteq P \cap P^0$, where P^0 is the dual lattice of P.*

Proof. Let $\alpha \in L_0$. Then $e^{2\pi i\alpha(0)} = \mathrm{id}_V$ if and only if $\alpha(0)$ has integral eigenvalues on V. This proves $\alpha \in P^0$. From (3.13), we obtain:

$$\psi_\alpha\beta(0) = (\beta(0) + \langle\alpha,\beta\rangle)\psi_\alpha, \quad (3.42)$$

$$\psi_\alpha L(0) = \left(L(0) + \alpha(0) + \frac{1}{2}\langle\alpha,\alpha\rangle\right)\psi_\alpha\,. \quad (3.43)$$

Then (3.42) implies that $P(V^{(\alpha)}) = -\alpha + P$. Since $V^{(\alpha)}$ is V-isomorphic to $V, P = P(V^{(\alpha)})$. Thus $\alpha \in P$. Let $u \in V^{(0,\lambda)}$ for $\lambda \in P$. Then from (3.43) the $L(0)$-weight of $\psi_{-\alpha}(u) \in V^{(\alpha)}$ is wt $u + \langle\alpha,\lambda\rangle + \frac{1}{2}\langle\alpha,\alpha\rangle$. In particular, the $L(0)$-weight of $\psi_{-\alpha}(\mathbf{1}) \in V^{(\alpha)}$ is $\frac{1}{2}\langle\alpha,\alpha\rangle$. Thus $\langle\alpha,\alpha\rangle \in 2\mathbb{Z}$. The proof is complete. □

Let $\alpha \in L_0$ and let $\bar{\pi}_\alpha$ be a fixed V-isomorphism from $V^{(\alpha)}$ onto V. If $\alpha = 0$, we choose $\bar{\pi}_0 = \mathrm{id}_V$. For any $\beta \in L$, considering the following composition map:

$$V^{(\beta)} \to V^{(\alpha)} \to V \to V^{(\beta-\alpha)}, \quad (3.44)$$

we obtain a linear isomorphism $f = \psi_{\alpha-\beta}\bar{\pi}_\alpha\psi_{\beta-\alpha}$ from $V^{(\beta)}$ onto $V^{(\beta-\alpha)}$. Then

$$f(Y(a,z)u) = Y(\Delta(\alpha-\beta,z)\Delta(\beta-\alpha,z)a,z)f(u) = Y(a,z)f(u) \quad (3.45)$$

for any $a \in V, u \in V^{(\beta)}$. Then we define an endomorphism f_α of U as follows:

$$f_\alpha u = e^{\langle\alpha,\beta-\alpha\rangle\pi i}\psi_{\alpha-\beta}\bar{\pi}_\alpha\psi_{\beta-\alpha}(u) \quad \text{for } u \in V^{(\beta)} \subseteq U\,. \quad (3.46)$$

Since $\psi_0 = \mathrm{id}_V$, we have $f_\alpha|_{V(\alpha)} = \bar{\pi}_\alpha$. Then we may use $\bar{\pi}_\alpha$ to denote the linear isomorphism f_α of U without any confusion. It is easy to see that $\bar{\pi}_\alpha$ satisfies the following condition:

$$\bar{\pi}_\alpha(Y(a,z)u) = Y(a,z)\bar{\pi}_\alpha(u) \quad \text{for } a \in V, u \in U\,. \tag{3.47}$$

Lemma 3.9. *For any $\alpha \in L_0, \beta \in L$, we have*

$$\psi_\beta \bar{\pi}_\alpha = e^{\langle \alpha, \beta \rangle \pi i} \bar{\pi}_\alpha \psi_\beta\,. \tag{3.48}$$

Proof. Let $\gamma \in L, u \in V^{(\gamma)}$. Then by (3.46) we have:

$$\begin{aligned}\psi_\beta \bar{\pi}_\alpha(u) &= \psi_\beta e^{\langle \alpha, \gamma-\alpha \rangle \pi i} \psi_{\alpha-\gamma} \bar{\pi}_\alpha \psi_{\gamma-\alpha}(u) \\ &= e^{\langle \alpha, \gamma-\alpha \rangle \pi i} \psi_{\alpha+\beta-\gamma} \bar{\pi}_\alpha \psi_{\gamma-\alpha}(u)\,. \end{aligned} \tag{3.49}$$

On the other hand, we have:

$$\bar{\pi}_\alpha \psi_\beta(u) = e^{\langle \alpha, \gamma-\beta-\alpha \rangle \pi i} \psi_{\alpha+\beta-\gamma} \bar{\pi}_\alpha \psi_{\gamma-\beta-\alpha} \psi_\beta(u) = e^{\langle \alpha, \gamma-\beta-\alpha \rangle \pi i} \psi_{\alpha+\beta-\gamma} \bar{\pi}_\alpha \psi_{\gamma-\alpha}(u)\,. \tag{3.50}$$

The result follows. □

Lemma 3.10. *For any $\alpha \in L_0$ we have*

$$\bar{\pi}_\alpha(Y(u,z)v) = Y(u,z)\bar{\pi}_\alpha(v) \quad \textit{for any } u, v \in U\,. \tag{3.51}$$

Proof. Without losing generality we assume that $u \in V^{(\beta)}, v \in V^{(\gamma)}$, where $\beta, \gamma \in L$. Then by Definition 3.3, (3.14) and Lemma 3.9 we obtain

$$\begin{aligned}\bar{\pi}_\alpha(Y(u,z)v) &= \bar{\pi}_\alpha \psi_{-\beta-\gamma} E^-(\beta,z) Y(\psi_\beta \Delta(\gamma,z)(u),z)\Delta(\beta,-z)\psi_\gamma(v) \\ &= e^{\langle \alpha, \beta \rangle \pi i} \psi_{-\beta-\gamma} E^-(\beta,z) Y(\psi_\beta \Delta(\gamma,z)(u),z)\Delta(\beta,-z)\psi_\gamma \bar{\pi}_\alpha(v) \\ &= e^{\langle \alpha, \beta \rangle \pi i} \psi_{\alpha-\beta-\gamma} \psi_{-\alpha} E^-(\beta,z) Y(\psi_\beta \Delta(\gamma,z)(u),z)\Delta(\beta,-z)\psi_\gamma \bar{\pi}_\alpha(v) \\ &= e^{\langle \alpha, \beta \rangle \pi i} \psi_{\alpha-\beta-\gamma} E^-(\beta,z) \\ &\quad \cdot Y(\Delta(-\alpha,z)\psi_\beta \Delta(\gamma,z)(u),z)\psi_{-\alpha}\Delta(\beta,-z)\psi_\gamma \bar{\pi}_\alpha(v) \\ &= e^{\langle \alpha, \beta \rangle \pi i} \psi_{\alpha-\beta-\gamma} E^-(\beta,z) z^{\langle \alpha, \beta \rangle} (-z)^{-\langle \alpha, \beta \rangle} \\ &\quad \cdot Y(\psi_\beta \Delta(\gamma-\alpha,z)(u),z)\Delta(\beta,-z)\psi_{\gamma-\alpha}\bar{\pi}_\alpha(v) \\ &= \psi_{\alpha-\beta-\gamma} E^-(\beta,z) Y(\psi_\beta \Delta(\gamma-\alpha,z)(u),z)\Delta(\beta,-z)\psi_{\gamma-\alpha}\bar{\pi}_\alpha(v) \\ &= Y(u,z)\bar{\pi}_\alpha(v)\,. \end{aligned} \tag{3.52}$$

This proves the lemma. □

Lemma 3.11. *For any* $u \in U, v \in V^{(\gamma)}, \alpha \in L_0, \gamma \in L$, *we have*

$$Y(\bar{\pi}_\alpha(u), z)v = e^{-\langle\alpha,\gamma\rangle\pi i}\bar{\pi}_\alpha Y(u,z)v\,. \tag{3.53}$$

Proof. By linearity we may assume that $u \in V^{(\beta)}$ for some $\beta \in L$. Using skew-symmetry, (3.15), Definition 3.3 and Lemma 3.9 we obtain

