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A Simple Geometric Representative for μ of a Point

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Abstract: For $SU(2)$ (or $SO(3)$) Donaldson theory on a 4-manifold X, we construct a simple geometric representative for μ of a point. Let p be a generic point in X. Then the set $\{[A] | F_A^-(p) \text{ is reducible}\}$, with coefficient $-1/4$ and appropriate orientation, is our desired geometric representative. The construction is an exercise in real algebraic geometry in the style of Ehresmann and Pontryagin.

1. Background and Statement of Results

In the past decade, an industry has developed studying the homology of moduli spaces, thereby shedding light on the topology or geometry of underlying manifolds. The best known example is Donaldson's work on gauge theory in 4 dimensions [DK]. Donaldson's polynomial invariants measure the fundamental classes of moduli spaces of anti-self-dual connections over an orientable 4-manifold, giving information about the differentiable structure of that manifold.

Let X be an oriented 4-manifold, let $G = SU(2)$ or $SO(3)$ and let $\mathcal{B}_k$ be the space of connections (up to gauge equivalence) on P_k, the principal G bundle of instanton number k over X. Let $\mathcal{B}_k^*$ (resp. $\tilde{\mathcal{B}}_k^*$) be the space of irreducible connections, (resp. irreducible framed connections) on P_k, modulo gauge equivalence. $\tilde{\mathcal{B}}_k^*$ is a principal $SO(3)$ bundle over $\mathcal{B}_k^*$.

Donaldson [D1, D2] defined a map $\tilde{\mu} : H_i(X, \mathbb{Q}) \to H^{4-i}(\tilde{\mathcal{B}}_k^*, \mathbb{Q})$, $i = 1, 2, 3$, whose image freely generates the rational cohomology of $\tilde{\mathcal{B}}_k^*$. For Σ a 1, 2, or 3-cycle in X, the class $\tilde{\mu}([\Sigma])$ descends to a cohomology class on $\mathcal{B}_k^*$, which is then denoted $\mu([\Sigma])$. The classes $\mu([\Sigma])$, together with an additional 4-dimensional class, freely generate the cohomology of $\mathcal{B}_k^*$. The additional class can be viewed as μ of the point class $[x] \in H_0(X)$. In this view, μ maps $H_i(X)$ to $H^{4-i}(\mathcal{B}_k^*)$, where i now ranges from 0 to 3, and the image of the μ map freely generates $H^*(\mathcal{B}_k^*, \mathbb{Q})$.

This gives a polynomial invariant on the homology of X, the action of μ of the elements of H_* on the "fundamental class" of $\mathcal{M}_k$. Formally, for elements

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$[\Sigma_1], \ldots, [\Sigma_n] \in H_*(X)$, we write

$$q([\Sigma_1], \ldots, [\Sigma_n]) = \mu([\Sigma_1]) \smile \cdots \smile \mu([\Sigma_n])[\mathcal{M}_k]. \tag{1}$$

The "fundamental class of $\mathcal{M}_k$" is usually not well defined, as $\mathcal{M}_k$ is typically not compact. To make sense of (1) one must compactify $\mathcal{M}_k$ and show that the classes $\mu([\Sigma])$ extend properly to the compactification of $\mathcal{M}_k$. This is usually done with geometric representatives. One finds finite-codimension varieties V_Σ in $\mathcal{B}$ that are, roughly speaking, Poincaré dual to $\mu([\Sigma])$. One then attempts to count points in $V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_n} \cap \mathcal{M}_k$. To make a topological invariant one must show that the intersection points can be bounded away from the ends of $\mathcal{M}_k$. This requires careful analysis of the bubbling-off phenomena that make $\mathcal{M}_k$ noncompact.

The success of such a program can depend on good choices of geometric representatives. For example, for 2-dimensional Yang-Mills theory, the generalized Newstead conjecture resisted abstract analysis until Weitsman [We] found a set of simple geometric representatives for the problem. Using these, it was fairly easy to characterize the points in $\cap_i V_{\Sigma_i} \cap \mathcal{M}$, compute the invariants, and prove the conjecture.

