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On the Deformability of Heisenberg Algebras

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Abstract: Based on the vanishing of the second Hochschild cohomology group of the Weyl algebra it is shown that differential algebras coming from quantum groups do not provide a non-trivial deformation of quantum mechanics. For the case of a q-oscillator there exists a deforming map to the classical algebra. It is shown that the differential calculus on quantum planes with involution, i.e., if one works in position-momentum realization, can be mapped on a q-difference calculus on a commutative real space. Although this calculus leads to an interesting discretization it is proved that it can be realized by generators of the undeformed algebra and does not possess a proper group of global transformations.

1. Introduction

It is known that the deformation of an algebra, either of Lie or associative type, is connected to its (Chevalley or Hochschild) cohomology [15]. More precisely, for an algebra $\mathbf{g}$ the second cohomology group $H^2(\mathbf{g},\mathbf{g})$ contains the information if a non-trivial deformation of it exists or not. In particular, if $H^2(\mathbf{g},\mathbf{g}) = 0$, then there exists no non-trivial deformation of $\mathbf{g}$.

This result can readily be applied to the case of quantum groups [8, 17]. Here one takes for example a finite-dimensional semisimple Lie algebra $\mathbf{g}$ and addresses the question of existence of deformations of its enveloping algebra $\mathscr{U}(\mathbf{g})$. It is well known that we have non-trivial deformations denoted by $\mathscr{U}_h(\mathbf{g})$ as long as one considers $\mathscr{U}_h(\mathbf{g})$ as being a Hopf algebra or at least a bialgebra. The non-triviality of this deformation comes from the fact that $H^2(\mathscr{U}(\mathbf{g}),\mathscr{U}(\mathbf{g}))_{\text{bialgebra}} \simeq \Lambda^2(\mathbf{g}) \neq 0$ [9, 18, Ch.18], where $\Lambda(\mathbf{g})$ denotes the exterior algebra.

In contrast if one would consider only the algebra part of $\mathscr{U}(\mathbf{g})$ the classical Whitehead lemma applies in this case. That lemma states that for a finite-dimensional semisimple Lie algebra $\mathbf{g}$ and a finite-dimensional left-$\mathbf{g}$-module M it holds that:

$$H^1(\mathbf{g},M) = H^2(\mathbf{g},M) = 0\,. \tag{1}$$

[1] We take for the deformation parameter $q = e^h > 1$ throughout this paper.

After carrying out some steps [18] towards $M = \mathscr{U}(\mathbf{g})$ this result gives rise to the following:

Theorem 1.1 [9]. *There exists an isomorphism*

$$\alpha\colon \mathscr{U}_h(\mathbf{g}) \to \mathscr{U}(\mathbf{g})[[h]] \tag{2}$$

of topological algebras, such that $\alpha \equiv \mathrm{id} \bmod h$.

Practically speaking this result means that there exist *deforming maps* in the sense of [7] which connect the generators of the deformed to the ones of the undeformed algebra respectively.

It has been shown in the fundamental paper [2] that quantum mechanics itself can be understood as a deformation of classical mechanics. Already in that work the question of stability of quantum mechanics with respect to further deformations has been addressed. This question can be answered affirmatively using a result which has been obtained some years ago in [10] that states that the second cohomology group of the Weyl algebra regarded as a bimodule over itself is zero. This means that (at least in the reasonable sense of star-products) quantum mechanics cannot be further deformed.

However, in recent years the possibility of q-deformations of the Heisenberg algebra has been studied extensively within the context of quantum groups, see e.g. [1, 3, 11, 14, 25, 26, 27] and references therein. The characteristic relation arising from these studies is of the type:

$$px - qxp = -i\,. \tag{3}$$

In general one has two possibilities of considering these algebras according to the inequivalent antilinear involutions one can choose on the algebra (3). The first choice is to take a Bargmann–Fock type conjugation, i.e., $\bar{x} = -ip$. In this case (3) becomes a q-oscillator algebra. Here the equivalence with the undeformed case is known, see e.g. [30].

There exists, however, a second possibility which has been addressed in [27]. In this case one wants to interpret the generators of (3) as being momentum and position operators respectively.

The aim of this paper is to show that due to the mentioned rigidity theorem for the Weyl algebra even the second approach (in any dimensions) does not provide a true deformation of the Heisenberg algebra.

The plan of the paper is as follows. In Sect. 2 Heisenberg and Weyl algebras are defined and the Hochschild cohomology is calculated. It is shown how the q-difference calculus arises from a trivial deformation of the Weyl algebra. The consequences for q-oscillator algebras in a Fock space representation are explained. We study the algebra (3) with position and momentum operators in Sect. 3. It turns out that this algebra does not have a proper group of global transformations containing the Weyl group of ordinary quantum mechanics as it should in order to have a quantum mechanical interpretation. In Sect. 4 q-difference algebras on almost commutative spaces with involution on both the coordinates and the q-difference operators will be considered. The existence of this involution is necessary for having a position-momentum interpretation of the generators. A uniqueness result for the calculus will be obtained. The more general case of a q-differential calculus on the real $SO_q(N)$ quantum plane is addressed in Sect. 5. It will be proved that there exists a deforming map which provides an isomorphism of this q-differential calculus to the q-difference calculus on a commutative space. Finally we summarize and comment on prospects in Sect. 6.

2. The Cohomology of the Weyl Algebra

The Weyl algebra A_n is the associative algebra which is generated by the set of generators $p_1, q^1; p_2, q^2; \ldots; p_n, q^n$ satisfying the relations with fixed center:

$$[q^i, p_j] = i\delta^i_j, \quad \text{and} \quad [q^i, q^j] = [p_i, p_j] = 0\,. \tag{4}$$

We note that by minor changes in the definitions one can also consider the Weyl algebra $A_n(k)$ over an arbitrary field k of characteristic zero. It is obvious that elements of the form $(p_1)^{i_1}(q^1)^{j_1}(p_2)^{i_2}(q^2)^{j_2}\cdots(p_n)^{i_n}(q^n)^{j_n}$ with $i_1, j_1, \ldots, i_n, j_n \in \mathbf{N}_0$ generate A_n as a vector space. We will refer to the relations (4) as Heisenberg algebra $\mathbf{h}_n$.

