Commun. Math. Phys. 178, 225–236 (1996)

Communications in
Mathematical
Physics

Čech Cocycles for Characteristic Classes

J.-L. Brylinski [1,*], **D.A. McLaughlin** [2,**]

[1] Mathematics Department, Pennsylvania State University, University Park, PA 16802, USA
[2] Mathematics Department, Fine Hall, Princeton University, Princeton, NJ 08544-1000, USA

Received: 19 May 1995

Abstract: We give general formulae for explicit Čech cocycles representing characteristic classes of real and complex vector bundles, as well as for cocycles representing Chern–Simons classes of bundles with arbitrary connections. Our formulae involve integrating differential forms over moving simplices inside homogeneous spaces. An important feature of our cocycles is that they take integer values (as opposed to real or rational values). We find in particular a formula for the instanton number of a connection over a closed four-manifold with arbitrary structure group. For flat connections, our formulae recover and generalize those of Cheeger and Simons. The methods of this paper apply also to the purely geometric construction of the Quillen line bundle with its metric.

A vector bundle $E \to M$ has characteristic classes (Chern, Pontryagin and Euler classes) in integral cohomology groups $H^p(M,\mathbb{Z})$. The Chern–Weil theory gives differential forms which represent the corresponding classes in the real cohomology groups $H^p(M,\mathbb{R})$. Gelfand posed the problem of finding a combinatorial formula for integer-valued singular cocycles representing the Pontryagin classes. This is considerably more difficult than finding a real-valued cocycle, which can easily be done using a partition of unity [5].

An explicit formula for a singular cocycle representing 24 times the first Pontryagin class $p_1(M)$ of a smooth manifold M was found by I. Gelfand, Gabrielov and Losik [16] and by MacPherson [20]. This formula, which involves the dilogarithm function, has had considerable influence in algebraic topology and in algebraic K-theory. More recently, Gelfand and MacPherson [17] gave a formula for a *rational* simplicial cocycle representing any Pontryagin class of a smooth polyhedral manifold.

In this paper we work with Čech cohomology instead of singular cohomology. We give a direct and explicit construction of *all* the *integer-valued* Čech cocycles

* The first author was supported in part by N.S.F. grant DMS-9203517.
** The second author was supported in part by N.S.F. grant DMS-9310433.

which represent a given characteristic class. We adopt the following point of view. A vector bundle may be completely described by a set $\{g_{ij}\}$ of transition cocycles, relative to some suitable open covering of M. We construct the integer-valued Čech cocycles representing a characteristic class directly from the data g_{ij}. We obtain Čech cocycles representing the Pontryagin classes $p_k \in H^{4k}(M, \mathbb{Z})$ and the Euler class $e \in H^n(M, \mathbb{Z})$ of a real vector bundle of rank n, and the Chern classes $c_k \in H^{2k}(M, \mathbb{Z})$ of a complex vector bundle (see Theorems 2, 3 and 4). The only denominators which are needed are some powers of 2 in the case of p_k for $k > 1$. These formulae generalize the formula for p_1 announced in [6].

Our formula may also be viewed as the completion of a project of Chern and Simons, who explain in the introduction to [12] that their secondary characteristic classes "grew out of an attempt to derive a purely combinatorial formula for the first Pontryagin number of a 4-manifold." The "boundary term" which Chern and Simons say "did not yield to simple combinatorial analysis" is incorporated in the formulae given in this paper.

All the characteristic classes above may be defined by transgression in an associated bundle, with fiber a Stiefel manifold [4]. We realize this transgression geometrically, by constructing "moving cycles," which are moving families of singular cycles on the Stiefel manifold, parametrized by various open subsets of M. This construction is contained in the first section, along with the statement of Theorems 2, 3 and 4.

The proof of these theorems is given in Sects. 2 and 3. In fact, much more is shown. In Sect. 2, by choosing a connection on $E \to M$, we exhibit lifts of p_k, e and c_k to characteristic classes in Cheeger–Simons cohomology – or in smooth Deligne cohomology (see Theorem 5). In Sect. 3, we show that these lifts agree with the classes of Chern–Cheeger–Simons [11, 12]. We also recover formulae of Cheeger–Simons [11] and of Dupont [13, 14] for flat bundles and we generalize them to the case where the g_{ij} are not in general position. In Sect. 4, we look at some interesting special cases of our formulae. For the second Chern class of a principal $SU(n)$-bundle, the formula simplifies considerably and involves computing the integral of the Chern–Simons 3-form v over a tetrahedron in $SU(n)$ whose vertices are the transition functions. The corresponding Chern–Cheeger–Simons class can then be written explicitly in terms of v. For a four-dimensional manifold, this gives a formula for the topological charge or instanton number. In the case of $SU(2)$, this was essentially known to Laursen, Schierholz and Wiese [19]. Thus we obtain an extension of their formulae for arbitrary classical compact groups. For the first Pontryagin class, we also obtain an explicit formula, but the tetrahedron has to be altered slightly due to the 2-torsion in $\pi_1 SO(n)$.

Finally, A. Goncharov [18] has constructed Chern classes in *grassmannian cohomology* and he has also constructed explicitly c_2 and c_3 in *motivic cohomology*. This in particular implies formulae for these classes in Deligne cohomology and in ordinary cohomology.

