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Witten's Invariants of Rational Homology Spheres at Prime Values of K and Trivial Connection Contribution

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Abstract: We establish a relation between the coefficients of asymptotic expansion of the trivial connection contribution to Witten's invariant of rational homology spheres and the invariants that T Ohtsuki extracted from Witten's invariant at prime values of K We also rederive the properties of prime K invariants discovered by H Murakami and T Ohtsuki We do this by using the bounds on Taylor series expansion of the Jones polynomial of algebraically split links, studied in our previous paper These bounds are enough to prove that Ohtsuki's invariants are of finite type The relation between Ohtsuki's invariants and trivial connection contribution is verified explicitly for lens spaces and Seifert manifolds

1. Introduction

Witten's invariant of 3d manifolds defined in [1] by a path integral over the $SU(2)$ connections A_μ on a 3d manifold M

$$Z(M,k) = \int [\mathcal{D}A_\mu] e^{\frac{ik}{2\pi} S_{\rm CS}[A_\mu]}\,, \tag{1 1}$$

$$S_{\rm CS} = \frac{1}{2}\,{\rm Tr}\,\varepsilon^{\mu\nu\rho} \int\limits_M d^3x \left(A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho \right) \tag{1 2}$$

($k \in \mathbb{Z}$, Tr is the trace taken in the fundamental representation of $SU(2)$) can also be calculated combinatorially with the help of the surgery formula. Let M be a 3d manifold constructed by $(p_j, 1)$ surgeries on the components $\mathcal{L}_j$ of an N-component link $\mathcal{L}$ in S^3. A $(p,1)$ surgery means that the meridian of the tubular neighborhood is glued to the parallel plus p meridians on the boundary of the knot complement (in other words, a Dehn's surgery is performed on a knot with framing number p) The invariant of M reduced to canonical 2-framing can be expressed in terms of

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the framing independent colored Jones polynomial $J_{\alpha_1,\ldots,\alpha_N}(\mathcal{L},k)$ of the link $\mathcal{L}$

$$\frac{Z(M,k)}{Z(S^3,k)}=\left(\frac{2}{K}\right)^{\frac{N}{2}}\exp\left[\frac{3}{4}\pi i\left(\frac{2}{K}-1\right)\sum_{j=1}^{N}\operatorname{sign}(p_j)\right]$$
$$\times\sum_{\substack{0\leqq\alpha_j\leqq K-1\\(1\leqq j\leqq N)}}J_{\alpha_1,\ldots,\alpha_N}(\mathcal{L},k)\exp\left(\frac{i\pi}{2K}\sum_{j=1}^{N}p_j(\alpha_j^2-1)\right)\prod_{j=1}^{N}\sin\left(\frac{\pi}{K}\alpha_j\right),\qquad(1.3)$$

$$Z(S^3,k)=\sqrt{\frac{2}{K}}\sin\frac{\pi}{K},\quad K=k+2\qquad(1.4)$$

The Jones polynomial $J_{\alpha_1,\ldots,\alpha_N}(\mathcal{L};k)$ is normalized in such a way that it is multiplicative for unlinked links and $J(\text{empty link},k)=1$, $J_\gamma(\text{unknot};k)=\sin\left(\frac{\pi}{K}\alpha\right)/\sin\left(\frac{\pi}{K}\right)$

Although N Reshetikhin and V Turaev proved [2] that Eq (1.3) indeed defines an invariant of M (*i e* the *l h s* of Eq (1.3) is invariant under Kirby moves), the topological origin of this invariant remains somewhat obscure The question is which of the "classical" topological invariants of M are contained inside $Z(M;k)$? Two distinct approaches to this problem have been tried The first one is to study $Z(M,k)$ for some particular values of K R Kirby and P. Melvin discovered [3] that if K is odd, then $Z(M,k)$ is proportional to $Z(M,1)$·

$$\frac{Z(M,k)}{Z(S^3,k)}=Z'(M,k)\times\begin{cases}\frac{Z(M,1)}{Z(S^3;1)}, & \text{if } K=-1\ (\mathrm{mod}\,4)\\ \overline{\frac{Z(M,1)}{Z(S^3;1)}}, & \text{if } K=1\ (\mathrm{mod}\,4)\end{cases}.\qquad(1.5)$$

If M is an integer homology sphere ($\mathbb{Z}$HS), then $Z(M,1)=Z(S^3,1)$ so that $Z'(M,k)=\frac{Z(M;\,k)}{Z(S^3,\,k)}$

The new invariant $Z'(M,k)$ can be calculated by the following surgery formula·

$$Z'(M,k)=K^{-\frac{N}{2}}\exp\left[-\frac{\pi i}{4}\left(3+\kappa-\frac{6}{K}\right)\sum_{j=1}^{N}\operatorname{sign}(p_j)\right]$$
$$\times\sum_{\substack{-K\leq\alpha_j\leq K\\ \alpha_j\in 2\mathbb{Z}-1\\(1\leq j\leq N)}}{}'J_{\alpha_1,\ldots,\alpha_N}(\mathcal{L},k)\exp\left(\frac{i\pi}{2K}\sum_{j=1}^{N}p_j(\alpha_j^2-1)\right)\prod_{j=1}^{N}\sin\left(\frac{\pi}{K}\alpha_j\right),\qquad(1.6)$$

here

$$\kappa=\begin{cases}-1 & \text{if } K=-1\ (\mathrm{mod}\,4)\\ +1 & \text{if } K=1\ (\mathrm{mod}\,4)\end{cases},\qquad(1.7)$$

while $\sum'$ means that we add an extra factor of $\frac{1}{2}$ to the terms corresponding to the boundary values of summation index ($\alpha_j=\pm K$ in this case) We changed slightly the original formula of [3] instead of taking a sum over $1\leqq\alpha_j\leqq\frac{K-1}{2}$ we sum over odd α_j between 1 and $K-1$. This allows us to get rid of some phase factors We also double the range of summation to $1-K\leqq\alpha_j\leqq K-1$ by using the fact that $J_{\alpha_1,\ldots,\alpha_N}(\mathcal{L},k)$ is an odd function of its indices (we use the $2K$ periodicity in α_j in order to extend $J_{\alpha_1,\ldots,\alpha_N}(\mathcal{L},k)$ to negative values of α_j) Note that the whole summand of Eq (1.6) has a periodicity of $2K$

S Garoufalidis [4] used nice properties of the gaussian sum $\sum_{\alpha=0}^{K-1} \exp\left(\frac{2\pi i}{K}\alpha^2\right)$ for prime values of K in order to study Witten's invariant of lens spaces and Seifert manifolds H Murakami and T Ohtsuki [5–8] carried out a detailed study of the invariant $Z'(M;k)$ of rational homology spheres (RHS) for prime K

Theorem 1.1 (*H Murakami,* [5, 6]). *For a RHS M and a prime $K > 2$,*

$$Z'(M,k) \in \mathbb{Z}[\check{q}], \quad \check{q} = e^{\frac{2\pi i}{K}}, \tag{1.8}$$

$\mathbb{Z}[\check{q}]$ being a cyclotomic ring

We need more notations in order to present the results of Ohtsuki's papers [7, 8] We introduce a new variable

$$x = \check{q} - 1 . \tag{1 9}$$

A polynomial from $\mathbb{Z}[\check{q}]$ can be reexpressed as a polynomial in x with integer coefficients It is defined modulo the polynomial

$$\frac{(1+x)^K - 1}{x} = \sum_{n=0}^{K-1} \frac{K(K-1) \quad (K-n)}{(n+1)!} x^n , \tag{1 10}$$

which is identically equal to zero for $x = e^{\frac{2\pi i}{K}} - 1$ All the coefficients of this polynomial except the one at x^{K-1}, are divisible by K As a result, all the coefficients at x^n, $n \leqq K-2$ for a polynomial of x coming from $\mathbb{Z}[\check{q}]$ are well defined modulo K We will limit our attention to the powers of x up to $x^{\frac{K-1}{2}}$ They are all well defined as elements of $\mathbb{Z}_K$ if $K \geqq 3$. Thus there is a homomorphism of rings·

$$^{\diamond} \quad \mathbb{Z}[\check{q}] \to \mathbb{Z}'_K[x] \stackrel{\text{def}}{=} \mathbb{Z}_K[x]/x^{\frac{K+1}{2}}\mathbb{Z}_K[x] \tag{1 11}$$

There is another homomorphism from polynomials (of maybe infinite degree) of x with rational coefficients whose denominators are not divisible by K to $\mathbb{Z}'_K[x]$:

$$^{\vee} \cdot \quad \mathbb{Q}[[x]] \to \mathbb{Z}'_K[x] \tag{1 12}$$

The action of the operation $^{\vee}$ on rational numbers was introduced in relation to Witten's invariants at prime values of K by S Garoufalidis [4]

$$^{\vee} \quad \mathbb{Q} \to \mathbb{Z}_K, \qquad (p/q)^{\vee} = pq^*, \tag{1 13}$$

here q^* is the inverse of q modulo K· $qq^* = 1 \pmod K$ The homomorphism $^{\vee}$ acts on polynomials (infinite series) by removing all powers of x higher than $\frac{K-1}{2}$ and converting the remaining coefficients to $\mathbb{Z}_K$.

Now we can present (a slightly stronger version of) Ohtsuki's results

Theorem 1.2 (*T Ohtsuki* [7, 8]). *For any RHS M there exists a sequence of rational numbers $\lambda_n(M) \in \mathbb{Z}\left[\frac{1}{2}, \frac{1}{3}, \quad , \frac{1}{2n}, \frac{1}{|H_1(M,\mathbb{Z})|}\right] \subset \mathbb{Q}$, $n \geqq 0$ so that for any prime number K such that $|H_1(M,\mathbb{Z})| \neq 0 \pmod K$*

$$\left[|H_1(M,\mathbb{Z})|\,(|H_1(M,\mathbb{Z})|/K)\,Z'(M,k)\right]^{\diamond} = \left[\sum_{n=0}^{\infty} \lambda_n(M) x^n\right]^{\vee} , \tag{1.14}$$

here $(\cdot/K)$ is the Legendre symbol for $p \in \mathbb{Z}$, $(p/K) = 1$ if there exists $p' \in \mathbb{Z}$ such that $p = (p')^2 \pmod K$ and $(p/K) = -1$ otherwise

We have slightly modified the theorem of [8]. Ohtsuki required that $K > |H_1(M,\mathbb{Z})|$, he estimated that $\lambda_n(M) \in \mathbb{Z}\left[\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{2n+1}, \frac{1}{|H_1(M,\mathbb{Z})|}\right] \subset \mathbb{Q}$, $n > 0$ and he used $\mathbb{Z}'_K[x] = \mathbb{Z}_K[x]/x^{\frac{K-1}{2}}\mathbb{Z}[x]$ instead of (1 11) (in other words, he did not fix the coefficient at $x^{\frac{K-1}{2}}$) Murakami showed that

$$\lambda_0 = 1, \qquad \lambda_1 = 3\lambda_{\mathrm{CW}}\,, \tag{1 15}$$

here λ_{CW} is the Casson-Walker invariant of RHS

The second approach to the search of the topological meaning of Witten's invariant $Z(M,k)$ is based on the path integral representation (1 1). According to quantum field theory, this integral can be calculated by stationary phase approximation when $K \to \infty$ The invariant is presented as a sum of contributions coming from connected components c of the moduli space of flat connections on the manifold M.

$$Z(M,k) = \sum_c Z^{(c)}(M,k)\,. \tag{1.16}$$

Each contribution has a general form

$$Z^{(c)}(M,k) = \left(\frac{4\pi^2}{K}\right)^{\frac{N_{\mathrm{zero}}}{2}} \frac{1}{\mathrm{Vol}(H_c)} \exp\left[2\pi i K S_{\mathrm{CS}}^{(c)} + \sum_{n=0}^{\infty} S_n^{(c)}(M)\left(\frac{i\pi}{K}\right)^n\right], \tag{1.17}$$

here H_c is the isotropy group, $N_{\mathrm{zero}} = \dim H_c^0 - \dim H_c^1$, $H_c^{0,1}$ being the cohomologies of 0,1-forms taking values in the adjoint $su(2)$ bundle, $S_{\mathrm{CS}}^{(c)}$ is the Chern–Simons action and $\sum_{n=0}^{\infty} S_n^{(c)}(M)\left(\frac{i\pi}{K}\right)^n$ is an asymptotic series The coefficients $S_n^{(c)}(M)$ are called $(n+1)$-loop corrections They might be related to "classical" topological invariants of M Indeed, the 1-loop correction $S_0^{(c)}$ is related to the Reidemeister-Ray-Singer torsion An attempt to relate the asymptotic properties of the surgery formula (1 3) for lens spaces and Seifert manifolds to the quantum field theory predictions of Eqs (1 16), (1 17) was initiated by D Freed and R Gompf [9] and carried out further by L. Jeffrey [10], S. Garoufalidis [4] and also in the papers [11, 12]. A complete agreement between the surgery formula and 1-loop predictions was observed

If the manifold M is a RHS, then the trivial connection is a separate point in the moduli space of flat connections According to quantum field theory, its contribution is of the form

$$Z^{(\mathrm{tr})}(M,k) = \frac{\sqrt{2}\pi}{K^{\frac{3}{2}}|H_1(M,\mathbb{Z})|^{\frac{3}{2}}} \exp\left[\sum_{n=1}^{\infty} S_n(M)\left(\frac{i\pi}{K}\right)^n\right] \tag{1.18}$$

(see *e g* [9] Eq (1 36), [10] Eq (5 1) and [11] Eq. (2.14)) A representation of the coefficients $S_n(M)$ in terms of $(n+1)$-loop Feynman diagrams was carried out by S Axelrod and I Singer [13], M Kontsevich [14], C. Taubes [15] and others We studied how the trivial connection contribution can be extracted from the surgery formula (1 3). We derived a knot surgery formula [16] and a link surgery formula [17] for it The knot formula allowed us to show that

$$S_1(M) = 6\lambda_{\mathrm{CW}} \tag{1.19}$$

The link surgery formula of [17] was much less explicit than the knot formula of [16], because it did not express $Z^{(\mathrm{tl})}(M,k)$ directly in terms of derivatives of the Jones polynomial $J_{\alpha_1,\ldots,\alpha_N}(\mathcal{L};k)$ However we derived an explicit surgery formula [18] for algebraically split links (a link is algebraically split (ASL) if all of its off-diagonal linking numbers are equal to zero)