For Donaldson theory, fairly simple geometric representatives have been found for the 1, 2, and 3-dimensional classes. In each case, the geometric representative of $\mu([\Sigma])$ is the set of connections that satisfy a simple condition when restricted to Σ. Until now, however, there has not been any similar description of $\mu([p])$, where p is a single point, in terms of data at that point. The purpose of this paper is to provide such a description. For any point $p \in X$, let $v_p = \{[A] \in \mathcal{B}_k^* | F_A^- \text{ is reducible at } p\}$. Here $F_A^- = (F_A - *F_A)/2$ is the anti-self-dual part of the curvature F_A, and by "reducible at p" we mean that the components $F_{ij}^-(p)$ are all colinear as elements of the Lie algebra of G. The main theorem is

Theorem 1. *v_p is a geometric representative of $-4\mu([p])$.*

The proof proceeds in stages. In Sect. 2, we review some classical real algebraic geometry and construct a simple representative of the first Pontryagin class p_1 of canonical $SO(3)$ bundles over Grassmannians of real oriented 3-planes. The construction is essentially due to Pontryagin [P] and Ehresmann [E], but their techniques seem to have been generally forgotten. In Sect. 3, we extend this analysis to $BSO(3)$ and construct an explicit isomorphism between a space of connections on a neighborhood of the point p and $ESO(3)$. Pulling the representative of $p_1(ESO(3))$ back by this isomorphism gives v_p, and fixes the orientation.

To be useful for Donaldson theory, v_p must be transverse to the moduli spaces $\mathcal{M}_k$ and extend to the compactification of $\mathcal{M}_k$. These issues are discussed in Sect. 4, where we also discuss a possible topological application of this representative.

2. Cohomology of Real Grassmannians

Let V_N be the space of real, rank 3, $3 \times N$ matrices. Equivalently, V_N is the Stiefel manifold of triples of linearly independent vectors in $\mathbb{R}^N$. Let V_N^0 be the triples of orthonormal vectors in $\mathbb{R}^N$. The group $SO(3)$ acts freely on both spaces by left multiplication. Let B_N be the quotient of V_N by $SO(3)$ and let G_N be the quotient of V_N^0 by $SO(3)$. G_N is the Grassmannian of oriented 3-planes in $\mathbb{R}^N$. We will denote by π both natural projections, from V_N to B_N and from V_N^0 to G_N. The Gram-Schmidt process gives a natural bundle map from V_N to V_N^0, which we denote ρ.

ρ itself defines trivial $\mathbb{R}^6$ bundles $V_N \to V_N^0$ and $B_N \to G_N$. Inclusion of V_N^0 in V_N defines a natural section. In short, we have the commutative diagram

$$\begin{array}{ccc} V_N & \xrightarrow{\rho} & V_N^0 \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ B_N & \xrightarrow{\rho} & G_N \end{array} \tag{2}$$

B_N and G_N have the same topology.

Theorem 2. *Let $v_N = \{m \in V_N \mid$ first 3 columns of m have rank $\leq 1\}$. Then $\pi(v_N)$ is Poincare dual to a generator of $H^4(B_N)$. By choosing orientations correctly, this generator may be taken to be the first Pontryagin class of the bundle $V_N \to B_N$.*

Proof. The proof is an application of some general computations of Pontryagin [P] and Ehresmann [E]. (Indeed, Theorem 2 was almost certainly known to Pontryagin). Within the 9 dimensional space of real 3×3 matrices, the rank ≤ 1 matrices form a closed codimension-4 set. $\pi(v_N)$ is thus a closed codimension-4 submanifold of B_N, and so is dual to some (possibly zero) element of H^4. We construct a generator of $H_4(B_N)$ and show it intersects $\pi(v_N)$ exactly once, establishing that $\pi(v_N)$ is a generator of H^4. The sign, relative to p_1, is determined separately.