We would like to thank Sasha Beilinson, Pierre Deligne, Israel Gelfand, Sasha Goncharov, David Kazhdan and Bob MacPherson for useful discussions.

1. The Formulae

In this section, we will write down an explicit $\mathbb{Z}$-valued Čech cocycle representing each of the following characteristic classes:

- the *Euler class* e and the *Pontryagin classes* p_k of a principal $SO(n)$-bundle;
- the *Chern classes* c_k of a principal $U(n)$-bundle

Let G be a Lie group, let M be a compact manifold and let $p : P \to M$ be a principal bundle with structure group G. By convention, we write the G-action on P as a right action, and use the notation $x \cdot g$ for the action of $g \in G$ on $x \in P$. We choose a *good* open covering $\mathscr{U} = (U_i)$ of M, i.e. we assume that all intersections $U_{i_1 \cdots i_k} := U_{i_1} \cap \cdots \cap U_{i_k}$ are contractible or empty (see [22]). Let $s_i : U_i \to P$ be a family of smooth local sections of p. Then the *transition cocycles* g_{ij} are the smooth functions $g_{ij} : U_{ij} \to G$ defined by the formula $s_j = s_i \cdot g_{ij}$.

Cycle associated to p_k. We start from the data of a principal $SO(n)$-bundle $p : P \to M$, where n is odd. We introduce the real *Stiefel manifold* $V_{n,q} = SO(n)/SO(q)$; for $g \in SO(n)$, we denote by $\bar{g}$ the image of g in the coset space $SO(n)/SO(q)$. The image $\bar{1}$ of 1 in $V_{n,q}$ is the base point of $V_{n,q}$. The reduced homology $\tilde{H}_i(V_{n,2k-1}, \mathbb{Z})$ is well known [3]. We have:

$$\tilde{H}_i(V_{n,2k-1}, \mathbb{Z}) = \begin{cases} \mathbb{Z}/2 \text{ for } i \text{ odd, } 2k-1 \leqq i \leqq 4k-2 \\ \mathbb{Z} \oplus (2 - \text{primary group}) \text{ for } i = 4k-1 \\ 0 \text{ if } i \leqq 4k-3,\ i \text{ even or } i \leqq 2k-2\,. \end{cases}$$

We will work with the groups $C_j(i_0, \ldots, i_m)$ of smooth singular j-chains for the manifold of smooth maps from $U_{i_0 \cdots i_m}$ to $V_{n,2k-1}$. Note that such a chain may be viewed as a linear combination of smooth mappings $\sigma : U_{i_0 \cdots i_m} \times \Delta^j \to V_{n,2k-1}$, where Δ^j is the standard j-simplex. We have the usual boundary map $\partial : C_j(i_0, \ldots, i_m) \to C_{j-1}(i_0, \ldots, i_m)$.

Lemma 1. *There exists a family of elements $\sigma^j_{i_0 \cdots i_j}$ of $C_j(i_0, \ldots, i_j)$, where j ranges over $\{0, 1, \ldots, 4k-1\}$, and $i_0, \ldots, i_j$ range over $j+1$-tuples of elements of I, which satisfies*

$$\sigma^0_i(y) = \bar{1} \quad \textit{for } i \in I,\ y \in U_i\,, \tag{1}$$

and

$$\partial \sigma^j_{i_0 \cdots i_j} = a_{j-1} \cdot \left[g_{i_0 i_1} \cdot \sigma^{j-1}_{i_1 \cdots i_j} + \sum_{l=1}^{j} (-1)^l \sigma^{j-1}_{i_0 \cdots \hat{i}_l \cdots i_j} \right] \quad \textit{for } j \geqq 1\,. \tag{2}$$

We explain in more detail the meaning of (2). In the right-hand side of (2), we let $g_{i_0 i_1}$ operate on smooth maps from $U_{i_0 \cdots i_j} \times \Delta^j \to V_{n,2k-1}$ via the action of $SO(n)$ on $V_{n,2k-1}$, and this is extended linearly to j-chains. We denote by the same letter a chain like $\sigma^{j-1}_{i_1 \cdots i_j}$ and its restriction to $\mathrm{Map}(U_{i_0 \cdots i_j}, V_{n,2k-1})$. Finally, the number a_j is the cardinality of the reduced homology group $\tilde{H}_j(V_{n,2k-1}, \mathbb{Z})$.

Proof. The chains $\sigma^j_{i_0 \cdots i_j}$ are constructed by induction on j. For $j = 0$, we define $\sigma^0_i(y) = \bar{1}$ so that (1) is verified. Given $j \geqq 1$ and $i_0, \ldots, i_j$, we may assume that $U_{i_0\cdots i_j}$ is not empty, hence contractible. The singular chain $g_{i_0 i_1} \cdot \sigma^{j-1}_{i_1\cdots i_j} + \sum_{l=1}^{j}(-1)^l \sigma^{j-1}_{i_0\cdots\hat{i}_l\cdots i_j}$ is seen to be a smooth singular cycle in the manifold $\mathrm{Map}(U_{i_0\cdots i_j}, V_{n,2k-1})$, which is homotopy equivalent to $V_{n,2k-1}$. This cycle, when multiplied by a_j, becomes the boundary of some chain $\sigma^j_{i_0\cdots i_j}$. □