In this paper we are going to prove the following·

Proposition 1.1. *Ohtsuki's invariants $\lambda_n(M)$ of Eq (1 14) and loop corrections to the trivial connection contribution $S_n(M)$ of Eq (1 18) can be expressed in terms of each other through the following relation·*

$$\sum_{n=0}^{\infty}\lambda_n(M)x^n = \exp\left[\sum_{n=1}^{\infty}(S_n(M)-S_n(S^3))\left(\frac{i\pi}{K}\right)^n\right]$$

$$\equiv \frac{\left(\frac{\pi}{K}\right)}{\sin\left(\frac{\pi}{K}\right)}\exp\left[\sum_{n=1}^{\infty}S_n(M)\left(\frac{i\pi}{K}\right)^n\right] \tag{1 20}$$

by substituting either

$$x = e^{\frac{2\pi i}{K}}-1 = \frac{2\pi i}{K}\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\left(\frac{2\pi i}{K}\right)^n, \tag{1 21}$$

or

$$\frac{i\pi}{K} = \frac{1}{2}\log(1+x) = \frac{x}{2}\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n+1}\,. \tag{1 22}$$

In other words, we will show that

$$\left[|H_1(M,\mathbb{Z})|\,(|H_1(M,\mathbb{Z})|/K)\,Z'(M,k)\right]^{\diamond} = \left[|H_1(M,\mathbb{Z})|^{\frac{3}{2}}\frac{Z^{(\mathrm{tl})}(M,k)}{Z(S^3,k)}\right]^{\vee} \tag{1 23}$$

In the process of doing this we will rederive the results of [5–8] in a more explicit way

The results of this paper are rigorously derived from the following two propositions·

Proposition 1.2. *Let $\mathcal{L}$ be an algebraically split link (ASL) in S^3 Then its framing-independent colored Jones polynomial has the following Taylor series expansion in powers of K*

$$J_{\alpha_1,\ldots,\alpha_N}(\mathcal{L},k) = \left(\prod_{j=1}^{N}\alpha_j\right)\sum_{n=0}^{\infty}\sum_{m\leqq\frac{3}{4}n}D_{m,n}(\alpha_1,\quad,\alpha_N)\left(\frac{i\pi}{K}\right)^n, \tag{1 24}$$

here $D_{m,n}(\alpha_1,\quad,\alpha_N)$ are even homogeneous polynomials of degree $2m$·

$$D_{m,n}(\alpha_1,\quad.,\alpha_N) = \sum_{\substack{m_1\\ m_1+\quad+m_N=m}}\sum_{m_N\geqq 0} D^{(m,n)}_{m_1,\ldots,m_N}\alpha_1^{2m_1}\quad\alpha_N^{2m_N} \tag{1 25}$$

and

$$m_j \leqq n-m\,,\qquad 1\leqq j\leqq N \tag{1 26}$$

Proposition 1.3. *Let M be a rational homology sphere (RHS) constructed by $(p_j,1)$ surgeries on the components of an N-component ASL $\mathcal{L}$ in S^3. Then the loop corrections to the trivial connection contribution* (1.18) *to Witten's invariant of M are given by the formula*

$$\exp\left[\sum_{n=1}^{\infty}(S_n(M)-S_n(S^3))\left(\frac{i\pi}{K}\right)^n\right] \tag{1.27}$$

$$=\exp\left[\frac{3}{4}\pi i\left(\frac{2}{K}-1\right)\sum_{j=1}^{N}\operatorname{sign}(p_j)\right]\exp\left[-\frac{i\pi}{2K}\sum_{j=1}^{N}\left(p_j+\frac{1}{p_j}\right)\right]\frac{i^N}{(2K)^{\frac{N}{2}}}\left|\prod_{j=1}^{N}p_j\right|^{\frac{3}{2}}$$

$$\times\int_{\substack{-\infty\\ |\alpha_j=0|}}^{+\infty}d\alpha_1\quad d\alpha_N\exp\left(\frac{i\pi}{2K}\sum_{j=1}^{N}p_j\alpha_j^2\right)J_{\alpha_1+\frac{1}{p_1},\ \ldots,\alpha_N+\frac{1}{p_N}}(\mathcal{L},k)$$

($S_n(S^3)$ *are defined by the relation* $\exp\left(\sum_{n=1}^{\infty}S_n(S^3)\left(\frac{i\pi}{K}\right)^n\right)=\frac{\sin\left(\frac{\pi}{K}\right)}{\left(\frac{\pi}{K}\right)}$)

The symbol $\int_{\substack{-\infty\\ |\alpha_j=0|}}^{+\infty}$ *means that the integral has to be calculated in the following way: first, an expansion* (1.24) *has to be substituted and then the gaussian integrals over α_j have to be calculated for each polynomial* $\left(\prod_{j=1}^{N}\left(\alpha_j+\frac{1}{p_j}\right)\right)D_{m,n}\left(\alpha_1+\frac{1}{p_1},\ \ldots,\alpha_N+\frac{1}{p_N}\right)$ *separately (for more details see* [18] *and* [20])

Proposition 1.2 was rigorously derived in [18] (Proposition 3.1) from Reshetikhin's formula for the colored Jones polynomial of a link. Reshetikhin's formula was introduced and "physically derived" in [19] with the help of Chern–Simons perturbation theory. The mathematically rigorous proof can be obtained [21] by using Kontsevich's integral representation of the Taylor series expansion of the Jones polynomial.

Proposition 1.3 presented in [18] (Proposition 3.2) is a mathematical conjecture. Its status is similar to that of Eqs. (1.16), (1.17) and (1.18). However we can proceed rigorously by declaring Eq. (1.27) a definition of an infinite sequence of RHS invariants $S_n(M)$ without mentioning their relation to the asymptotic properties of Witten's invariant $Z(M,k)$[2]. By using Reshetikhin's formula of [19] we can prove that $S_n(M)$ as defined by Eq. (1.27) are invariant under the Kirby moves (see [20] for details). Now Proposition 1.1 becomes a rigorous statement about the relation between Ohtsuki's invariants $\lambda_n(M)$ and invariants $S_n(M)$ defined by Eq. (1.27).

In Sect. 2 we modify the surgery formula (1.6) and prove Theorem 1.1. Our main tool is the observation that the gaussian sum $\sum_{\gamma=0}^{K-1}\check{q}^{\gamma^2}$ is proportional to $x^{\frac{K-1}{2}}$, while the sum $\sum_{\gamma=0}^{K-1}\check{q}^{\gamma^2}\alpha^{2m}$ for $m\leqq\frac{K-1}{2}$ is only proportional to $x^{\frac{K-1}{2}-m}$. This is similar to the behavior of gaussian integrals: each two extra powers of α in the integral $\int_{-\infty}^{+\infty}e^{\frac{2\pi i}{K}\alpha^2}\alpha^{2m}\,d\alpha$ bring a power of K to denominator. In Sect. 3 we prove Proposition 1.1 and thus also Theorem 1.2. We use again the similarities of the formulas for $\sum_{\gamma=0}^{K-1}\check{q}^{\gamma^2}\alpha^{2m}$ and $\int_{-\infty}^{+\infty}e^{\frac{2\pi i}{K}\alpha^2}\alpha^{2m}\,d\alpha$. These similarities are due to

[2] This is similar to the approach of N. Reshetikhin and V. Turaev [2] who treated Eq. (1.2) as a definition of the invariant $Z(M;k)$ rather than as a property of the path integral (1.1) derived "physically" by E. Witten in [1].

the fact that both the sum $\sum_{\alpha=0}^{K-1} \check{q}^{\gamma^2+2n\alpha}$ and the integral $\int_{-\infty}^{+\infty} e^{\frac{2\pi i}{K}(\gamma^2+2n\alpha)}\, d\alpha$ can be calculated by completing the square in the exponents. In Sect. 4 we derive a rational surgery formula for $Z'(M,k)$ which is similar to the formula (4.1) of [10] for the original Witten's invariant $Z(M,k)$. We use this formula to verify Proposition 1.1 for lens spaces and Seifert manifolds which are rational homology spheres. In Sect. 5 we discuss the properties of Ohtsuki's invariants $\lambda_n(M)$ as related to the properties of invariants $S_n(M)$ studied in [18].

2. Gaussian Sums and Divisibility in Cyclotomic Ring

We start by modifying the surgery formula (1.6). Since $\frac{1}{4}(\alpha_j^2 - 1) \in \mathbb{Z}$, then

$$\exp\left[\frac{i\pi}{2K}\sum_{j=1}^{N} p_j(\alpha_j^2 - 1)\right] = \check{q}^{\frac{1}{4}\sum_{j=1}^{N} p_j(\alpha_j^2-1)} = \check{q}^{4^*\sum_{j=1}^{N} p_j(\alpha_j^2-1)} \tag{2.1}$$

Also since $\frac{1}{2}(\alpha_j \pm \operatorname{sign}(p_j)) \in \mathbb{Z}$,

$$\sin\left(\frac{\pi}{K}\alpha_j\right) = \frac{i}{2}\sum_{\mu_j=\pm 1} \mu_j e^{-\frac{i\pi}{K}\mu_j\alpha_j} = \frac{i}{2}\sum_{\mu_j=\pm 1} \mu_j \check{q}^{-2^*\mu_j\alpha_j}\check{q}^{(2^*-\frac{1}{2})\operatorname{sign}(p_j)} \tag{2.2}$$

The Jones polynomial $J_{\alpha_1,\ldots,\alpha_N}(\mathcal{L},k)$ is odd and $e^{\frac{i\pi}{2K}p_j\alpha_j^2}$ is even as a function of α_j. Therefore we can drop the factor $\frac{1}{2}$ and put $\mu_j = 1$ in Eq. (2.2) upon substituting it into Eq. (1.6)

$$\begin{aligned} Z'(M,k) \;=\; & K^{-\frac{N}{2}} i^N e^{-\frac{i\pi}{4}(3+\kappa)\sum_{j=1}^{N}\operatorname{sign}(p_j)} \check{q}^{(\frac{1}{4}+2^*)\sum_{j=1}^{N}\operatorname{sign}(p_j)} \check{q}^{-4^*\sum_{j=1}^{N} p_j} \\ & \times \sum_{\substack{-K\le\alpha_j\le K \\ \alpha_j\in 2\mathbb{Z}+1 \\ (1\le j\le N)}}{}' \check{q}^{\sum_{j=1}^{N}(4^* p_j\alpha_j^2 - 2^*\alpha_j)} J_{\alpha_1,\ldots,\alpha_N}(\mathcal{L},k)\,. \end{aligned} \tag{2.3}$$

After completing the square

$$4^* p_j\alpha_j^2 - 2^*\alpha_j = 4^* p_j(\alpha_j - p_j^*)^2 - 4^* p_j^* \pmod K\,, \tag{2.4}$$

we shift the summation variable α_j by p_j^* (we assume that p_j^* is even in order to preserve the parity of α_j, we can always make such choice of p_j^* since K is odd). Then

$$\begin{aligned} Z'(M,k) \;=\; & K^{-\frac{N}{2}} i^N e^{-\frac{i\pi}{4}(3+\kappa)\sum_{j=1}^{N}\operatorname{sign}(p_j)} \check{q}^{(\frac{1}{4}+2^*)\sum_{j=1}^{N}\operatorname{sign}(p_j)} \check{q}^{-4^*\sum_{j=1}^{N}(p_j+p_j^*)} \\ & \times \sum_{\substack{-K\le\alpha_j\le K \\ \alpha_j\in 2\mathbb{Z}+1 \\ (1\le j\le N)}}{}' \check{q}^{4^*\sum_{j=1}^{N} p_j\alpha_j^2} J_{\alpha_1+p_1^*,\ldots,\alpha_N+p_N^*}(\mathcal{L},k)\,. \end{aligned} \tag{2.5}$$

Next we use the identities

$$ie^{-\frac{i\pi}{2}\operatorname{sign}(p_j)} = \operatorname{sign}(p_j)\,, \tag{2.6}$$

$$\frac{1-\kappa K}{4} = 4^* \tag{2.7}$$

in order to rearrange the phase factors preceding the sum in Eq. (2.5)

$$Z'(M,k) = K^{-\frac{\lambda}{2}} e^{\frac{i\pi}{4}(\kappa-1)\sum_{j=1}^{\lambda}\operatorname{sign}(p_j)} \left(\prod_{j=1}^{N}\operatorname{sign}(p_j)\right) \check{q}^{4^*\sum_{j=1}^{\lambda}(3\operatorname{sign}(p_j)-p_j-p_j^*)}$$

$$\times \sum_{\substack{-K\leqq\alpha_j\leqq K \\ \alpha_j\in 2\mathbb{Z}+1 \\ (1\leqq j\leqq\lambda)}}{}' \check{q}^{4^*\sum_{j=1}^{\lambda}p_j\alpha_j^2} J_{\alpha_1+p_1^*,\ \ldots,\alpha_\lambda+p_\lambda^*}(\mathcal{L};k) \tag{2.8}$$

Since the values of the Jones polynomial $J_{\alpha_1,\ \ldots,\alpha_\lambda}(\mathcal{L},k)$ belong to the cyclotomic ring $\mathbb{Z}[\check{q}]$ when all the indices α_j are odd, we can apply to it a combination of Proposition 1.2 and Lemma 2.3 of [5]