If we take $M := A_n/A_n(p_1, p_2, \ldots, p_n)$ as an A_n-module we get the following realizations of the generators of A_n by differential operators:

$$q^k \sim x^k, \qquad p_k \sim -i\frac{\partial}{\partial x^k}\,. \tag{5}$$

Since we will compute and use the cohomology of $A_n(k)$ we introduce the Hochschild cohomology of an associative k-algebra B with values in some B-module M.

Considering $f \in \mathrm{Hom}_k(B^{\otimes n}, M)$ the Hochschild differential d is given by [4]:

$$\begin{aligned} &\mathrm{d}f(b_1 \otimes b_2 \otimes \cdots \otimes b_{n+1}) \\ &\quad = b_1 f(b_2 \otimes \cdots \otimes b_n) + \sum_{i=1}^{n}(-1)^i f(b_1 \otimes \cdots \otimes b_i b_{i+1} \otimes \cdots \otimes b_n) \\ &\quad\quad + (-1)^{i+1} f(b_1 \otimes \cdots \otimes b_n) b_{n+1}\,. \end{aligned} \tag{6}$$

For example a 2-cocycle ρ is defined by

$$\mathrm{d}\rho(b_1, b_2, b_3) = b_1\rho(b_2 \otimes b_3) - \rho(b_1 b_2 \otimes b_3) + \rho(b_1 \otimes b_2 b_3) - \rho(b_1 \otimes b_2) b_3\,, \tag{7}$$

while if ρ is a 2-coboundary there exists a $f \in \mathrm{Hom}_k(B, M)$ such that

$$\rho(b_1 \otimes b_2) = b_1 f(b_2) + f(b_1) b_2 - f(b_1 b_2)\,. \tag{8}$$

We can now state the following theorem on the cohomology $H^\star(A_n(k), A_n(k))$.

Theorem 2.1 [10]. *It holds that*:

$$H^m(A_n(k), A_n(k)) = k\delta_{m,0}\,. \tag{9}$$

Due to the importance of this result we sketch the proof.

Proof. One first proves a more general result. Let $\mathbf{a}$ be a finite-dimensional nilpotent Lie algebra over k and B its enveloping algebra. We denote by $\mathbf{b}$ an ideal of $\mathbf{a}$ and by λ a character of $\mathbf{b}$. For $\xi \in \mathbf{b}$ the set of elements in B of the form $\xi - \lambda(\xi)$ is denoted by $\mathbf{b}_\lambda$. Obviously $\mathbf{b}_\lambda$ is a sub-vector space of B stable under the adjoint action of $\mathbf{a}$. Moreover we define $B_\lambda := B/B\mathbf{b}_\lambda$. Using the *inverse process* of homological algebra (see e.g. [4]) one can show that for a B_λ-bimodule X the following statement is true:

$$H^m(\mathbf{a}/\mathbf{b}, X) \simeq H^m(B_\lambda, X), \quad \forall m \geqq 0 . \tag{10}$$

We can now pass to the special case of the Weyl algebra. In this case $\mathbf{a}$ is the Heisenberg algebra $\mathbf{h}_n$ and $\mathbf{b} = \mathbf{z}$ its center which is of course trivial. Obviously it holds that $B_\lambda = A_n(k)$ with $2n+1$ being the dimension of $\mathbf{a}$. In order to prove the theorem we set $X = A_n(k)$. By (10) we are enabled to take $H^\star(A_n(k), A_n(k)) = H^\star(\mathbf{h}_n/k\mathbf{z}, A_n(k))$. Using the realization of the generators of $\mathbf{h}_n$ in terms of differential operators (5) the problem of calculating $H^\star(\mathbf{h}_n/k\mathbf{z}, A_n(k))$ is mapped to a problem in de Rham cohomology. The observation that all the derivations in $A_n(k)$ are interior ones completes the proof of the theorem. □

It will now be shown how this result on the cohomology can be applied to show the triviality of q-deformed differential calculi. Therefore we have to introduce some facts about the deformation theory of an associative algebra [15].

Let h be a parameter. A deformation of an associative algebra A over a field k is a topological algebra A_h over $k[[h]]$ such that A_h is isomorphic to $A[[h]]$ as a $k[[h]]$-module, i.e., $a_h = a_0 + a_1 h + a_2 h^2 + \cdots$, with $a_h \in A_h$ and $a_0, a_1, a_2, \ldots \in A$. The product in A_h is given by a family $(\mu_i)_{i \in \mathbf{N}_0}$ of bilinear maps from $A \times A$ into A. We take μ_0 to be the ordinary multiplication in A. We write:

$$\mu_h(a_h \otimes b_h) := a_h * b_h := a_0 b_0 + \sum_{j,k,l=0, i=1}^{\infty} \mu_j(a_k \otimes b_l) h^i . \tag{11}$$

The sum is restricted to $j + k + l = i$. Since μ_h is a $k[[h]]$-module map its properties are determined by its values on elements of the algebra A.

If we require that the deformed algebra should still be associative, i.e., $(a_h * b_h) * c_h = a_h * (b_h * c_h)$, the multiplication maps μ_i have to satisfy:

$$\sum_{i+j=n} \mu_i(\mu_j(a \otimes b) \otimes c) = \sum_{i+j=n} \mu_i(a \otimes \mu_j(b \otimes c)), \quad a, b, c \in A . \tag{12}$$

In particular, for μ_1 we get

$$\mu_1(ab \otimes c) + \mu_1(a \otimes b)c = a\mu_1(b \otimes c) + \mu_1(a \otimes bc) , \tag{13}$$

which means that in order to preserve associativity μ_1 has to be Hochschild 2-cocycle. The conditions on μ_i with $i > 1$ impose obstructions for the integrability of the deformation. However, if the third cohomology group vanishes all obstructions vanish automatically.