We now define the smooth singular $4k-1$ cycle $X_{i_0\cdots i_{4k}}$ in $\mathrm{Map}(U_{i_0\cdots i_{4k}}, V_{n,2k-1})$ by the equality

$$X_{i_0\cdots i_{4k}} = g_{i_0 i_1} \cdot \sigma^{4k-1}_{i_1\cdots i_{4k}} + \sum_{l=1}^{4k}(-1)^l \sigma^{4k-1}_{i_0\cdots\hat{i}_l\cdots i_{4k}} . \tag{3}$$

We may think of $X_{i_0\cdots i_{4k}}$ as a smooth family of cycles $X_{i_0\cdots i_{4k}}(y)$ on $V_{n,2k-1}$, parametrized by $y \in U_{i_0\cdots i_{4k}}$.

Let Ω be the closed $SO(n)$-invariant form generating $H^{4k-1}(V_{n,2k-1};\mathbb{Z})$ modulo torsion. The Pontryagin class p_k of the bundle $p: P \to M$ is defined to be the transgression of $2\cdot\Omega$ in the associated bundle $\pi: P\times_{SO(n)} V_{n,2k-1} \to M$ [4]. For $y \in U_{i_0\cdots i_{4k}}$, we let $\int_{X_{i_0\cdots i_{4k}(y)}} \Omega$ be the number obtained by integration of Ω on the cycle $X_{i_0\cdots i_{4k}}(y)$.

Theorem 2. (I) *For each $i_0,\ldots,i_{4k} \in I$, the function $y \mapsto \int_{X_{i_0\cdots i_n}(y)} \Omega$ is a constant function from $U_{i_0\cdots i_{4k}}$ to $\mathbb{Z}$, denoted by $\int_{X_{i_0\cdots i_{4k}}} \Omega$.*

(II) *The Čech $4k$-cochain $\int_{X_{i_0\cdots i_{4k}}} \Omega$ is a $\mathbb{Z}$-valued cocycle, whose cohomology class is equal to $-2^{k-1}\cdot p_k$.*

(III) *Any $\mathbb{Z}$-valued Čech cocycle which represents $-2^{k-1}\cdot p_k$ is obtained by this procedure, for a suitable choice of $\sigma^j_{i_0\cdots i_j}$.*

We will now give the similar constructions which are appropriate for the Euler class and for the Chern classes.

Cycle associated to e. Again let $p: P \to M$ be an $SO(n)$-bundle, but we now assume n even. As in Lemma 1, we can construct a family $\gamma^j_{i_0\cdots i_j}$ of smooth singular j-chains in $\mathrm{Map}(U_{i_0\cdots i_j}, S^{n-1})$, defined for $j \leqq n-1$, satisfying the analog of (1) as well as

$$\partial\gamma^j_{i_0\cdots i_j} = g_{i_0 i_1} \cdot \gamma^{j-1}_{i_1\cdots i_j} + \sum_{l=1}^{j}(-1)^l \gamma^{j-1}_{i_0\cdots\hat{i}_l\cdots i_j} . \tag{4}$$

Then we define $Y_{i_0\cdots i_n}$ to be the smooth singular n-cycle of $\mathrm{Map}(U_{i_0\cdots i_n}, S^{n-1})$ given by

$$Y_{i_0\cdots i_n} = g_{i_0 i_1} \cdot \gamma^{n-1}_{i_1\cdots i_n} + \sum_{l=1}^{n}(-1)^l \gamma^{n-1}_{i_0\cdots\hat{i}_l\cdots i_n} . \tag{5}$$

Let Λ be the $SO(n)$-invariant form generating $H^{n-1}(S^{n-1};\mathbb{Z})$. The Euler class e may be defined as the transgression of Λ in the sphere bundle associated to $P \to M$.

Theorem 3. (I) *For each $i_0,\ldots,i_n \in I$, the function $y \mapsto \int_{Y_{i_0\cdots i_n(y)}} \Lambda$ is a constant function from $U_{i_0\cdots i_n}$ to $\mathbb{Z}$, denoted by $\int_{Y_{i_0\cdots i_n}} \Lambda$.*

(II) *The Čech n-cochain* $\int_{Y_{i_0\cdots i_n}} \Lambda$ *is a* $\mathbb{Z}$*-valued cocycle, whose cohomology class is equal to* $(-1)^{\frac{n}{2}+1}\cdot e$.

(III) *Any* $\mathbb{Z}$*-valued Čech cocycle which represents* $(-1)^{\frac{n}{2}+1}\cdot e$ *is obtained by this procedure, for a suitable choice of* $\gamma^j_{i_0\cdots i_j}$.