Proposition 2.1. *For odd values of its indices α_j, the unframed colored Jones polynomial of an N-component ASL $\mathcal{L}$ in S^3 can be presented as the following sum*

$$J_{\alpha_1,\ \ldots,\alpha_\lambda}(\mathcal{L},k) = \left(\prod_{j=1}^{N}\alpha_j\right) \sum_{n=0}^{(N+1)\frac{k-1}{2}} \sum_{m\leqq\frac{3}{4}n} \tilde{D}_{m,n}(\alpha_1,\ \ldots,\alpha_N)x^n$$

$$+x^{(N+1)\frac{k-1}{2}+1} J^{(\mathrm{res})}_{\alpha_1,\ \ldots,\alpha_\lambda}(\mathcal{L};k), \quad J^{(\mathrm{res})}_{\alpha_1,\ \ldots,\alpha_\lambda}(\mathcal{L},k)\in\mathbb{Z}[\check{q}]\,, \tag{2.9}$$

here $\tilde{D}_{m,n}(\alpha_1,\ \ldots,\alpha_N)$ are homogeneous polynomials of degree $2m$

$$\tilde{D}_{m,n}(\alpha_1,\ \ldots,\alpha_N) = \sum_{\substack{m_1+\ldots \\ m_1+\ldots+m_\lambda=m}} \tilde{D}^{(m,n)}_{m_1,\ \ldots,m_\lambda}\alpha_1^{2m_1}\ \ldots\ \alpha_N^{2m_\lambda}, \tag{2.10}$$

$$m_j\leqq n-m\,, \tag{2.11}$$

and the polynomials $\left(\prod_{j=1}^{N}\alpha_j\right)\sum_{m\leqq\frac{3}{4}n}\tilde{D}_{m,n}(\alpha_1,\ \ldots,\alpha_N)$ take integer values when α_j are odd

The latter property of the polynomials $\tilde{D}_{m,n}$ allows us to express them in terms of "binomial coefficient" polynomials:

$$\left(\prod_{j=1}^{N}\alpha_j\right)\sum_{m\leqq\frac{3}{4}n}\tilde{D}_{m,n}(\alpha_1,\ \ldots,\alpha_N) = \sum_{\substack{1\leqq m_j\leqq 2\bar{m}_j+1 \\ (1\leqq j\leqq\lambda)}} C^{(n)}_{m_1,\ \ldots,m_\lambda} P_{m_1}\left(\frac{\alpha_1-1}{2}\right)\ \ldots\ P_{m_\lambda}\left(\frac{\alpha_N-1}{2}\right),$$

$$C^{(n)}_{m_1,\ \ldots,m_\lambda}\in\mathbb{Z}\,, \tag{2.12}$$

here

$$P_m(\alpha) = \frac{\alpha(\alpha-1)\ \ldots\ (\alpha-m+1)}{m!} \tag{2.13}$$

and $\bar{m}_j$ are the maximum values of m_j in the representation (2.10) of all the polynomials $\tilde{D}_{m,n}$ appearing in the $l.h.s.$ of Eq. (2.12)

The following proposition is a reflection of the inequality (2.11) for the representation (2 12):

Proposition 2.2. *There is an upper bound on the indices of the coefficients* $C^{(n)}_{m_1,\ldots,m_N}$ *of Eq* (2 12)

$$\left[\frac{m_j}{2}\right] \leqq n - \sum_{i=1}^{N} \left[\frac{m_i}{2}\right], \tag{2.14}$$

here $\left[\frac{m}{2}\right]$ *denotes the integer part of* $\frac{m}{2}$

The proof is completely similar to that of Proposition 3 4 of [18] Suppose that there is a coefficient $C^{(n)}_{m_1,\ldots,m_N}$ for which (2.14) is not true, say, for m_1 If all indices m_j of $C^{(n)}_{m_1,\ldots,m_N}$ are odd, then the highest degree monomial of the corresponding polynomial

$$C^{(n)}_{m_1,\ldots,m_N} P_{m_1}\left(\frac{\alpha_1 - 1}{2}\right) \cdot \quad P_{m_N}\left(\frac{\alpha_N - 1}{2}\right) \tag{2.15}$$

violates the inequality (2.11). Therefore it has to be canceled by monomials of other polynomials

$$C^{(n)}_{m'_1,\ldots,m'_N} P_{m'_1}\left(\frac{\alpha_1 - 1}{2}\right) \quad P_{m'_N}\left(\frac{\alpha_N - 1}{2}\right) \tag{2 16}$$

for which

$$m'_j \geqq m_j, \quad 1 \leqq j \leqq N, \qquad \sum_{j=1}^{N} m'_j > \sum_{j=1}^{N} m_j \,. \tag{2.17}$$

If some m_j are even, then the highest degree monomial of the polynomial (2.15) is incompatible with the structure of the $l\,h\,s$ of Eq. (2.12) and it also has to be canceled The inequalities (2 17) show that the index m_1 of the polynomials (2.15) again violates (2.14), so we need to go to higher values of $\sum_{j=1}^{N} m_j$ for further cancellation Since $\sum_{j=1}^{N} m_j \leqq n$, this process can not be completed This contradiction proves the proposition

Now we begin to prove Theorem 1.1. Following [5], we use the relations

$$\sum_{\alpha=0}^{K-1} \check{q}^{\alpha^2} = e^{i\frac{\pi}{4}(1-\kappa)}\sqrt{K}\,, \tag{2 18}$$

$$\sum_{\alpha=0}^{K-1} \check{q}^{\gamma^2} = x^{\frac{\kappa-1}{2}} u^{-1}, \quad u,\, u^{-1} \in \mathbb{Z}[\check{q}] \tag{2 19}$$

(see *e g* [22], chapter 6) in order to present $K^{-\frac{N}{2}}$ in the following form

$$K^{-\frac{N}{2}} = e^{i\frac{\pi}{4}N(1-\kappa)} \frac{u^N}{x^{N\frac{K-1}{2}}}\,. \tag{2 20}$$

Substituting this expression into Eq (2 8) we find that

$$Z'(M,k) = e^{\frac{i\pi}{4}(\kappa-1)\sum_{j=1}^{N}(\mathrm{sign}(p_j)-1)}\left(\prod_{j=1}^{N}\mathrm{sign}\,(p_j)\right)\check{q}^{4^*\sum_{j=1}^{\lambda}(3\,\mathrm{sign}(p_j)-p_j-p_j^*)}$$
$$\times\frac{u^N}{x^{N\frac{K-1}{2}}}\sum_{\substack{-K\leqq\alpha_j\leqq K\\ \alpha_j\in 2\mathbb{Z}+1\\ (1\leqq j\leqq\lambda)}}{}'\check{q}^{4^*\sum_{j=1}^{\lambda}p_j\alpha_j^2}J_{\alpha_1+p_1^*,\ \ldots,\alpha_\lambda+p_\lambda^*}(\mathcal{L},k) \tag{2 21}$$

Since $\frac{1}{4}(\kappa-1)(\mathrm{sign}\,(p_j)-1)\in\mathbb{Z}$, we conclude that to prove Theorem 1 1 it is enough to show that

$$\sum_{\substack{-K\leqq\alpha_j\leqq K\\ \alpha_j\in 2\mathbb{Z}+1\\ (1\leqq j\leqq\lambda)}}{}'\check{q}^{4^*\sum_{j=1}^{N}p_j\alpha_j^2}J_{\alpha_1+p_1^*,\ \ldots,\alpha_\lambda+p_\lambda^*}(\mathcal{L},k)=x^{N\frac{K-1}{2}}w,\quad w\in\mathbb{Z}[\check{q}] \tag{2 22}$$

We will substitute the expansion (2.9) and check the property (2 22) for every polynomial $\left(\prod_{j=1}^{N}\alpha_j\right)\sum_{m\leqq\frac{3}{4}n}\tilde{D}_{m,n}(\alpha_1,\ldots,\alpha_N)$ separately The remainder term $x^{(N+1)\frac{K-1}{2}}J^{(\mathrm{res})}_{\alpha_1,\ldots,\alpha_\lambda}(\mathcal{L};k)$ obviously satisfies Eq (2.22) Moreover, for some $w\in\mathbb{Z}[\check{q}]$,

$$\left[\frac{1}{x^{N\frac{K-1}{2}}}\sum_{\substack{-K\leqq\alpha_j\leqq K\\ \alpha_j\in 2\mathbb{Z}+1\\ (1\leqq j\leqq\lambda)}}{}'\check{q}^{4^*\sum_{j=1}^{\lambda}p_j\alpha_j^2}J^{(\mathrm{res})}_{\alpha_1,\ldots,\alpha_\lambda}(\mathcal{L};k)x^{(N+1)\frac{K-1}{2}+1}\right]^{\diamond}=\left[x^{\frac{K+1}{2}}w\right]^{\diamond}=0\,, \tag{2 23}$$

so that we can neglect the contribution of this term in all further calculations

To estimate the contribution of a polynomial $\left(\prod_{j=1}^{N}\alpha_j\right)\sum_{m\leqq\frac{3}{4}n}\tilde{D}_{m,n}(\alpha_1,\ldots,\alpha_N)x^n$ we need the following simple lemma:

Lemma 2.1. *For* $p,m\in\mathbb{Z}$, $m\geqq 0$

$$\sum_{\substack{-K\leqq\alpha\leqq K\\ \alpha\in 2\mathbb{Z}+1}}{}'\check{q}^{p\alpha^2}\alpha^{2m}=x^{\max\{0,\frac{K-1}{2}-m\}}w,\quad w\in\mathbb{Z}[\check{q}] \tag{2.24}$$

The lemma needs a proof only for $m<\frac{K-1}{2}$ To prove that an element $u\in\mathbb{Z}[\check{q}]$ is divisible by x^n, $n\leqq K-1$ one may present it as an integer coefficient polynomial of x and check that the coefficients in front of all $x^{n'}$, $n'<n$ are divisible by K. We substitute $\check{q}=x+1$ in Eq (2.24) and express the powers of $\check{q}$ in terms of "binomial" polynomials

$$\sum_{\substack{-K\leqq\alpha\leqq K\\ \alpha\in 2\mathbb{Z}+1}}{}'\check{q}^{p\alpha^2}\alpha^{2m}=\sum_{n\geqq 0}\sum_{\substack{-K\leqq\alpha\leqq K\\ \alpha\in 2\mathbb{Z}+1}}{}'P_n(p\alpha^2)\alpha^{2m}x^n$$
$$=\sum_{n\geqq 0}\sum_{0\leqq m'\leqq n}\sum_{\substack{-K\leqq\alpha\leqq K\\ \alpha\in 2\mathbb{Z}+1}}{}'\frac{C_{n,m'}}{n!}p^{2m'}\alpha^{2(m+m')}x^n, \tag{2.25}$$

here

$$\alpha(\alpha-1)\quad(\alpha-n+1)=\sum_{m'\leqq n}C_{n,m'}\alpha^{m'},\quad C_{n,m'}\in\mathbb{Z} \tag{2.26}$$

It is well known in number theory that

$$\sum_{\substack{-K\le\alpha\le K\\ \alpha\in 2\mathbb{Z}+1}}{}' \alpha^{2m} = 0 \pmod K \quad \text{for } 0 \le m < K-2\,. \tag{2.27}$$

Therefore the numerator of the contribution of a term $\frac{C_{n,m'}}{n!}p^{2m'}\alpha^{2(m+m')}x^n$ will be divisible by K for all $m+m' < \frac{K-1}{2} \le K-2$, that is, for all $n < \frac{K-1}{2} - m$ The denominator $n!$ is harmless, because since $n < K$, it is not divisible by K and can not cancel the factors of K coming from the sum over α This proves the lemma.

This lemma can be easily generalized to the "binomial" polynomials (2 13)

Lemma 2.2. *For* $p, p', m \in \mathbb{Z}$, $p' \in 2\mathbb{Z}$, $m \ge 0$,

$$\sum_{\substack{-K\le\alpha\le K\\ \alpha\in 2\mathbb{Z}+1}}{}' \check{q}^{p\alpha^2} P_m\left(\frac{\alpha+p'-1}{2}\right) = x^{\max\{0,\frac{K-1}{2}-\left[\frac{m}{2}\right]\}}w, \quad w\in\mathbb{Z}[\check{q}] \tag{2.28}$$

The proof is similar to that of the previous lemma The choice of summation range for α makes obvious the fact that odd powers of α in $P_m\left(\frac{\alpha+p'-1}{2}\right)$ can be ignored The numerators of the contributions of even powers of α are divisible by K. The coefficients of P_m have a denominator $m!$, but we may assume that $m < K-1$ (otherwise Eq (2.28) is obvious) so that the denominator does not cancel the factors of K

Let $\bar{m}(n)$ be the maximum value of m appearing in the $l\,h\,s$ of Eq (2.12) Then for every coefficient $C^{(n)}_{m_1,\ldots,m_N}$ from the $r\,h\,s$ of that equation

$$m_1 + \quad + m_N \le \bar{m}(n) \tag{2 29}$$

Therefore we can combine Eqs (2 12) and (2 28) into the following estimate of the contribution of polynomials $\tilde{D}_{m,n}$ to the sum (2 22)·

$$\sum_{\substack{-K\le\alpha\le K\\ \alpha\in 2\mathbb{Z}+1}}{}' \check{q}^{4^*\sum_{j=1}^{N}p_j\alpha_j^2}\left(\prod_{j=1}^{N}(\alpha_j+p_j^+)\right)\sum_{m\le\frac{3}{4}n}\tilde{D}_{m,n}(\alpha_1+p_1^*,\ldots,\alpha_N+p_N^*)x^n$$
$$= x^{N\frac{K-1}{2}+n-\bar{m}(n)}w, \qquad w\in\mathbb{Z}[\check{q}] \tag{2 30}$$

Since $\bar{m}(n) \le \frac{3}{4}n \le n$, this estimate is enough to prove Eq (2.22) and also Theorem 1 1 Note that the proof required only a weaker bound $m \le n$ for $\tilde{D}_{m,n}$ rather than a stronger bound $m \le \frac{3}{4}n$ of [18] However the bound $m \le \frac{3}{4}n$ is necessary to prove that only a finite number of polynomials $\tilde{D}_{m,n}$ contribute to the coefficients of $x^{n'}$, $n' \le \frac{K-1}{2}$ in the expansion of $[Z'(M,k)]^\diamond$. Indeed, since $\bar{m}(n) \le \frac{3}{4}n$, then $n-\bar{m}(n) \ge \frac{1}{4}n$ and Eq (2.30) suggests that we may limit our attention to only those polynomials (2 15) for which

$$n \le 2(K-1) \tag{2.31}$$

3. Gaussian Sums and Integrals

We are going to derive a surgery formula for $[Z'(M,k)]^\diamond$ which would express it in terms of the derivatives $\tilde{D}_{m,n}$ of the colored Jones polynomial As we will see, this requires a calculation of the gaussian sum

$$G(p,q,m)=\frac{e^{i\frac{\pi}{4}(\kappa-1)}}{\sqrt{K}}\sum_{\substack{-K\le\gamma\le K\\ \gamma\in 2\mathbb{Z}+1}}{}'\check{q}^{q^*p\gamma^2}\alpha^{2m}x^m,\qquad m\le\frac{K-1}{2} \tag{3.1}$$

More precisely, we need to find only $[G(p,q,m)]^\diamond$. We already know that $G(p,q,m)\in\mathbb{Z}[\check{q}]$.

