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On the Lieb–Thirring Constants $L_{\gamma,1}$ for $\gamma \geqq 1/2$

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Abstract: Let $E_i(H)$ denote the negative eigenvalues of the one-dimensional Schrödinger operator $Hu := -u'' - Vu$, $V \geqq 0$, on $L_2(\mathbb{R})$. We prove the inequality

$$\sum_i |E_i(H)|^\gamma \leqq L_{\gamma,1} \int_{\mathbb{R}} V^{\gamma+1/2}(x)dx\,, \tag{1}$$

for the "limit" case $\gamma = 1/2$. This will imply improved estimates for the best constants $L_{\gamma,1}$ in (1) as $1/2 < \gamma < 3/2$.

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Let $H = -\Delta - V$ denote the Schrödinger operator in $L_2(\mathbb{R}^d)$. If the potential $V \geqq 0$ decreases sufficiently fast at infinity, the negative part of the spectrum of H is discrete. Let $\{E_i(H)\}$ be the corresponding increasing sequence of negative eigenvalues, each eigenvalue occurs with its multiplicity. This sequence is either finite or tends to zero.

Estimates on the behavior of the sequence of eigenvalues in terms of the potential have been in the focus of research for many years. In the earlier papers the main attention was paid to bounds on the number of negative eigenvalues ([2, 4, 18, 16, 7, 14, 12, 6]). In [15] Lieb and Thirring proved inequalities of the type

$$\sum_i |E_i(H)|^\gamma \leqq L_{\gamma,d} \int_{\mathbb{R}^d} V^{\gamma+\kappa}(x)dx, \quad \kappa = d/2\,. \tag{2}$$

Since then these estimates and the corresponding constants $L_{\gamma,d}$ have been studied intensively (e.g. [13, 9, 10]). Up to now it was known that (2) holds for all $\gamma \geqq 0$ if $d \geqq 3$, for $\gamma > 0$ if $d = 2$, and for $\gamma > 1/2$ if $d = 1$. On the contrary (2) fails for $\gamma = 0, d = 2$ and for $\gamma < 1/2, d = 1$. In this paper we prove (2) for the remaining case $d = 1, \gamma = 1/2$, which does not seem to have been settled so far. This result will imply an essential improvement for the estimates on the constants $L_{\gamma,1}$, $1/2 < \gamma < 3/2$. Moreover we deduce a new integral bound on the transmission coefficient of the corresponding scattering problem.

In conclusion the author expresses his gratefulness to M.Sh. Birman, who introduced him to the topic of negative bound states of Schrödinger operators. Moreover I am grateful to A. Laptev, under whose intensive supervision this paper was written.

1

In this subsection we provide some auxiliary results on the negative spectrum of the Neumann problem for the Sturm–Liouville-operator

$$(L_I^N u)(x) = -u''(x) - V(x)u(x)\,,$$

$$x \in I = [0,l]\,, \quad u'(0) = u'(l) = 0\,, \quad 0 \leqq V(x) \in L_1(I)\,.$$

Let $N_I(V,E)$ be the number of eigenvalues $E_i(L_I^N)$ of L_I^N below $E < 0$. According to the Birman–Schwinger principle ([4, 18]), the value of $N_I(V,E)$ does not exceed the square of the Hilbert–Schmidt norm of the integral operator

$$(Q_E u)(x) = \sqrt{V(x)}\int_0^l G(x,y,E)\sqrt{V(y)}u(y), \quad x \in I\,.$$

Here

$$G(x,y,E) = \begin{cases} \frac{\cosh(\lambda x)\cosh(\lambda(y-l))}{\lambda\sinh(\lambda l)} & x \leqq y \\ \frac{\cosh(\lambda y)\cosh(\lambda(x-l))}{\lambda\sinh(\lambda l)} & y \leqq x \end{cases}, \quad \lambda = \sqrt{|E|},\ E < 0,\ x,y \in I\,,$$

denotes the Green function of the problem $-u''-Eu$, $u'(0)=u'(l)=0$ on I. In view of

$$|G(x,y,E)| \leqq \frac{\coth(\lambda l)}{\lambda}\,,$$

one obtains the inequality

$$N_I(V,E) \leqq \frac{\coth^2(\lambda l)}{\lambda^2}\left(\int_I V(x)dx\right)^2, \quad \lambda = \sqrt{|E|},\ E < 0\,. \tag{3}$$

We apply (3) to the lowest eigenvalue $E_1(L_I^N)$, and find

$$\vartheta(\lambda_1 l) \leqq l\int_I V(x)dx\,, \quad \lambda_1 = \sqrt{|E_1(L_I^N)|} > 0\,, \quad \vartheta(x) := x\tanh x\,. \tag{4}$$

The function $\vartheta(x) = x\tanh x$ is strongly increasing in $x \geqq 0$. Let $\varsigma(y)$ be the inverse function of $\vartheta(x) = y$, $x,y > 0$. From (4) we immediately conclude

Lemma 1. *Let $E_1(L_I^N)$ be the lowest eigenvalue of the Neumann problem L_I^N on $I = [0,l]$. Assume $0 \leqq V \in L_1(I)$. Then the estimate*

$$\lambda_1 \leqq \varsigma\Big(l\int_I V(x)dx\Big)/l\,, \qquad \lambda_1 = \sqrt{|E_1(L_I^N)|} \geqq 0\,, \tag{5}$$

holds.