$$\begin{aligned}
&Y(\bar{\pi}_\alpha(u),z)v \\
&= \psi_{\alpha-\beta-\gamma}E^-(\beta-\alpha,z)Y(\psi_{\beta-\alpha}\Delta(\gamma,z)\bar{\pi}_\alpha(u),z)\Delta(\beta-\alpha,-z)\psi_\gamma(v) \\
&= \psi_{\alpha-\beta-\gamma}E^-(\beta-\alpha,z)e^{zL(-1)}Y(\Delta(\beta-\alpha,-z)\psi_\gamma(v),-z)\psi_{\beta-\alpha}\Delta(\gamma,z)\bar{\pi}_\alpha(u) \\
&= \psi_{\alpha-\beta-\gamma}E^-(\beta-\alpha,z)e^{zL(-1)}\psi_{-\alpha}Y(\Delta(\beta,-z)\psi_\gamma(v),-z)\psi_\beta\Delta(\gamma,z)\bar{\pi}_\alpha(u) \\
&= e^{\langle\alpha,\beta\rangle\pi i}\psi_{\alpha-\beta-\gamma}E^-(\beta-\alpha,z)e^{zL(-1)}\psi_{-\alpha}\bar{\pi}_\alpha Y(\Delta(\beta,-z)\psi_\gamma(v),-z)\psi_\beta\Delta(\gamma,z)(u) \\
&= e^{\langle\alpha,\beta\rangle\pi i}\psi_{\alpha-\beta-\gamma}E^-(\beta-\alpha,z)e^{zL(-1)}\psi_{-\alpha}\bar{\pi}_\alpha \\
&\quad\cdot e^{-zL(-1)}Y(\psi_\beta\Delta(\gamma,z)(u),z)\Delta(\beta,-z)\psi_\gamma(v) \\
&= e^{\langle\alpha,\beta\rangle\pi i}\psi_{\alpha-\beta-\gamma}E^-(\beta-\alpha,z)e^{zL(-1)}\psi_{-\alpha}e^{-zL(-1)}\bar{\pi}_\alpha \\
&\quad\cdot Y(\psi_\beta\Delta(\gamma,z)(u),z)\Delta(\beta,-z)\psi_\gamma(v) \\
&= e^{\langle\alpha,\beta\rangle\pi i}\psi_{\alpha-\beta-\gamma}E^-(\beta-\alpha,z)E^-(\alpha,z)\psi_{-\alpha}\bar{\pi}_\alpha Y(\psi_\beta\Delta(\gamma,z)(u),z)\Delta(\beta,-z)\psi_\gamma(v) \\
&= e^{\langle\alpha,\beta\rangle\pi i}\psi_{\alpha-\beta-\gamma}E^-(\beta,z)\psi_{-\alpha}\bar{\pi}_\alpha Y(\psi_\beta\Delta(\gamma,z)(u),z)\Delta(\beta,-z)\psi_\gamma(v) \\
&= e^{\langle\alpha,\beta\rangle\pi i}\psi_{-\beta-\gamma}E^-(\beta,z)\bar{\pi}_\alpha Y(\psi_\beta\Delta(\gamma,z)(u),z)\Delta(\beta,-z)\psi_\gamma(v) \\
&= e^{\langle\alpha,\beta\rangle\pi i}e^{-\langle\alpha,\beta+\gamma\rangle\pi i}\bar{\pi}_\alpha\psi_{-\beta-\gamma}E^-(\beta,z)Y(\psi_\beta\Delta(\gamma,z)(u),z)\Delta(\beta,-z)\psi_\gamma(v) \\
&= e^{-\langle\alpha,\gamma\rangle\pi i}\bar{\pi}_\alpha Y(u,z)v\,. \qquad \square
\end{aligned} \tag{3.54}$$

Remark 3.12. Let I be the sum of all subspaces $(\bar{\pi}_\alpha - 1)U$ of U for $\alpha \in L_0$. Define $\bar{U}$ to be the quotient space U/I. Then the multiplicity of the σ_β-twisted V-module $V^{(\beta)}$ in $\bar{U}$ is exactly one for $\beta \in L$. It follows from Lemma 3.10 that I is a left ideal of the generalized vertex algebra U. But from Lemma 3.11, I is not necessarily a right ideal, i.e., for $u \in I, v \in U, Y(u,z)v$ may not be in $I\{z\}$ unless $\langle\alpha,\beta\rangle \in 2\mathbb{Z}$ for any $\alpha \in L_0, \beta \in L$ (Lemma 3.11). In general, $\bar{U}$ is a U-module, but it is not a quotient generalized vertex algebra of U.

Before we modify the definition of vertex operator $Y(\,\cdot\,,z)$ (3.23) to get an abelian intertwining algebra we consider a special case. Let L_1 be an integral sublattice of L such that $L_0 \subseteq L_1 \subseteq L$ and

$$\langle\lambda,\beta\rangle \in \mathbb{Z}, \langle\alpha,\beta\rangle \in 2\mathbb{Z} \quad \text{for any } \lambda \in P, \alpha \in L_0, \beta \in L_1\,. \tag{3.55}$$

Set $U_1 = \bigoplus_{\beta\in L_1} V^{(\beta)}$ and $\bar{U}_1 = U_1/I$. Then it follows from Theorem 3.5, Lemmas 3.10 and 3.11 that $\bar{U}_1$ is a generalized vertex algebra. By Lemma 3.8, for any $\alpha_j \in L_1, \lambda_j \in P$ for $j = 1,2$ we have:

$$\eta((\alpha_1,\lambda_1),(\alpha_2,\lambda_2)) = -\langle\alpha_1,\alpha_2\rangle - \langle\alpha_1,\lambda_2\rangle - \langle\alpha_2,\lambda_1\rangle \in \mathbb{Z}/2\mathbb{Z}, \tag{3.56}$$

$$C((\alpha_1,\lambda_1),(\alpha_2,\lambda_2)) = (-1)^{\langle\alpha_1,\lambda_2\rangle+\langle\alpha_2,\lambda_1\rangle} . \tag{3.57}$$

Then

$$C((\alpha_1,\lambda_1),(\alpha_2,\lambda_2))z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)\left(\frac{z_2-z_1}{z_0}\right)^{\eta((\alpha_1,\lambda_1),(\alpha_2,\lambda_2))}$$

$$= (-1)^{\langle\alpha_1,\alpha_2\rangle}z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right) . \tag{3.58}$$

Therefore for any $u \in V^{(\alpha_1,\lambda_1)}, v \in V^{(\alpha_2,\lambda_2)}, w \in V^{(\alpha_3,\lambda_3)}, (\alpha_j,\alpha_j) \in L_1 \times P$, the generalized Jacobi identity (3.26) becomes the following super Jacobi identity:

$$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y(u,z_1)Y(v,z_2)w - (-1)^{\langle\alpha_1,\alpha_2\rangle}z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y(v,z_2)Y(u,z_1)w$$

$$= z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)Y(Y(u,z_0)v,z_2)w . \tag{3.59}$$

Then we have:

Corollary 3.13. *Let L_1 be an integral sublattice of L satisfying* (3.55). *Then U_1 is a vertex superalgebra with I as an ideal so that $\bar{U}_1$ is a quotient vertex superalgebra.*

Continuing with Corollary 3.13, let M be an irreducible V-module such that $\alpha(0)$ has rational eigenvalues on M for any $\alpha \in L_1$. Let γ be an H-weight of M. Then $P(M) = \gamma + P$. Set $W_1 = \bigoplus_{\alpha\in L_1} M^{(\alpha)}$. For any $(\alpha_1,\lambda_1) \in L_1 \times P, (\alpha_3,\lambda_3) \in L_1 \times P(M) = L_1 \times (\gamma + P)$, since $\langle\alpha_1,\alpha_3\rangle, \langle\alpha_3,\lambda_1\rangle \in \mathbb{Z}$ and $\langle\alpha_1,\lambda\rangle \in \mathbb{Z}$ for any $\lambda \in P$, we have:

$$\eta((\alpha_1,\lambda_1),(\alpha_3,\lambda_3)) = -\langle\alpha_1,\alpha_3\rangle - \langle\alpha_1,\lambda_3\rangle - \langle\alpha_3,\lambda_1\rangle$$

$$= -\langle\alpha_1,\gamma\rangle \in \frac{1}{T}\mathbb{Z}/\mathbb{Z} . \tag{3.60}$$

Then we have the following twisted Jacobi identity:

$$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y(u,z_1)Y(v,z_2)w$$

$$- (-1)^{\langle\alpha_1,\alpha_2\rangle}z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y(v,z_2)Y(u,z_1)w$$

$$= z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)\left(\frac{z_2+z_0}{z_1}\right)^{-\langle\gamma,\alpha_1\rangle}Y(Y(u,z_0)v,z_2)w . \tag{3.61}$$

Therefore, for any $u \in V^{(\alpha_1,\lambda_1)}, v \in V^{(\alpha_2,\lambda_2)}, w \in M^{(\alpha_3,\lambda_3)}, (\alpha_1,\lambda_1),(\alpha_2,\lambda_2) \in L_1 \times P$, $(\alpha_3,\lambda_3) \in L_1 \times P(M)$, we have the following super Jacobi identity:

$$\begin{aligned} &z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y(u,z_1)Y(v,z_2)w \\ &\quad - (-1)^{\langle\alpha_1,\alpha_2\rangle}z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y(v,z_2)Y(u,z_1)w \\ &= z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)\left(\frac{z_2+z_0}{z_1}\right)^{\eta((\alpha_1,\lambda_1),(\alpha_3,\lambda_3))} Y(Y(u,z_0)v,z_2)w\,. \end{aligned} \tag{3.62}$$

It is clear that $\sigma_\gamma = e^{-2\pi i\gamma(0)}$ is an automorphism of $\bar{U}_1$. Then W_1 is a σ_γ-twisted $\bar{U}_1$-module with $I(W_1)$ as a submodule, where $I(W_1)$ is defined as the subspace linearly spanned by $(\pi_\alpha - 1)W_1$ for $\alpha \in L_0$. Summarizing the previous arguments we have:

Corollary 3.14. *Let L_1 be an integral sublattice of L satisfying* (3.55). *Let M be an irreducible V-module such that $\alpha(0)$ has rational eigenvalues on M for any $\alpha \in L_1$. Then $W_1 = \bigoplus_{\beta\in L_1} M^{(\beta)}$ with $M^{(0)} = M$, defined as in Theorem* 3.6, *is a σ_γ-twisted $\bar{U}_1$-module with a submodule $I(W_1)$, so that $\bar{W}_1 = W_1/I(W_1)$ is a quotient σ_γ-twisted $\bar{U}_1$-module.*

Corollary 3.15. *Under the conditions of Corollary* 3.14, *assume that there is a τ-twisted $\bar{U}_1$-module E containing M as a V-submodule for some finite-order automorphism τ of $\bar{U}_1$. Then $\tau = \sigma_\gamma$.*

Proof. Since $\bar{U}_1$ is simple and $V^{(\alpha)}$ is a simple current for any $\alpha \in L_1, (M^{(\alpha)}, Y_{\bar{W}_1}(\,\cdot\,,z))$ is a tensor product of $V^{(\alpha)}$ with M. Let E^α be the subspace of E linearly spanned by u_nM for $u \in V^{(\alpha)}, n \in \mathbb{Q}$. Similarly, $(E^\alpha, Y_E(\,\cdot\,,z))$ is also a tensor product of $V^{(\alpha)}$ with M. Therefore, there is a $c_\alpha \in \mathbb{C}^*$ such that $Y_{\bar{W}_1}(u,z)w = c_\alpha Y_E(u,z)w$ for any $u \in V^{(\alpha)}, w \in M$. By definition of a twisted module, τ and σ_γ have the same order. Then $\tau|_{V^{(\alpha)}} = \sigma_\gamma|_{V^{(\alpha)}}$ for any $\alpha \in L_1$. Thus $\tau = \sigma_\gamma$. □

As mentioned in Remark 3.12, in general $\bar{U} = U/I$ is not a generalized vertex algebra. Next we shall modify Definition (3.23) and prove that $\bar{U} = U/I$ is an abelian intertwining algebra.