We begin with a cell decomposition of G_N. Consider the set of $3 \times N$ matrices of the form

$$\begin{pmatrix} x_1 & x_2 & \dots & x_{i-1} & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ y_1 & y_2 & \dots & y_{i-1} & 0 & y_{i+1} & \dots & y_{j-1} & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ z_1 & z_2 & \dots & z_{i-1} & 0 & z_{i+1} & \dots & z_{j-1} & 0 & z_{j+1} & \dots & z_{k-1} & 1 & 0 & \dots & 0 \end{pmatrix}. \tag{3}$$

That is, a matrix with pivots $x_i = y_j = z_k = 1$, $i < j < k$, $y^i = z^i = z^j = 0$, and with no nonzero entries to the right of the pivots. Each oriented 3-plane corresponds to a unique matrix of this form, or to minus such a matrix. For fixed i, j, k we denote the set of matrices of this type as $e_+(i,j,k)$, and the set of negatives of these matrices as $e_-(i,j,k)$. The closures of the sets $e_\pm(i,j,k)$, called Schubert cycles, give a cellular decomposition of G_N.

The cell $e_+(i,j,k)$ has dimension $i+j+k-6$. We give it the orientation $dx^1 \cdots dx^{i-1} dy^1 \cdots dy^{j-1} dz^1 \cdots dz^{k-1}$, where of course the variables y^i, z^i, z^j are skipped in this list. We orient $e_-(i,j,k)$ so the map $-1 : e_\pm(i,j,k) \to e_\mp(i,j,k)$ is orientation-preserving. The boundary map is then

$$\begin{aligned} \partial e_\pm(i,j,k) = {} & (-1)^i e_\pm(i-1,j,k) - e_\mp(i-1,j,k) \\ & + (-1)^{i+j+1} e_\pm(i,j-1,k) + (-1)^i e_\mp(i,j-1,k) \\ & + (-1)^{i+j+k+1} e_\pm(i,j,k-1) + (-1)^{i+j} e_\mp(i,j,k-1) \end{aligned} \tag{4}$$

This formula is of course independent of N.

$H_4(G_N)$ is then easily computed. It is $\mathbb{Z}$, and is generated by $S_N = e_+(1,4,5) + e_+(1,3,6) - e_+(1,2,7)$. The cycle $\rho(\pi(v_N))$ doesn't intersect $e_+(1,3,6)$ or $e_+(1,2,7)$, and hits $e_+(1,4,5)$ at exactly one point, namely

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots & 0 \end{pmatrix}, \tag{5}$$

and the intersection is transverse. Thus $\rho(\pi(v_N))$ is a generator of $H^4(G_N)$. Pulling back we get that $\pi(v_N)$ is a generator of $H^4(B_N)$. All that remains is to fix the orientation such that $\pi(v_N)$ represents p_1.

To fix the orientation we consider the natural embedding $i : G_N \to G^{\mathbb{C}}_{3,N}$, where $G^{\mathbb{C}}_{3,N}$ is the Grassmannian of complex 3-planes in $\mathbb{C}^N$. The Pontryagin classes on G_N are pullbacks of Chern classes on $G^{\mathbb{C}}_{3,N}$. In particular, $p_1 = -i^* c_2$ [MS]. We therefore have only to compute the intersection number in $G^{\mathbb{C}}_{3,N}$ of $i(S_N)$ with a cycle representing c_2. If W is a complex codimension-2 subspace of $\mathbb{C}^N$, then c_2 is represented by $Y \subset G^{\mathbb{C}}_{3,N}$, the set of 3-planes in $\mathbb{C}^N$ whose intersections with W have (complex) dimension 2 or greater [GH].