Proposition 3.1. *The sum of Eq* (3.1) *is related to the gaussian integral For* $p,q\in\mathbb{Z}$, $p,q\neq 0\ (\mathrm{mod}\,K)$, $0\le m\le\frac{K-1}{2}$,

$$[G(p,q,m)]^\diamond=(pq^*/K)\left[e^{-i\frac{\pi}{4}\,\mathrm{sign}\left(\frac{p}{q}\right)}\left(\frac{2}{K}\right)^{\frac{1}{2}}\left|\frac{p}{q}\right|^{\frac{1}{2}}\int_{-\infty}^{+\infty}d\alpha\, e^{\frac{2\pi i}{K}\frac{p}{q}\gamma^2}\alpha^{2m}x^m\right]^\vee$$
$$+x^{\frac{K+1}{2}-m}w,\quad w\in\mathbb{Z}[\check{q}] \tag{3.2}$$

To prove the proposition we calculate the following sum

$$\frac{e^{i\frac{\pi}{4}(\kappa-1)}}{\sqrt{K}}\sum_{\substack{-K\le\gamma\le K\\ \gamma\in 2\mathbb{Z}+1}}{}'\check{q}^{pq^*\alpha^2}\check{q}^{2n\alpha}=\frac{e^{i\frac{\pi}{4}(\kappa-1)}}{\sqrt{K}}\check{q}^{-p^*qn^2}\sum_{\substack{-K\le\gamma\le K\\ \gamma\in 2\mathbb{Z}+1}}{}'\check{q}^{pq^*(\alpha+np^*q)^2} \tag{3.3}$$

Since (for $p^*\in\mathbb{Z}$)

$$\sum_{\substack{-K\le\gamma\le K\\ \gamma\in 2\mathbb{Z}+1}}{}'\check{q}^{pq^*(\alpha+np^*q)^2}=\sum_{\substack{-K\le\gamma\le K\\ \gamma\in 2\mathbb{Z}+1}}{}'\check{q}^{pq^*\gamma^2}=\sum_{\frac{3-K}{2}<\tilde{\gamma}\le\frac{K-1}{2}}\check{q}^{pq^*(2\tilde{\alpha}+1)^2}$$
$$=\sum_{\frac{3-K}{2}<\tilde{\alpha}\le\frac{K-1}{2}}\check{q}^{4pq^*(\tilde{\alpha}+2^*)^2}=\sqrt{K}e^{i\frac{\pi}{4}(1-\kappa)}(pq^*/K), \tag{3.4}$$

we find that

$$\frac{e^{i\frac{\pi}{4}(\kappa-1)}}{\sqrt{K}}\sum_{\substack{-K\le\gamma\le K\\ \gamma\in 2\mathbb{Z}+1}}{}'\check{q}^{pq^*\alpha^2}\check{q}^{2n\gamma}=(pq^*/K)\check{q}^{-p^*qn^2} \tag{3.5}$$

We substitute $\check{q}=1+x$ in $\check{q}^{2n\gamma}$ and $\check{q}^{-p^*qn^2}$ After going from $\mathbb{Z}[\check{q}]$ to the factor-ring $\mathbb{Z}'_K[x]=\mathbb{Z}_K[x]/x^{\frac{K+1}{2}}\mathbb{Z}_K[x]$ and using the "checked binomial polynomial"

$$P_m^\vee(\alpha)=(m!)^*\alpha(\alpha-1)\quad(\alpha-m+1)=(m!)^*\sum_{l=0}^m C_{m,l}\alpha^l, \tag{3.6}$$

we find that

$$\left[\sum_{m=0}^{K-1}\sum_{l=0}^{\left[\frac{m}{2}\right]}(m!)^*C_{m,2l}G(p,q;l)4^ln^{2l}x^{m-l}\right]^\diamond$$
$$=(pq^*/K)\left[\sum_{m=0}^{\frac{K-1}{2}}x^m\sum_{l=0}^m(m!)^*C_{m,l}(-1)^l(p^*q)^ln^{2l}\right]^\diamond. \tag{3.7}$$

We limited the sum over m in the $l.h.s.$ of this equation to $m \leqq K-1$ because for $m \geqq K$ the minimum power of x^{m-l} is greater than $\frac{K-1}{2}$. Note that since $m \leqq K-1$, then $(m!)^*$ is well defined.

If we substitute the expansion

$$[G(p,q,l)]^{\diamond} = \sum_{m=0}^{\frac{K-1}{2}-l} G_m(p,q,l)\, x^m + x^{\frac{K+1}{2}-l} w, \quad w \in \mathbb{Z}[\check{q}] \tag{3.8}$$

into Eq. (3.7), then we can find all the coefficients $G_m(p,q;l)$ by equating the coefficients of $l.h.s.$ and $r.h.s.$ of Eq. (3.7) at equal powers of x and n. These coefficients have to be equal due to the following simple lemma.

Lemma 3.1. *If a degree of a polynomial $P(n) \in \mathbb{Z}_K[n]$ is less than K and $P(n) = 0$ for all $n \in \mathbb{Z}_K$, then all the coefficients of $P(n)$ are zero modulo K.*

The proof follows from the fact that the $K \times K$ Vandermonde determinant in $\mathbb{Z}_K$ is non-zero.

Each gaussian sum $G(p,q,l)$ appears in the $l.h.s.$ of Eq. (3.7) with its own power of n: n^{2l}. Therefore the coefficients $G_{m'}(p,q;l)$ of Eq. (3.8) can be calculated by "dividing" the polynomial $(-1)^l (p^*q)^l \sum_{m=l}^{\frac{K-1}{2}} (m!)^* C_{m,l} x^m$ appearing at n^{2l} in the $r.h.s.$ of Eq. (3.7) by the polynomial $\sum_{m=2l}^{K-1} C_{m,2l} 4^l x^{m-l}$ appearing in the $r.h.s.$ of that equation at the same power on n. The division is not quite well-defined, hence the indeterminacy in the elements w of Eq. (3.8).

This whole calculation of dividing the polynomials can be made more explicit if we go back to Eq. (3.5) and make the following substitutions

$$\check{q}^{2n\alpha} = e^{2n\alpha \log(1+x)} = \sum_{l=0}^{\infty} \frac{(2n\alpha)^l}{l!} (\log(1+x))^l , \tag{3.9}$$

$$\check{q}^{-pq^*n^2} = e^{-pq^*n^2 \log(1+x)} = \sum_{l=0}^{\infty} (-1)^l \frac{(pq^*n^2)^l}{l!} (\log(1+x))^l \tag{3.10}$$

After "checking" the logarithm

$$\log^{\vee}(1+x) = x \sum_{n=0}^{K-2} (-1)^n (n+1)^* x^n , \tag{3.11}$$

we see that Eq. (3.7) transforms into

$$\left[\sum_{l=0}^{\frac{K-1}{2}} G(p,q;l)(2n)^{2l} ((2l)!)^* (\log^{\vee}(1+x))^l \left(\frac{\log^{\vee}(1+x)}{x} \right)^l \right]^{\diamond}$$

$$= (pq^*/K) \left[\sum_{l=0}^{\frac{K-1}{2}} (-1)^l n^{2l} (p^*q)^l (l!)^* (\log^{\vee}(1+x))^l \right]^{\diamond} \tag{3.12}$$

Thus we find that

$$[G(p,q,m)]^{\diamond} = (-1)^m (pq^*/K)(2^*)^{2m} (2m)! (m!)^* (p^*q)^m \left[\left(\frac{x}{\log^{\vee}(1+x)} \right)^m \right]^{\diamond}$$

$$+ x^{\frac{K+1}{2}-m} w, \quad w \in \mathbb{Z}[\check{q}] \tag{3.13}$$

Consider now the following identity which is an integral analog of Eq (3.5)·

$$\int_{-\infty}^{+\infty} d\alpha\, \check{q}^{\frac{p}{q}\alpha^2} \check{q}^{2n\alpha} = e^{i\frac{\pi}{4}\operatorname{sign}\left(\frac{p}{q}\right)} \left(\frac{K}{2}\right)^{\frac{1}{2}} \left|\frac{q}{p}\right|^{\frac{1}{2}} \check{q}^{-\frac{q}{p}n^2} \tag{3.14}$$

After substituting Eqs (3 9), (3 10) we find that

$$\begin{aligned}\int_{-\infty}^{+\infty} d\alpha\, \check{q}^{\frac{p}{q}\alpha^2} \alpha^{2m} x^m &= e^{i\frac{\pi}{4}\operatorname{sign}\left(\frac{p}{q}\right)} \left(\frac{K}{2}\right)^{\frac{1}{2}} \left|\frac{q}{p}\right|^{\frac{1}{2}} \\ &\quad \times \frac{(-1)^m (2m)!}{4^m m!} \left(\frac{q}{p}\right)^m \left(\frac{x}{\log(1+x)}\right)^m \end{aligned} \tag{3.15}$$

Equation (3.2) follows from comparing Eq (3.15) to Eq. (3 13).

The formula (3 2) can be generalized to the type of summands that appear in Eq. (2.8) after the substitutions (2 9) and (2 12).

Proposition 3.2. *For* $p \neq 0 \pmod K$, $0 \leqq m \leqq K$,

$$\begin{aligned}&\left[\frac{e^{i\frac{\pi}{4}(K-1)}}{\sqrt{K}} \sum_{\substack{-K \leqq \gamma \leqq K \\ \gamma \in 2\mathbb{Z}+1}}{}' \check{q}^{4^* p\gamma^2} P_m\left(\frac{\alpha - 1 + p^*}{2}\right) x^{\left[\frac{m}{2}\right]}\right]^{\diamond} \\ &= (p/K)\left[e^{-i\frac{\pi}{4}\operatorname{sign}(q)} \left(\frac{|p|}{2K}\right)^{\frac{1}{2}} \int_{-\infty}^{+\infty} d\alpha\, e^{\frac{i\pi}{2K} p\alpha^2} P_m\left(\frac{\alpha - 1 + \frac{1}{p}}{2}\right) x^{\left[\frac{m}{2}\right]}\right]^{\diamond} \\ &\quad + x^{\frac{K+1}{2} - \left[\frac{m}{2}\right]} w, \quad w \in \mathbb{Z}[\check{q}]\,. \end{aligned} \tag{3.16}$$

To prove the proposition for $m < K$ we substitute $P_m^{\vee}(2^*(\alpha - 1 + p^*))$ for $P_m\left(\frac{\alpha - 1 + p^*}{2}\right)$ in the *l h s* of this equation and then take the sum for each monomial of $P_m^{\vee}(2^*(\alpha - 1 + p^*))$ separately If m is odd then the highest power α^m does not contribute to the sum, therefore the factor $x^{\left[\frac{m}{2}\right]}$ is enough to apply Eq. (3 2). We also used the multiplicativity of Legendre symbol and $(4^*/K) = 1$, since $4^* = (2^*)^2$.

The case of $m = K$ requires special care. We start with the *l h s* of Eq (3.16) We can use the symmetry of the summation range and gaussian exponent in order to substitute

$$P_K^{(\mathrm{ev})}(\alpha) = \frac{1}{2}\left(P_K\left(\frac{\alpha - 1 + p^*}{2}\right) + P_K\left(\frac{-\alpha - 1 + p^*}{2}\right)\right) \tag{3 17}$$

instead of $P_K\left(\frac{\alpha - 1 + p^*}{2}\right)$. The even polynomial (3 17) takes integer values for odd α and its degree is equal to $K - 1$ Therefore the highest divisor of denominators of its coefficients is $K - 1$ and we can apply all our previous results to the calculation of the contribution of its monomials Equation (3 16) indicates that we need to determine only the terms of order x^0, hence we are interested only in the contribution of the highest degree monomial

$$\frac{p^* - K}{2^K (K-1)!} \alpha^{\frac{K-1}{2}}\,, \tag{3 18}$$

which is determined with the help of Eq (3 13). Consider now the *r h s* of Eq (3 16). Again we substitute the even polynomial

$$P_K^{(cv)}(\alpha) = \frac{1}{2}\left(P_K\left(\frac{\alpha - 1 + \frac{1}{p}}{2}\right) + P_K\left(\frac{-\alpha - 1 + \frac{1}{p}}{2}\right)\right) \tag{3 19}$$

Some of its denominators may have K as a divisor, but according to Eq (3.16) we are interested only in the contribution of the highest power of α

$$\frac{\frac{1}{p} - K}{2^K (K-1)!}\alpha^{\frac{K-1}{2}} \tag{3 20}$$

Comparing it to monomial (3.18) and applying Eq (3.2) to their contributions we arrive at Eq (3 16) This ends the proof of Proposition 3 2

Next we move to the polynomials $\tilde{D}_{m,n}$ which participate in the expansion Eq. (2.9).