For any $\alpha, \beta \in L_0$, we have a V-isomorphism $\bar{\pi}_\alpha\bar{\pi}_\beta\bar{\pi}_{\alpha+\beta}^{-1}$ from V onto V. Since V is a simple vertex operator algebra, by Schur's lemma there is a nonzero complex number $A_0(\alpha,\beta)$ such that

$$\bar{\pi}_{\alpha+\beta} = A_0(\alpha,\beta)\bar{\pi}_\alpha\bar{\pi}_\beta\,, \tag{3.63}$$

where both sides are considered as V-isomorphisms $V^{(\alpha+\beta)}$ onto V. For any $\gamma \in L$ and for any $u \in V^{(\gamma)}$, we have:

$$\begin{aligned} \bar{\pi}_{\alpha+\beta}(u) &= \bar{\pi}_{\alpha+\beta}\psi_{\alpha+\beta-\gamma}\psi_{\gamma-\alpha-\beta}(u) \\ &= e^{-\langle\alpha+\beta,\alpha+\beta-\gamma\rangle\pi i}\psi_{\alpha+\beta-\gamma}\bar{\pi}_{\alpha+\beta}\psi_{\gamma-\alpha-\beta}(u) \\ &= e^{-\langle\alpha+\beta,\alpha+\beta-\gamma\rangle\pi i}A_0(\alpha,\beta)\psi_{\alpha+\beta-\gamma}\bar{\pi}_\alpha\bar{\pi}_\beta\psi_{\gamma-\alpha-\beta}(u) \\ &= A_0(\alpha,\beta)\bar{\pi}_\alpha\bar{\pi}_\beta(u)\,. \end{aligned} \tag{3.64}$$

Then (3.63) holds when both sides are considered as operators on U. It is easy to see that the following 2-cocycle condition hold:

$$A_0(\alpha_1+\alpha_2,\alpha_3)A_0(\alpha_1,\alpha_2)=A_0(\alpha_1,\alpha_2+\alpha_3)A_0(\alpha_2,\alpha_3) \tag{3.65}$$

for any $\alpha_i \in L_0, i=1,2,3$.

Next we define

$$C_0(\alpha,\beta)=A_0(\alpha,\beta)A_0(\beta,\alpha)^{-1} \quad \text{for } \alpha,\beta \in L_0\,. \tag{3.66}$$

Then $C_0(\cdot\,,\,\cdot)$ satisfies the properties (3.22) and

$$\bar{\pi}_\beta\bar{\pi}_\alpha = C_0(\alpha,\beta)\bar{\pi}_\alpha\bar{\pi}_\beta \quad \text{for } \alpha,\beta \in L_0\,. \tag{3.67}$$

Since L_0 is a sublattice of L, there is a basis $\{\beta_1,\beta_2,\dots,\beta_n\}$ for L and a basis $\{\alpha_1,\alpha_2,\dots,\alpha_n\}$ for L_0 such that each α_i is an integral multiple of β_i. It is easy to find a $\mathbb{Z}$-bilinear function $A_1(\cdot\,,\,\cdot)$ on L with values in $\mathbb{C}^*$ satisfying the following condition:

$$A_0(\alpha_i,\alpha_j)=A_1(\alpha_i,\alpha_j)^2 \quad \text{for any } 1 \leqq i,j \leqq n\,. \tag{3.68}$$

Fixing such an $A_1(\cdot\,,\,\cdot)$, we define $C_i(\cdot\,,\,\cdot)$ on $L\times L$ as follows:

$$C_1(\alpha,\beta)=A_1(\alpha,\beta)A_1(\beta,\alpha)^{-1} \quad \text{for any } \alpha,\beta \in L\,. \tag{3.69}$$

Then $C_1(\cdot\,,\,\cdot)$ satisfies the following conditions:

$$C_1(\beta,\beta)=1, C_1(\beta_1,\beta_2)C_1(\beta_2,\beta_1)=1, C_1(\beta_1+\beta_2,\beta_3)=C_1(\beta_1,\beta_3)C_1(\beta_2,\beta_3) \tag{3.70}$$

for any $\beta,\beta_1,\beta_2,\beta_3 \in L$.

Next, for any $\alpha \in L_0$, we define a linear automorphism π_α on U as follows:

$$\pi_\alpha(u)=C_1(\beta,\alpha)\bar{\pi}_\alpha(u) \quad \text{for any } u \in V^{(\beta)} \subseteq U\,. \tag{3.71}$$

Then for any $\alpha_1,\alpha_2 \in L_0, \beta \in L$, we have:

$$\begin{aligned}\pi_{\alpha_1}\pi_{\alpha_2}(u) &= C_1(\beta-\alpha_2,\alpha_1)C_1(\beta,\alpha_2)\bar{\pi}_{\alpha_1}\bar{\pi}_{\alpha_2}(u)\\ &= C_1(\beta-\alpha_2,\alpha_1)C_1(\beta,\alpha_2)C_0(\alpha_2,\alpha_1)\bar{\pi}_{\alpha_2}\bar{\pi}_{\alpha_1}(u)\\ &= C_1(\beta-\alpha_2,\alpha_1)C_1(\beta,\alpha_2)C_0(\alpha_2,\alpha_1)C_1(\alpha_2,\beta-\alpha_1)C_1(\alpha_1,\beta)\pi_{\alpha_2}\pi_{\alpha_1}(u)\\ &= \pi_{\alpha_2}\pi_{\alpha_1}(u)\end{aligned} \tag{3.72}$$

for any $u \in V^{(\beta)}$. Thus

$$\pi_{\alpha_1}\pi_{\alpha_2}=\pi_{\alpha_2}\pi_{\alpha_1} \quad \text{for any } \alpha_1,\alpha_2 \in L_0\,. \tag{3.73}$$

Remark 3.16. If $L=\mathbb{Z}\alpha$ is of rank one, then $L_0=kL$ for some positive integer k. We can fix $\bar{\pi}_{k\alpha}$ first, then we define $\bar{\pi}_{nk\alpha}=\bar{\pi}_{k\alpha}^n$ for any $n \in \mathbb{Z}$. Then $C_0(\cdot\,,\,\cdot)\equiv 1$. So we can take $C_1(\cdot\,,\,\cdot)\equiv 1$.

Lemma 3.17. *For any $\alpha \in L_0, \beta \in L$, we have*

$$\psi_\beta\pi_\alpha = e^{\langle\alpha,\beta\rangle\pi i}C_1(\beta,\alpha)\pi_\alpha\psi_\beta\,. \tag{3.74}$$

Proof. Let $\gamma \in L, u \in V^{(\gamma)}$. Then by Definition (3.71) of π_α we have:

$$\begin{aligned}\psi_\beta \pi_\alpha(u) &= C_1(\gamma,\alpha)\psi_\beta \bar{\pi}_\alpha(u) = C_1(\gamma,\alpha)e^{\langle\alpha,\beta\rangle\pi i}\bar{\pi}_\alpha\psi_\beta(u) \\ &= C_1(\gamma,\alpha)C_1(\alpha,\gamma-\beta)e^{\langle\alpha,\beta\rangle\pi i}\pi_\alpha\psi_\beta(u) \\ &= e^{\langle\alpha,\beta\rangle\pi i}C_1(\beta,\alpha)\pi_\alpha\psi_\beta(u). \qquad \square \end{aligned} \tag{3.75}$$

Lemma 3.18. *For any $\alpha \in L_0$ we have*

$$\pi_\alpha(Y(u,z)v) = C_1(\beta,\alpha)Y(u,z)\pi_\alpha(v) \quad \text{for any } u \in V^{(\beta)} \subseteq U\,. \tag{3.76}$$

Proof. Let $u \in V^{(\beta)}, v \in V^{(\gamma)}$ with $\beta,\gamma \in L$. Then by Definition (3.71) we have:

$$\begin{aligned}\pi_\alpha(Y(u,z)v) &= C_1(\beta+\gamma,\alpha)\bar{\pi}_\alpha(Y(u,z)v) = C_1(\beta+\gamma,\alpha)Y(u,z)\bar{\pi}_\alpha v \\ &= C_1(\beta+\gamma,\alpha)C_1(\gamma,\alpha)^{-1}Y(u,z)\pi_\alpha v = C_1(\beta,\alpha)Y(u,z)\pi_\alpha(v)\,. \end{aligned} \tag{3.77}$$