If $w_1, \dots w_N$ are the natural coordinates on $\mathbb{C}^N$, we choose $W = \{w_1 + iw_4 = w_2 + iw_3 = 0\}$. A 3-plane spanned by the rows of

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & \dots \\ y_1 & y_2 & y_3 & y_4 & \dots \\ z_1 & z_2 & z_3 & z_4 & \dots \end{pmatrix}, \tag{6}$$

is in Y if and only if the complex 3-vectors $(x_1 + ix_4,\ y_1 + iy_4,\ z_1 + iz_4)$ and $(x_2 + ix_3,\ y_2 + iy_3,\ z_2 + iz_3)$ are (complex) colinear. This is never the case in the closures of $e_+(1,3,6)$ or $e_+(1,2,7)$.

Matrices in $e_+(1,4,5)$ take the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & y_2 & y_3 & 1 & 0 & 0 & \dots & 0 \\ 0 & z_2 & z_3 & 0 & 1 & 0 & \dots & 0 \end{pmatrix}, \tag{7}$$

Y intersects $e_+(1,4,5)$ at the single point $y_2 = y_3 = z_2 = z_3 = 0$, and the intersection number is easily computed to be $+1$.

Thus for a cycle on G_N (or B_N) to represent p_1, it must be oriented to intersect S_N (or its image under the natural section) negatively. This completes the proof of Theorem 2.

3. Evaluation of $\mu(p)$

The finite-dimensional results of Sect. 2 cannot be directly applied to gauge theory. We need to extend them to appropriate infinite-dimensional spaces. Let H be an infinite-dimensional Banach space. Pick an infinite sequence of linearly independent vectors in H. Then there are natural inclusions

$$\mathbb{R}^N \overset{i}{\hookrightarrow} \mathbb{R}^{N+1} \overset{i}{\hookrightarrow} \dots \overset{i}{\hookrightarrow} \mathbb{R}^\infty \overset{i}{\hookrightarrow} H, \tag{8}$$

where $\mathbb{R}^\infty$ is the direct limit of the spaces $\mathbb{R}^N$. This induces a sequence of inclusions

$$V_N \overset{i}{\hookrightarrow} V_{N+1} \overset{i}{\hookrightarrow} \dots \overset{i}{\hookrightarrow} V_\infty \overset{i}{\hookrightarrow} V_H \tag{9}$$

and corresponding inclusions for V^0, B and G. For N large, these inclusions induce isomorphisms in H_4 (see e.g. [MS]), sending S_N to S_{N+1} to ... to S_∞ to S_H. $\pi(v_\infty)$ is closed and intersects S_∞ once, and $\pi(v_H)$ is closed and intersects S_H once. By the same argument as before, we have

Theorem 3. *$\pi(\nu_\infty)$, oriented so as to intersect S_∞ negatively, represents p_1 of the bundle $V_\infty \to B_\infty$, and $\pi(\nu_H)$, oriented to intersect S_H negatively, represents p_1 of $V_H \to B_H$.*

An equivalent description of p_1 is as follows. Let W be a codimension-3 subspace in H. Let Y_W be the set of 3-frames whose span, intersected with W, is at least 2-dimensional. When $W = \{x_1 = x_2 = x_3 = 0\}$, Y_W is the same as ν_H. But, since G_{H^*} is connected, the choice of W cannot affect the topology of Y_W. Thus Y_W, oriented to intersect S_H negatively, represents p_1 for any choice of W.

We are now able to construct μ of a point. Let p be a point on the manifold X, let D be a geodesic ball around p, let $\mathcal{A}_D$ be the $SU(2)$ (or $SO(3)$) connections on D within the Sobolev space L^q_k (the choice of q and k is not important), let $\mathcal{G}^0$ be the gauge transformations in L^q_{k+1} that leave the fiber at p fixed, and let $\mathcal{G}$ be all gauge transformations in L^q_{k+1}. Define $\mu_D(p)$ to be $-\frac{1}{4}p_1$ of the $SO(3)$ bundle $\mathcal{A}_D/\mathcal{G}^0 \to \mathcal{A}_D/\mathcal{G}$. $\mu(p)$ is the pullback of $\mu_D(p)$ to $\mathcal{B}(X)$ via the map that restricts connections on a bundle over X to a bundle over D.