Next we recall a criteria, providing the existence of not more than one negative eigenvalue of the operator L_I^N.

First notice that for functions $u \in C^\infty(I)$, satisfying the orthogonality condition $\int_I u dx = 0$, the inequality

$$|u(x)|^2 \leqq \frac{l}{3}\int_I |u'|^2 dx, \quad x \in I\,, \tag{6}$$

holds. Indeed, we have

$$lu(x_0) = \int_0^{x_0} u'(x) x dx - \int_{x_0}^{l} u'(x)(l-x)dx\,.$$

This gives

$$|u(x_0)|^2 \leqq \frac{(x_0^{3/2} + (l-x_0)^{3/2})^2}{3l^2}\int_0^l |u'(x)|^2 dx\,.$$

Passing to the upper bound in $x_0 \in I$ we find (6). The constant $l/3$ in (6) is sharp.

Lemma 2. *Assume that for the non-trivial potential* $0 \leqq V$ *the estimate*

$$l\int_I V(x)dx \leqq 3 \tag{7}$$

holds. Then the Neumann problem L_I^N *on* $I = [0, l]$ *has exactly one negative eigenvalue.*

Proof. The existence of the eigenvalue is obvious. By (6) we find

$$\int_I |u'|^2 dx - \int_I V(x)|u|^2 dx \geqq 0\,, \quad u \in C^\infty([0,l])\,, \ \int_I u dx = 0\,. \tag{8}$$

The inequality (8) holds on a set of functions of codimension one with respect to the domain of the quadratic form of the Neumann problem L_I^N. Thus L_I^N itself has not more than one negative eigenvalue.

2

We turn now our attention to the one-dimensional Schrödinger operator

$$Hu = -u'' - V(x)u, \quad x \in \mathbb{R}\,, \ 0 \leqq V \in L_1(\mathbb{R})\,,$$

realized as a self-adjoint operator on $L_2(\mathbb{R})$ in the form sum sense. Let H_+ and H_- denote the self-adjoint operators on $L_2(\mathbb{R}_\pm)$, corresponding to the Neumann problem on the positive and negative semi-axes respectively.

Assume $V \not\equiv 0$ on $\mathbb{R}_+$. Fix the point $l_0 = 0$, and by iteration construct the sequence $l_k, k \in \mathbb{K} \subset \mathbb{N}$,

$$l^{(k)} \int_{l_k}^{l_{k+1}} V(x)dx = 3\,, \quad l^{(k)} := l_{k+1} - l_k\,. \tag{9}$$

If it occurs that $\int_{l_n}^{\infty} V(x)dx = 0$, we formally choose $l_{n+1} = +\infty$. For the elements of the sequence $l^{(k)}$ we have the bound $l^{(k)} \geqq 3/\int V(x)dx > 0$. Hence the intervals $I_k := [l_k, l_{k+1}]$, $k \geqq 0$, cover $\mathbb{R}_+$.

On each interval we consider the Neumann problem $L_{I_k}^N u = -u'' - V(x)u, u'(l_k) = u'(l_{k+1}) = 0$. Let $H_+^N = \oplus_{k\in\mathbb{K}} L_{I_k}^N$ denote the orthogonal sum of these operators. We have $H_+^N \leqq H_+$. For the ordered sequence of the respective negative eigenvalues this implies

$$E_i(H_+^N) \leqq E_i(H_+)\,. \tag{10}$$

In case of a semi-infinite interval the potential is identically zero on this interval, the respective Neumann problem has no negative spectrum. Therefore it will not play any role in our considerations.

By Lemma 2 the Neumann problem $L_{I_k}^N$ on the finite intervals I_k has exactly one negative eigenvalue. Because of (9) the bound (5) for $\lambda_1(I_k) := \sqrt{|E_1(L_{I_k}^N)|}$ turns into $\lambda_1(I_k) \leqq \varsigma(3)/l^{(k)}$, or equivalently [1]

$$\lambda_1(I_k) \leqq \frac{\varsigma(3)}{3}\int_{I_k} V(x)dx\,. \tag{11}$$

Since $V \in L_1(\mathbb{R}_+)$, the sequence $\int_{I_k} V(x)dx$ tends to zero as $k \to \infty$. Thus both operators H_+^N and H_+ are semibounded and their negative spectra are discrete. The negative spectrum of H_+^N coincides (as set and in its multiplicity) with the sequence of eigenvalues $\{E_1(L_{I_k}^N)\} = \{-\lambda_1^2(I_k)\}$. By (10) we have $|E_i(H_+)| \leqq |E_i(H_+^N)|$. Together with $0 \leqq V \in L_1(\mathbb{R}_+)$ this implies