This proves the assertion. $\square$

Lemma 3.19. *For any $u \in V^{(\beta)}, v \in V^{(\gamma)}, \alpha \in L_0, \gamma \in L$, we have*

$$Y(\pi_\alpha(u),z)v = e^{-\langle\alpha,\gamma\rangle\pi i}C(\alpha,\gamma)\pi_\alpha Y(u,z)v\,. \tag{3.78}$$

Proof. We may assume that $u \in V^{(\beta)}$ for some $\beta \in L$. Then by definition we have:

$$\begin{aligned}Y(\pi_\alpha(u),z)v &= C_1(\beta,\alpha)Y(\bar{\pi}_\alpha(u),z)v = e^{-\langle\alpha,\gamma\rangle\pi i}C_1(\beta,\alpha)\bar{\pi}_\alpha Y(u,z)v \\ &= e^{-\langle\alpha,\gamma\rangle\pi i}C_1(\beta,\alpha)C_1(\beta+\gamma,\alpha)^{-1}\pi_\alpha Y(u,z)v \\ &= e^{-\langle\alpha,\gamma\rangle\pi i}C_1(\alpha,\gamma)\pi_\alpha Y(u,z)v\,. \end{aligned} \tag{3.79}$$

Then the proof is complete. $\square$

Remark 3.20. Let $\alpha_1,\ldots,\alpha_n$ be a basis of L_0. Then we can define a $\mathbb{Z}$-linear map π' from L_0 to Aut (U) as follows:

$$\pi'_\alpha = \pi_{\alpha_1}^{k_1}\pi_{\alpha_2}^{k_2}\cdots\pi_{\alpha_n}^{k_n} \tag{3.80}$$

for any $\alpha = k_1\alpha_1 + k_2\alpha_2 + \cdots + k_n\alpha_n \in L_0$. Since all π_{α_i}'s commute each other, π' is well-defined. It is easy to see that

$$\pi'_{\alpha+\beta} = \pi'_\alpha\pi'_\beta = \pi'_\beta\pi'_\alpha \quad \text{for any } \alpha,\beta \in L_0\,. \tag{3.81}$$

It is also easy to see that Lemmas 3.17, 3.18 and 3.19 still hold. By slightly abusing the notion, from now on we will use π' for π.

Let $\bar{A} = L \times P/D$, where $D = \{(\alpha,-\alpha)|\alpha \in L_0\}$ is a subgroup of $A = L \times P$. Let $\{\lambda_i | i \in \bar{A}\}$ be a (complete) set of representatives in A. Then we define $\bar{V} = \bigoplus_{i\in\bar{A}} V^{(\lambda_i)}$. For any $u \in V^{(\lambda_i)}, v \in V^{(\lambda_j)}$, we define $\bar{Y}(u,z)v \in V^{(\lambda_{i+j})}\{z\}$ as follows:

$$\bar{Y}(u,z)v = C_1(\lambda_j,\lambda_i)\pi_{\lambda_i+\lambda_j-\lambda_{i+j}}Y(u,z)v\,. \tag{3.82}$$

Because $L(-1)$ commutes with π_α for $\alpha \in L_0$ and $L(-1)V^{(\beta)} \subseteq V^{(\beta)}$ for $\beta \in L$, the $L(-1)$-derivative property (Proposition 3.4) still holds.

We define a function $h(i,j,k)$ from $\bar{A} \times \bar{A} \times \bar{A}$ to $\mathbb{C}^*$ as follows:

$$h(i,j,k) = e^{-\langle \lambda_i+\lambda_j-\lambda_{i+j},\lambda_k\rangle \pi i} C_1(\lambda_i + \lambda_j - \lambda_{i+j}, \lambda_k)^2 \quad \text{for } i,j,k \in \bar{A}\,. \tag{3.83}$$

Next we shall prove that $h(i,j,k)$ is a 3-cocycle. Observe that $h(i,j,k)$ is symmetric in the first two variables with the third variable being fixed. Following [DL], we prove that $h(i,j,k)$ is a 3-cocycle by proving that with the third variable k being fixed, $h(i,j,k)$ is a 2-cocycle, i.e.,

$$h(i,j,k)h(i,j+r,k)^{-1}h(i+j,r,k)h(j,r,k)^{-1} = 1 \quad \text{for } i,j,r \in \bar{A}\,. \tag{3.84}$$

For $i,j,r,k \in \bar{A}$ we have:

$$e^{-\langle \lambda_i+\lambda_j-\lambda_{i+j},\lambda_k\rangle \pi i} e^{\langle \lambda_i+\lambda_{j+r}-\lambda_{i+j+r},\lambda_k\rangle \pi i} = e^{\langle \lambda_{i+j}+\lambda_r-\lambda_{i+j+r},\lambda_k\rangle \pi i} e^{\langle \lambda_j+\lambda_r-\lambda_{j+r},\lambda_k\rangle \pi i} \tag{3.85}$$

and

$$\begin{aligned} &C_0(\lambda_i + \lambda_j - \lambda_{i+j}, \lambda_k)C_0(\lambda_i + \lambda_{j+r} - \lambda_{i+j+r}, \lambda_k)^{-1} \\ &= C_0(\lambda_{i+j} + \lambda_r - \lambda_{i+j+r}, \lambda_k)C_0(\lambda_j + \lambda_r - \lambda_{j+r}, \lambda_k)\,. \end{aligned} \tag{3.86}$$

Combining (3.85) with (3.86) we obtain (3.84). Then from Proposition 12.13 of [DL], $h(i,j,k)$ is a 3-cocycle. Next we define

$$\bar{C}((\lambda_1,h_1),(\lambda_2,h_2)) = C((\lambda_1,h_1),(\lambda_2,h_2))C_1(\lambda_1,\lambda_2) \tag{3.87}$$

for any $(\lambda_i \times P) \in L \times P$ (recall the definition of $C(\cdot\,,\,\cdot)$ from (3.20)). Then $\bar{C}$ satisfies (3.22).

Theorem 3.21. *For any $u \in V^{(\lambda_i,h_1)}$, $v \in V^{(\lambda_j,h_2)}$, $w \in V^{(\lambda_k,h_3)}$, the following generalized Jacobi identity holds:*

$$\begin{aligned} &z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)\left(\frac{z_1-z_2}{z_0}\right)^{\eta((\lambda_i,h_1),(\lambda_j,h_2))}\bar{Y}(u,z_1)\bar{Y}(v,z_2)w \\ &- \bar{C}((\lambda_i,h_1),(\lambda_j,h_2))z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)\left(\frac{z_2-z_1}{z_0}\right)^{\eta((\lambda_i,h_1),(\lambda_j,h_2))} \\ &\cdot \bar{Y}(v,z_2)\bar{Y}(u,z_1)w \\ &= z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)\left(\frac{z_2+z_0}{z_1}\right)^{\eta((\lambda_i,h_1),(\lambda_k,h_3))} h(i,j,k)\bar{Y}(\bar{Y}(u,z_0)v,z_2)w\,. \end{aligned} \tag{3.88}$$

Therefore $(\bar{U},\mathbf{1},\omega,\bar{Y},T,\bar{A},\eta,\bar{C})$ is an abelian intertwining algebra.

Proof. Since $Y(v,z)w \in V^{(\lambda_i+\lambda_j,h_2+h_3)}\{z\}$, using (3.82) and Lemma 3.18 we obtain

$$\begin{aligned}
&\bar{Y}(u,z_1)\bar{Y}(v,z_2)w \\
&= C_1(\lambda_{j+k},\lambda_i)\pi_{\lambda_i+\lambda_{j+k}-\lambda_{i+j+k}}Y(u,z_1)\bar{Y}(v,z_2)w\,, \\
&= C_1(\lambda_{j+k},\lambda_i)C_1(\lambda_k,\lambda_j)\pi_{\lambda_i+\lambda_{j+k}-\lambda_{i+j+k}}Y(u,z_1)\pi_{\lambda_j+\lambda_k-\lambda_{j+k}}Y(v,z_2)w \\
&= C_1(\lambda_{j+k},\lambda_i)C_1(\lambda_k,\lambda_j)C_1(\lambda_j+\lambda_k-\lambda_{j+k},\lambda_i)\pi_{\lambda_i+\lambda_{j+k}-\lambda_{i+j+k}}\pi_{\lambda_i+\lambda_{j+k}-\lambda_{i+j+k}} \\
&\quad\cdot \pi_{\lambda_j+\lambda_k-\lambda_{j+k}}Y(u,z_1)Y(v,z_2)w \\
&= C_1(\lambda_k,\lambda_j)C_1(\lambda_j+\lambda_k,\lambda_i)\pi_{\lambda_i+\lambda_j+\lambda_k-\lambda_{i+j+k}}Y(u,z_1)Y(v,z_2)w\,. \qquad (3.89)
\end{aligned}$$

Symmetrically, we have:

$$\begin{aligned}
&\bar{Y}(v,z_2)\bar{Y}(u,z_1)w \\
&\quad= C_1(\lambda_k,\lambda_i)C_1(\lambda_i+\lambda_k,\lambda_j)\pi_{\lambda_i+\lambda_j+\lambda_k-\lambda_{i+j+k}}Y(v,z_2)Y(u,z_1)w\,. \qquad (3.90)
\end{aligned}$$