Two deformations, A_h and A'_h say, are equivalent if there exists an isomorphism $f_h : A_h \to A'_h$ over $k[[h]]$ of topological algebras. This isomorphism is of the form $f_h = id + f_1 h + f_2 h^2 + \cdots$. The existence of f_h implies for the multiplications $\mu'_h = f_h \mu_h (f_h^{-1} \otimes f_h^{-1})$. Evaluating this again for μ_1 we get:

$$\mu'_1(a \otimes b) = \mu_1(a \otimes b) + f_1(ab) - af_1(b) - f(a)b, \quad a, b \in A . \tag{14}$$

Using the Hochschild cohomology it can easily be seen using (8) that if μ_1 is a 2-coboundary f_h can be chosen in a way such that $\mu'_1 \equiv 0$. This argument can be extended to all orders in h (e.g. [18]) giving the result that if the second Hochschild cohomology group is zero then every deformation of the algebra A within the category of associative algebras is trivial which means equivalent to the undeformed algebra.

Finally we can construct a deformed Lie algebra from the associative algebra A_h by setting $[a,b]_* := \sum_{i=0}^{\infty} h^i(\mu_i(a,b) - \mu_i(b,a)) =: \sum_{i=0}^{\infty} h^i L_i(a,b)$ for $a,b \in A$. The cohomological considerations from above directly apply to the so defined deformed Lie algebra. The Hochschild cohomology obviously gives rise to a Chevalley cohomology and L_1 decides about the triviality of the deformation.

It is now clear that Theorem 2.1 yields the following result:

Corollary 2.2. *Within the category of associative algebras there does not exist a non-trivial deformation of the Weyl algebra.*

In particular if one regards the Heisenberg algebra as providing the generating relations of the Weyl algebra, the corollary states that even the latter algebra does not possess a non-trivial deformation in this context.

Nevertheless we will now point out how the differential calculus on quantum planes as introduced in [29] arises from a trivial deformation of the Weyl algebra. This will be done explicitly only in the one-dimensional case, but it will be pointed out how this analysis generalizes to higher dimensions.

We first assume that we had a non-trivial deformation of the Weyl algebra, its generators being D and X. Writing this down we have in mind as above a power series in the deformation parameter h of the form

$$D = D_0 + hD_1 + h^2 D_2 + \cdots, \qquad X = X_0 + hX_1 + h^2 X_2 + \cdots , \tag{15}$$

with $D_0 = \partial$ and $X_0 = x$ being the generators of the undeformed Weyl algebra (4) with relation $[\partial, x] = 1$.

Next we allow a deformed associative product for the new generators:

$$D * X = \sum_{k+m+l=i} h^i \mu_k(D_l, X_m) . \tag{16}$$

To be more accurate let us write down this expression up to order h^2:

$$\begin{aligned} D * X = \partial x + h\,(D_1 x + \partial X_1 + \mu_1(\partial, x)) + h^2\,(D_2 x + D_1 X_1 \\ + \partial X_2 + \mu_2(\partial, x)\mu_1(D_1, x) + \mu_1(\partial, X_1)) + \cdots . \end{aligned} \tag{17}$$

As outlined above this product allows to construct the associated Lie algebra with deformed Lie brackets L_i:

$$[D, X]_* := D * X - X * D = \sum_{l+k+m=i} h^i L_k(D_l, X_m) . \tag{18}$$

The q-calculus by which in one dimension we mean a relation of the type (3) can be obtained by the following requirement. Let the quantities D_i and X_i for $i > 0$ be given by a polynomial of degree i in the reflection element ∂x times ∂ and x respectively. This choice seems reasonable since D_i and X_i then possess the same dimension as ∂ and x. However, this choice is not unique, but for the present purposes this does not cause any problems.

We now use Theorem 2.1 which states that every 2-cocycle has to be a 2-coboundary which means that μ_1 in (17) is of the form (8). If we take the function f in (8) to be of the form $f(\alpha) = (c_1 \partial x + c_2)\alpha$ for α in the undeformed

Weyl algebra and constants c_1, c_2, one can easily show that the term of order h in the deformed Lie bracket (18) vanishes after adjusting some constants.

Since f is a polynomial of order 1 in ∂x it follows that $\mu_1(\partial,x)$ and $\mu_1(x,\partial)$ are polynomials of order 2 in that quantity. Using this, the associativity condition (12), and the 2-coboundary property of μ_1, we can see that $\mu_i(\partial,x)$ is a polynomial of order $i+1$ in ∂x. It is therefore already clear at this stage that the q-calculus can only come from a special closure in the h-adic topology of the undeformed Weyl algebra depending on the reflection element ∂x.

Again by adjusting the coefficients properly one can show the terms in any order in h in (18) do actually vanish. Therefore we are left with the first term and hence:

$$[D,X]_* = \partial x - x\partial = 1 . \tag{19}$$

Now the q-calculus can be shown to arise from the Hochschild 2-coboundary defined above. The above choices of the parameters lead to the following expansion which can be summed up:

$$\begin{aligned} D*X &= \partial x + \frac{1}{2}\partial x(\partial x - 1)h + \frac{1}{6}\partial x\left(\frac{1}{2} - \frac{3}{2}\partial x + (\partial x)^2\right)h^2 + \cdots \\ &= \frac{1}{x}\frac{1-\exp(hx\partial)}{1-\exp(h)}x := \partial_q x . \end{aligned} \tag{20}$$

In the same way one can directly show that

$$X*D = \exp(h)x\partial_q . \tag{21}$$

Combining the previous equations and taking $q = e^h$ we arrive at

$$\partial_q x - qx\partial_q = 1 , \tag{22}$$

which is exactly the q-calculus we have been looking for.