$$\begin{aligned}\sum_i \sqrt{|E_i(H_+)|} &\leqq \sum_i \sqrt{|E_i(H_+^N)|} = \sum_k \lambda_1(I_k)\\ &\leqq \frac{\varsigma(3)}{3}\sum_k \int_{I_k} V(x)dx = \frac{\varsigma(3)}{3}\int_0^{\infty} V(x)dx\,,\end{aligned}$$

and we find the claimed result for the negative eigenvalues of the Neumann operator on the semi-axes

$$\sum_i \sqrt{|E_i(H_+)|} \leqq L_{\frac{1}{2},1}^{+} \int_{\mathbb{R}_+} V(x)dx\,, \quad L_{\frac{1}{2},1}^{+} \leqq \frac{\varsigma(3)}{3} < 1.005. \tag{12}$$

Naturally the analogous bound with the same constant holds for the operator H_-. Because of $H_- \oplus H_+ \leqq H$ we obtain the analog estimate on the negative eigenvalues of the Schrödinger operator H on $\mathbb{R}$,

$$\sum_i \sqrt{|E_i(H)|} \leqq L_{\frac{1}{2},1} \int_{\mathbb{R}} V(x)dx\,, \quad L_{\frac{1}{2},1} \leqq \frac{\varsigma(3)}{3} < 1.005\,. \tag{13}$$

We recall the reverse estimate for the operator H (see and [15, 9]). The first sum rule of Faddeev–Zakharov [8] states

$$\int V(x)dx = 4\sum_i \sqrt{|E_i(H)|} + \pi^{-1}\int \ln(1-|R(k)|^2)dk\,, \tag{14}$$

[1] On the other hand for $u(x) = l^{-1/2}$ one has $E_1(L_{I_k}^N) \leqq -l^{-1}\int_{I_k} V(x)dx$, and $\lambda_1(I_k) \geqq \sqrt{1/3}$ $\int_{I_k} V(x)dx$.

for (not necessary sign-defined) potentials $V \in C_0^\infty(\mathbb{R})$. In this $R(k) \in [0,1]$ is the reflection coefficient of the operator H. The integrand on the right hand side is negative, hence

$$\sum_i \sqrt{|E_i(H)|} \geqq \frac{1}{4}\int V(x)dx \,. \tag{15}$$

This bound can be closed to all potentials $V \in L_1(\mathbb{R})$.

The estimate from below on the constant $L_{1/2,1}$ can be improved. For a potential $0 \leqq V \in L_1(\mathbb{R})$ the number $N(V,E)$ of eigenvalues $E_i(H) < E < 0$ is bounded by

$$N(V,E) \leqq \frac{1}{2\sqrt{|E|}}\int V dx \,,$$

(see (3.7) in [5]). For the lowest eigenvalue this gives

$$\sqrt{|E_1(H)|} \leqq \frac{1}{2}\int V dx \,. \tag{16}$$

The constant in this estimate is sharp. Indeed, if the non-trivial potential $0 \leqq V \in C_0^\infty(\mathbb{R})$ is supplied with a sufficiently small coupling constant $\alpha > 0$, the operator $H_\alpha u = -u'' - \alpha V u$ has exactly one negative eigenvalue $E_1(H_\alpha)$. This eigenvalue obeys the asymptotics (see [17])

$$\sqrt{|E_1(H_\alpha)|} = (\alpha/2)\int V dx + o(\alpha)\,, \quad \alpha \to 0 \,.$$

We conclude $L_{1/2,1} \geqq 1/2$.

The previous arguments can be adapted to the problem on the semi-axes. Assume that $0 \leqq V$ is continuous on $\mathbb{R}_+$ up to the point zero, and has compact support. We supply this potential with a small coupling constant $\alpha > 0$, and consider the lowest eigenvalue $E_1(H_{+,\alpha})$ of the respective Neumann problem on $\mathbb{R}_+$. Let $u_\alpha(x)$ denote the corresponding eigenfunction. The even extension $u_\alpha(x) = u_\alpha(-x)$ is an eigenfunction of the operator H_α with the extended potential $V(x) = V(-x)$ on $\mathbb{R}$. The corresponding eigenvalue is $E_1(H_\alpha) = E_1(H_{+,\alpha})$. Since the operators H_α and $H_{+,\alpha}$ have only one negative eigenvalue for sufficiently small $\alpha > 0$, we find

$$\sqrt{|E_1(H_+)|} = \alpha \int_0^\infty V dx + o(\alpha)\,, \quad \alpha \to 0 \,.$$

We obtain $1 \leqq L_{1/2,1}^+ < 1.005$, our bound on the constant for the Neumann problem on the semi-axes is almost sharp!