Cycle associated to c_k. Now let $p : P \to M$ be a principal $U(n)$-bundle. We introduce the complex Stiefel manifold $W_{n,q} = U(n)/U(q)$. The homology of $W_{n,q}$ is zero in degrees $\leqq 2q$ and is equal to $\mathbb{Z}$ in degree $2q+1$ [4]. We construct a family $\tau^j_{i_0\cdots i_j}$ of smooth j-chains in $\mathrm{Map}(U_{i_0\cdots i_j}, W_{n,k-1})$ which satisfy the analog of (1) as well as

$$\partial\tau^j_{i_0\cdots i_j} = g_{i_0 i_1}\cdot\tau^{j-1}_{i_1\cdots i_j} + \sum_{l=1}^{j}(-1)^l\tau^{j-1}_{i_0\cdots\hat{i}_l\cdots i_j}\,. \tag{6}$$

We let $Z_{i_0\cdots i_{2k}}$ be the smooth $2k-1$-cycle in $\mathrm{Map}(U_{i_0\cdots i_{2k}}, W_{n,k-1})$ defined by

$$Z_{i_0\cdots i_{2k}} = g_{i_0 i_1}\cdot\tau^{2k-1}_{i_1\cdots i_{2k}} + \sum_{l=1}^{2k}(-1)^l\tau^{2k-1}_{i_0\cdots\hat{i}_l\cdots i_{2k}}\,. \tag{7}$$

Let Θ be the $SU(n)$-invariant form generating $H^{2k-1}(W_{n,k-1};\mathbb{Z})$. The Chern class c_k may be defined as the transgression of Θ in the associated bundle $P\times_{SU(n)} W_{n,k-1}\to M$ [3].

Theorem 4. (I) *For each* $i_0,\ldots,i_{2k}\in I$, *the function* $y\mapsto\int_{Z_{i_0\cdots i_{2k}}(y)}\Theta$ *is a constant function from* $U_{i_0\cdots i_{2k}}$ *to* $\mathbb{Z}$, *denoted by* $\int_{Z_{i_0\cdots i_{2k}}}\Theta$.

(II) *The Čech* $2k$*-cochain* $\int_{Z_{i_0\cdots i_{2k}}(y)}\Theta$ *is a* $\mathbb{Z}$*-valued cocycle, whose cohomology class is equal to* $(-1)^{k-1}\cdot c_k$.

(III) *Any* $\mathbb{Z}$*-valued Čech cocycle which represents* $(-1)^{k-1}\cdot c_k$ *is obtained by this procedure, for a suitable choice of the* $\tau^j_{i_0\cdots i_j}$.

Remarks

1. Volumes of simplices also appear in [11]. These are geodesic simplices defined on the sphere bundle associated to a *flat* bundle.

2. There is a combinatorial formula for the integral first Pontryagin class of a simplicial manifold [17, 20], but we do not know whether our approach sheds any new light on this. We note the combinatorial formula of [10] for the signature of a closed oriented manifold.

3. A geometric proof of the formula for p_1 was given in [8]. This proof involved the notion of a 2-gerbe. It is expected that the general result can be proved using n-gerbes, although this has not yet been formulated precisely.

2. The Proof

We need only give the proof of Theorem 2, the other proofs being similar (and in fact slightly simpler). Let $\beta_{i_0\cdots i_{4k}} = \int_{X_{i_0\cdots i_{4k}}}\Omega$ denote the Čech cocycle which is our candidate for $-2^{k-1}p_k$. Choose a connection on $p : P \to M$, and let α be the $4k$-form which is the Chern–Weil representative of p_k. Let $S = P \times_{SO(n)} V_{n,2k-1}, \pi : S \to M$ the projection. The pullback of α to S is exact, so there is a $(4k-1)$-form $\tilde{\Omega}$ on S with $d\tilde{\Omega} = \frac{\pi^*\alpha}{2}$. In fact, for a given connection, there is a natural choice of a form $\tilde{\Omega}$ with this property [10, 11]. The restriction of $\tilde{\Omega}$ to the fiber is Ω.

We denote by $\bar{s}_i$ the trivialization $\bar{s}_i : U_i \times V_{n,2k-1} \tilde{\to} \pi^{-1}(U_i)$ induced by the section s_i of P over U_i. Define

$$f_{i_0 \cdots i_{4k-1}}(y) = \exp\left(2\pi i \cdot \int_{\sigma^{4k-1}_{i_0 \cdots i_{4k-1}}(y)} \Omega\right), \quad \text{for } y \in U_{i_0 \cdots i_{4k-1}} . \tag{8}$$

This is a degree $(4k-1)$ cocycle with values in the sheaf $\underline{\mathbb{T}}$ of smooth circle-valued functions. Its image under the boundary map in the exponential exact sequence is $2\pi i \cdot \beta_{i_0 \cdots i_{4k}}$. Using the trivialization $\bar{s}_i$, we associate to any smooth map $\sigma : U_{i_0 \cdots i_l} \times \Delta^l \to V_{n,2k-1}$ a smooth map $\bar{\sigma} : U_{i_0 \cdots i_l} \times \Delta^l \to S$ such that $\pi \circ \bar{\sigma}$ is the first projection $U_{i_0 \cdots i_j} \times \Delta^l \to M$. We extend this linearly to smooth singular chains. Thus to $\sigma^l_{i_0 \cdots i_l}$ we associate a smooth singular l-chain $\tilde{\sigma}^l_{i_0 \cdots i_l}$ in $\mathrm{Map}(U_{i_0 \cdots i_l}, S)$. We have the following expression for the boundary of these chains:

$$\partial \tilde{\sigma}^j_{i_0 \cdots i_j} = a_j \left[\tilde{\sigma}^{j-1}_{i_1 \cdots i_j} + \sum_{l=1}^{j} (-1)^l \tilde{\sigma}^{j-1}_{i_0 \cdots \hat{i}_l \cdots i_j}\right] \quad \text{for } j \geqq 1 . \tag{9}$$