In higher dimensions an analogous calculation can be carried out. The only difference then is that more than one reflection element exists. Therefore the functions in the 2-coboundaries can depend in various ways on these elements. The result of this consideration is that even in higher dimensions the q-calculus can be obtained formally by a trivial deformation of the classical Weyl algebra.

It is now evident that q-deformations of Heisenberg algebras are trivial from the general deformation theory point of view. This could have been guessed for some time if one considers q-deformed oscillator algebras in some representation. In one dimension this algebra takes the form:

$$aa^\dagger - qa^\dagger a = 1 . \tag{23}$$

Studying its representations one finds the same Fock space as in the undeformed case. The only difference between the classical and the q-case are the norms of the operators. The undeformed oscillator algebra is given by:

$$AA^\dagger - A^\dagger A = 1, \qquad N := A^\dagger A . \tag{24}$$

If one chooses a certain completion of these algebras in the h-adic topology it is possible to introduce an element of the form $\exp(hN) = q^N$. In the sense of Theorems 1.1 and 2.1 we then get the following deforming maps valid in the usual

Fock space representation:

$$a = q^{\frac{N}{4}} A \sqrt{\frac{[N]}{N}}, \qquad a^{\dagger} = q^{\frac{N}{4}} \sqrt{\frac{[N]}{N}} A^{\dagger}, \quad [N] := \frac{q^{\frac{N}{2}} - q^{-\frac{N}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} . \tag{25}$$

We note that deforming maps are not unique in general.

This deforming map can be applied to higher dimensional q-boson algebras as well. For instance it has been shown in [20] that the q-differential calculus on quantum planes belonging to quantum groups $SL_q(N)$ and $SO_q(N)$ (see Sect. 5 for more details) can be mapped into a tensor product of mutually commuting algebras of the form (23). Related observations have also been made in [19, 12, 6]. However, this result applies only to cases without any reality conditions either on the quantum plane or on the differential operators. This means that the mentioned map in [20] is not at all compatible with the natural involution (real form) coming from the quantum group.

3. Remarks on "q-Deformed Quantum Mechanics"

As mentioned in the introduction there exists another approach to the deformation of Heisenberg algebras which comes from the Wess–Zumino differential calculus [29]. In this approach one considers the quantum plane which is a certain comodule of a quantum group and the q-differential calculus on it. The differential relations are interpreted as q-Heisenberg relations.

In contrast to the q-boson algebras the involution in this approach is not of Bargmann–Fock type. As mentioned above it has been shown in [20] that with an involution of Bargmann–Fock type the full q-differential algebra in the case of $SL_q(N)$ and $SO_q(N)$ can be transformed into a tensor product of mutually commuting algebras similar to (23).

The authors of [27] investigate the following algebra:

$$px - qxp = -i . \tag{26}$$

As in ordinary quantum mechanics the generators p and x ought to be interpreted as momentum and position operators on some Hilbert space. Since q is taken to be real, one cannot find an antilinear involution which allows for taking both generators to be real under involution. However, one can take $\bar{p} = p$ and introduce an additional generator $\bar{x}$ together with the obvious relations:

$$p\bar{x} - q^{-1}\bar{x}p = -iq^{-1}, \qquad x\bar{x} = q\bar{x}x . \tag{27}$$

For convenience an additional object is introduced by using the usual commutators:

$$r = i[p,x], \qquad \bar{r} = i[p,\bar{x}] . \tag{28}$$

The aim is to get a q-Heisenberg algebra with real formal objects. If p is interpreted as real momentum an obvious choice for a real position would be $\xi := x + \bar{x}$. This definition results in the algebra:

$$\xi p - q^{-1} p\xi = (q^{-1} + 1)i\bar{r}, \qquad \xi p - qp\xi = (q^{-1} + 1)iqr . \tag{29}$$

The problem with this algebra is that even though r q-commutes with p it has rather involved relations with $\tilde{\xi}$. To circumvent this problem one uses the observation that r and $\bar{r}$ can be decomposed, however non-uniquely, into a formal real and a quasi unitary object.

Denoting $T := r\bar{r}$ one is allowed to write $\bar{r} := \sqrt{q}uT^{1/2}$ and $r := \sqrt{q}T^{1/2}\bar{u}$ which of course implies $u\bar{u} = q^{-1} = \bar{u}u$. Applying these decompositions to the algebra (29) and redefining the position to be

$$\xi := \frac{\sqrt{q}}{q+1}(T^{-1/2}x + \bar{x}T^{-1/2}), \tag{30}$$

yields the following algebra:

$$\xi p - q^{-1}p\xi = iu, \qquad \xi p - qp\xi = iu^{-1}, \qquad up = q^{-1}pu, \quad u\xi = q^{-1}\xi u\,. \tag{31}$$

Although this algebra suggests to be interpreted as a deformation of the Heisenberg algebra it turns out that all relations can be entirely realized within a certain completion of the enveloping algebra of the usual Heisenberg algebra $[x_c, p_c] = i$.

One has the freedom to interpret the momentum p to be the usual momentum generator p_c. This gives:

$$u = \exp(-ihp_c x_c), \qquad \bar{u} = \exp(ihx_c p_c) = q^{-1}u^{-1}\,. \tag{32}$$

The following realization follows directly from (31):

$$\xi = \frac{i}{p_c}\frac{u - u^{-1}}{q - q^{-1}} \equiv x_c \operatorname{mod} h\,. \tag{33}$$

It is shown in [27, 28] that the spectra of p and ξ are discrete. In momentum representation the Hilbert space states are given by vectors $|n\rangle_{\pi_0}$, where $n \in \mathbf{Z}$ and the continuous parameter $\pi_0 \in [1, q)$ labels the different irreducible representations of the algebra (31). One then has:

$$p|n\rangle_{\pi_0} = \pi_0 q^n |n\rangle_{\pi_0}, \qquad \xi|n\rangle_{\pi_0} = \frac{i}{\pi_0 q^n(q - q^{-1})}(q^{\frac{1}{2}}|n-1\rangle_{\pi_0} - q^{-\frac{1}{2}}|n+1\rangle_{\pi_0})\,. \tag{34}$$

By a proper Fourier transformation [28] it is possible to show that one can also construct irreducible representations in which ξ is diagonal. Its spectrum then is similar to that of p in (34). Actually the momentum eigenstates $|n\rangle_{\pi_0}$ can be realized by ordinary functions. Up to normalization we have:

$$|n\rangle_{\pi_0} \sim \exp(iq^n \pi_0 x_c)\,. \tag{35}$$

The operator $p = p_c$ then acts on these states by ordinary Schrödinger representation although the application of x_c does lead out of the irreducible representations of the algebra (31).