Finally we remark the analog of (15) for the operator H_+. For a summable potential $V(x) = V(-x)$ it holds

$$\int_0^\infty V(x)dx \leqq 2\sum_i \sqrt{|E_i(H)|} \leqq 2\sum_i \sqrt{|E_i(H_- \oplus H_+)|} = 4\sum_i \sqrt{|E_i(H_+)|}\,. \tag{17}$$

The results of this subsection we summarize in

Theorem 1. 1. *The inclusion* $0 \leqq V \in L_1(\mathbb{R}_+)$ *implies the inequality*

$$\sum_i \sqrt{|E_i(H_+)|} \leqq L^+_{\frac{1}{2},1} \int_0^\infty V(x)dx \,. \tag{18}$$

For the best constant $L^+_{\frac{1}{2},1}$ *in* (18) *we have the estimate* $1 \leqq L^+_{\frac{1}{2},1} \leqq \varsigma(3)/3 < 1.005$. *Reversely, a priori assuming* $0 \leqq V \in L_1^{loc}(\mathbb{R}_+)$, *the discreteness of the negative spectrum together with the convergence of the sum in* (18) *imply* $V \in L_1$ *and* (17).

2. *The inclusion* $0 \leqq V \in L_1(\mathbb{R})$ *implies the inequality*

$$\sum_i \sqrt{|E_i(H)|} \leqq L_{\frac{1}{2},1} \int V(x)dx \,. \tag{19}$$

For the best constant $L_{\frac{1}{2},1}$ *in* (19) *we have the estimate* $1/2 \leqq L_{\frac{1}{2},1} \leqq \varsigma(3)/3 < 1.005$. *Reversely, a priori assuming* $0 \leqq V \in L_1^{loc}(\mathbb{R})$, *the discreteness of the negative spectrum together with the convergence of the sum in* (19) *imply* $V \in L_1$ *and* (15).

Remark. As usual one can drop the assumption $V \geqq 0$. One has to ensure that the corresponding operators H, H_+ are defined in the form sum sense, and the integrand in (18) and (19) has to be replaced by $V_+(x) := \max\{0, V(x)\}$.

Notice that (19) and (14) together with $1/2 \leqq L_{1/2,1} < \infty$ imply

Theorem 2. *Assume* $V \in C_0^\infty(\mathbb{R})$, $2V_\pm = |V| \pm V$, *and let* $R(k)$ *be the reflection coefficient for the corresponding one-dimensional Schrödinger operator* $Hu = -u'' - Vu$ *on* $L_2(\mathbb{R})$. *Then the integral estimate*

$$\frac{1}{\pi}\int |\ln(1 - |R(k)|^2)|dk \leqq \int V_- dx + (4L_{\frac{1}{2},1} - 1)\int V_+ dx \leqq (4L_{\frac{1}{2},1} - 1)\|V\|_{L_1(\mathbb{R})}$$

holds.

3

We turn now to the case $\gamma > 1/2$. We restrict our considerations to the operator H on $L_2(\mathbb{R})$. Here the inequalities

$$\sum_i |E_i(H)|^\gamma \leqq L_{\gamma,1} \int_{\mathbb{R}} V^{\gamma+1/2}(x)dx \,, \tag{20}$$

are well established, but we will give an essential improvement of the estimates for the corresponding constants $L_{\gamma,1}$. For $\gamma \geqq 3/2$ in [1] it has been proven that $L_{\gamma,1} = L^{cl}_{\gamma,1}$. The last notation stands for the classical constant

$$L^{cl}_{\gamma,1} = \frac{\Gamma(\gamma+1)}{2\sqrt{\pi}\Gamma(\gamma+\frac{3}{2})} \,.$$

Hence we will stress the case $1/2 < \gamma < 3/2$. We shall compare our results with the bounds of Lieb and Thirring,

$$L_{\gamma,1} \leqq L^{LT}_{\gamma,1} := \frac{\gamma^{\gamma+1}}{\sqrt{2}(\gamma - 1/2)^{\gamma+1/2}(\gamma + 1/2)} \,, \tag{21}$$

and their improvements by Glaser, Grosse and Martin ([9]) $L_{\gamma,1} \leqq L^{GGM}_{\gamma,1}$, with

$$L^{GGM}_{\gamma,1} := \inf_{1<m<3/2} \frac{(m-1)^{m-1}\Gamma(2m)\gamma^{\gamma+1}\Gamma(\gamma+\frac{1}{2}-m)}{2^{2m-1}m^{m-1}\Gamma(m)\Gamma(\gamma+\frac{3}{2})(m-\frac{1}{2})^{m-\frac{1}{2}}(\gamma+\frac{1}{2}-m)^{\gamma+\frac{1}{2}-m}} \tag{22}$$