Using the invariance of Ω, the cocycle $f_{i_0 \cdots i_{4k-1}}$ can be rewritten as

$$f_{i_0 \cdots i_{4k-1}}(y) = \exp\left(2\pi i \int_{\tilde{\sigma}^{4k-1}_{i_0 \cdots i_{4k-1}}(y)} \tilde{\Omega}\right) . \tag{10}$$

We now lift $f_{i_0 \cdots i_{4k-1}}$ to a Čech cocycle with values in the complex of sheaves

$$\underline{\mathbb{T}}^d \overset{\log}{\to} i \cdot \underline{A}^1_M \to \cdots \to i \cdot \underline{A}^{4k-1}_M , \tag{11}$$

where $\underline{A}^j_M$ is the sheaf of smooth real j-forms on M. Note that there is the exponential exact sequence $0 \to 2\pi i \cdot \mathbb{Z} \to i \cdot \underline{A}^0_M \overset{\exp}{\to} \underline{\mathbb{T}} \to 0$, hence the complex of sheaves (10) is quasi-isomorphic to the complex of sheaves

$$2\pi i \cdot \mathbb{Z} \to i \cdot \underline{A}^0_M \to \cdots \to i \cdot \underline{A}^{4k-1}_M , \tag{12}$$

shifted by one degree to the left. The complex of sheaves (12) is called the *smooth Deligne complex* and is denoted by $\mathbb{Z}(4k)_D$; it is the smooth analog of the holomorphic Deligne complex [1].

Recall that the Čech hypercohomology of a complex of sheaves $K^\bullet$ with respect to $\mathcal{U} = (U_i)$ is the total cohomology of the double complex $C^p(\mathcal{U}, K^q)$ of Čech p-cochains with coefficients in K^q, with total differential equal to $\delta + (-1)^p d$, where d is the differential of $K^\bullet$ and δ is the Čech differential (see [5]).

The next result gives an explicit Čech cocycle with coefficients in the complex of sheaves $\mathbb{Z}(4k)_D$. We will use the following notations. If we have a product fibration $\Delta^q \times U \to U$ and a smooth mapping $\sigma : \Delta^q \times U \to G$, then for α a k-form on $U \times G$ we will denote by $\int_\sigma \alpha$ the $(k-q)$-form on U obtained by fiber integration of the pull-back of α by the mapping $(p_2, \sigma) : \Delta^q \times U \to U \times G$. We extend this construction linearly to any smooth q-chain in $\mathrm{Map}(U, G)$.

Theorem 5. (I) *For $l \geqq 1$, define an l-form $\omega_{i_0 \cdots i_{4k-(l+1)}}$ over $U_{i_0 \ldots i_{4k-(l+1)}}$ by*

$$\omega^l_{i_0 \cdots i_{4k-(l+1)}} = (-1)^{[\frac{l}{2}]} 2^{r(l)} 2\pi i \int_{\tilde{\sigma}^{4k-(l+1)}_{i_0 \cdots i_{4k-(l+1)}}} \tilde{\Omega}\,, \tag{13}$$

where $[m]$ is the greatest integer $\leqq m$, and $r(l)$ is $[\frac{l}{2}]$, if $l \leqq 2k$, and k if $l \geqq 2k$.

Then $(f, \omega^1, \ldots, \omega^{4k-1})$ is a Čech $4k$-cocycle of the covering (U_ι), with values in the complex of sheaves $\mathbb{Z}(4k)_D$. Moreover, $d\omega^{4k-1} = -2^{k-1} \cdot 2\pi i \alpha$.

(II) *The cohomology class of $(f, \omega^1, \ldots, \omega^{4k-1})$ in $H^{4k}(M, \mathbb{Z}(4k)_D)$ depends only on the principal bundle and the connection over it.*

We note that Theorem 5 implies the fact that the Čech cohomology class of $\delta f = 2\pi i \cdot \beta$ corresponds to the de Rham cohomology class of $-2^{k-1} \cdot 2\pi i \cdot \alpha$, which yields statements (I) and (II) in Theorem 2. Then statement (III) of Theorem 2 will follow because at the last stage of our construction, we have the freedom of adding to each $\sigma^{4k-1}_{i_1 \cdots i_{4k}}$ a smooth singular $(4k-1)$-cycle in $V_{n,2k-1}$, say $\beta_{i_1 \cdots i_{4k}}$. The effect of this operation on the Čech cocycle $X_{i_0 \cdots i_{4k}}$ is to add to it the coboundary of the Čech cochain $\int_{\beta_{i_1 \cdots i_{4k}}} \Omega$; this follows from the definition (3) of $X_{i_0 \cdots i_{4k-1}}$ and from the fact that Ω is an invariant form. Therefore Theorem 2 is a consequence of Theorem 5.

We thus turn to the proof of Theorem 5. To prove (I), we must show that $d \log s = \delta\omega^1$ and that $d\omega^l = (-1)^l \delta\omega^{l+1}$. Both equations are straightforward applications of the boundary relation (2), and of Stokes' theorem.