Although it seems quite interesting to interpret the algebra (31) to be a q-deformation of the Heisenberg algebra which provides a discretization quite similar to the one arising from ordinary lattice quantum mechanics, we have the following:

Lemma 3.1. *The algebra* (31) *does not allow for a group of global transformations which contains the Weyl group of usual quantum mechanics consistent with the irreducible representations of that algebra.*

Proof. We have seen that all generators of the algebra (31) can be realized in terms of the generators of the classical Heisenberg algebra. Strictly speaking (31) is not generated by the classical Heisenberg algebra itself which is nilpotent but by the minimal solvable extension of it. This means that on the algebra level we have an additional generator equivalent to $p_c x_c$. The enveloping algebras of these algebras are identical since the additional generator lies in the vector space spanned by elements of the form $(p_c)^i (x_c)^j$ with $i, j \in \mathbf{N}$.

If we denote the classical nilpotent Heisenberg algebra by H_1 the corresponding group is the Weyl group W_1. Obviously H_1 is a subalgebra and moreover a Lie bi-ideal of its minimal solvable extension. The usual uniform lattice discretization corresponds to considering the Weyl group W_1 not over the real numbers but over the integers $\mathbf{Z}$. The group corresponding to the minimal solvable extension of H_1 is some product of W_1 with a group of dilatations generated by $p_c x_c$ which we will call D. We denote this fact by $WD_1 = W_1 \cdot D$.

The spectra of (31) would correspond – by the conformal invariance of the real line – to fix the group parameters in the pure D part of WD_1 to the set $h\mathbf{Z}$. For consistency the space of group parameters for the full group WD_1 has to be $h\mathbf{Z}$. It is evident that the global transformations corresponding to this set is inconsistent with the spectrum of (31). The only thing one has to do is to apply a translation of the form $\exp(\alpha p_c)$ to the Hilbert space of the algebra (31) with $\alpha/h \in \mathbf{Z}$. This completes our proof. □

This result can be applied to higher dimensional cases. In [16] for example the case of $SO_q(N)$ covariant quantum mechanics has been investigated. It turned out that the total Hilbert space of the theory is a tensor product of two Hilbert spaces. One of them corresponds to $\mathcal{U}_q(su(2))$ and the other is the one coming from (31). In the sense of Theorem 1.1 and of Lemma 3.1, the models do not provide a true deformation of quantum mechanics.

4. q-Derivatives on Almost Commutative Spaces

In this section q-difference calculi on commutative or almost commutative spaces with involution are considered. By almost commutative (sometimes also referred to as q-commutative) we mean two objects A and B having a commutation relation of the form $AB = q^r BA$, where r might be any number different from zero. Having the background of quantum groups we require that for $q = 1$ we obtain the usual continuous calculus.

Let us first assume that the configuration space is a finite-dimensional commutative algebra generated by objects x^α. We will talk about these generators freely as coordinates. For the basis we choose the following convention. If the dimension of the space is odd ($\dim = 2n+1$), then $\alpha \in \mathcal{I}_{2n+1} := \{-n, \ldots, -1, 0, 1, \ldots, n\}$; if it is even ($\dim = 2n$) we have that $\alpha \in \mathcal{I}_{2n} := \{-n, \ldots, -1, 1, \ldots, n\}$. When writing $\alpha \in \mathcal{I}$ it is assumed that α is either in $\mathcal{I}_{2n+1}$ or in $\mathcal{I}_{2n}$.

An (anti-linear) involution on this space is introduced by the rule $\overline{x^\alpha} := x^{-\alpha}$. This is just the standard involution on an euclidean space in a lightcone-like basis. We remark that our results are of course basis-independent. Since we later want to make contact with the quantum group case the chosen basis is convenient.

Let D_α be q-partial derivatives acting on x^α. For the application we have in mind and hence in analogy to (31) we require these partial derivatives to have the standard conjugation property $\overline{D_\alpha} = -D_{-\alpha}$. As has been mentioned in the introduction the existence of an involution of this kind is essential for our considerations because in the sense of the previous section we have in mind a position-momentum interpretation of the differential algebra.

For the remainder of this paper no summation over repeated indices is assumed. By $\mathrm{Diff}_{q^{k(\alpha)}}(x^\alpha)$ we denote the algebra generated by $x^\alpha, D_\alpha, u_\alpha$ and u_α^{-1}, where $k(\alpha)$ is some number not equal to zero belonging to the index α. The ideal of relations in $\mathrm{Diff}_{q^{k(\alpha)}}(x^\alpha)$ is generated by:

$$D_\alpha x^\alpha - q^{k(\alpha)} x^\alpha D_\alpha = u_\alpha^{-k(\alpha)}, \qquad u_\alpha x^\alpha = q x^\alpha u_\alpha \,,$$
$$D_\alpha x^\alpha - q^{-k(\alpha)} x^\alpha D_\alpha = u_\alpha^{k(\alpha)}, \qquad u_\alpha D_\alpha = q^{-1} D_\alpha u_\alpha \,. \tag{36}$$