On the other hand, using (3.82) and Lemma 3.19 we get:

$$\begin{aligned}
&\bar{Y}(\bar{Y}(u,z_0)v,z_2)w \\
&= C_1(\lambda_k,\lambda_{j+k})\pi_{\lambda_{i+j}+\lambda_k-\lambda_{i+j+k}}Y(\bar{Y}(u,z_0)v,z_2)w \\
&= C_1(\lambda_k,\lambda_{j+k})C_1(\lambda_j,\lambda_i)\pi_{\lambda_{i+j}+\lambda_k-\lambda_{i+j+k}}Y(\pi_{\lambda_i+\lambda_j-\lambda_{i+j}}Y(u,z_0)v,z_2)w \\
&= C_1(\lambda_k,\lambda_{j+k})C_1(\lambda_j,\lambda_i)C_1(\lambda_i+\lambda_j-\lambda_{i+j},\lambda_k)e^{-\langle\lambda_i+\lambda_j-\lambda_{i+j},\lambda_k\rangle\pi i} \\
&\quad\cdot \pi_{\lambda_{i+j}+\lambda_k-\lambda_{i+j+k}}\pi_{\lambda_i+\lambda_j-\lambda_{i+j}}Y(Y(u,z_0)v,z_2)w \\
&= C_1(\lambda_k,\lambda_{j+k})C_1(\lambda_j,\lambda_i)C_1(\lambda_i+\lambda_j-\lambda_{i+j},\lambda_k)e^{-\langle\lambda_i+\lambda_j-\lambda_{i+j},\lambda_k\rangle\pi i} \\
&\quad\cdot \pi_{\lambda_i+\lambda_j+\lambda_k-\lambda_{i+j+k}}Y(Y(u,z_0)v,z_2)w\,. \qquad (3.91)
\end{aligned}$$

Multiplying the generalized Jacobi identity (3.26) by $C_1(\lambda_k,\lambda_j)C_1(\lambda_j+\lambda_k,\lambda_i)$, applying $\pi_{\lambda_i+\lambda_j+\lambda_k-\lambda_{i+j+k}}$, then using (3.89)–(3.91) we obtain

$$\begin{aligned}
&z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)\left(\frac{z_1-z_2}{z_0}\right)^{\eta((\alpha,h_1),(\beta,h_2))}\bar{Y}(u,z_1)\bar{Y}(v,z_2)w \\
&- C((\lambda_i,h_1),(\lambda_j,h_2))C_1(\lambda_i,\lambda_j)^{-2}z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)\left(\frac{z_2-z_1}{z_0}\right)^{\eta((\lambda_i,h_1),(\lambda_j,h_2))} \\
&\quad\cdot \bar{Y}(v,z_2)\bar{Y}(u,z_1)w \\
&= e^{-\langle\lambda_i+\lambda_j-\lambda_{i+j},\lambda_k\rangle\pi i}C_1(\lambda_i+\lambda_j-\lambda_{i+j},\lambda_k)^2 \\
&\quad\cdot z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)\left(\frac{z_2+z_0}{z_1}\right)^{\eta((\lambda_i,h_1),(\lambda_k,h_3))}\bar{Y}(\bar{Y}(u,z_0)v,z_2)w\,. \qquad (3.92)
\end{aligned}$$

This gives the generalized Jacobi identity (3.88). The proof is complete. □

4. Rationality for Certain Extensions of Vertex Operator Algebras

In this section we study the rationality of certain extensions of rational vertex operator algebras. Such an extension $V = \bigoplus_{g\in G} V^g$ graded by a finite abelian group G can be characterized by the properties listed below. We first obtain the complete reducibility of a G-graded module $M = \bigoplus_{g\in G} M^g$ with M^0 being an irreducible V^0-module. Then we study the complete reducibility of a canonical class of V-modules. We apply our results to some special cases.

From the last section (Corollaries 3.13, 3.14 and 3.15) under certain conditions we obtain vertex operator (super)algebras satisfy the following conditions:

(1) $V = \bigoplus_{g\in G} V^g$, where G is a finite abelian group.

(2) V^0 is a simple rational vertex operator subalgebra and each V^g is a simple current V^0-module.

(3) $a_n V^h \subseteq V^{g+h}$ for any $a \in V^g, n \in \mathbb{Z}, g, h \in G$.

(4) For any subgroup H of G, $V^{(H)} =: \bigoplus_{h\in H} V^{(h)}$ is a simple vertex (operator) algebra and $V^{(g+H)} =: \bigoplus_{h\in H} V^{(h+g)}$ for any $g \in G$ is a simple current $V^{(H)}$-module.

(5) Let M^0 be any irreducible V^0-module which is a V^0-submodule of some V-module W. Then there is a V-module $M = \bigoplus_{g\in G} M^g$ satisfying the condition: $a_n M^h \subseteq M^{g+h}$ for any $a \in V^g$, $n \in \mathbb{Z}, g, h \in G$.

Proposition 4.1. *Let $M = \bigoplus_{g\in G} M^g$ be a V-module satisfying the following conditions*: (a) *M^0 is an irreducible V^0-module.* (b) *$a_n M^h \subseteq M^{g+h}$ for any $a \in V^g$, $n \in \mathbb{Z}, g, h \in G$. Then M is a direct sum of irreducible V-modules.*

First we prove the following two special cases:

Lemma 4.2. *Let $M = \bigoplus_{g\in G} M^g$ be a V-module satisfying the conditions* (a) *and* (b) *of Proposition* 4.1. *Suppose M^g and M^h are not isomorphic V^0-modules for $g \neq h$. Then M is an irreducible V-module.*

Proof. Let M_1 be any nonzero V-submodule of M. We must show that $M = M_1$. Since each M^g generates M by the action of V, it suffices to prove that M_1 contains some M^g. Because M is a direct sum of irreducible V^0-modules, M_1 is also a direct sum of irreducible V^0-modules. For any $g \in G$, let P_g be the projection of M onto M^g. Then P_g is a V^0-homomorphism. Let W be any irreducible V^0-submodule of M_1. Then the restriction of P_g to W is either zero or a V^0-isomorphism onto M^g. Since M^g and M^h are not V^0-isomorphic for $g \neq h$, there is $g \in G$ such that $P_g(W) = M^g$ and $P_h(W) = 0$ for $h \neq g$. Therefore $W = M^g$ for some $g \in G$. Thus M_1 contains some M^g, as required. □

Lemma 4.3. *Let $M = \bigoplus_{g\in G} M^g$ be a V-module satisfying conditions* (a) *and* (b) *of Proposition* 4.1. *Suppose that G is a cyclic group and that all M^g $(g \in G)$ are isomorphic irreducible V^0-modules. Then M is a direct sum of $|G|$ irreducible V-modules, each of which is isomorphic to M^0 as a V^0-module.*

Proof. For any $g \in G$, let f_g be a fixed V^0-isomorphism from M^0 onto M^g. If $g = 0$, we choose $f_0 = \mathrm{Id}_{M^0}$. We shall extend f_g to be a V-automorphism of M. Let $h \in G$. Then $(M^h, Y(\cdot, z))$ is a tensor product for (V^h, M^0) by Corollary 2.9. Since $Y(\cdot, Z) f_g$ is an intertwining operator of type $\binom{M^{g+h}}{V^h, M^0}$, there is a (unique)

V^0-homomorphism $f_{g,h}$ from M^h to M^{g+h} satisfying

$$Y(a,z)f_g(u) = f_{g,h}Y(a,z)u \quad \text{for any } a \in V^h, u \in M^0 \tag{4.1}$$

(see Definition 2.3 for the tensor product of modules). If $h = 0$, we have $f_{g,0} = f_g$. Then we extend f_g to be a linear endomorphism of M by defining:

$$f_g(u) = f_{g,h}(u) \quad \text{for any } h \in G, u \in M^h \subseteq M\ . \tag{4.2}$$

Thus

$$Y(a,z)f_g(u) = f_gY(a,z)u \quad \text{for any } a \in V, u \in M^0\ . \tag{4.3}$$

Let $a, b \in V$, $g \in G$, $u \in M^0$ and let k be a positive integer such that the following associativities hold:

$$(z_0 + z_2)^k Y(a, z_0 + z_2)Y(b, z_2)u = (z_0 + z_2)^k Y(Y(a, z_0), b, z_2)u\ , \tag{4.4}$$

$$(z_0 + z_2)^k Y(a, z_0 + z_2)Y(b, z_2)f_gu = (z_0 + z_2)^k Y(Y(a, z_0), b, z_2)f_gu\ . \tag{4.5}$$

Then by (4.3)–(4.5) we have:

$$\begin{aligned}
&(z_0 + z_2)^k f_gY(a, z_0 + z_2)Y(b, z_2)u \\
&= (z_0 + z_2)^k f_gY(Y(a, z_0)b, z_2)u \\
&= (z_0 + z_2)^k Y(Y(a, z_0)b, z_2)f_g(u) \\
&= (z_0 + z_2)^k Y(a, z_0 + z_2)Y(b, z_2)f_g(u) \\
&= (z_0 + z_2)^k Y(a, z_0 + z_2)f_g(Y(b, z_2)u)\ .
\end{aligned} \tag{4.6}$$

Multiplying by $(z_0 + z_2)^{-k}$ we obtain

$$f_gY(a, z_0 + z_2)Y(b, z_2)u = Y(a, z_0 + z_2)f_g(Y(b, z_2)u)\ . \tag{4.7}$$

Thus

$$f_gY(a, z_1)Y(b, z_2)u = Y(a, z_1)f_g(Y(b, z_2)u)\ . \tag{4.8}$$

Since $V \cdot M^0 = M$, we get

$$f_gY(a, z_1)v = Y(a, z_1)f_gv \quad \text{for any } a \in V,\ v \in M\ . \tag{4.9}$$

That is, f_g is a V-endomorphism of M.

