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Erratum

Global Existence of Solutions of the Spherically Symmetric Vlasov–Einstein System with Small Initial Data

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In the above paper the authors proved a local existence result for the spherically symmetric Vlasov–Einstein system (Theorem 3.1). Unfortunately, the proof contains an error: To estimate $\dot{f}_n$ in the proof of Lemma 3.3 we had in mind to differentiate the relation (3.4)

$$f_n(t,x,v) = \mathring{f}((X_n, V_n)(0,t,x,v))$$

with respect to t, and use the boundedness of the right-hand side of the characteristic system (3.3) and the "fact" that $(X_n, V_n)(s,t,x,v)$ is symmetric in s,t in the sense that $(X_n, V_n)(0,t,x,v)$ as a function of t solves (3.3) with the signs of the right-hand side reversed. This "fact" is wrong, it would be correct only if (3.3) were autonomous. In the following we indicate the main arguments which have to be added to the analysis in the above paper in order to set things right. A detailed exposition of the arguments can be obtained from the first author.

As a first step we prove Lemma 3.3. By (3.26) and (3.27) we have to bound $\|p_n'(t)\|_\infty$ and $\|\dot{\rho}_n(t)\|_\infty$. Using the Vlasov equation to express $\dot{f}_n$ in

$$\dot{\rho}_n(t,x) = \int \sqrt{1+v^2}\,\dot{f}_n(t,x,v)\,dv\,,$$

integrating the term with $\partial_v f_n$ by parts and using Lemma 3.2 we get

$$\|p_n'(t)\|_\infty, \|\dot{\rho}_n(t)\|_\infty \leqq C_1(t)(1 + \|\partial_x f_n(t)\|_\infty), \quad t \in [0,T[\,,$$

where $C_1(t)$ depends on the functions z_1, z_2 introduced in Lemma 3.2. Differentiating (3.3) with respect to x and using a Gronwall argument yields the estimate

$$|\partial_x X_{n+1}(0,t,z)| + |\partial_x V_{n+1}(0,t,z)| \leqq \exp \int_0^t C_1(s)(1 + \|\partial_x f_n(s)\|_\infty)\,ds\,,$$

where $z = (x, v)$. Substituting this into (3.4), differentiated with respect to x, gives

$$\|\partial_x f_{n+1}(t)\|_\infty \leqq \|\partial_z \overset{\circ}{f}\|_\infty \exp \int_0^t C_1(s)(1 + \|\partial_x f_n(s)\|_\infty)\, ds \,.$$

By induction, $\|\partial_x f_n(t)\|_\infty$ is for all n bounded by the solution z_3 of

$$z_3(t) = \|\partial_z \overset{\circ}{f}\|_\infty \exp \int_0^t C_1(s)(1 + z_3(s))\, ds \,,$$

which exists on some interval $[0, T'[\subset [0, T[$. This proves Lemma 3.3, but on a possibly shorter time interval, the length of which is determined as the maximal existence interval of the functions z_1, z_2, and z_3.

On $[0, T'[$ Lemma 3.4 holds, but the proof of the regularity of the limit obtained (Proof of Theorem 3.1) contains the same error again. Here the correction requires an essential new idea, which is contained in the following lemma:

Lemma. *Let $\lambda, \mu, \tilde{\lambda} : [0, T[\times [0, \infty[$ be sufficiently regular so that all derivatives appearing below exist. Define*

$$F(s,x,v) = e^{\mu-\lambda} \frac{v}{\sqrt{1+v^2}},\; G(s,x,v) = -\left(\frac{x \cdot v}{r} \tilde{\lambda} + e^{\mu-\lambda}\sqrt{1+v^2}\mu'\right)\frac{x}{r}, \quad x, v \in \mathbb{R}^3 \,,$$

and let $(X, V)(s, t, z) = (X, V)(s)$ be the solution of

$$\dot{x} = F(s,x,v), \qquad \dot{v} = G(s,x,v)$$

with $(X, V)(t, t, z) = z$. Define

$$\xi_j(s) = \partial_{z_j} X(s,t,z) \,,$$

$$\eta_j(s) = \partial_{z_j} V(s,t,z) + \sqrt{1 + V^2(s)}e^{(\lambda-\mu)(s,X(s))}\tilde{\lambda}(s, X(s))\frac{X(s)}{|X(s)|}\frac{X(s)}{|X(s)|} \cdot \xi_j(s) \,.$$

Then

$$\dot{\xi}_j = c_1(s, X(s), V(s))\xi_j + c_2(s, X(s), V(s))\eta_j \,,$$

$$\dot{\eta}_j = (c_3 + c_5)(s, X(s), V(s))\xi_j + c_4(s, X(s), V(s))\eta_j \,,$$

where $c_1, \ldots, c_4$ contain only $\lambda, \lambda', \tilde{\lambda}, \mu, \mu'$, and

$$(c_5(s,x,v))_{i,k} = -e^{\mu+\lambda}\sqrt{1+v^2} \cdot \left(e^{-2\lambda}\left(\mu'' + (\mu' - \lambda')\left(\mu' + \frac{1}{r}\right)\right) - e^{-2\mu}(\dot{\tilde{\lambda}} + \tilde{\lambda}(\dot{\lambda} - \dot{\mu}))\right)\frac{x_i x_k}{r^2} \,.$$