Proposition 4.1. *Under the previous assumptions together with* $k(\alpha) = -k(-\alpha)$ *and* $\alpha \in \mathscr{I}$ *it holds that: Every linear q-difference calculus on a commutative space, obeying the formal hermiticity conditions* $\overline{x^\alpha} = x^{-\alpha}$ *and* $\overline{iD_\alpha} = iD_{-\alpha}$ *is equivalent to*

$$\bigotimes_{\alpha \in \mathscr{I}} \mathrm{Diff}_{q^{k(\alpha)}}(x^\alpha)\,. \tag{37}$$

The tensor product here implies that all the off-diagonal relations are commutative. Before proving the proposition we state the following

Corollary 4.2. *The usual continuous partial derivatives on the coordinates are given by* $\partial_\alpha x^\beta - x^\beta \partial_\alpha = \delta_\alpha^\beta$. *The formal hermiticity of the relations* (36) *restricts* u_α *to be of the form* $u_\alpha = \exp(h\, x^\alpha \partial_\alpha)$. *Moreover we have* $\overline{u_\alpha} = q^{-1} u_{-\alpha}^{-1}$.

Proof. If we assume that the proposition is true, i.e., relations (36) hold for all $\alpha \in \mathscr{I}$, the corollary is a direct consequence of these relations. Since we require that for $q \longrightarrow 1$ the derivatives D should turn into the continuous derivatives, it must hold for all α may be up to some normalization that $D \equiv \partial \bmod h$. Together with the linearity of the calculus and the formal hermiticity of the q-derivatives this requires that in the realization of D_α some linear combination of the $u_\alpha^{k(\alpha)}$ and $(u_\alpha^{k(\alpha)})^{-1}$ must appear. Hence, a realization of any D_α is of the form:

$$D_\alpha \sim \frac{1}{x^\alpha} \frac{u_\alpha^{k(\alpha)} - (u_\alpha^{k(\alpha)})^{-1}}{q^{k(\alpha)} - q^{-k(\alpha)}} \prod_\beta u_\beta^{r(\beta)} + \lambda I\,. \tag{38}$$

The product appearing in this expression is taken over some u's such that the hermiticity of iD is not spoiled. λ is a number tending to zero as $q \longrightarrow 1$. I is a polynomial in x's, D's, and may be classical ∂'s and is required to have the dimension of a partial derivative. It is easy to show that the q-derivative in the above expression can be shifted and rescaled. This means that according to the assumptions of the proposition D_α has the following realization:

$$D_\alpha = \frac{1}{x^\alpha} \frac{u_\alpha^{k(\alpha)} - (u_\alpha^{k(\alpha)})^{-1}}{q^{k(\alpha)} - q^{-k(\alpha)}}\,. \tag{39}$$

Now it is easy to conclude. The only thing which has to be done is to calculate the commutation relations of D_α realized as in (39) with all other generators. The relation with x^α is just (36) while it commutes with D_β and x^β for $\alpha \neq \beta$. $\square$

We note that the considerations of Sect. 3 apply to each component of the tensor product (37).

Remark 1. It can easily be seen that if the coordinate algebra is almost commutative the corresponding q-difference calculus can be transformed into the form described in Proposition 4.1. To illustrate this we use a simple three-dimensional example which, however, is generic. We take commutative coordinates x^{-1}, x^0 and x^1 subject to the above mentioned conjugation rule. We can make these coordinates q-commuting by using the q-shift operators u_α. In the simplest case we set:

$$x_q^{-1} := x^{-1}, \qquad x_q^0 := u_{-1}(u_1)^{-1}x^0, \qquad x_q^1 := x^1 . \tag{40}$$

By Corollary 4.2 this unitary transformation preserves the real structure of the coordinates and leads to the commutation relations:

$$x_q^{-1}x_q^0 = qx_q^0x_q^{-1}, \quad x_q^0x_q^{-1} = q^{-1}x_q^0x_q^1 \quad x_q^{-1}x_q^1 = x_q^1x_q^{-1} . \tag{41}$$

The q-derivative with respect to x^0 has to be rescaled as well by $D_0^q := (u_{-1}(u_1)^{-1})^{-1} D_0$, while the other ones remain the same. The outcome is a q-difference calculus in which the diagonal relations are almost the same as (36) while the off-diagonal ones are q-commutative.

Although the Hilbert spaces of the commutative and of the almost commutative case are not identical the algebras can simply be related. These almost commutative calculi appear for example by reduction of a GL_q-quantum group to some lower dimensional orthogonal quantum group (see e.g. [5]).

Remark 2. It is a well known problem in the study of inhomogeneous quantum groups (e.g. [22]) that it is difficult to find a coproduct which preserves the formal antihermiticity of q-derivatives. It can be read off the algebra (36) that the comultiplication of the formal antihermitian q-difference operator necessarily involves either the quantity u or u^{-1}. Due to the conjugation property stated in Corollary 4.2 it is clear that the comultiplication of a q-difference operator can hardly preserve the formal anti-hermiticity.

Remark 3. Another problem with the differential calculus on quantum spaces is the nonlinear conjugation rule of the derivatives [21]. Although we will treat this case in the next section in some detail, already at this stage some comments are necessary. The q-partial derivatives as they turn out of the Wess–Zumino calculus [29] are unsymmetric in the sense that they produce q-shifts in only one direction in contrast to the case considered in (34). It can be shown that the nonlinear conjugation rule occurs already in simpler cases.