We now prove that the cohomology class of $(f, \omega^1, \ldots, \omega^{4k-1})$ is independent of the choice of the chains σ^j and also of the given covering. For the first point, we begin by showing that two homotopic families of j-chains σ^j give the same cohomology class. So, for a given open covering, consider another choice of sections over U_i and another choice $\tau^j_{i_0 \cdots i_j}$ of smooth chains, satisfying the same assumptions as $\sigma^j_{i_0 \cdots i_j}$, and such that there are smooth homotopies $P^j_{i_0 \cdots i_j}$ between $\sigma^j_{i_0 \cdots i_j}$ and $\tau^j_{i_0 \cdots i_j}$, which are compatible with the face relations among these chains. Each $P^j_{i_0 \cdots i_j}$ is then a smooth singular cycle in $\mathrm{Map}(U_{i_0 \cdots i_j} \times [0,1], V_{n,2k-1})$. As before, $\tilde{P}^j_{i_0 \cdots i_j}$ will denote the singular cycle in $\mathrm{Map}(U_{i_0 \cdots i_j} \times [0,1], S)$ obtained from $P^j_{i_0 \cdots i_j}$ by means of the trivialization $\bar{s}_{i_0}$. We then define a degree Čech $(4k-1)$-cochain $(g, \alpha^1, \cdots, \alpha^{4k-1})$ with values in $\mathbb{Z}(4k)_D$ as follows:

$$g_{i_0 \cdots i_{4k-2}} = \exp\left(2\pi i \cdot \int_{\tilde{P}_{i_0 \cdots i_{4k-2}}} \tilde{\Omega}\right),$$

$$\alpha^j_{i_0 \cdots i_{4k-2-j}} = (-1)^{[\frac{j}{2}]} 2^{r(j)} 2\pi i \cdot \int_{\tilde{P}_{i_0 \cdots i_{4k-2-j}}} \tilde{\Omega}\,.$$

It is then easy to show, using the boundary relations (2) and Stokes' theorem, that the boundary of this Čech $(4k-1)$-cochain is the difference between the two cocycles corresponding to the two choices of smooth chains.

To finish the proof of independence of the choice of the σ^j, we still have to study the effect of modifying each $\sigma^{4k-1}_{i_0\cdots i_{4k-1}}$ by an integral $(4k-1)$-cycle; this however has no influence on $f_{i_0\cdots i_{4k-1}}$, as Ω has integral periods. It is also clear that the construction is invariant under a refinement of the open covering, which finishes the proof of (II).

This concludes the proof of Theorem 5. □

3. Relation with Differential Characters

Let $\tilde{p}_k$ be the lift of the Pontryagin class constructed in Theorem 5, and denote by $\tilde{e}$ and $\tilde{c}_k$ the similar lifts for the Euler and Chern classes. They are classes in the Čech hypercohomology of a smooth Deligne complex of sheaves $\mathbb{Z}(m)_D$ (where m is the degree of the characteristic class). Then each of these classes depends on the connection chosen in the construction, so it is natural to compare them to the *Chern–Cheeger–Simons Characteristic classes* $\hat{p}_k, \hat{e}$ and $\hat{c}_k$, which take values in the ring of differential characters [11, 12]. We show in this section that all these classes agree.

To make the comparison, we follow [15, 23] and consider the complex

$$\mathrm{Cone}\{A^{\geqq m}(M) \xrightarrow{\overline{T}} S^\bullet_{sm}(M; \mathbb{R}/\mathbb{Z})\} . \tag{14}$$

The notation $A^{\geqq m}(M)$ means truncation from below in the de Rham complex $A^\bullet(M), S^\bullet_{sm}(M; \mathbb{R}/\mathbb{Z})$ denotes the complex of smooth singular cochains, and $\overline{T}$ is integration, followed by reduction mod $\mathbb{Z}$. The degree $m-1$ cohomology of this complex is exactly $\hat{H}^{m-1}(M)$, the group of degree $m-1$ differential characters, introduced by Cheeger and Simons. The complex (14) is quasi-isomorphic to

$$\mathrm{Cone}\{A^{\geqq m}(M) \to \mathrm{Cone}\{S^\bullet_{sm}(M; \mathbb{Z}) \hookrightarrow S^\bullet_{sm}(M; \mathbb{R})\}\} .$$

From a purely algebraic fact about cones, the latter complex identifies with

$$\mathrm{Cone}\{A^{\geqq m}(M) \oplus S^\bullet_{sm}(M; \mathbb{Z}) \to S^\bullet_{sm}(M; \mathbb{R})\} . \tag{15}$$

On the other hand, the Deligne complex of sheaves $\mathbb{Z}(m)_D$ is quasi-isomorphic to

$$\mathrm{Cone}\{\mathbb{Z} \oplus \underline{A}^{\geqq m}_M \to \underline{A}^\bullet_M\}[-1] , \tag{16}$$

where $[-1]$ denotes translation of a complex by one step to the right. The quasi-isomorphism in question is given by dividing by $2\pi i$. The global hypercohomology of the Deligne complex (16) may be computed from a good open covering $\mathscr{U}$ of M. If $C^\bullet(\mathscr{U}, -)$ denotes the complex of Čech cochains of this covering with values in a complex of sheaves, we realize the Deligne cohomology $H^*(M, \mathbb{Z}(m)_D)$ as the cohomology of the complex

$$\mathrm{Cone}\{C^\bullet(\mathscr{U}, \mathbb{Z}) \oplus C^\bullet(\mathscr{U}, \underline{A}^{\geqq m}_M) \to C^\bullet(\mathscr{U}, \underline{A}^\bullet_M)\}[-1] . \tag{17}$$