Our proof of Theorem 1 can be generalized to the case $\gamma \geqq 1/2$. However, this direct approach gives the bound $L_{\gamma,1} \leqq (\varsigma(3))^{2\gamma}/3^{\gamma+1/2}$, which is not very sharp. A better bound can be found using the fact that the ratio $L_{\gamma,1}/L^{cl}_{\gamma,1}$ is non-increasing in γ, see [1]. We find

$$L_{\gamma,1} \leqq L^*_{\gamma,1} := 4\varsigma(3)L^{cl}_{\gamma,1}/3 = \frac{2\varsigma(3)\Gamma(\gamma+1)}{3\sqrt{\pi}\Gamma(\gamma+\frac{3}{2})}\,.$$

This bound is sharper than (21) and (22) for all $1/2 \leqq \gamma \leqq 3/2$. In particular, $L^*_{1,1} < 0.853$, while $L^{LT}_{1,1} = 4/3$ and $L^{GGM}_{1,1} = 1.269$. [2]

If we consider only potentials $\tilde{V}$ proportional to a characteristic function of a set $M \subset \mathbb{R}$ of finite measure,

$$\tilde{V}(x) = v\chi_M(x), \quad \chi_M(x) = \begin{cases} 1 & x \in M \\ 0 & x \notin M \end{cases}, \quad v > 0\,,$$

we can find a better constant by "interpolating" between the cases $\gamma = 1/2$ and $\gamma = 3/2$. Indeed, the ratio

$$\psi(\gamma,\tilde{V}) := \frac{\sum_i |E_i(H)|^\gamma}{\int \tilde{V}^{\gamma+1/2}dx}$$

is analytic and continuous up to the boundary for complex γ in the strip $1/2 < \Re\gamma < 3/2$. On the boundary we have the estimates

$$|\psi(\gamma,\tilde{V})| \leqq L_{\frac{1}{2},1} \leqq \frac{\varsigma(3)}{3}, \quad \text{as } \Re\gamma = \frac{1}{2}\,, \qquad |\psi(\gamma,\tilde{V})| \leqq L_{\frac{3}{2},1} = \frac{3}{16}, \quad \text{as } \Re\gamma = \frac{3}{2}\,.$$

By the Hadamard Lemma we obtain

$$\psi(\gamma,\tilde{V}) \leqq \tilde{L}^*_{\gamma,1} := \left(\frac{\varsigma(3)}{3}\right)^{\frac{3}{2}-\gamma}\left(\frac{3}{16}\right)^{\gamma-\frac{1}{2}}\,, \quad \frac{1}{2} < \gamma < \frac{3}{2}\,. \tag{23}$$

In particular, $\tilde{L}^*_{1,1} < 0.4341$. We notice, that (23) is sharper than the results for characteristic functions by A. Laptev in [11] for the case of dimension one .

For completeness we recall the estimate from below on the constants $L_{\gamma,1}$, obtained in [15]. To do so we consider the best constants $L^1_{\gamma,1}$ in the inequalities

$$|E_1(H)|^\gamma \leqq L^1_{\gamma,1}\int V^{\gamma+1/2}dx, \quad \gamma \geqq \frac{1}{2}\,. \tag{24}$$

Obviously $L_{\gamma,1} \geqq L^1_{\gamma,1}$. For $\gamma > 1/2$ the corresponding variational equation can be solved analytically and one obtains[3]

$$L^1_{\gamma,1} = \pi^{-1/2}\frac{1}{\gamma-1/2}\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1/2)}\left(\frac{\gamma-1/2}{\gamma+1/2}\right)^{\gamma+1/2} = 2L^{cl}_{\gamma,1}\left(\frac{\gamma-1/2}{\gamma+1/2}\right)^{\gamma-1/2}, \tag{25}$$

[2] One can apply an argument of Glaser, Grosse and Martin [9], to deduce a bound on $L^{sph}_{0,3}$ for *spherical symmetric* potentials from $L_{1,1}$. Although one considers only a special class of potentials, even the new bound on $L_{1,1}$ is not sharp enough to reach Lieb's result for $L_{0,3}$ by this method.

[3] In particular this gives $0.2451 < L_{1,1} < 0.853$.

(see [15]). Moreover in the previous subsection we showed that (25) remains true for $\gamma = 1/2$ and $L^1_{1/2,1} = 1/2$. For $\gamma \geqq 3/2$ it holds $L^1_{\gamma,1} \leqq L^{cl}_{\gamma,1}$. For $\gamma < 3/2$ we have $L^1_{\gamma,1} > L^{cl}_{\gamma,1}$, this implies $L_{\gamma,1} > L^{cl}_{\gamma,1}$ as $1/2 \leqq \gamma < 3/2$ (see [15] and also [10]).