Let us use the commutative three dimensional space introduced in Remark 1, and introduce unsymmetric but commuting q-difference operators by:

$$\partial_{-1} := \frac{u_0}{x^{-1}}\frac{1-(u_{-1})^2}{1-q^2}, \qquad \partial_0 := \frac{u_{-1}u_1}{x^0}\frac{1-(u_0)}{1-q}, \qquad \partial_1 := \frac{u_0}{x^1}\frac{1-(u_1)^2}{1-q^2} . \tag{42}$$

The action of these derivatives on the coordinates looks almost like the ones coming from the $SO_q(3)$ covariant calculus. The diagonal relations are for instance of the form $\partial_0 x^0 = u_{-1}u_1 + qx^0\partial_0$. The off diagonal relations are almost commutative. If one introduces a formal antilinear involution, denoted as above by a bar, on the derivatives we get e.g. $\hat{\partial}_0 x^0 = (u_{-1}u_1)^{-1} + q^{-1}x^0\hat{\partial}_0$, where $\hat{\partial}_0 := -q^{-3}\overline{\partial_0}$. The relation between $\hat{\partial}_0$ and ∂_0 is established introducing the quantity $\Lambda := u_{-1}^2 u_0^2 u_1^2$. We get:

$$\hat{\partial}_0 \sim \Lambda^{-1}\partial_0 - \frac{q+1}{q^{-2}-1}\frac{(u_{-1}u_1)^{-1}}{x^0}((u_0)^{-2}-1)\,. \tag{43}$$

Analogous relations hold also for the other q-partial derivatives. Thus, the aim of this remark is that the nonlinear conjugation rule is not surprising when considering unsymmetric q-difference operators in the sense of (42).

5. q-Differential Algebras Coming from Orthogonal Quantum Groups

In this section we want to extend the results obtained in the previous section to the case of differential calculi on quantum planes coming from orthogonal quantum groups. These quantum planes fit well in our treatment since the compact form of an orthogonal quantum group $SO_q(N)$ naturally induces a real structure on the corresponding quantum plane. In order to present the main result of this section we define the ingredients and fix the notation. For further details see [21] and references therein.

The $\hat{R}$-matrix for the orthogonal quantum group in N dimensions, $SO_q(N)$, possesses a decomposition into the following projection operators: the q-analogs of the symmetrizer P^+, the antisymmetrizer P^-, and the trace projector P^0. The latter defines the q-analogue of the metric tensor g_{ij} by $P^{0\,ij}_{kl} = c g^{ij} g_{kl}$, where c is some constant. We then have:

$$\hat{R} = qP^+ - q^{-1}P^- + q^{1-N}P^0\,. \tag{44}$$

The quantum plane corresponding to $SO_q(N)$ is the algebra generated by N generators x^i with i running through the index sets $\mathscr{I}_{2N+1}$ or $\mathscr{I}_{2N}$ which have been defined at the beginning of the previous section. The ideal of relations in this algebra is generated by $\sum_{k,l} P^{-ij}_{kl} x^k x^l = 0$. The so-defined algebra will be denoted by $V_q(N)$. By definition $V_q(N)$ is a $SO_q(N)$-comodule. The metric defines a $SO_q(N)$-invariant object (by the comodule mapping) $L := (1+q^{N-2})^{-1}(\sum_{i,j} g_{ij}x^i x^j)$ which is central in the algebra $V_q(N)$.

Due to the real form of the quantum group there exists an antilinear involution on $V_q(N)$ given by:

$$\overline{x^i} = \sum_j g_{ji} x^j\,. \tag{45}$$

It has been shown in [29] that it is possible to construct an algebra of q-partial derivatives on $V_q(N)$. We denote these derivatives by ∂_i with $i \in \mathscr{I}$. The set of ∂_i spans a $SO_q(N)$-comodule algebra (just another quantum plane) with relations dual to the ones in $V_q(N)$, namely:

$$\sum_{i,j} P^{-ij}_{kl}\partial_i\partial_j = 0\,. \tag{46}$$

The element $\Delta := (1+q^{N-2})^{-1}(\sum_{i,j} g^{ij}\partial_i\partial_j)$ is central in the algebra of the q-derivatives and invariant under $SO_q(N)$-coaction.

The action of the q-partial derivatives on the generators of $V_q(N)$ is given by:

$$\partial_i x^j = \delta_i^j + q\sum_{k,l}\hat{R}^{jk}_{il}x^l\partial_k\,. \tag{47}$$

This action is unsymmetric by its definition in the sense of Remark 3 in the previous section.

The involution on $V_q(N)$ as defined in (45) cannot be extended to the q-differential algebra. Another copy of partial derivatives has to be introduced. Using the notation of (45) we have $\hat{\partial}_i := -q^N \sum_{k,l} g_{ik} g^{kl} \overline{\partial^l}$. Although the algebra of these derivatives is generated by relations identical to (46) the action of the conjugated derivatives is given by:

$$\hat{\partial}_i x^j = \delta_i^j + q^{-1} \sum_{k,l} \hat{R}_{il}^{-1\,jk} x^l \hat{\partial}_k \,. \tag{48}$$

For consistency the q-derivatives have to satisfy the relation $\partial\hat{\partial} = q^{-1}\hat{R}^{-1}\hat{\partial}\partial$. Both kinds of q-derivatives can be connected using an element which is similar to the one introduced in Remark 3 of the previous section. Using the classical analogue of $V_q(N)$ with commutative generators x_c^i and the usual continuous partial derivatives ∂_i^c $(i \in \mathscr{I})$ we define a classical Euler element $E_c := \sum_{i\in\mathscr{I}} x_c^i \partial_i^c$. This definition gives rise to introduce:

$$\Lambda := \exp(2hE_c), \qquad \overline{\Lambda} = q^{-2N}\Lambda^{-1} \,. \tag{49}$$

This element can be expressed in terms of the generators of $V_q(N)$ and the partial derivatives [21]. A straightforward calculation gives the following almost commutative relations:

$$\Lambda x_{(c)}^k = q^2 x_{(c)}^k \Lambda, \qquad \Lambda \partial_k^{(c)} = q^{-2}\partial_k^{(c)}\Lambda \,. \tag{50}$$

The notation (c) means that these commutation relations hold for both the generators coming from the quantum groups and for their classical analogs. Equation (50) shows that the element Λ is well defined in the q-algebra and in the classical algebra as well. Its coproduct is grouplike but not consistent with the involution on the quantum group [22]. Using this element the original and the conjugated derivatives can be related by the formula:

$$\hat{\partial}_k = \Lambda^{-1}(\partial_k + q^{N-1}(q - q^{-1}) x_k \Delta) \,. \tag{51}$$

This equation should be compared with (43) in the previous section.