We denote by $\underline{S}^\bullet_{sm}(A)$ the sheaf of smooth singular chains with values in some abelian group A. There is a natural map of complexes of sheaves from $\underline{A}^\bullet_M$ to

$\underline{S}^{\bullet}_{sm}(\mathbb{R})$, given again by integration. Then there is a natural map of complexes from (17) to the complex

$$\mathrm{Cone}\{C^{\bullet}(\mathcal{U},\underline{S}^{\bullet}_{sm}(\mathbb{Z}))\oplus C^{\bullet}(\mathcal{U},\underline{A}^{\geqq m}_{M})\to C^{\bullet}(\mathcal{U},\underline{S}^{\bullet}_{sm}(\mathbb{R}))\}[-1]\,. \tag{18}$$

This is a quasi-isomorphism from (17) to (18).

There is a natural map of complexes from (15) to (18), obtained simply by mapping the global sections of the complex $\mathrm{Cone}\{\underline{S}^{\bullet}_{sm}(\mathbb{Z})\oplus\underline{A}^{\geqq m}_{M}\to\underline{S}^{\bullet}_{sm}(\mathbb{R})\}$ to the corresponding Čech double complex. This is again a quasi-isomorphism.

The upshot of this discussion is that both $\hat{H}^{m-1}(M)$ and $H^m(M;\mathbb{Z}(m)_D)$ map isomorphically to the degree m cohomology of (18). All the characteristic classes can then be compared in this complex, and it is enough to show that they agree universally. Narasimhan and Ramanan showed that there is a universal object for principal bundles with connection [21]. In the universal case, all three cohomology groups are isomorphic to the group of closed m-forms with integral periods, the isomorphism being induced by exterior differentiation. Since all classes are defined so that their exterior derivative is the appropriate Chern–Weil representative, they must therefore agree. Collecting these results, we have shown:

Proposition 6. *The classes $\hat{p}_k,\hat{e},\hat{c}_k$ constructed in Theorem* 5 *agree with the Chern–Cheeger–Simons classes $\tilde{p}_k,\hat{e}$ and $\tilde{c}_k$, respectively, in the cohomology of* (18).

4. Special Cases of the Formula

We will specialize the formula of Theorem 4 to the second Chern class $c_2(P)$ of a principal $SU(n)$-bundle $p:P\to M$. If $\dim(M)=4$, the number $\langle c_2(P),M\rangle$ is called the topological charge or instanton number. There is a canonical choice for an invariant 3-form Θ on $SU(n)$ generating $H^3(SU(n),\mathbb{Z})$, namely the bi-invariant form $\Theta=\frac{1}{24\pi^2}\cdot\mathrm{Tr}(g^{-1}dg\wedge g^{-1}dg\wedge g^{-1}dg)$. For a point $y\in U_{ijklm}$, the 3-cycle $Z_{ijklm}(y)$ is constructed as follows. For each pair of indices (i,j), we choose a path $\tau^1_{ij}(y)$ from 1 to $g_{ij}(y)$ in $SU(n)$; we assume that $\tau^1_{ij}(y)$ is a smooth function of y. It follows from the cocycle condition $g_{ij}g_{jk}=g_{ik}$ that the composition $\tau^1_{ij}(y)*g_{ij}(y)\cdot\tau^1_{jk}(y)*\tau^1_{ik}(y)^{-1}$ is a loop. Since $SU(n)$ is simply-connected, this loop bounds some 2-simplex $\tau^2_{ijk}(y)$, which we may take to depend smoothly on $y\in U_{ijk}$. Then we see that the linear combination

$$g_{ij}(y)\cdot\tau^2_{jkl}(y)-\tau^2_{ikl}(y)+\tau^2_{ijl}(y)-\tau^2_{ijk}(y)$$

is a 2-cycle. Since $SU(n)$ is 2-connected this 2-cycle bounds some 3-simplex $\tau^3_{ijkl}(y)$. This may be pictured as a tetrahedron in $SU(n)$ whose vertices are $1,g_{ij}(y),g_{ik}(y),g_{il}(y)$.

Finally $Z_{ijklm}(y)$ is defined to be the formal linear combination $-\tau^3_{ijkl}(y)+\tau^3_{ijkm}(y)-\tau^3_{ijlm}(y)+\tau^3_{iklm}(y)-g_{ij}\cdot\tau^3_{jklm}(y)$.

This is a 3-cycle in $SU(n)$ and our formula for an integral cocycle representing $c_2(P)$ becomes

$$\frac{1}{24\pi^2}\cdot\int_{Z_{ijklm}(y)}\mathrm{Tr}\left(g^{-1}dg\wedge g^{-1}dg\wedge g^{-1}dg\right)\,.$$

Remarks

(i) If we set

$$f_{ijkl}(y) = \exp\left(2\pi i \cdot \int_{\tau^3_{ijkl(y)}} \frac{1}{24\pi^2} \cdot \mathrm{Tr}(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg)\right),$$

then f is a degree 3 Čech cocycle with coefficients in the sheaf $\underline{\mathbb{T}}$. The coboundary of f_{ijkl} in the exponential exact sequence is equal to $2\pi i$ times our Čech cocycle for c_2. For $n = 2$, the integer associated by our formula to every fivefold intersection of open sets is exhibited as the "winding number" of f. This is entirely analogous to the situation which arises with the first Chern class.