We proved

Theorem 3. *For the numerical values of the best possible constants* $L_{\gamma,1}, 1/2 \leqq \gamma \leqq 3/2$ *in* (20) *the estimate*

$$2L^{cl}_{\gamma,1}\left(\frac{\gamma-\frac{1}{2}}{\gamma+\frac{1}{2}}\right)^{\gamma-\frac{1}{2}} \leqq L_{\gamma,1} \leqq L^*_{\gamma,1} = \frac{4\varsigma(3)}{3}L^{cl}_{\gamma,1} = \frac{2\varsigma(3)\Gamma(\gamma+1)}{3\sqrt{\pi}\Gamma(\gamma+\frac{3}{2})}\,,\quad \frac{1}{2} \leqq \gamma \leqq \frac{3}{2}\,,$$

holds. For potentials $\tilde{V}$ *proportional to characteristic functions, the constant* $L_{\gamma,1}$ *in the Lieb–Thirring inequality can be replaced by* $\tilde{L}^*_{\gamma,1}$ *from* (23).

Notice that the bound $L^*_{\gamma,1}$ on $L_{\gamma,1}$ does not tend to $L_{\frac{3}{2},1} = L^{cl}_{\frac{3}{2},1} = 3/16$ as $\gamma \to 3/2 - 0$. For γ near 3/2 the estimate on $L_{\gamma,1}$ can be improved. To do so we shall recall some auxiliary material from real interpolation theory.

4

Let ℓ_p denote the ideal of p-summable sequences $\{u_n\}_{n\in\mathbb{N}}$, equipped by the standard quasi-norm

$$\|\{u_n\}\|^p_{\ell_p} := \sum_n |u_n|^p\,,\quad p > 0\,.$$

For a sequence $\{u_n\}_{n\in\mathbb{N}} \in \ell_{p_0} + \ell_{p_1}$ one can define the $(p_0,p_1) - K$-function

$$K(\{u_n\},t,p_0,p_1) := \inf_{\substack{u_n = u^{(0)}_n + u^{(1)}_n \\ u^{(\iota)}_n \in \ell_{p_\iota}}} \left(\|u^{(0)}_n\|^{p_0}_{\ell_{p_0}} + t\|u^{(1)}_n\|^{p_1}_{\ell_{p_1}}\right)\,,\quad t > 0\,.$$

For a function $f \in L_{p_0} + L_{p_1}$ one may use the analogous definition

$$K(f,t,p_0,p_1) := \inf_{\substack{f = f_0 + f_1 \\ f_\iota \in L_{p_\iota}}} \left(\|f_0\|^{p_0}_{L_{p_0}} + t\|f_1\|^{p_1}_{L_{p_1}}\right)\,,\quad t > 0\,.$$

On functions $h : (0,\infty) \to [0,\infty)$ we define the functionals

$$\Phi_{\eta,q}[h] = \left(\int_0^\infty (t^{-\eta}h(t))^q \frac{dt}{t}\right)^{1/q}\,,\quad \eta \in (0,1)\,,\ 0 < q < \infty\,,$$

$$\Phi_{\eta,\infty}[h] = \sup_{t>0} t^{-\eta}h(t)\,,\quad \eta \in (0,1)\,.$$

Notice that $h_1(t) \leqq h_2(t)$ implies $\Phi_{\eta,q}[h_1] \leqq \Phi_{\eta,q}[h_2]$. According to the "power theorem" of real interpolation theory, see [3], it holds

$$\Phi_{\eta,q}[K(\{u_n\},\ \cdot\ ,p_0,p_1)] \asymp \|\{u_n\}\|^p_{\ell^{p,r}}\,, \tag{26}$$

$$\Phi_{\eta,q}[K(f,\ \cdot\ ,p_0,p_1)] \asymp \|f\|^p_{L^{p,r}}\,, \tag{27}$$

$$p = (1-\eta)p_0 + \eta p_1\,,\quad r = pq\,,\quad \eta \in (0,1)\,,$$
$$0 < q \leqq \infty\,,\quad 0 < p_0\,,\quad p_1 < \infty\,,\quad p_0 \neq p_1\,.$$

The quasi-norms on the right hand side denote the Lorentz scale of sequence ideals $\ell^{p,r}$ or function spaces $L^{p,r}$, respectively. For the definition of these ideals see, e.g., [3] or [19]. We just point out, that $\ell_p = \ell^{p,p}$ and $L_p = L^{p,p}$.