Proposition 3.3. *The gaussian sums and integrals of polynomials $\tilde{D}_{m,n}$ are related by the equation*

$$\left[\frac{e^{i\frac{\pi}{4}N(K-1)}}{K^{\frac{N}{2}}} \sum_{\substack{-K \leqq \gamma_j \leqq K \\ \gamma_j \in 2\mathbb{Z}+1 \\ (1 \leqq j \leqq N)}}{}' \check{q}^{4^* \sum_{j=1}^N p_j \gamma_j^2} \left(\prod_{j=1}^N (\alpha_j + p_j^*)\right) \sum_{m \leqq \frac{3}{4}n} \tilde{D}_{m,n}(\alpha_1 + p_1^*, \quad , \alpha_N + p_N^*) x^n\right]^{\diamond}$$

$$= \left(\prod_{j=1}^N p_j / K\right)\left[e^{-i\frac{\pi}{4}\sum_{j=1}^N \operatorname{sign}(p_j)}(2K)^{-\frac{N}{2}}\left|\prod_{j=1}^N p_j\right|^{\frac{1}{2}}\right.$$

$$\left. \times \int_{-\infty}^{+\infty} d\alpha\, e^{\frac{i\pi}{2K}\sum_{j=1}^N p_j \alpha_j^2} \left(\prod_{j=1}^N \left(\alpha_j + \frac{1}{p_j}\right)\right) \sum_{m \leqq \frac{3}{4}n} \tilde{D}_{m,n}\left(\alpha_1 + \frac{1}{p_1}, \quad , \alpha_N + \frac{1}{p_N}\right)\right]^{\diamond} \tag{3 21}$$

To prove this proposition we rearrange the representation (2 12) in the following form:

$$\left(\prod_{j=1}^N \alpha_j\right) \sum_{m \leqq \frac{3}{4}n} \tilde{D}_{m,n}(\alpha_1, \quad , \alpha_N) x^n = x^{n - \sum_{j=1}^N \left[\frac{m_j}{2}\right]} \sum_{\substack{1 \leqq m_j \leqq 2\tilde{m}_j + 1 \\ (1 \leqq j \leqq N)}} C^{(n)}_{m_1, \ , m_N} x^{\left[\frac{m_1}{2}\right]}$$

$$\times P_{m_1}\left(\frac{\alpha_1 - 1}{2}\right) \quad x^{\left[\frac{m_N}{2}\right]} P_{m_N}\left(\frac{\alpha_N - 1}{2}\right) \tag{3 22}$$

We know from Lemma 2 2 that the contribution of each polynomial $x^{\left[\frac{m_j}{2}\right]} P_{m_j}\left(\frac{\alpha_j - 1}{2}\right)$ to the *l h s* of Eq (3 21) belongs to $\mathbb{Z}[\check{q}]$ Therefore the contribution of the whole expression (3 22) starts at $x^{n - \sum_{j=1}^N \left[\frac{m_j}{2}\right]}$. Hence we may assume that

$$n - \sum_{j=1}^N \left[\frac{m_j}{2}\right] \leqq \frac{K-1}{2}, \tag{3.23}$$

otherwise the contribution of the polynomial

$$x^{n-\sum_{j=1}^{\lambda}\left[\frac{m_j}{2}\right]} C^{(n)}_{m_1,\ \ldots,m_\lambda} x^{\left[\frac{m_1}{2}\right]} P_{m_1}\left(\frac{\alpha_1-1}{2}\right) \quad \ldots \quad x^{\left[\frac{m_\lambda}{2}\right]} P_{m_\lambda}\left(\frac{\alpha_N-1}{2}\right) \tag{3.24}$$

is annihilated by the homomorphism $^\diamond$.

The inequalities (2 14) and (3 23) mean that $m_j \leqq \frac{K-1}{2}$, so we can apply Proposition 3 2 to every polynomial $x^{\left[\frac{m_j}{2}\right]} P_{m_j}\left(\frac{\alpha_j-1}{2}\right)$ The terms $x^{\frac{K+1}{2}-\left[\frac{m_j}{2}\right]} w$ of Eq (3 16) can be neglected because

$$\left[x^{n-\sum_{j=1}^{\lambda}\left[\frac{m_j}{2}\right]} x^{\frac{K+1}{2}-\left[\frac{m_j}{2}\right]} w\right]^\diamond = \left[x^{\frac{K+1}{2}+\left(n-\sum_{j=1}^{\lambda}\left[\frac{m_j}{2}\right]-\left[\frac{m_j}{2}\right]\right)} w\right]^\diamond = 0 \tag{3 25}$$

in view of the inequality (2.14) This proves Proposition 3 2

Now we can prove Proposition 1 1. We substitute Eq. (2 9) into Eq (2 5), apply the homomorphism $^\diamond$ and retain only the relevant terms from the sum of Eq (2 9) Since the contribution of the *l h s* of Eq. (3 22) starts at $x^{n-\sum_{j=1}^{\lambda}\left[\frac{m_j}{2}\right]}$ and $\sum_{j=1}^{N}\left[\frac{m_j}{2}\right] \leqq m \leqq \frac{3}{4}n$, it is enough to retain only the terms with $n \leqq 2(K-1)$.

$$\begin{aligned}
[Z'(M;k)]^\diamond = &\left[e^{i\frac{\pi}{4}(k-1)\sum_{j=1}^{\lambda}(\mathrm{sign}(p_j)-1)}\left(\prod_{j=1}^{N}\mathrm{sign}\,(p_j)\right)\right.\\
&\times \breve{q}^{4^*\sum_{j=1}^{\lambda}(3\mathrm{sign}(p_j)-p_j-p_j^*)}\,\frac{e^{i\frac{\pi}{4}N(k-1)}}{K^{\frac{N}{2}}}\sum_{\substack{-k\leq\gamma_j\leq k\\ \gamma_j\in 2\mathbb{Z}+1\\ (1\leq j\leq N)}}{}' \breve{q}^{4^*\sum_{j=1}^{N}p_j\gamma_j^2}\left(\prod_{j=1}^{N}(\alpha_j+p_j^*)\right)\\
&\left.\times\sum_{n=0}^{2(K-1)}\sum_{m\leqq\frac{3}{4}n}\tilde{D}_{m,n}(\alpha_1+p_1^*,\ \ldots,\alpha_N+p_N^*)x^n\right]^\diamond\\
=&\left(\prod_{j=1}^{N}p_j/K\right)e^{i\frac{\pi}{4}(k-1)\sum_{j=1}^{\lambda}(\mathrm{sign}(p_j))-1)}\left[e^{-i\frac{\pi}{4}\sum_{j=1}^{\lambda}\mathrm{sign}(p_j)}\left(\prod_{j=1}^{N}\mathrm{sign}\,(p_j)\right)\right.\\
&\times\breve{q}^{\frac{1}{4}\sum_{j=1}^{\lambda}(3\mathrm{sign}(p_j)-p_j-p_j^*)}\frac{\left|\prod_{j=1}^{N}p_j\right|^{\frac{1}{2}}}{(2K)^{\frac{N}{2}}}\int_{-\infty}^{+\infty}d\alpha_1\quad\ldots\quad d\alpha_N\,e^{\frac{i\pi}{2K}\sum_{j=1}^{\lambda}p_j\alpha_j^2}\\
&\left.\times\left(\prod_{j=1}^{N}\left(\alpha_j+\frac{1}{p_j}\right)\right)\sum_{n=0}^{2(K-1)}\sum_{m\leqq\frac{3}{4}n}\tilde{D}_{m,n}\left(\alpha_1+\frac{1}{p_1},\ \ldots,\alpha_N+\frac{1}{p_N}\right)x^n\right]^\vee
\end{aligned} \tag{3.26}$$

Here we used an identity

$$\left[\breve{q}^{4^*\sum_{j=1}^{\lambda}(3\,\mathrm{sign}(p_j)-p_j-p_j^*)}\right]^\diamond = \left[\breve{q}^{\frac{1}{4}\sum_{j=1}^{\lambda}(3\,\mathrm{sign}(p_j)-p_j-p_j^*)}\right]^\vee \tag{3 27}$$

We can extend the sum over n in the *r h s* of Eq (3.27) to all $n \geqq 0$ because the contribution of $\tilde{D}_{m,n}x^n$ with $n > 2(K-1)$ starts above $x^{\frac{K-1}{2}}$ As a result, we

obtain the full Jones polynomial. Now using the identities

$$e^{i\frac{\pi}{4}(K-1)(\mathrm{sign}(p)-1)}(p/K)=(|p|/K)\ , \tag{3 28}$$

$$\left|\prod_{j=1}^{N} p_j\right| = |H_1(M,\mathbb{Z})|\ , \tag{3 29}$$

$$\mathrm{sign}\,(p) = ie^{-i\frac{\pi}{2}\mathrm{sign}(p)}\ , \tag{3 30}$$

we get the following formula

$$\begin{aligned}[Z'(M,k)]^{\diamond} &= (|H_1(M,\mathbb{Z})|/K)\\ &\times\left[e^{-\frac{3}{4}i\pi\sum_{j=1}^{N}\mathrm{sign}(p_j)}\frac{i^N}{(2K)^{\frac{N}{2}}}e^{\frac{i\pi}{2K}\sum_{j=1}^{N}\left(\mathrm{sign}(p_j)-p_j-\frac{1}{p_j}\right)}\left|\prod_{j=1}^{N}p_j\right|^{\frac{1}{2}}\right.\\ &\times\left.\int_{\substack{-\infty\\ |\alpha_j=0|}}^{+\infty} d\alpha_1\quad d\alpha_N\, e^{\frac{i\pi}{2K}\sum_{j=1}^{N}p_j\alpha_j^2}J_{\alpha_1+\frac{1}{p_1},\ \ldots,\alpha_N+\frac{1}{p_N}}(\mathcal{L};k)\right]^{\vee}\end{aligned} \tag{3 31}$$

Combining it with Eqs. (1.27) and (1 18) and using Eq (3.29) again we easily arrive at Eq. (1.23). Note that Eq (3.29) guarantees that $p_j\neq 0\ (\mathrm{mod}\,K)$ if $|H_1(M,\mathbb{Z})|\neq 0\ (\mathrm{mod}\,K)$ This proves Proposition 1 1.

4. A General Rational Surgery Formula

Up until this point we were working only with surgeries of the type $(p,1)$. This was enough to prove the theorems of Sect 1, because H Murakami and T Ohtsuki showed [6] that any RHS M can be constructed by $(p_j,1)$ surgeries on an ASL in S^3 up to a connected sum of lens spaces $L_{p'_j,1}$, that is, instead of M one might end up with $M\#L_{p'_1,1}\#\quad\#L_{p'_n,1}$. However from the technical point of view it would be better to have a formula for the invariant $Z'(M,k)$ of a manifold constructed by general rational surgeries (p_j,q_j) on the components of an N-component link $\mathcal{L}$ in S^3.

The formula for Witten's invariant $Z(M,k)$ was derived by L Jeffrey [10]

$$\begin{aligned}Z(M,k) &= Z(S^3,k)\exp\left[i\frac{\pi}{4}\frac{K-2}{K}\left(\sum_{j=1}^{N}\Phi(U^{(p_j,q_j)})-3\,\mathrm{sign}\,(L)\right)\right]\\ &\times\sum_{1\leqq\alpha_1,\ \ldots,\alpha_N\leqq K-1}J_{\alpha_1,\ \ldots,\alpha_N}(\mathcal{L},k)\prod_{j=1}^{N}\tilde{U}^{(p_j,q_j)}_{\alpha_j,1}\ ,\end{aligned} \tag{4.1}$$

here L is the linking matrix of $\mathcal{L}$, $\frac{p_j}{q_j}$ being the self-linking numbers The matrices

$$U^{(p_j,q_j)}=\begin{pmatrix}p_j & r_j\\ q_j & s_j\end{pmatrix}\in SL(2,\mathbb{Z}) \tag{4 2}$$

describe the surgeries (a meridian on the tubular neighborhood is glued to p_j(meridian) + q_j(parallel) of the link complement),

$$\Phi\begin{bmatrix} p & r \\ q & s \end{bmatrix} = \frac{p+s}{q} - 12s(p,q)\,, \tag{4.3}$$

$s(p,q)$ being the Dedekind sum, and

$$\begin{aligned}\tilde{U}^{(p,q)}_{\gamma\beta} &= i\frac{\mathrm{sign}\,(q)}{\sqrt{2K|q|}}e^{-i\frac{\pi}{4}\Phi(U^{(p,q)})}\sum_{n=0}^{q-1}\sum_{\mu=\pm 1}\mu \\ &\times\exp\left[\frac{i\pi}{2Kq}(p\gamma^2 - 2\gamma(2Kn+\mu\beta)+s(2Kn+\mu\beta)^2)\right]\end{aligned} \tag{4.4}$$

(see [10], Propositions 2.7 and 2.8)

Let us introduce some notations. A rational (p_j,q_j) surgery on $\mathcal{L}_j$ can be presented as a combination of $(m_t^{(j)},1)$, $1\leqq t\leqq \vec{t}^{(j)}$ surgeries on a chain of unknots simply linked to $\mathcal{L}_j$ (see *e.g.* [9, 10] and references therein) such that

$$U^{(p_j,q_j)} = T^{m^{(j)}_{\vec{t}^{(j)}}}ST^{m^{(j)}_{\vec{t}^{(j)}-1}}S\quad\cdots\quad T^{m^{(j)}_1}S\,, \tag{4.5}$$

here

$$S=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\qquad T=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \tag{4.6}$$

We denote this chain (including $\mathcal{L}_j$ itself) as $\tilde{\mathcal{L}}_j$ and all the chains of $\mathcal{L}$ as $\tilde{\mathcal{L}}$. For $1\leqq t\leqq \vec{t}^{(j)}$ we set

$$U^{(p_t^{(j)},q_t^{(j)})} = T^{m_t^{(j)}}S\quad\cdots\quad T^{m_1^{(j)}}S\,, \tag{4.7}$$

so that $p^{(j)}_{\vec{t}^{(j)}} = p_j$, $q^{(j)}_{\vec{t}^{(j)}} = q_j$. From now on we assume for simplicity that none of the numbers $q_t^{(j)}$ is divisible by K. Then we are going to prove the following:

Proposition 4.1. *Let M be a manifold constructed by (p_j,q_j) surgeries on an N-component link $\mathcal{L}$ in S^3. Then*

$$\begin{aligned}Z'(M,k) &= \left(\left|\prod_{j=1}^N q_j\right|/K\right)\frac{\prod_{j=1}^N \mathrm{sign}\,(q_j)}{K^{\frac{N}{2}}}e^{-i\frac{\pi}{4}K\,\mathrm{sign}(L)}e^{-i\pi\frac{3}{4}\frac{k-2}{K}\,\mathrm{sign}(L)} \\ &\times\check{q}^{-4^*\sum_{j=1}^N\Phi(U^{(p_j,q_j)})}\check{q}^{(2^*-\frac{1}{2})\sum_{j=1}^N \mathrm{sign}\left(\frac{p_j}{q_j}\right)} \\ &\times\sum_{\substack{-K\leqq\gamma_j\leqq K\\ \gamma_j\in 2\mathbb{Z}-1\\ (1\leqq j\leqq N)}}{}' J_{\gamma_1,\ldots,\gamma_N}(\mathcal{L},k)\,\check{q}^{4^*\sum_{j=1}^N q_j^*(p_j\gamma_j^2+s_j)}\prod_{j=1}^N\left(\frac{i}{2}\sum_{\mu_j=\pm 1}\mu_j\check{q}^{-2^*q_j^*\mu_j\gamma_j}\right)\end{aligned} \tag{4.8}$$

(s_j *come from Eq.* (4.2))

We could use the general surgery formula (4.8) instead of the $(p_j, 1)$ surgery formula (1.6) throughout Sects. 2 and 3 in order to produce a somewhat more flexible proof of Theorems 1.1, 1.2 and Proposition 1.1.