The space $\mathcal{A}_D/\mathcal{G}^0$ is isomorphic to the set of connections in radial gauge with respect to the point p. In such a gauge the connection form A vanishes in the radial direction but is otherwise unconstrained. In particular, $A(p) = 0$, so the curvature at p, $F_A(p) = dA(p) + A(p) \wedge A(p) = dA(p)$, is a linear function of A.

Let H be the space of (scalar valued) 1-forms with no radial component. A connection in radial gauge is defined by a triple of elements of H, one for each direction in the Lie Algebra. Deleting the infinite-codimension set for which these elements are linearly dependent we get V_H. Thus $\mu_D(p)$ is $-1/4p_1$ of $V_H \to B_H$, which we have already computed. Let $W = \{\alpha \in H | d^-\alpha(p) = 0\}$. Thus Y_W is the set of connections over D, in radial gauge, for which the three components of $F^-_A(p)$ span a 1 (or 0) dimensional subspace of the Lie algebra. In other words, for which $F^-_A(p)$ is reducible. Pulling $\mu_D(p)$ back by the restriction map we get the connections on X for which $F^-_A(p)$ is reducible, i.e. ν_p. This completes the proof of Theorem 1.

4. Transversality and Extension to the Boundary

We have shown that for any point p in our manifold, the cycle ν_p is Poincaré dual to p_1 of the base point fibration, as a class in $\mathcal{B}^*(X)$. However, to do Donaldson theory we need more than this. Ideally, we want ν_p to intersect the moduli space $\mathcal{M}_k$ transversely and to extend in a well-behaved way to the compactification of moduli space. Had we chosen ν_p to depend on F^+_A rather than F^-_A, it would still have been dual to p_1, but would have been useless as a geometric representative of $-4\mu(p)$, insofar as F^+_A is identically zero on $\mathcal{M}_k$.

Even with our definition of ν_p, it is unrealistic to expect ν_p to intersect $\mathcal{M}_k$ transversely for all points p. For example, if $\mathcal{M}_k$ has dimension $d < 4$, then transversality would imply that $\nu_p \cap \mathcal{M}_k = \emptyset$. However, there is a $d+4$ dimensional set of pairs (A, p) for which $F^-_A(p)$ might be reducible. Since reducibility is a codimension-4 condition, we should expect reducibility at a d-dimensional set of pairs. Thus for p in a d-dimensional subset of X, ν_p would not intersect $\mathcal{M}_k$ transversely. There is no reason to suppose that this d-dimensional set is always empty.

The most we can reasonably expect is the following:

Conjecture. *Pick $k > 0$ and a generic metric on X, and let $\mathcal{M}'_k$ be either $\mathcal{M}_k$ cut down by standard Donaldson varieties, or $\mathcal{M}_k$ itself. Then, for generic points p, the intersection of v_p with $\mathcal{M}'_k$ is transverse.*

Should this conjecture prove true, then non-transverse intersection points (for generic metrics) can always be resolved by moving p. If the conjecture is not true, then we will require more subtle means of perturbing v_p, $\mathcal{M}_k$, or the other Donaldson varieties. For many purposes, one wishes to perturb $\mathcal{M}_k$ anyway (e.g. modeling connections near the ends of $\mathcal{M}_k$ as m concentrated charges glued by a particular formula to connections in $\mathcal{M}_{k-m}$). For such purposes, the utility of the representative v_p does not depend on the conjecture.

Next we consider the extension of v_p to the compactification of $\mathcal{M}_k$. The boundary of $\mathcal{M}_k$ consists of strata where m instantons have pinched off, leaving a solution of charge $k - m$ behind. These take the form $\mathcal{M}_{k-m} \times S^m(X)$, where $m > 0$. These boundary strata have lower dimension than $\mathcal{M}_k$, so they *should* not contribute to Donaldson invariants. To ensure that they do not contribute, v_p must remain a codimension-4 set on the boundary.