For any $g, h \in G$, both f_{g+h} and f_gf_h are V^0-homomorphisms from M^0 to M^{g+h}. Since M^0 is an irreducible V^0-module, f_{g+h} is a constant multiple of f_gf_h. That is, there is $A(g,h) \in \mathbb{C}^*$ such that $f_{g+h} = A(g,h)f_gf_h$ from M^0 to M^{g+h}. Since each f_g is a V-endomorphism of M and M^0 generates M by $V, f_{g+h} = A(g,h)f_gf_h$ holds on M. It is clear that $A(g,h)$ is a 2-cocycle.

Since G is cyclic, let $G = \langle g \rangle$ with $o(g) = k$. Since f_g^k is a V^0-endomorphism of M^0 and M^0 is an irreducible V^0-module, there is a complex number α such that $f_g^k(u) = \alpha u$ for any $u \in M^0$. Then we modify f_g by multiplying a kth root of α, we have: $f_g^k(u) = u$ for any $u \in M^0$. Since f_g commutes with all vertex operators $Y(a,z)$ for $a \in V$ and M^0 generates M by V, we have $f_g^k = \mathrm{Id}_M$. Using f_g we obtain a representation of G on M. For any nonzero $u \in M^{(0)}$, $\mathbb{C}u \oplus \mathbb{C}f_g u \oplus \cdots \oplus \mathbb{C}f_g^{k-1}u$ is isomorphic to the regular representation of G. For any character $\chi \in \hat{G}$, let $M(\chi)$ be the χ-homogeneous subspace of M. Then $M = \bigoplus_{\chi\in\hat{G}} M(\chi)$ and $M(\chi) \neq 0$ for any $x \in \hat{G}$. Since G commutes with all vertex operators $Y(a,z)$ for $a \in V$, each $M(\chi)$ is a V-module. Since $M(\chi) \neq 0$ for any $\chi \in \hat{G}$ and M is a direct sum of $|G|$ irreducible V^0-modules, then each $M(\chi)$ must be an irreducible V-module, so that it is also an irreducible V^0-module. □

Proof of Proposition 4.1. We are going to prove Proposition 4.1 by using induction on $|G|$. If $|G| = 1$, there is nothing to prove. Suppose that Proposition 4.1 is true for any finite abelian group with less than n elements. Suppose that $|G| = n$ with $n > 1$. Let H be a subgroup of G of prime order. Set $V^{(H)} = \bigoplus_{h\in H} V^{(h)}$ and $V^{(g+H)} = \bigoplus_{h\in H} V^{(g+h)}$ for $g \in G$. Then $V^{(H)}$ is a simple vertex operator algebra and each $V^{(g+H)}$ is a simple current for $V^{(H)}$. Similarly set $M^{(H)} = \bigoplus_{h\in H} M^{(h)}$. Let H_0 be the subset of H consisting of h such that $M^{(h)}$ is isomorphic to $M^{(0)}$. Then it is clear that H_0 is a subgroup of H. Consequently, either $H_0 = H$ or $H_0 = 0$. By Lemmas 4.2 and 4.3, $M^{(H)}$ is a direct sum of irreducible $V^{(H)}$-modules. Let $M^{(H)} = W_1 \oplus \cdots \oplus W_m$, where W_j are irreducible $V^{(H)}$-modules. Let M^j be the V-module generated by W_j. Denote the span of $\{a_n w | a \in V^g,\ n \in \mathbb{Z},\ w \in W_j\}$ by $V^g W_j$ for $g \in G$. Then from the proof of Lemmas 4.2 and 4.3,

$$M_j = \sum_{g\in G} V^g W_j = \bigoplus_{k\in G/H} V^k W_j$$

and $M = \sum_{j=1} M^j$. The $V^{(H)}, G/H, M_j$ satisfy the assumptions of Proposition 4.1. By the inductive assumption, each M_j is a direct sum of irreducible V-modules, hence so too is M. □

Theorem 4.4. *Suppose that V^0 is rational and that for any irreducible V^0-module W^0, if there is a V-module W such that W^0 is a V^0-submodule of W, then W^0 can be lifted to be a V-module $M = \bigoplus_{g\in G} M^g$ with $M^0 = W^0$. Then V is rational.*

Proof. Let W be any V-module. Then W is a completely reducible V^0-module. Thus it suffices to prove that any irreducible V^0-submodule W^0 of W generates a completely reducible V-submodule of W. By assumption, there is a V-module $M = \bigoplus_{g\in G} M^g$ such that

$$M^0 = W^0,\ a_n M^h \subseteq M^{g+h} \quad \text{for any } a \in V^g,\ n \in \mathbb{Z}, g, h \in G\,. \tag{4.10}$$

From Proposition 4.1, M is a completely reducible V-module. So it sufficient to prove that the V-submodule $\langle W^0\rangle$ of W generated by W^0 is a V-homomorphism image of M. It is easy to see that $(M, Y(\cdot, z))$ is a tensor product of V^0-modules V with $M^0 = W^0$. Therefore there is a V^0-homomorphism f from M to W such that

$$Y_W(a,z)u = fY_M(a,z)u \quad \text{for any } a \in V,\ u \in M^0 = W^0\,. \tag{4.11}$$

Using the same argument used in the proof of Lemma 4.3 we obtain:

$$Y_W(a,z)f(u) = fY_M(a,z)u \quad \text{for any } a \in V,\ u \in M\,. \tag{4.12}$$

Then f is a V-homomorphism from M to W. Then $\langle W^0\rangle$ is a completely reducible V-module. Therefore M is a completely reducible V-module. □

Theorem 4.5. *Suppose that V^0 is rational and that for any $g \in G$, there is an $h_g \in V_1$ satisfying condition* (2.18) *and such that V^g is V^0-isomorphic to $(V^0, Y(\Delta(h_g,z)\cdot,z))$. Then V is rational.*

Proof. This follows from Theorem 4.4 and Corollaries 3.14–15 immediately. □

5. Applications to Affine Lie Algebras

This section is devoted to the study of affine Kac–Moody algebras and their representations. It is well known that $L(l\Lambda_0)$ is a vertex operator algebra (see the definition below) and any weak module which is truncated below is a direct sum of standard modules of level l (cf. [DL and FZ]). In this section we improve this result by showing that under a mild assumption, any weak module is a direct sum of standard modules of level l. Then we discuss the simple currents for vertex operator algebras $L(l\Lambda_0)$ and various extensions of $L(l\Lambda_0)$ as applications of results obtained in previous sections.

Let $\mathbf{g}$ be a finite-dimensional simple Lie algebra with a fixed Cartan subalgebra H and let $\{e_i, f_i, \alpha_i^\vee | i = 1,\dots,n\}$ be the Chevalley generators. Let $(\cdot,\cdot)$ be the normalized Killing form on $\mathbf{g}$ such that the square norm of the longest root is 2. Let $\mathbb{Q} = \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_n$, where $\alpha_1,\dots,\alpha_n$ are all simple roots. Notice that $\alpha_1^\vee,\dots,\alpha_n^\vee$ form a basis for H. Let $h_i \in H$ such that $\alpha_i(h_j) = \delta_{i,j}$ for $i,j = 1,\dots,n$. Let $\theta = \sum_{i=1}^n a_i\alpha_i$ be the highest positive root. Let $\lambda_i (i = 1,\dots,n)$ be the fundamental weights for $\mathbf{g}$ (cf. [H]). We shall denote the fundamental weights of its dual algebra by $\lambda_i^\vee$. A dominant integral weight λ is called a *minimal* weight (cf. [H]) if there is no dominant integral weight γ satisfying $\lambda - \gamma \in \mathbb{Q}_+$. Weight λ_i is called *cominimal* if $\lambda_i^\vee$ is a minimal weight (see [FG]). Then λ_i is minimal if and only if $a_i = 1$, and all minimal dominant integral weights are given as follows (cf. [H]):

$$\begin{aligned}
&A_n : \lambda_1,\dots,\lambda_n\\
&B_n : \lambda_n\\
&C_n : \lambda_1\\
&D_n : \lambda_1, \lambda_{n-1}, \lambda_n\\
&E_6 : \lambda_1, \lambda_6\\
&E_7 : \lambda_7 .
\end{aligned} \tag{5.1}$$

Let $\tilde{\mathbf{g}}$ be the affine Lie algebra [K] with Chevalley generators $\{e_i, f_i, \alpha_i^\vee | i = 0,\dots,n\}$. Then each λ_i for $1 \leqq i \leqq n$ is naturally extended to a fundamental weight Λ_i for $\tilde{\mathbf{g}}$. Let Λ_0 be the fundamental weight for $\tilde{\mathbf{g}}$ defined by $\Lambda_0(\alpha_i^\vee) = \delta_{i,0}$ for $0 \leqq i \leqq n$ (cf. [K]). Then Λ_i is of level one if and only if $a_i^\vee = 1$ (see [K] for the definition of $a_i^\vee$). Let $\lambda \in H^*$ and let ℓ be any complex number. Then we denote by $L(\ell,\lambda)$ the highest weight $\tilde{\mathbf{g}}$-module of level ℓ with lowest weight λ. It is well known (cf. [DL, FZ, Li1]) that $L(\ell,0)$ is a vertex operator algebra. One can identify $\mathbf{g}$ as a subspace of $L(\ell,0)$ through the linear map ϕ: $u \mapsto u_{-1}\mathbf{1}$. Using this formulation, it was proved in [Li4] that if λ_i is cominimal, then for any $\ell, L(\ell\Lambda_i)$ (or $L(\ell,\lambda_i)$) is an irreducible (weak) $L(\ell,0)$-module and it is a simple current. The following proposition is proved in [Li4]. Since the information obtained is useful (see Remarks 5.2 and 5.3), we repeat the short proof here.