We begin the proof of Proposition 4.1 by recalling the Kirby-Melvin formula ([3], Corollary 8.9) which expresses $Z'(M,k)$ in terms of data associated to $\tilde{\mathcal{L}}$.

$$Z'(M,k) = e^{\iota\frac{\pi}{4}\frac{K-2}{K}\left[\sum_{l=1}^{N}\Phi(U^{(p_l,q_l)})-3\,\mathrm{sign}(L)\right]} e^{-\iota\frac{\pi}{4}K\,\mathrm{sign}(\tilde{L})} \times \sum_{\substack{-K\le\alpha_j\le K \\ \alpha_j\in 2\mathbb{Z}+1 \\ (1\le j\le N)}}{}' J_{\alpha_1,\ldots,\alpha_N}(\mathcal{L},k)\prod_{j=1}^{N}\check{U}^{(p_j,q_j)}_{\alpha_j 1}, \tag{4.9}$$

here (we drop the index j in Eq. (4.5))

$$\check{U}^{(p,q)}_{\alpha\beta} = \left(\check{T}^{m_{\tilde{r}}}\check{S}\,\check{T}^{m_{\tilde{r}-1}}\check{S}\quad\cdots\quad\check{T}^{m_1}\check{S}\right)_{\alpha\beta}, \tag{4.10}$$

$$\check{T}_{\alpha\beta} = e^{-\iota\frac{\pi}{4}}\check{q}^{\frac{1}{4}\alpha^2}\delta_{\alpha\beta}, \qquad \check{S}_{\alpha\beta} = \frac{1}{\sqrt{K}}\sin\left(\frac{\pi}{K}\alpha\beta\right), \tag{4.11}$$

and $\tilde{L}$ is the linking matrix of the "expanded" link $\tilde{\mathcal{L}}$.

The following lemma presents an explicit expression for $\check{U}^{(p,q)}_{\alpha\beta}$, which is similar to that of Eq. (4.4).

Lemma 4.1.

$$\check{U}^{(p,p)}_{\alpha\beta} = (|q|/K)\frac{\mathrm{sign}\,(q)}{\sqrt{K}}e^{-\iota\frac{\pi}{4}\Phi(U^{(p,q)})}e^{-\iota\frac{\pi}{4}K\sum_{t=0}^{\tilde{r}-1}\mathrm{sign}\left(\frac{p_t}{q_t}\right)} \times\check{q}^{\left(\frac{1}{4}-4^*\right)\sum_{t=1}^{\tilde{r}}m_t+\left(2^*-\frac{1}{2}\right)\sum_{t=1}^{\tilde{r}}\mathrm{sign}\left(\frac{p_t}{q_t}\right)}\check{q}^{4^*q^*(p\alpha^2+s\beta^2)}\frac{i}{2}\sum_{\mu=\pm1}\mu\,\check{q}^{-2^*q^*\mu\alpha\beta}. \tag{4.12}$$

To prove the lemma we slightly change Eq. (4.11)

$$\check{T}_{\alpha\beta} = e^{-i\frac{\pi}{4}}\check{q}^{\frac{1}{4}-4^*}\check{q}^{4^*\alpha^2}\delta_{\alpha\beta}, \qquad \check{S}_{\alpha\beta} = \frac{1}{\sqrt{K}}\frac{i}{2}\check{q}^{\pm\left(2^*-\frac{1}{2}\right)}\sum_{\mu=\pm1}\mu\,\check{q}^{-2^*\mu\alpha\beta}, \tag{4.13}$$

the choice of sign in $\check{q}^{\pm\left(2^*-\frac{1}{2}\right)}$ is arbitrary.

We prove Eq. (4.12) by induction on $\tilde{t}$. If $\tilde{t}=1$, that is, if $U^{(p,q)}=T^mS$, then the check is trivial if we recall that $\Phi(T^mS)=m$. It is also easy to check Eq. (4.12) for $U^{(p+mq,q)}=T^mU^{(p,q)}$. It remains to check Eq. (4.12) for $U^{(-q,p)}=SU^{(p,q)}$.

We have to calculate the sum

$$\check{U}^{(-q,p)}_{\alpha\beta} = \sum_{\substack{-K\le\alpha_\iota\le K \\ \in 2\mathbb{Z}+1}}{}'\check{S}_{\alpha\iota}\check{U}^{(p,q)}_{\iota\beta} \tag{4.14}$$

The following gaussian sum is at the center of this calculation

$$\begin{aligned}
&\sum_{\substack{-K\le s_1\le K\\ s_1\in 2\mathbb{Z}+1}}{}' \sum_{\mu_1\mu_2=\pm 1} \mu_1\mu_2\, \breve{q}^{4^* q^* p s_1^2 - 2^* s_1 (q^*\mu_1\beta+\mu_2\alpha)+4^* q^* s\beta^2} \\
&= \sum_{\substack{-K\le s_2\le K\\ s_2\in 2\mathbb{Z}+1}}{}' \sum_{\mu_1\mu_2=\pm 1} \mu_1\mu_2\, \breve{q}^{4^* q^* p (s_2 - p^*\mu_1\beta - p^* q\mu_2\alpha)^2 - 4^* q^* p^* (\mu_1\beta+q\mu_2\alpha)^2 + 4^* q^* s\beta^2} \\
&= 2\sqrt{K} e^{i\frac{\pi}{4}(\kappa-1)} \left(pq^*/K\right) \breve{q}^{4^* p^* (-q\alpha^2+s\beta^2)} \sum_{\mu=\pm 1} \mu\, \breve{q}^{-2^* p^* \mu\alpha\beta}\,, \qquad (4.15)
\end{aligned}$$

here we used the following relations:

$$\sum_{\substack{-K\le s\le K\\ s\in 2\mathbb{Z}+1}}{}' \breve{q}^{4^* pq^* s^2} = e^{i\frac{\kappa}{4}(\kappa-1)} \sqrt{K}\left(pq^*/K\right), \qquad (4.16)$$

$$s - p^* = p^* qr\,, \qquad (4.17)$$

the latter relation follows from $ps - qr = 1$. To complete the verification of Eq. (4.12) we recall the following identities.

$$\Phi(SU^{(p,q)}) = \Phi(U^{(p,q)}) - 3\,\mathrm{sign}\left(\frac{p}{q}\right), \qquad (4.18)$$

$$i\,\mathrm{sign}\,(q) = e^{i\frac{\pi}{2}\,\mathrm{sign}\left(\frac{p}{q}\right)}\,\mathrm{sign}\,(p), \qquad (4.19)$$

$$e^{i\frac{\pi}{4}(\kappa-1)}\left(|q|/K\right)\left(pq^*/K\right) = e^{i\frac{\pi}{4}(\kappa-1)\,\mathrm{sign}\left(\frac{p}{q}\right)}\left(|p|/K\right) \qquad (4.20)$$

This ends the proof of the lemma.

To finish the proof of Proposition 4.1 we rearrange some phase factors of Eqs. (4.9) and (4.12). We substitute a relation

$$\mathrm{sign}\left(\tilde{L}\right) = \mathrm{sign}\,(L) + \sum_{j=1}^{N} \sum_{t=1}^{l^{(j)}-1} \mathrm{sign}\left(\frac{p_t^{(j)}}{q_t^{(j)}}\right) \qquad (4.21)$$

into the factor $e^{-i\frac{\pi}{4}\kappa\,\mathrm{sign}(\tilde{L})}$ of Eq. (4.9). Then we calculate the combination of phases coming from that factor and from Eq. (4.12) (we drop the index j):

$$\begin{aligned}
&\breve{q}^{\left(\frac{1}{4}-4^*\right)\sum_{t=1}^{\tilde{l}} m_t + \left(2^*-\frac{1}{2}\right)\sum_{t=1}^{\tilde{l}}\mathrm{sign}\left(\frac{p_t}{q_t}\right)} e^{-i\frac{\pi}{2}\kappa\sum_{t=1}^{\tilde{l}-1}\mathrm{sign}\left(\frac{p_t}{q_t}\right)} \\
&= \breve{q}^{\left(\frac{1}{4}-4^*\right)\sum_{t=1}^{\tilde{l}} m_t + \left(2^*-\frac{1}{2}\right)\mathrm{sign}\left(\frac{p}{q}\right)} \breve{q}^{\left(2^*-\frac{1}{2}-\frac{\kappa K}{4}\right)\sum_{t=1}^{\tilde{l}-1}\mathrm{sign}\left(\frac{p_t}{q_t}\right)} \\
&= \breve{q}^{\left(\frac{1}{4}-4^*\right)\left[\sum_{t=1}^{\tilde{l}} m_t - 3\sum_{t=1}^{\tilde{l}-1}\mathrm{sign}\left(\frac{p_t}{q_t}\right)\right]} \breve{q}^{\left(2^*-\frac{1}{2}\right)\mathrm{sign}\left(\frac{p}{q}\right)} \\
&= \breve{q}^{\left(\frac{1}{4}-4^*\right)\Phi(U^{(p,q)})}\, \breve{q}^{\left(2^*-\frac{1}{2}\right)\mathrm{sign}\left(\frac{p}{q}\right)}. \qquad (4.22)
\end{aligned}$$

We used here Eq. (2.7) and the formula

$$\Phi(U^{(p,q)}) = \sum_{t=1}^{\tilde{l}} m_t - 3\sum_{t=1}^{\tilde{l}-1}\mathrm{sign}\left(\frac{p_t}{q_t}\right) \qquad (4.23)$$

(see [10] and references therein)

A combination of Eqs (4.9), (4.12), (4.21) and (4.22) leads to Eq. (4.8). This concludes the proof of Proposition 4.1.

As an application of Proposition 4.1 let us calculate the invariant of a lens space $L_{p,q}$. Its Witten's invariant was calculated originally by L. Jeffrey in [10] (Theorems 3.4 and 3.7).

This manifold is constructed by a $U^{(-p,q)}$ surgery on an unknot in S^3. Since

$$J_\alpha(\text{unknot}, k) = \frac{\sin\left(\frac{\pi}{K}\alpha\right)}{\sin\left(\frac{\pi}{K}\right)}, \tag{4.24}$$

we can say that $L_{p,q}$ is constructed by a chain surgery $U^{(-q,-p)} = SU^{(-p,q)}$ applied to an empty knot, times a factor $\frac{\sqrt{K}}{\sin\left(\frac{\pi}{K}\right)}$. Then we can read the result directly from Eq. (4.8) by setting there $N = 1$, $\text{sign}(L) = 0$, $\alpha = 1$ and

$$U^{(-q,-p)} = \begin{pmatrix} -q & s \\ -p & r \end{pmatrix} \tag{4.25}$$

instead of $U^{(p,q)}$.

$$\begin{aligned} Z'(L_{p,q}) &= (|p|/K)\,\text{sign}(p)\,\check{q}^{4^*(p^*(q-1)-\Phi(U^{(-q,-p)}))}\,\check{q}^{2^*}\,\frac{\check{q}^{2^*p^*} - \check{q}^{-2^*p^*}}{\check{q} - 1} \\ &= (|p|/K)\,\text{sign}(p)\,\check{q}^{3s^\vee(q,p)}\,\check{q}^{2^*}\,\frac{\check{q}^{2^*p^*} - \check{q}^{-2^*p^*}}{\check{q} - 1} \\ &= (|p|/K)\,\text{sign}(p)\,\check{q}^{3s^\vee(q,p)}\,(-1)^{p^*+1}\frac{\check{q}^{\frac{p^*}{2}} - \check{q}^{-\frac{p^*}{2}}}{\check{q}^{\frac{1}{2}} - \check{q}^{-\frac{1}{2}}}, \end{aligned} \tag{4.26}$$

here $s^\vee(q,p)$ is the "checked" Dedekind sum, that is, its denominator is inverted modulo K as in Eq. (1.13). We used the relation $\check{q}^{2^*} = -\check{q}^{\frac{1}{2}}$ in order to derive the last expression in this equation. Although it might look simpler than the previous one, it obscures the fact that $Z'(L_{p,q}) \in \mathbb{Z}[\check{q}]$. Also note that the expression $\check{q}^{\frac{p^*}{2}}$ by itself is ambiguous since p^* is defined only modulo K.

Comparing the second expression in the $r.h.s.$ of Eq. (4.26) with the formula for the trivial connection contribution to Witten's invariant of the lens space

$$Z^{(\text{tr})}(L_{p,q}, k) = Z(S^3; k)\frac{\text{sign}(p)}{\sqrt{|p|}}\check{q}^{3s(p,q)}\frac{\check{q}^{\frac{p}{2}} - \check{q}^{-\frac{p}{2}}}{\check{q}^{\frac{1}{2}} - \check{q}^{-\frac{1}{2}}} \tag{4.27}$$

derived in [10] (take the term $n = p$ in the formula of the Theorem 3.4 of [10]), we can easily check the relation (1.20).