In general it is difficult to trace the constants in the two-side estimates in (26), (27) . However for the special case $q = 1$ one has the *equalities* (see [3], p. 111, proof of Theorem 5.2.2).

$$\Phi_{\eta,1}[K(\{u_n\}, \cdot, p_0, p_1)] = \Theta(\eta, p_0, p_1)\|\{u_n\}\|^p_{\ell_p}, \tag{28}$$

$$\Phi_{\eta,1}[K(f, \cdot, p_0, p_1)] = \Theta(\eta, p_0, p_1)\|f\|^p_{L_p}, \tag{29}$$

$$p = (1-\eta)p_0 + \eta p_1, \quad \eta \in (0,1), \quad 0 < p_0, \quad p_1 < \infty, \quad p_0 \neq p_1,$$

where

$$\Theta(\eta, p_0, p_1) = \int_0^\infty t^{-\eta-1} \inf_{y_0+y_1=1}(|y_0|^{p_0} + t|y_1|^{p_1})dt .$$

Below we shall use these identities for improving the bounds on $L_{\gamma,1}$ for certain $\gamma \in (1/2, 3/2)$.

5

In this subsection we consider the Schrödinger operator

$$H = -\Delta - V(x), \quad V \geqq 0, \quad x \in \mathbb{R}^d,$$

in arbitrary dimensions $d \geqq 1$. We assume that this operator is semibounded from below and that its negative spectrum is discrete. Let $\{E_n(H)\}$ be the non-decreasing sequence of negative eigenvalues of the operator H, each eigenvalue appears with its multiplicity.

Let us start from the Ky-Fan inequality for the discrete negative spectrum. If $V = V_0 + V_1$, and the operators

$$H_0 = -\theta\Delta - V_0, \quad H_1 = -(1-\theta)\Delta - V_1, \quad \theta \in (0,1),$$

have discrete negative spectrum, then the inequality

$$|E_{m+n-1}(H)| \leqq |E_n(H_0)| + |E_m(H_1)|$$

holds for all $m, n = 1, 2, \ldots$ We construct the sequences

$$a_k := E_s(H_0), \quad s = 1 + \left[\frac{k}{N+1}\right],$$

$$b_k := E_l(H_1), \quad l = N\left[\frac{k}{N+1}\right] + (k \bmod N+1), \quad N, k, l, s \in \mathbb{N},$$

and obtain

$$E_k(H) \leqq a_k + b_k, \quad k \in \mathbb{N}. \tag{30}$$

Assume now $V_i \in L_{p_i+\kappa}(\mathbb{R}^d)$, $\kappa = d/2, 0 < p_i < \infty$ for $d \geqq 2$ and $1/2 \leqq p_i < \infty$ if $d = 1$. From (30) and (2) it follows, that

$$\begin{aligned} K(\{E_k(H)\}, t, p_0, p_1) &\leqq \|\{a_k\}\|_{\ell_{p_0}}^{p_0} + t\|\{b_k\}\|_{\ell_{p_1}}^{p_1} \\ &\leqq (1+N)\sum_n |E_n(H_0)|^{p_0} + t(1+N^{-1})\sum_m |E_m(H_1)|^{p_1} \\ &\leqq (1+N)\theta^{-\kappa} L_{p_0,d}\|V_0\|_{L_{p_0+\kappa}(\mathbb{R}^d)}^{p_0+\kappa} + t(1+N^{-1})(1-\theta)^{-\kappa} L_{p_1,d}\|V_1\|_{L_{p_1+\kappa}(\mathbb{R}^d)}^{p_1+\kappa} . \end{aligned}$$

Interchanging the definitions of the sequences $\{a_k\}$ and $\{b_k\}$ one can see that in the previous expression the role of N and $1/N$ can be interchanged. Thus we can assume that N is of the form k or $1/k$, $k \in \mathbb{N}$. Passing to the lower bound over all suitable decompositions $V = V_0 + V_1$ one finds

$$K(\{E_k(H)\}, t, p_0, p_1)$$

$$\leqq \frac{(1+N)L_{p_0,d}}{\theta^\kappa} K\left(V, t\frac{(1+N^{-1})(1-\theta)^{-\kappa}L_{p_1,d}}{(1+N)\theta^{-\kappa}L_{p_0,d}}, p_0+\kappa, p_1+\kappa \right), \tag{31}$$

$$N = \cdots, \frac{1}{3}, \frac{1}{2}, 1, 2, 3, \cdots \quad \kappa = \frac{d}{2},$$

with $0 < p_i < \infty$ for $d \geqq 2$ and $1/2 \leqq p_i < \infty$ for $d = 1$. This relation allows one to apply interpolation methods *directly* to the sequences of negative bound states, although the mapping $V \mapsto \{E_n(H)\}$ is strongly non-linear.