We denote the full algebra generated by $\{x^i, \partial_i, \hat{\partial}_i, \Lambda, \Lambda^{-1} | i \in \mathscr{I}\}$ together with their algebraic relations by $\mathrm{Diff}^c_{SO_q}(N)$.

In the spirit of Sects. 3 and 4 the task is to construct a differential calculus on $V_q(N)$ consisting of formal anti-hermitian q-derivatives. The immediate guess for an object possessing this property is (cf. [13]):

$$D_i = \partial_i + q^{-N}\hat{\partial}_i, \qquad \overline{D_i} = -D_{-i} \,. \tag{52}$$

The relations of these newly introduced objects D_i are identical to (46). D_i is a linear combination of q-derivatives acting either via $\hat{R}$ (47) or via $\hat{R}^{-1}$ (48) on the generators of $V_q(N)$. Therefore one has two possibilities for writing the diagonal actions:

$$D_i x^i - q^2 x^i D_i = r_i, \qquad D_i x^i - q^{-2} x^i D_i = \tilde{r}_i \,. \tag{53}$$

This relation is valid for any $i \in \mathscr{I}$ except for x^0 in the odd dimensional case due to the properties of the $\hat{R}$-matrix. x^0 requires a q rather than a q^2 in the diagonal relation. This should be noted for all the remaining formulae in this section. The objects r_i and $\tilde{r}_i$ can be calculated without any difficulties using (47) and (48) respectively. The off-diagonal commutation relations are involved. Moreover the algebra of r_i and $\tilde{r}_i$ with all generators of $\mathrm{Diff}^c_{SO_q}(N)$ is quite complicated but their explicit form is not needed for our purposes.

Due to the properties of the involution (45) and (51) one finds:

$$q^{-2}\overline{r_{-i}} = \tilde{r}_i, \quad \forall i \in \mathscr{I} . \tag{54}$$

We now proceed in analogy to the one-dimensional case in Sect. 3 and split the objects r_i. For the decomposition we have in mind the quantities u_i as they have been obtained in Corollary 4.2 are required. Of course, the so-defined u_i is not supposed to obey any managable commutation relation with the generators of $\mathrm{Diff}^c_{SO_q}(N)$.

Introducing for any $i \in \mathscr{I}$ elements ρ_i we make the ansatz:

$$r_i := u_i^{-2}\rho_i \Rightarrow \tilde{r}_i = \overline{\rho_{-i}}u_i^2 . \tag{55}$$

The second expression in (55) is a consequence of (54). It is clear that this decomposition is not unique. However, its existence is guaranteed. Note that in the odd dimensional case it holds for the zero component $r_0 = u_0^{-1}\rho_0$.

This ansatz can now be inserted in (53) yielding:

$$u_i^{-2} = (D_i x^i - q^2 x^i D_i)\rho_i^{-1}, \qquad u_i^2 = \overline{\rho_{-i}}^{-1}(D_i x^i - q^{-2} x^i D_i) . \tag{56}$$

Now we can apply Proposition 4.1 and Eq. (36). The statement there is that the quantities u_i, u_i^{-1}, or any power of them are defined by q-difference relations on commutative spaces. Thus the factors ρ_i^{-1} and $\overline{\rho_{-i}}^{-1}$ in (56) provide deforming maps and we have the following

Theorem 5.1. *The q-differential algebra* $\mathrm{Diff}^c_{SO_q}(N)$ *is isomorphic to the q-difference calculus of the same dimension* $\bigotimes_{\alpha\in\mathscr{I}} \mathrm{Diff}_{q^{k(i)}}(x^i)$ *with* $k(i)=2$ *for* $i>0$, $k(i)=-2$ *for* $i<0$, *and* $k(0)=1$ *as outlined in Proposition* 4.1. □

We note that this result is not covered by the treatment in [20]. As mentioned above, the differential calculus on a commutative space obtained in that paper is not consistent with an involution on the coordinate algebra. In contrast, that calculus is only consistent with a Bargmann–Fock type conjugation rule on the full differential algebra.

We have preferred to show the triviality of the real differential calculus coming from the quantum group $SO_q(N)$ in a rather explicit way. However, its triviality can also be proven using a cohomological argument as follows: Applying the Whitehead lemma and Theorem 1.1 to the present case it is clear that $\mathscr{U}_q(so(N))$ is a trivial deformation. In [13] it was shown that the quantum Lie algebra (not the bialgebra) of $SO_q(N)$, i.e., $\mathscr{U}_q(so(N))$, can be realized by the generators of $\mathrm{Diff}^c_{SO_q}(N)$. Since we know the triviality of $\mathscr{U}_q(so(N))$ it can immediately be shown that every realization of it must be trivial. Hence, $\mathrm{Diff}^c_{SO_q}(N)$ is trivial.

6. Conclusions

We have pointed out that coming from the differential calculus on quantum groups one is led to two a priori non-equivalent approaches towards a q-deformation of quantum mechanics. The first one is based on q-oscillator algebras. The second approach arises from the q-differential calculus on involutive quantum planes and focusses on position-momentum interpretation of the generators of the q-differential algebra.

Although both approaches might have interesting applications, e.g. either in statistical mechanics or in the theory of generalized hypergeometric series, it has been shown that due to the rigidity Theorem 2.1 for the Heisenberg algebra both approaches do not yield a true deformation of quantum mechanics. Moreover it has been shown that it is hardly possible to find a hermiticity preserving comultiplication on the generators of both the q-difference calculus and of the q-differential calculus.

For these reasons one has to think carefully if quantum planes, although they have interesting features [23, 24], provide a reasonable base space for quantum field theories.

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