(ii) In the case of $SU(2)$, things simplify further since $SU(2)$ is diffeomorphic to S^3. Then $f_{ijkl}(y)$ can be taken to be the volume of a spherical tetrahedron $\tau^3_{ijkl}(y)$. This was found by Laursen, Schierholz and Wiese some years ago [19]. Recall that the formula for the volume of a spherical tetrahedron involves the dilogarithm function.

Next we will describe explicitly the lift $\hat{c}_2$ to the smooth Deligne complex of sheaves $\mathbb{Z}(4)_D$. This class is represented by a cocycle $(f_{ijkl}, \omega^1_{ijk}, \omega^2_{ij}, \omega^3_i)$. We have here the 0-cochain f_{ijkl}, the 1-cochain ω^1_{ijk}, etc., considered in gauge theory. They are obtained here by specializing the formula in Theorem 5, and taking into account the simplifications which occur for $\hat{c}_2$. To make contact with the gauge theory literature, we denote by $CS(A)$ the Chern–Simons 3-form on P associated to some connection A. We introduce simplices $\tilde{\tau}^j_{i_0\cdots i_j}(y)$ in P by setting

$$\tilde{\tau}^j_{i_0\cdots i_j}(y) = s_{i_0}(y) \cdot \tau^j_{i_0\cdots i_j}(y).$$

These are simplices in P whose vertices are the values $s_i(y)$ of the local sections; they are fully symmetric with respect to permutations of indices. We define differential forms as follows:

$$\omega^1_{ijk}(\xi)_y = 2\pi i \cdot \int_{\tilde{\tau}^2_{ijk}(y)} i_{\tilde{\xi}} \cdot CS(A),$$

$$\omega^2_{ij}(\xi_1, \xi_2)_y = -2\pi i \cdot \int_{\tilde{\tau}^1_{ij}(y)} i_{\tilde{\xi}_1} \cdot i_{\tilde{\xi}_2} \cdot CS(A),$$

$$\omega^3_i(\xi_1, \xi_2, \xi_3)_y = -2\pi i \cdot CS(A)(\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3)_{s_i(y)}.$$

In these formulae, for a vector field ξ over an open set in M, $\tilde{\xi}$ denotes some lift of ξ to $\tilde{\tau}^2_{ijk}$, resp. $\tilde{\tau}^1_{ij}$, resp. the image of s_i.

Again, in the case of $SU(2)$, these formulae were essentially known to Laursen, Schierholz and Wiese [19].

(iii) A similar formula for the first Pontryagin class of an $SO(n)$-bundle is given in [6]. The only essential difference lies in the construction of the 3-cycle $X_{ijklm}(y)$. Again one chooses paths $\sigma^1_{ij}(y)$ from 1 to $g_{ij}(y)$, but this time the loop $\sigma^1_{ij}(y) * g_{ij}(y)\sigma^1_{jk}(y) * \sigma^1_{ik}(y)^{-1}$ need not bound. However, since $\pi_1(SO(n)) = \mathbb{Z}_2$ for $n \geqq 3$, twice that loop will bound some 2-simplex. The construction then proceeds as in the case of c_2, but instead of a 3-simplex one needs a singular 3-chain.

(iv) The case of the first Pontryagin class is very important geometrically. The formulae in that case (and some holomorphic refinements) are used in [7] and in

[9] to give a geometric construction of a Quillen metric on some determinant line bundles.

(v) Finally, in the case of flat bundles, we recover the formulae of [11, Sect. 8]. The transition functions are then constant and if one chooses the flat connection, the lifts to the smooth Deligne complexes of Proposition 6 reduce to just one component, namely $f_{i_0 \cdots i_j}$ which is a Čech cocycle with values in the constant sheaf $\mathbb{T}$. This component is the exponential of $2\pi i$ times the volume of some simplex in a Stiefel manifold, and this simplex may be chosen to be totally geodesic. In the case of $\hat{c}_2$ we obtain

$$\exp\left(\frac{i}{12\pi} \cdot \int_{\tau^3_{ijkl}(y)} \mathrm{Tr}(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg)\right) ,$$

where $\tau^3_{ijkl}(y)$ is in fact independent of $y \in U_{ijkl}$.

In the case of the Euler class of a flat real vector bundle, the $(n-1)$-simplex $\gamma^{n-1}_{i_0,\ldots,i_{n-1}}(y)$, may be chosen to be a geodesic simplex in S^{n-1}, independent of y. The formula for the $\mathbb{T}$-valued cocycle becomes

$$\exp(2\pi i \cdot \mathrm{Vol}(\gamma^{n-1}_{i_0,\ldots,i_{n-1}})) ,$$

which is precisely the formula in [11]. The formula of the present paper is more general since it does not require the cocycles g_{ij} to be in general position.

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Communicated by H. Araki