Proposition 5.1. *Suppose that λ_i is cominimal. Let ℓ be any complex number which is not equal to $-\Omega$, where Ω is the dual Coxeter number of* **g**. *Then $L(\ell\Lambda_i)$ is isomorphic to $(L(\ell,0),Y(\Delta(h_i,z)\cdot,z))$ as a $\tilde{\mathbf{g}}$-module. Consequently, $L(\ell\Lambda_i)$ is a simple current.*

Proof. Note that $\theta(h_i)=a_i=1$. By definition we have:

$$\Delta(h_i,z)\alpha_j^\vee=\alpha_j^\vee+\ell\delta_{i,j}z^{-1},\quad \Delta(h_i,z)e_i=ze_i,\quad \Delta(h_i,z)f_i=z^{-1}f_i\,, \tag{5.2}$$

$$\Delta(h_i,z)e_j=e_j,\quad \Delta(h_i,z)f_j=f_j,\quad \Delta(h_i,z)f_\theta=z^{-1}f_\theta\quad \text{for } j\neq i\,. \tag{5.3}$$

In other words, the corresponding automorphism ψ of $U(\tilde{\mathbf{g}})$ or $U(L(\ell,0))$ satisfies the following conditions:

$$\psi(\alpha_i^\vee(n))=\alpha_i^\vee(n)+\delta_{n,0}\ell,\quad \psi(e_i(n))=e_i(n+1),\quad \psi(f_i(n))=f_i(n-1)\,; \tag{5.4}$$

$$\psi(\alpha_j^\vee(n))=\alpha_j^\vee(n),\quad \psi(e_j(n))=e_j(n),\psi(f_j(n))=f_j(n)\quad \text{for } j\neq i, n\in\mathbb{Z}\,, \tag{5.5}$$

and

$$\psi(f_\theta(n))=f_\theta(n-1)\quad \text{for } n\in\mathbb{Z}\,. \tag{5.6}$$

Then the vacuum vector **1** in $(L(\ell,0),Y(\Delta(h_i,z)\cdot,z))$ is a highest weight vector of weight $\ell\Lambda_i$. Thus $(V,Y(\Delta(h_i,z)\cdot,z))$ is isomorphic to $L(\ell\Lambda_i)$ as a $\tilde{\mathbf{g}}$-module. By Proposition 2.12, $L(\ell\Lambda_i)$ is a simple current. □

Remark 5.2. It has been shown [FG] by calculating the four point functions that if λ_i is a cominimal weight and ℓ is a positive integer, then $L(\ell\Lambda_i)$ is a simple current. Moreover if **g** is of type E_8, $L(\Lambda_7)$ is a simple current of level 2 which is not isomorphic to $L(2,0)$. It has also been proved [F] that these are all simple currents. In string theory, simple currents are useful for constructing modular invariants.

Remark 5.3. From the proof of Proposition 5.1 we see that the vacuum **1** becomes a lowest weight vector of $L(\ell\Lambda_i)$. The lowest weight of $L(\ell\Lambda_i)$ is $\frac{\ell}{2}\langle h_i,h_i\rangle$ because $L(0)$ acts on $L(\ell\Lambda_i)$ $(=L(\ell\Lambda_0))$ as $L(0)+h_i(0)+\frac{\ell}{2}\langle h_i,h_i\rangle$. In general, if $h\in H$ satisfies $\alpha_i(h)\in\mathbb{Z}$ for $i=1,\dots,n$, it follows from the proof of Proposition 5.1 that the vacuum vector is a lowest weight vector for $\tilde{\mathbf{g}}$ if and only if either $h=0$ or h corresponds to a cominimal weight. Then for some $h\in H$, $(L(\ell,0),Y(\Delta(h,z)\cdot,z))$ might not be a highest weight $\tilde{\mathbf{g}}$-module.

Next we shall prove that if ℓ is a positive integer, then for any $h\in H$ satisfying $\alpha_i(h)\in\mathbb{Z}$ for $i=1,\dots,n$ and for any $L(\ell,0)$-module $(M,Y_M(\cdot,z))$, $(M,Y_M(\Delta(h,z)\cdot,z))$ is an ordinary $L(\ell,0)$-module. *From now on, we assume that* **g** *is a fixed finite-dimensional simple Lie algebra and ℓ is a fixed positive integer.*

The following lemma easily follows from Proposition 13.16 in [DL] (see [Li1 or MP] for a proof).

Lemma 5.4. *Let M be any weak $L(\ell,0)$-module and let $e\in\mathbf{g}_\alpha$, where α is any root of* **g**. *Then $Y_M(e,z)^{\ell+1}=0$ if α is a long root and $Y_M(e,z)^{3\ell+1}=0$ for any α.*

Lemma 5.5. *Let M be any nonzero weak $L(\ell,0)$-module on which $t\mathbb{C}[t]\otimes H$ acts locally nilpotently. Then M contains a standard $\tilde{\mathbf{g}}$-module of level ℓ.*

Proof. Set $\tilde{\mathbf{g}}_+ = t\mathbb{C}[t] \otimes \mathbf{g}$. We define $M^0 = \{u \in M \mid \tilde{\mathbf{g}}_+ u = 0\}$. Since $[\mathbf{g}, \tilde{\mathbf{g}}_+] \subseteq \tilde{\mathbf{g}}_+$, $\mathbf{g}M^0 \subseteq M^0$. Let $0 \neq e \in \mathbf{g}_\theta$. Applying $Y_M(e,z)^{\ell+1}$ to M^0 and extracting the coefficient of $z^{-\ell-1}$, we obtain $e(0)^{\ell+1}M^0 = 0$. By Proposition 5.1.2 in [Li1] (see also [KW]), M^0 is a direct sum of finite-dimensional irreducible $\mathbf{g}$-modules. If $M^0 \neq 0$, let u be a highest weight vector for $\mathbf{g}$ in M^0. Then u is a highest weight vector for $\tilde{\mathbf{g}}$. Extracting the constant from $Y_M(e,z)^{\ell+1}u = 0$ we obtain $e(-1)^{\ell+1}u = 0$. Then u generates a standard $\tilde{\mathbf{g}}$-module. So it suffices to prove that M^0 is nonzero. For any $u \in M$, it follows from the definition of a weak $L(\ell,0)$-module that $\tilde{\mathbf{g}}_+ u$ is finite-dimensional. Let u be a nonzero vector of M such that $\tilde{\mathbf{g}}_+ u$ has minimal dimension. If the minimal dimension is zero, then we are done. Suppose the minimal dimension is not zero. Let k be an integer such that $\mathbf{g}(n)u = 0$ for $n > k$ and that there is a nonzero $a \in \mathbf{g}$ such that $a(k)u \neq 0$. Without loss of generality we may assume that $a \in \mathbf{g}_\alpha$ for some root α or $a \in H$. By assumption, $k > 0$. If $a \in \mathbf{g}_\alpha$ for some root α, then $Y_M(a,z)^{3\ell+1}u = 0$ implies that $a(k)^{3\ell+1}u = 0$. If $a \in H$, by assumption $a(k)$ locally nilpotently acts on u. Therefore, there is a nonnegative integer r such that $a(k)^r u \neq 0$ and $a(k)^{r+1}u = 0$. Let $v = a(k)^r u$. If $b(n)u = 0$ for some $b \in \mathbf{g}, n > 1$, then $b(n)v = 0$. Then $\dim \tilde{\mathbf{g}}_+ v \leqq \dim \tilde{\mathbf{g}}_+ u - 1$, contradiction. The proof is complete. □

Proposition 5.6. *Let M be a weak $L(\ell\Lambda_0)$-module on which $t\mathbb{C}[t] \otimes H$ acts locally nilpotently. Then M is a direct sum of standard $\tilde{\mathbf{g}}$-modules of level ℓ.*

Proof. Let M_1 be a direct sum of standard submodules of M. We have to prove that $M = M_1$. Otherwise, the quotient module $\bar{M} = M/M_1$ is not zero. By Lemma 5.5, there is a standard submodule of $\bar{M}$, say W/M_1, where W is a submodule of M containing M_1. It follows from Theorem 10.7 in [K] that W is a direct sum of standard modules, so that $W = M_1$, contradiction. □

Corollary 5.7. *Let $h \in H$ such that $\alpha(h) \in \mathbb{Z}$ for any root α of $\mathbf{g}$ and let $(M, Y_M(\cdot,z))$ be any $L(\ell,0)$-module. Then $(M, Y(\Delta(h,z)\cdot,z))$ is also an $L(\ell,0)$-module.*

Proof. Since any $L(\ell,0)$-module M is a direct sum of finitely many standard $\tilde{\mathbf{g}}$-modules and a direct sum of finitely many $L(\ell,0)$-modules is a module, it suffices to prove the corollary for an irreducible module M. Since $\Delta(h,z)$ is invertible, it is clear that $(M, Y(\Delta(h,z)\cdot,z))$ is still irreducible as a $\tilde{\mathbf{g}}$-module. It follows from the proof of Proposition 2.15 that $(M, Y(\Delta(h,z)\cdot,z))$ is still a direct sum of highest weight modules for the Heisenberg Lie algebra $\hat{H}$. By Proposition 5.6, $(M, Y(\Delta(h,z)\cdot,z))$ is a direct sum of standard $\tilde{\mathbf{g}}$-modules of level ℓ. Consequently, M is a standard $\tilde{\mathbf{g}}$-module of level ℓ. The proof is complete. □

Remark 5.8. If we just consider the action of the affine Lie algebra $\tilde{\mathbf{g}}$, it is easy to see that the transition from $L(\ell,0)$ to $L(\ell,\lambda_i)$ is due to a Dynkin diagram automorphism of $\tilde{\mathbf{g}}$. The automorphism group of the Dynkin diagram of the affine Lie algebra is commonly called the outer automorphism group. To a certain extent we have realized them explicitly as "inner automorphisms" in terms of exponentials of certain elements of $\tilde{\mathbf{g}}$.