The formula (4.8) also allows us to check Proposition 1.1 for Seifert manifolds which are rational homology spheres. Consider an $(N+1)$-component link $\mathcal{L}$ in S^3 consisting of N unknots $\mathcal{L}_j$, $1 \leqq j \leqq N$ simply linked to a single unknot $\mathcal{L}_0$. A Seifert manifold $X\left(\frac{p_1}{q_1}, \quad , \frac{p_n}{q_n}\right)$ is constructed by performing (p_j, q_j) surgeries on the components $\mathcal{L}_j$ and a $(0,1)$ surgery on $\mathcal{L}_0$ (see, $e.g.$ [9]). The Jones polynomial

of $\mathcal{L}$ is known [1] to be equal to

$$J_{\beta,\alpha_1,\ldots,\alpha_N}(\mathcal{L},k)=\frac{1}{\sin\left(\frac{\pi}{K}\right)}\frac{\prod_{j=1}^{N}\sin\left(\frac{\pi}{K}\beta\alpha_j\right)}{\sin^{N-1}\left(\frac{\pi}{K}\beta\right)}\,, \tag{4.28}$$

here β is the color of $\mathcal{L}_0$ and α_j are the colors of $\mathcal{L}_j$ The signature of $\mathcal{L}$ is equal to

$$\mathrm{sign}\,(L)=-\,\mathrm{sign}\left(\frac{H}{P}\right)+\sum_{j=1}^{N}\mathrm{sign}\left(\frac{q_j}{p_j}\right) \tag{4.29}$$

Here we introduced notations

$$P=\prod_{j=1}^{N}p_j,\qquad H=P\sum_{j=1}^{N}\frac{q_j}{p_j}\,, \tag{4.30}$$

so that

$$\left|H_1\left(X\left(\frac{p_1}{q_1},\ldots,\frac{p_n}{q_n}\right)\right)\right|=|H| \tag{4.31}$$

Proposition 4.1 provides the following expression for the invariant of the Seifert manifold $X\left(\frac{p_1}{q_1},\ldots,\frac{p_n}{q_n}\right)$.

$$\begin{aligned}Z'(X,k)\;=\;&\frac{i}{2\sqrt{K}}e^{\imath\frac{\pi}{4}K\,\mathrm{sign}\left(\frac{H}{P}\right)}e^{\imath\pi\frac{3}{4}\frac{K-2}{K}\,\mathrm{sign}\left(\frac{H}{P}\right)}\check{q}^{\left(\frac{1}{2}-2^*\right)\mathrm{sign}\left(\frac{H}{P}\right)}\\&\times\sum_{\substack{-K\le\beta\le K\\ \beta\in 2\mathbb{Z}+1}}{}'(\check{q}^{-2^*\beta}-\check{q}^{2^*\beta})Z'_\beta(L^{(\mathrm{con})},k)\,,\end{aligned} \tag{4.32}$$

here

$$Z'_\beta(L^{(\mathrm{con})},K)\;=\;\left(\frac{\sin\left(\frac{\pi}{K}\right)}{\sin\left(\frac{\pi}{K}\beta\right)}\right)^{N-1}\prod_{j=1}^{N}Z'_\beta(L_{-p_j,q_j};K)\,, \tag{4.33}$$

$$Z'_\beta(L_{-p_j,q_j},K)\;=\;(|q_j|/K)\,\frac{\mathrm{sign}\,(q_j)}{\sqrt{K}}e^{-\imath\frac{\pi}{K}K\,\mathrm{sign}\left(\frac{q_j}{p_j}\right)}e^{-\imath\pi\frac{3}{4}\frac{K-2}{K}\,\mathrm{sign}\left(\frac{q_j}{p_j}\right)} \tag{4.34}$$

$$\begin{aligned}&\times\check{q}^{-4^*\Phi(U^{(p_j\,q_j)})}\check{q}^{\left(2^*-\frac{1}{2}\right)\mathrm{sign}\left(\frac{q_j}{p_j}\right)}\sum_{\substack{-K\le\nu_j\le K\\ \nu_j\in 2\mathbb{Z}+1\\ (1\le j\le N)}}{}'\frac{\sin\left(\frac{\pi}{K}\beta\alpha_j\right)}{\sin\left(\frac{\pi}{K}\right)}\check{q}^{4^*q_j^*(p_j\nu_j^2+\nu_j)}\\&\times\frac{i}{2}\left(\check{q}^{-2^*q_j^*\alpha_j}-\check{q}^{2^*q_j^*\alpha_j}\right)\end{aligned} \tag{4.35}$$

In these formulas $Z'_\beta(L_{-p_j,q_j},k)$ is an invariant of the β-colored link $\mathcal{L}_0$ inside the lens space L_{-p_j,q_j} constructed by the (p_j,q_j) surgery on the unknot $\mathcal{L}_j$ $Z'_\beta(L^{(\mathrm{con})},k)$ is the invariant of the β-colored knot $\mathcal{L}_0$ inside the connected sum of lens spaces

$$L^{(\mathrm{con})}=L_{-p_1,q_1}\#\quad\#L_{-p_N,q_N} \tag{4.36}$$

The calculation of invariants $Z'_\beta(L_{-p_j,q_j},k)$ runs similar to that of $Z'(L_{p,q},k)$ in Eq (4.26) The only difference is that by taking a sum over α_j we go from $\tilde{U}^{(p_j,q_j)}_{\alpha_j 1}$ to $\tilde{U}^{(-q_j,p_j)}_{\beta 1}$ rather than to $\tilde{U}^{(-q_j,p_j)}_{11}$ As a result,

$$\begin{aligned} Z'_\beta(L_{-p_j,q_j},k) \;=\; & \big(|p_j|/K\big)\,\mathrm{sign}\,\big(p_j\big)\,\check{q}^{4^* p_j^* q_j - 3s^\vee(q_j,p_j)} \\ & \times \check{q}^{-4^* p_j^* q_j \beta^2}\,\frac{\check{q}^{-2^*p_j^*\beta}-\check{q}^{2^*p_j^*\beta}}{\check{q}^{-2^*}-\check{q}^{2^*}} \end{aligned} \tag{4.37}$$

and

$$\begin{aligned} Z'(X,k) \;=\; & \frac{i}{2\sqrt{K}}\,e^{l\frac{\pi}{4}\kappa\,\mathrm{sign}\left(\frac{H}{P}\right)}\,e^{i\pi\frac{3}{4}\frac{K-2}{K}\,\mathrm{sign}\left(\frac{H}{P}\right)}\,\check{q}^{\left(\frac{1}{2}-2^*\right)\mathrm{sign}\left(\frac{H}{P}\right)} \\ & \times\big(|P|/K\big)\,\mathrm{sign}\,(P)\,\check{q}^{4^*P^*H-3\sum_{j=1}^{N}s^\vee(q_j,p_j)} \\ & \times\frac{1}{\check{q}^{-2^*}-\check{q}^{2^*}}\sum_{\substack{-K\le\beta\le K\\ \beta\in 2\mathbb{Z}+1}}{}'\;\check{q}^{-4^*P^*H\beta^2}\,\frac{\prod_{j=1}^{N}(\check{q}^{-2^*p_j^*\beta}-\check{q}^{2^*p_j^*\beta})}{(\check{q}^{-2^*\beta}-\check{q}^{2^*\beta})^{N-2}} \end{aligned} \tag{4.38}$$

The preexponential factor of Eq (4.38) can be put in the form

$$J(\beta)(\check{q}^{-2^*\beta}-\check{q}^{2^*\beta})\,, \tag{4.39}$$

where

$$J(\beta)=\frac{\prod_{j=1}^{N}(\check{q}^{-2^*p_j^*\beta}-\check{q}^{2^*p_j^*\beta})}{(\check{q}^{-2^*}-\check{q}^{2^*})(\check{q}^{-2^*\beta}-\check{q}^{2^*\beta})^{N-1}}\,. \tag{4.40}$$

It is easy to check that the function $J(\beta)$ belongs to $\mathbb{Z}[\check{q}]$ and satisfies the properties of the Jones polynomial described in Proposition 1.2. Therefore the full machinery of Sect 3 could be applied to the sum of Eq (4.38) in order to convert it to the integral and ultimately prove Proposition 1.1 for Seifert manifolds However there is an easier way. The numbers p_j^*, $1\le j\le N$ determine a set of positive integer numbers $\Lambda(p_1^*,\ \ ,p_N^*)\in\mathbb{N}$ and multiplicity factors $C_n(p_1^*,\ \ ,p_N^*)\in\mathbb{Z}$, $n\in\Lambda(p_1^*,\ldots,p_N^*)$ such that

$$\frac{\prod_{j=1}^{N}(\check{q}^{-2^*p_j^*\beta}-\check{q}^{2^*p_j^*\beta})}{(\check{q}^{-2^*\beta}-\check{q}^{2^*\beta})^{N-1}}=\sum_{n\in\Lambda(p_1^*,\ \ ,p_N^*)}C_n(p_1^*,\ \ ,p_N^*)(\check{q}^{-2^*n\beta}-\check{q}^{2^*n\beta})\,. \tag{4.41}$$

Now we can apply Eq (3.5) to the calculation of the sum

$$\begin{aligned} & \frac{1}{\check{q}^{-2^*}-\check{q}^{2^*}}\sum_{\substack{-K\le\beta\le K\\ \beta\in 2\mathbb{Z}+1}}{}'\;\check{q}^{-4^*P^*H\beta^2}\,\frac{\prod_{j=1}^{N}(\check{q}^{-2^*p_j^*\beta}-\check{q}^{2^*p_j^*\beta})}{(\check{q}^{-2^*\beta}-\check{q}^{2^*\beta})^{N-2}} \\ & =\frac{2}{\check{q}^{-2^*}-\check{q}^{2^*}}\sum_{n\in\Lambda}C_n\sum_{\substack{-K\le\beta\le K\\ \beta\in 2\mathbb{Z}+1}}{}'\;\check{q}^{-4^*P^*H\beta^2}(\check{q}^{-2^*(n+1)\beta}-\check{q}^{-2^*(n-1)\beta}) \\ & =2\sqrt{K}\,(P^*H/K)\,e^{l\frac{\pi}{4}(\kappa-1)}\sum_{n\in\Lambda}C_n\frac{\check{q}^{4^*PH^*(n+1)^2}-\check{q}^{4^*PH^*(n-1)^2}}{\check{q}^{-2^*}-\check{q}^{2^*}} \\ & =2\sqrt{K}\,(P^*H/K)\,e^{l\frac{\pi}{4}(\kappa-1)}\sum_{n\in\Lambda}C_n\check{q}^{4^*PH^*(n^2+1)}\frac{\check{q}^{2^*PH^*n}-\check{q}^{-2^*PH^*n}}{\check{q}^{-2^*}-\check{q}^{2^*}} \end{aligned} \tag{4.42}$$

Combining Eqs (4.38) and (4.42) we find the formula

$$Z'(X,k) = (|H|/K)\,\mathrm{sign}\,(H)\,\check{q}^{4^*P^*H-3\,\cdot\,4^*\,\mathrm{sign}\left(\frac{H}{P}\right)-3\sum_{l=1}^{N}s^\vee(q_l,p_l)} \times \sum_{n\in\Lambda} C_n \check{q}^{4^*PH^*(n^2+1)}\frac{\check{q}^{-2^*PH^*n}-\check{q}^{2^*PH^*n}}{\check{q}^{-2^*}-\check{q}^{2^*}}\,, \tag{4.43}$$

which demonstrates that $Z'(X;k)\in\mathbb{Z}[\check{q}]$.

Now we come back to Eqs (4.41), (4.42) and use the fact that for $n\in\mathbb{Z}$,

$$\sum_{\substack{-K\le\beta\le K\\ \beta\in 2\mathbb{Z}+1}}{}' \check{q}^{-4^*P^*H\beta^2-2^*n\beta} = \sqrt{K}\,(P^*H/K)\,e^{i\frac{\pi}{4}(\kappa-1)}\check{q}^{4^*PH^*n^2}, \tag{4.44}$$

$$\int_{-\infty}^{+\infty} d\beta\,\check{q}^{-\frac{\beta^2}{4H^*P}-2^*\beta n} = e^{-i\frac{\pi}{4}\,\mathrm{sign}\left(PH^*\right)}\sqrt{2K|PH^*|}\,\check{q}^{4^*PH^*n^2}, \tag{4.45}$$

so that

$$\sum_{\substack{-K\le\beta\le K\\ \beta\in 2\mathbb{Z}+1}}{}' \check{q}^{-4^*P^*H\beta^2-2^*n\beta} = (P^*H/K)\,e^{i\frac{\pi}{4}(\kappa-1)}\frac{e^{i\frac{\pi}{4}\,\mathrm{sign}\left(PH^*\right)}}{\sqrt{2|PH^*|}}\int_{-\infty}^{+\infty} d\beta\,\check{q}^{-\frac{\beta^2}{4H^*P}-2^*\beta n}\,. \tag{4.46}$$

This equation allows us to convert the sum over β in Eq (4.42) into an integral Then by using Eq. (4.41) backwards we arrive at Eq. (4.38) with the integral instead of a sum

$$Z'(X,k) = (|H|/K)\,\mathrm{sign}\,(H)\,\check{q}^{4^*P^*H-3\,\cdot\,4^*\,\mathrm{sign}\left(\frac{H}{P}\right)-3\sum_{l=1}^{\Lambda}s^\vee(q_l,p_l)}I(\check{q})\,, \tag{4.47}$$

$$I(\check{q}) = \frac{1}{\check{q}^{2^*}-\check{q}^{-2^*}}\frac{e^{i\frac{\pi}{4}\,\mathrm{sign}\left(PH^*\right)}}{2\sqrt{2K|PH^*|}}\int_{-\infty}^{+\infty} d\beta\,\check{q}^{-\frac{\beta^2}{4PH^*}}\frac{\prod_{l=1}^{N}(\check{q}^{-2^*p_l^*\beta}-\check{q}^{2^*p_l^*\beta})}{(\check{q}^{-2^*\beta}-\check{q}^{2^*\beta})^{N-2}} \tag{4.48}$$