6

Let us return to the one-dimensional case and choose $p_0 = 1/2$ and $p_1 = 3/2$. Applying the functional $\Phi_{\eta,1}$ to both sides of this inequality, by (28) and (29) we obtain

$$\sum_k |E_k(H)|^\gamma \leqq L_{\gamma,1} \int V^{\gamma+\frac{1}{2}} dx, \quad \frac{1}{2} < \gamma < \frac{3}{2},$$

where

$$L_{\gamma,1} \leqq \frac{\Theta(\eta,1,2)}{\Theta(\eta,\frac{1}{2},\frac{3}{2})}(1+N)^{1-\eta}\theta^{-(1-\eta)/2}L_{1/2,1}^{(1-\eta)}(1+N^{-1})^\eta(1-\theta)^{-\eta/2}L_{p_1,1}^\eta, \tag{32}$$

$$\gamma = \frac{(1-\eta)}{2} + \frac{3\eta}{2}, \quad N = \ldots, \frac{1}{3}, \frac{1}{2}, 1, 2, 3, \ldots .$$

Let $M(\eta)$ be the minimum of the sequence

$$(1+N)^{1-\eta}(1+N^{-1})^\eta, \quad N = \ldots, \frac{1}{3}, \frac{1}{2}, 1, 2, 3, \ldots .$$

It occurs that $M(\eta) \to 1$ as $\eta \to 0, 1$. If we minimize (32) in $\theta \in (0,1)$, we find $\theta(\eta) = 1 - \eta$, and

$$L_{\gamma,1} \leqq L_{\gamma,1}^{**} := C(\eta)\left(\frac{\varsigma(3)}{3}\right)^{(1-\eta)}\left(\frac{3}{16}\right)^\eta, \quad \gamma = \frac{1}{2} + \eta, \tag{33}$$

$$C(\eta) = \frac{\Theta(\eta,1,2)}{\Theta(\eta,\frac{1}{2},\frac{3}{2})} \frac{M(\eta)}{\sqrt{\eta^\eta(1-\eta)^{1-\eta}}} . \tag{34}$$

The involved functions Θ can be evaluated as

$$\Theta(\eta, 1, 2) = \frac{2^\eta}{\eta(1-\eta)(1+\eta)} ,$$

and

$$\Theta\left(\eta, \frac{1}{2}, \frac{3}{2}\right) = \frac{\left(\frac{2}{3}\sqrt{1+2/\sqrt{3}}\right)^{1-\eta}}{1-\eta} + \sqrt{\frac{1}{2}}\left(\frac{3}{2}\right)^\eta \left(I_0(\eta) + \frac{2}{3} I_1(\eta)\right) ,$$

$$I_0(\eta) = \int_{u_0}^{1} u(1-u)^{\frac{\eta-2}{2}}(1+u)^{\frac{\eta-1}{2}} du ,$$

$$I_1(\eta) = \int_{u_0}^{1} u(1-u)^{\frac{\eta}{2}}(1+u)^{\frac{\eta-3}{2}} du ,$$

$$u_0 = \sqrt{\frac{2}{2+\sqrt{3}}} .$$

Notice, that $C(\eta) \to 1$ as $\eta \to 1$, thus $L^{**}_{\gamma,1} \to 3/16$ as $\gamma \to 3/2$ and $L^{**}_{\gamma,1} < L^{*}_{\gamma,1}$ as $\gamma \to 3/2$.

Theorem 4. *For the constant $L_{\gamma,1}$ in* (19) *the bound*

$$L_{\gamma,1} \leqq \min\{L^{*}_{\gamma,1}, L^{**}_{\gamma,1}\}, \quad 1/2 < \gamma < 3/2,$$

holds.

7

Let $\{\phi_i\}$ be some $L_2(\mathbb{R}^d)$-orthonormal system, $\phi_i \in W_2^1(\mathbb{R}^d)$. Then (2) implies ([15, 13])

$$\sum_{i=1}^{n} \int |\nabla \phi_i|^2 dx \geqq K_{p,d}\left(\int \rho_\phi^{p/(p-1)} dx\right)^{2(p-1)/d} , \tag{35}$$

$$\rho_\phi(x) := \sum_{i=1}^{n} |\phi_i(x)|^2,$$

$$\max\{d/2, 1\} \leqq p < 1 + d/2, \quad \text{excluding } p = 1 \text{ for } d = 2,$$

with suitable constants $K_{p,d}$. In case of $d = 1$ and $p = 3/2$ this turns into

$$\sum_{i=1}^{n} \int |\phi_i'|^2 dx \geqq K_{3/2,1} \int \rho_\phi^3 dx . \tag{36}$$

The constant $K_{3/2,1}$ is related to $L_{1,1}$ via the formula

$$L_{1,1} = 2/\sqrt{27 K_{3/2,1}} .$$

Our improved estimate on $L_{1,1}$ implies $K_1 \geqq 0.203$, compare with $K_1 \geqq 1/12$ in [13].

We also point out the case $p = d = 1$. Then

$$\sum_{i=1}^{n} \int |\phi_i'|^2 dx \geqq K_{1,1} \| \rho_\phi \|^2_{L_\infty(\mathbb{R})} , \tag{37}$$

with a constant $1 \geqq K_{1,1} \geqq 1/(2L_{1/2,1})$, see (3.27) in [15]. Thus we find (37) with $1 \geqq K_{1,1} > 0.497$.

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