Remark 5.9. It is very special for E_8 that there is a simple current other than the vacuum representation when $\ell = 2$, but there is no outer automorphism. Let $h \in H$

be the element uniquely determined by $\alpha_i(h) = \delta_{i,7}$ for $1 \leqq i \leqq 7$. By Corollary 5.7, $(L(2,0), Y(\Delta(h,z)\cdot, z))$ is still a standard module and it is a simple current. It is interesting to ask what this module is. If this module is $L(2,\lambda_7)$, then all simple currents can be constructed in terms of $\Delta(h,z)$.

Let L be the $\mathbb{Z}$-span of all cominimal weights of $\mathbf{g}$. Then it follows from Theorem 3.21 that for any positive integral level ℓ, the direct sum of all simple currents of $L(\ell,0)$ is an abelian intertwining algebra.

Theorem 5.10. *Let* $\mathbf{g}$ *be a finite-dimensional simple Lie algebra, but not of type* E_8 *with a fixed Cartan subalgebra* H *and let* ℓ *be any positive integer. Then the direct sum of all (simple currents)* $L(\ell,0)$*-modules* $L(\ell,\lambda_i)$ *for all cominimal weights* λ_i *is an abelian intertwining algebra.*

Example 5.11. Let $\mathbf{g}$ be the type A_n. From [H] we have:

$$\lambda_i = \frac{1}{n+1}((n-i+1)\alpha_1 + 2(n-i+1)\alpha_2 + \cdots + (i-1)(n-i+1)\alpha_{i-1})$$
$$+ \frac{1}{n+1}(i(n-i+1)\alpha_i + i(n-i)\alpha_{i+1} + \cdots + i\alpha_n)\,. \tag{5.7}$$

Then

$$h_i = \frac{1}{n+1}((n-i+1)\alpha_1^\vee + 2(n-i+1)\alpha_2^\vee + \cdots + (i-1)(n-i+1)\alpha_{i-1}^\vee)$$
$$+ \frac{1}{n+1}(i(n-i+1)\alpha_i^\vee + i(n-i)\alpha_{i+1}^\vee + \cdots + i\alpha_n^\vee)\,.$$

By a simple calculation we get $(h_i,h_i) = \frac{i(n+1-i)}{n+1}$ for $1 \leqq i \leqq n$. let $L = L_1 = \mathbb{Z}h_1$. Notice that (from (3.7))

$$\langle h,h'\rangle \mathbf{1} = h(1)h' = h(1)h'(-1)\mathbf{1} = \ell(h,h') \tag{5.8}$$

for $h,h' \in H$. Then $\langle h_i,h_i\rangle = \frac{\ell i(n+1-i)}{n+1}$. By definition, P is the root lattice with the bilinear form $\ell(\cdot,\cdot)$. If $\ell \in 2(n+1)\mathbb{Z}_+$, L_1 satisfies condition (3.56). Suppose that $L_0 = kL_1$ for some positive integer k. By Corollaries 3.13 and 2.9, $L(l,0) \oplus \bigoplus_{i=0}^{k-1} L(\ell,\lambda_1)^{\boxtimes i}$ is a vertex operator algebra if $\ell \in 2(n+1)\mathbb{Z}_+$. In fact, $L_0 = (n+1)L_1$. By Theorem 4.5, it is rational. For other types, there are similar arguments, so we just briefly state the result.

Example 5.12. If $\mathbf{g}$ is of type, B_n, then (cf. [H])

$$\lambda_n = \frac{1}{2}(\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n)\,. \tag{5.9}$$

Thus $h_n = \frac{1}{2}(\alpha_1^\vee + 2\alpha_1^\vee + \cdots + n\alpha_n^\vee)$. Let $L = L_1 = \mathbb{Z}h_n$. Since $L(\ell,\lambda_n)$ is isomorphic to its contragredient module, we have a nonzero intertwining operator of type $\begin{pmatrix} L(\ell,0) \\ L(\ell,\lambda_n)L(\ell,\lambda_n)\end{pmatrix}$. Then $2h_n \in L_0$. Thus $L_0 = 2L_1$. Since $(h_n,h_n) = 1$, we have $\langle\lambda_n,\lambda_n\rangle = \ell$. Then by Corollary 3.13 and Theorem 4.5, $L(\ell,0) \oplus L(\ell,\lambda_n)$ is a rational vertex operator algebra for ℓ is even and $L(\ell,0) \oplus L(\ell,\lambda_n)$ is a rational vertex operator superalgebra if ℓ is odd. If $\ell = 1$, the lowest weight of $L(1,\lambda_n)$ is $\frac{1}{2}$. It follows from the super Jacobi identity that $L(1,\lambda_n)_{\frac{1}{2}}$ generates a Clifford algebra. This result explains why one can realize $\tilde{B}_n$ [LP] in terms of the representations of a Clifford algebra.

Example 5.13. If $\mathbf{g}$ is of type C_n, then

$$\lambda_1 = \alpha_1 + \cdots + \alpha_{n-1} + \frac{1}{2}\alpha_n \,. \tag{5.10}$$

Thus $h_1 = \alpha_1^\vee + \cdots + \alpha_{n-1}^\vee + \frac{1}{2}\alpha_n^\vee$. Since $(h_1, h_1) = 1$, we have: $\langle h_1, h_1\rangle = \ell$. Thus $L(\ell,0)\oplus L(\ell,\lambda_n)$ if ℓ is even is a rational vertex operator algebra and $L(\ell,0)\oplus L(\ell,\lambda_n)$ is a rational vertex operator superalgebra if ℓ is odd.

Example 5.14. If $\mathbf{g}$ is of type D_n, then

$$\begin{aligned}
\lambda_1 &= \alpha_1 + \cdots + \alpha_{n-2} + \frac{1}{2}(\alpha_{n-1} + \alpha_n)\,,\\
\lambda_{n-1} &= \frac{1}{2}\left(\alpha_1 + 2\alpha_2 + \cdots + (n-2)\alpha_{n-2} + \frac{n}{2}\alpha_{n-1} + \frac{n-2}{2}\alpha_n\right)\,,\\
\lambda_n &= \frac{1}{2}\left(\alpha_1 + 2\alpha_2 + \cdots + (n-2)\alpha_{n-2} + \frac{n-2}{2}\alpha_{n-1} + \frac{n}{2}\alpha_n\right)\,.
\end{aligned} \tag{5.11}$$

Then

$$\begin{aligned}
h_1 &= \alpha_1^\vee + \cdots + \alpha_{n-2}^\vee + \frac{1}{2}(\alpha_{n-1}^\vee + \alpha_n^\vee)\,,\\
h_{n-1} &= \frac{1}{2}\left(\alpha_1^\vee + 2\alpha_2^\vee + \cdots + (n-2)\alpha_{n-2}^\vee + \frac{n}{2}\alpha_{n-1}^\vee + \frac{n-2}{2}\alpha_n^\vee\right)\,,\\
h_n &= \frac{1}{2}\left(\alpha_1^\vee + 2\alpha_2^\vee + \cdots + (n-2)\alpha_{n-2}^\vee + \frac{n-2}{2}\alpha_{n-1}^\vee + \frac{n}{2}\alpha_n^\vee\right)\,.
\end{aligned} \tag{5.12}$$

By a simple calculation we obtain:

$$(h_1, h_1) = 1, \qquad (h_{n-1}, h_{n-1}) = (h_n, h_n) = \frac{n}{4}\,. \tag{5.13}$$

Then

$$\langle h_1, h_1\rangle = \ell, \qquad \langle h_{n-1}, h_{n-1}\rangle = \langle h_n, h_n\rangle = \frac{n\ell}{4}\,. \tag{5.14}$$

Let $L = L_1 = \mathbb{Z}h_1 \oplus \mathbb{Z}h_{n-1} \oplus \mathbb{Z}h_n$. It is not difficult to see that $L_0 = 2L_1$. Thus $L(\ell,0)\oplus L(\ell,\lambda_1)\oplus L(\ell,\lambda_{n-1})\oplus L(\ell,\lambda_n)$ is an abelian intertwining algebra with $\mathbb{Z}_2\times\mathbb{Z}_2$ as its grading group. (This has been proved in [DL].) Furthermore, $L(\ell,0)\oplus L(\ell,\lambda_1)$ is a rational vertex operator superalgebra. Another special case is when $n\ell$ is divisible by 8 [DM]: $L(\ell,0)\oplus L(\ell,\lambda_{n-1})$ and $L(\ell,0)\oplus L(\ell,\lambda_n)$ are (holomorphic) vertex operator algebras.

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