The integral over β is well defined in view of Eq (4.43) (actually, one might add a regularizing factor $\lim_{\varepsilon\to 0}e^{-\varepsilon\beta^2}$) It can also be calculated by expanding the preexponential factor in powers of $x=q-1$ and integrating their coefficients, which are polynomials in β, with the gaussian exponential $\check{q}^{-\frac{\beta^2}{4PH^*}}$ This procedure leads to the following relation:

$$\begin{aligned}
&[I(\check{q})]^\diamond \\
&= \left[\frac{1}{\check{q}^{\frac12}-\check{q}^{-\frac12}}\frac{e^{i\frac{\pi}{4}\,\mathrm{sign}\left(\frac{H}{P}\right)}}{2\sqrt{2K}}\sqrt{\left|\frac{H}{P}\right|}\int_{\substack{-\infty\\ [\beta=0]}}^{+\infty} d\beta\,\check{q}^{-\frac14\frac{H}{P}\beta^2}\frac{\prod_{l=1}^{\Lambda}\left(\check{q}^{-\frac{\beta}{2p_l}}-\check{q}^{\frac{\beta}{2p_l}}\right)}{\left(\check{q}^{-\frac{\beta}{2}}-\check{q}^{\frac{\beta}{2}}\right)^{N-2}}\right]^\vee \\
&= \left[\frac{1}{\sqrt{\frac{2}{K}}\sin\left(\frac{\pi}{K}\right)}\frac{\mathrm{sign}\left(\frac{H}{P}\right)}{K}e^{i\frac34\pi\,\mathrm{sign}\left(\frac{H}{P}\right)}\sqrt{\left|\frac{H}{P}\right|}\int_{\substack{-\infty\\ [\beta=0]}}^{+\infty} d\beta\,e^{-\frac{i\pi}{2K}\frac{H}{P}\beta^2}\frac{\prod_{l=1}^{\Lambda}\sin\left(\frac{\pi}{K}\frac{\beta}{p_l}\right)}{\sin^{N-2}\left(\frac{\pi}{K}\beta\right)}\right]^\vee
\end{aligned} \tag{4.49}$$

Since

$$\left[\check{q}^{4^*P^*H-3\cdot 4^*\operatorname{sign}\left(\frac{H}{P}\right)-3\sum_{j=1}^{\nu}s^{\vee}(q_j,p_j)}\right]^{\diamond}=\left[\check{q}^{\frac{1}{4}\frac{H}{P}-\frac{3}{4}\operatorname{sign}\left(\frac{H}{P}\right)-3\sum_{j=1}^{N}s(q_j,p_j)}\right]^{\vee}, \tag{4.50}$$

and according to [11, 12, 16] (see, e g Eq (4 9) of [16])

$$\begin{aligned} Z^{(\mathrm{u})}(X,k) &= \frac{e^{i\pi\frac{3}{4}\operatorname{sign}\left(\frac{H}{P}\right)}}{K}\frac{\operatorname{sign}(P)}{\sqrt{|P|}}e^{\frac{i\pi}{2K}\left[\frac{H}{P}-3\operatorname{sign}\left(\frac{H}{P}\right)-12\sum_{j=1}^{\nu}s(q_j,p_j)\right]} \\ &\quad\times\int_{\substack{-\infty\\|\beta=0|}}^{+\infty}d\beta\, e^{-\frac{i\pi}{2K}\frac{H}{P}\beta^2}\frac{\prod_{j=1}^{\nu}\sin\left(\frac{\pi}{K}\frac{\beta}{p_j}\right)}{\sin^{\nu-2}\left(\frac{\pi}{K}\beta\right)}, \end{aligned} \tag{4 51}$$

we conclude that Eq (1 23) holds for Seifert manifolds which are rational homology spheres

Ohtsuki's invariants of a particular class of 3-fibered Seifert manifolds constructed by an integer surgery on a $(2,n)$ torus knot were studied recently by R Lawrence [25] She came to a conclusion that these invariants were integer if a manifold was a homology sphere and that Ohtsuki series (*i e* the *l h s* of Eq (1.20)) converged K-adicly to the total invariant $Z'(M,k)$ This reinforcement of Ohtsuki's results demonstrates a very interesting interplay between the differential geometry (see the calculations of [13] which should presumably produce the invariants $S_n(M)$) and number theory

5. Discussion

We have already mentioned in the Introduction that if we view Eq. (1 27) as a definition of an infinite sequence of invariants $S_n(M)$ of a RHS, then Proposition 1.1 becomes a mathematically rigorous (modulo the proof of Reshetikhin's formula that we will provide in [21]) statement about a relation between $S_n(M)$ and Ohtsuki's invariants $\lambda_n(M)$ This relation combined with Theorem 1 2 can be expressed by one formula

$$\begin{aligned} &\left[|H_1(M,\mathbb{Z})|\,(|H_1(M,\mathbb{Z})|/K)\,Z'(M,k)\right]^{\diamond} \\ &\quad=\left[|H_1(M,\mathbb{Z})|^{\frac{3}{2}}\exp\left((S_n(M)-S_n(S^3))\left(\frac{i\pi}{K}\right)^n\right)\right]^{\vee}, \end{aligned} \tag{5 1}$$

which we have proved in Sect. 3

If we conjecture that the invariants $S_n(M)$ are related to the trivial connection contribution to the total Witten's invariant $Z(M,k)$ by Eq. (1 18) then we face a much more interesting equation (1 23). The *l h s* there comes from the number-theoretic manipulations with $Z'(M,k)$, while the *r h s* should be calculable by the differential-geometric methods of [13]

In our previous paper [18] we studied the properties of $S_n(M)$ as they follow from Eq (1 27), Proposition 1.2 and some other properties of Reshetikhin's formula [19] Now we use Eq (1 20) in order to extend them to $\lambda_n(M)$

Proposition 5.1. *The invariants $\lambda_n(M)$ are finite type invariants of RHS as defined in* [23] *and* [24] *An invariant $\lambda_n(M)$ is of Ohtsuki order $3n$, Ohtsuki' ([18]) order $2n$ and at most of Garoufalidis order n Also*

$$2^{4n}\, n!\,(2n)!\,(9n)!\,|H_1(M,\mathbb{Z})|^n \lambda_n(M) \in \mathbb{Z}\,, \tag{5 2}$$

$$|H_1(M,\mathbb{Z})|^n \lambda_n(M) \in \mathbb{Z}\left[\frac{1}{2},\frac{1}{3},\ \ .,\frac{1}{2n}\right] \tag{5 3}$$

To see that $\lambda_n(M)$ is of exactly Ohtsuki order $3n$ one has to find an n-component link $\mathcal{L}$ such that the alternating sum

$$\sum_{\mathcal{L}'\subset\mathcal{L}} (-1)^{\#\mathcal{L}'} \lambda_n(\chi_{\mathcal{L}'}(S^3)) \tag{5 4}$$

is non-zero (here $\chi_{\mathcal{L}}(S^3)$ denotes a manifold (RHS) constructed by $(1,1)$ surgeries on the components of a link $\mathcal{L}$ in S^3) Recall that according to Eq. (1.20),

$$\lambda_n(M) = \sum_{\substack{m_1,\ \ m_n\geq 0\\ m_1+2m_2+\ \ +nm_n=n}} C^n_{m_1,\ \ ,m_n} S_1^{m_1}(M)\ \ \ S_n^{m_n}(M) \tag{5 5}$$

for some numbers $C_{m_1,\ \ ,m_n}$ Consider the n-loop diagram consisting of $(n-1)$ small loops sitting on one big loop, and the corresponding [23] link $\mathcal{L}$ This diagram has no subdiagrams with only trivalent vertices. Then according to [18], for any $S_{n'}(M)$, $n'<n$ the alternating sums are equal to zero

$$\sum_{\mathcal{L}''\subset\mathcal{L}'} (-1)^{\#\mathcal{L}''} S_{n'}(\chi_{\mathcal{L}''}(M)) = 0\,, \tag{5 6}$$

if $\mathcal{L}'\subset L$ and $\#\mathcal{L}'\geqq n'$. Therefore an invariant $S_{n'}(M)$ is of Ohtsuki order less than n' with respect to $\mathcal{L}$ and its subdiagrams (of course, there exist other links for which the $l\,h\,s$ of Eq (5 6) is non-zero if $\#\mathcal{L}'=n'$). As a result, only the term $C_{0,\ \ ,0,1}S_n(M)$ in Eq (5 5) matters for the calculation of the alternating sum (5 4) Since, according to [18], the sum

$$\sum_{\mathcal{L}'\subset\mathcal{L}} (-1)^{\#\mathcal{L}'} S_n(\chi_{\mathcal{L}'}(S^3)) \tag{5 7}$$

is non-zero, we conclude that the sum (5 4) is also non-zero Hence $\lambda_n(M)$ is of exactly Ohtsuki order $3n$

Finally, let us comment on the relation (1 23) between the invariant $Z'(M;k)$ at prime values of K and the trivial connection contribution to Witten's invariant This relation is not an obvious result in the sense that the contributions of non-trivial connections to $Z'(M,k)$ do not seem to cancel at prime K (one might expect that there is some cancellation leaving only the contribution of the trivial connection) This can be seen at the example of a lens space $L_{p,q}$ for which

$$Z'(L_{p,q}) = \mathrm{sign}\,(p)\,(|p|/K)\; e^{\frac{6\pi i}{K}s^{\vee}(q,p)}\,(-1)^{p^*+1}\,\frac{\sin\left(\frac{\pi}{K}p^*\right)}{\sin\left(\frac{\pi}{K}\right)}\,, \tag{5 8}$$

$$\frac{Z^{(\mathrm{tr})}(L_{p,q},k)}{Z(S^3,k)} = \frac{\mathrm{sign}\,(p)}{\sqrt{|p|}}\; e^{\frac{6\pi i}{K}s(q,p)}\,\frac{\sin\left(\frac{\pi}{K}\frac{1}{p}\right)}{\sin\left(\frac{\pi}{K}\right)} \tag{5.9}$$

Although these two expressions have many common features, which ultimately lead to the relation (1.23), still their numerical values are quite different. The p and K dependence of $Z'(L_{p,q})$ is somewhat typical of contributions of $U(1)$-reducible connections. Besides, the Dedekind sum $s(q, p)$ is generally a fraction, so $s^{\vee}(q, p) \neq s(q, p)$.

We established the relation (1.23) by comparing directly the surgery formulas. It would be much better to have a conceptual explanation for this phenomenon. One might speculate that it would come from number theory and perhaps p-adic quantum field theories.

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References

1. Witten, E.: Commun. Math. Phys. **121**, 351 (1989)
2. Reshetikhin, N., Turaev, V.: Invent. Math. **103**, 547 (1991)
3. Kirby, R., Melvin, P.: Invent. Math. **105**, 473 (1991)
4. Garoufalidis, S.: Relations among 3-Manifold Invariants. Ph.D. Thesis, University of Chicago, August (1992)
5. Murakami, H.: Math. Proc. Camb. Phil. Soc. **115**, 253 (1994)
6. Murakami, H.: Quantum $SO(3)$-Invariants Dominate the $SU(2)$-Invariant of Casson and Walker. Preprint, to be published in Math. Proc. Cam. Phil. Soc.
7. Ohtsuki, T.: Math. Proc. Camb. Phil. Soc. **117**, 83 (1995)
8. Ohtsuki, T.: A Polynomial Invariant of Rational Homology 3-Spheres. Preprint UTMS 94-49, August 12, 1994
9. Freed, D., Gompf, R.: Commun. Math. Phys. **141**, 79 (1991)
10. Jeffrey, L.: Commun. Math. Phys. **147**, 563 (1992)
11. Rozansky, L.: A Large k asymptotics of Witten's Invariant of Seifert Manifolds. Preprint UTTG-06-93, hep-th/9303099, to be published in Commun. Math. Phys.
12. Rozansky, L.: Residue Formulas for the Large k Asymptotics of Witten's Invariants of Seifert Manifolds. The Case of $SU(2)$. Preprint UMTG-179-94, hep-th/9412075
13. Axelrod, S., Singer, I.: Chern-Simons Perturbation Theory. Proceedings of XXth Conference on Differential Geometric Methods in Physics, Baruch College, C.U.N.Y., NY, NY
14. Kontsevich, M.: Feynman Diagrams and Low-Dimensional Topology. Preprint, Max-Planck Institut für Mathematik, Bonn
15. Taubes, C.: Unpublished
16. Rozansky, L.: A Contribution of the Trivial Connection to the Jones Polynomial and Witten's Invariant of 3d Manifolds I. Preprint UMTG-172-93, UTTG-30-93, hep-th/9401061, to be published in Commun. Math. Phys.
17. Rozansky, L.: A Contribution of the Trivial Connection to the Jones Polynomial and Witten's Invariant of 3d Manifolds II. Preprint UMTG-187-94, hep-th/9403021, to be published in Commun. Math. Phys.
18. Rozansky, L.: The Trivial Connection Contribution to Witten's Invariant and Finite Type Invariants of Rational Homology Spheres. Preprint UMTG-182-95, q-alg/9503011
19. Rozansky, L.: J. Math. Phys. **35**, 5219 (1994)
20. Rozansky, L.: On Finite Type Invariants of Links and Rational Homology Spheres Derived from the Jones Polynomial and Witten-Reshetikhin-Turaev Invariant. Preprint q-alg/9511025
21. Rozansky, L.: In preparation
22. Ireland, K., Rosen, M.: A Classical Introduction to Modern Number Theory. Second edition, Berlin Heidelberg-New York: Springer-Verlag, 1990

23 Ohtsuki, T : Finite Type Invariants of Integral Homology 3-Spheres Preprint UTMS 94-42, June 1, 1994
24 Garoufalidis, S : On Finite Type 3-Manifold Invariants I Preprint, February 3, 1995
25 Lawrence, R : Asymptotic Expansions of Witten-Reshetikhin-Turaev Invariants for Some Simple 3-Manifolds Preprint IHES/M/95/39, to appear in J Mod Phys

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