Theorem 4. *The intersection of the closure of v_p with the m-th stratum of $\partial\mathcal{M}_k$ is contained in the union of $(v_p \cap \mathcal{M}_{k-m}) \times S^m(X)$ and $\mathcal{M}_{k-m} \times \{p\} \times S^{m-1}(X)$.*

Proof. Consider a sequence of connections $[A_i] \in \mathcal{M}_k \cap v_p$ converging to $[A'] \times \{x_1, \ldots, x_m\}$, where $[A'] \in \mathcal{M}_{k-m}$. If $p \notin \{x_i\}$, then $F^-_{A_i}(p)$ converges, after suitable gauge transformations, to $F^-_{A'}(p)$. Since the set of rank ≤ 1 matrices is closed and invariant under left multiplication by $SO(3)$ (i.e. gauge transformations), $F^-_{A'}$ has rank at most 1, and we have the first set. If $p \in \{x_i\}$ we are in the second set. QED.

The first set is manifestly codimension-4. If the conjecture holds, then, for $m < k$ and generic p, the second set is codimension-4 as well. What remains is to consider the first set for $m = k$. This poses two difficulties. First, $\mathcal{M}_0$ contains the trivial connection (and other reducible connections if $H_1(X) \neq 0$), and so is not contained in $\mathcal{B}^*_0$. This complication is independent of the choice of representative of $\mu(x)$ and is not discussed here.

(The existence of the trivial connection is also the reason that, for $SU(2)$ theory, Donaldson invariants are only well defined for k sufficiently large, in the "stable range." For $SO(3)$ theory with nontrivial w_2, $\mathcal{M}_0$ is empty, and this complication disappears.)

The second complication is that every flat connection is in v_p, so that v_p cannot possibly intersect $\mathcal{M}_0$ transversely. To resolve this we must perturb $\mathcal{M}_0$. If $\pi_1 = 0$, so that $\mathcal{M}_0$ is just the trivial connection, this is easy. We just add a small connection that is zero outside a small neighborhood of p. One can always find a connection for which $F^-_A(p)$ will be irreducible, so v_p will miss the perturbed $\mathcal{M}_0$ entirely. If $\pi_1 \neq 0$ and $\mathcal{M}_0$ contains a representation variety of dimension 4 or greater, it may happen that one cannot lift $\mathcal{M}_0$ entirely off v_p. In that case we must interpret "$\mathcal{M}_0 \cap v_p$" as the intersection points that remain after a fixed (but generic) infinitesimal perturbation of $\mathcal{M}_0$.

Finally, we consider what must be done if the conjecture fails. In that case we would need to construct perturbations $\mathcal{M}'_k$ of the moduli spaces $\mathcal{M}_k$ such that each

$\mathcal{M}'_k$ intersects v_p transversely, and such that the boundary of $\mathcal{M}'_k$ consists of strata $\mathcal{M}'_{k-m} \times S^m(X)$. An analog of Theorem 4, for $\mathcal{M}'$, would then follow, and the discussion following Theorem 4 would also apply.

We close with a sketch of a topological application of this geometric representative. The Donaldson invariants of all known orientable 4-manifolds with $b_+ > 1$ satisfy a recursion relation called "simple type." This relation roughly says that, given two points p and q, $\mathcal{M}_k \cap v_p \cap v_q$ has the same fundamental class as $64\mathcal{M}_{k-1}$. For p and q close and A in $\mathcal{M}_{k-1}$, one can count the ways to glue in a concentrated instanton near p and q so as to make the curvature at p and q reducible. This number is well short of 64, indicating that simple type is not just a property of the ends of $\mathcal{M}_k$, but involves the topology of the interior as well. The results of this investigation will appear elsewhere [GS].

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