The proof is only a lengthy calculation and therefore omitted. Note that, if $\tilde{\lambda} = \dot{\lambda}$, then the term in brackets in c_5 is precisely the $(2,2)$-component of the Einstein tensor and can therefore be replaced by the $(2,2)$-component of the energy-momentum tensor, if λ, μ satisfy the full set of field equations. The geometric interpretation of this lemma is that the relative motion of geodesics which are close to each other is governed by the geodesic deviation equation, in which only those combinations of derivatives of Christoffel symbols appear, which appear in the Riemann curvature tensor, cf. also Eq. (4.15) and the discussion there.

To apply the lemma we first observe that for the limit obtained in Lemma 3.4,

$$\dot{\lambda} = -4\pi r e^{\mu+\lambda} j \,.$$

This follows from the identity

$$\dot{\lambda}_n = e^{\mu_{n-1}-\lambda_{n-1}-\mu_n+\lambda_n}\tilde{\lambda}_n + \frac{e^{2\lambda_n}}{r}\int_{|x|\leqq r}(e^{\mu_{n-1}+\lambda_{n-1}-\mu_n-\lambda_n}\tilde{\lambda}_n(\rho_{n-2}+p_{n-2})$$

$$-\dot{\lambda}_{n-1}(\rho_{n-1}+p_{n-1}))\,dx \,,$$

where

$$\tilde{\lambda}_n = -4\pi r e^{\mu_n+\lambda_n} j_{n-1} \to -4\pi r e^{\mu+\lambda} j, \quad n\to\infty \,,$$

and

$$j_n(t,x) = \int \frac{x\cdot v}{r} f_n(t,x,v)\,dv \,.$$

Passing to the limit in this identity gives the claim.

If we define F_n, G_n as indicated in the lemma then the convergence already established implies that the corresponding solutions $(X^n, V^n)(s,t,z)$ converge to $(X,V)(s,t,z)$, locally uniformly on $[0,T'[^2\times\mathbb{R}^6$. We use the lemma to show that the derivatives of the (X^n,V^n) with respect to z form a Cauchy sequence, and this proves the desired regularity of $(X,V)(s,t,z)$, and thus of all the limiting quantities obtained in Lemma 3.4. Note that for the iterates all quantities needed in the lemma are sufficiently regular, which is not true for the limiting quantities obtained in Lemma 3.4.

From Lemmas 3.2, 3.3, and 3.4 it follows that the coefficients $c_{n,i}$, $i = 1,\dots,4$, are bounded, together with $\partial_z c_{n,i}$, uniformly in n and locally uniformly on $[0,T'[\times\mathbb{R}^6$, and form Cauchy sequences. The crucial term is $c_{n,5}$. Here a lengthy calculation shows that

$$(c_{n,5})_{i,k} - e^{\mu_n+\lambda_n}\sqrt{1+v^2}\,4\pi q_{n-1}\frac{x_i x_k}{r^2} \to 0, \quad n\to\infty \,,$$

locally uniformly on $[0,T'[\times\mathbb{R}^6$, where

$$q_n(t,x) = \int \left|\frac{x\times v}{r}\right|^2 f_n(t,x,v)\,\frac{dv}{\sqrt{1+v^2}} \,.$$

Since q_n is twice the $(2,2)$-component of the energy-momentum tensor corresponding to f_n, this means that the $(2,2)$-component of the field equations holds approximately for the iterates. By what we know from Lemmas 3.2, 3.3, and 3.4 the quantities q_n are uniformly bounded together with q_n', and form a Cauchy sequence. Thus

$$\partial_{z_j} X^n(s,t,z),\ \partial_{z_j} V^n(s,t,z)$$

form a Cauchy sequence, locally uniformly on $[0,T'[^2\times\mathbb{R}^6$, and the proof of Theorem 3.1 is complete.

However, now a new problem comes up: From the above it follows that we need to bound the quantity $\partial_x f(t)$ in addition to the quantities $P(t)$ and $Q(t)$ considered in Theorem 3.2, if we want to extend a local solution. In the proof of Theorem 3.2 we showed that a bound on $P(t)$ implies a bound on $Q(t)$. Using the above lemma again, but now with

$$\tilde{\lambda} = \dot{\lambda} = -4\pi r e^{\mu+\lambda} j, \qquad (c_5)_{i,k} = -e^{\mu+\lambda}\sqrt{1+v^2}4\pi q \frac{x_i x_k}{r^2},$$

shows that a bound on $P(t)$ also implies a bound on $(\partial_x X, \partial_x V)(0,t,z)$ and thus on $\|\partial_x f(t)\|_\infty$; note that by Theorem 2.1 we now have the full set of field equations at our disposal. Therefore both the existence result, Theorem 3.1, and the continuation criterion, Theorem 3.2, remain correct as they stand.

As a final remark we mention that the above lemma can also be employed to avoid the use of the Jacobi fields and the geodesic deviation equation in the proof of the global existence result Theorem 4.1.

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