Non-linear Structures in the Non-critical NSR String

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Abstract: We investigate the Ward identities of the $\mathscr{W}_\infty$ symmetry in the super-Liouville theory coupled to the super-conformal matter of central charge $\hat{c}_M = 1 - 2(p - q)^2/pq$. The theory is classified into two chiralities. For the positive chirality, all gravitationally dressed scaling operators are generated from the $q - 1$ gravitational primaries by one of the ring generators in the R-sector acting on them repeatedly. After fixing the normalizations of the dressed scaling operators, we find that the Ward identities are expressed in the form of the *usual* $\mathscr{W}_q$ algebra constraints as in the bosonic case: $\mathscr{W}_n^{(k+1)}\tau = 0$, $(k = 1, \ldots, q - 1; \; n \in \mathbf{Z}_{\geq 1-k})$, where the equations for even and odd n come from the currents in the NS- and the R-sector respectively. The non-linear terms come from the anomalous contributions at the boundaries of moduli space. The negative chirality is defined by interchanging the roles of p and q. Then we get the $\mathscr{W}_p$ algebra constraints.

1. Introduction

One of the prominent features of the non-critical string, or 2D quantum gravity is the appearance of the non-linear structures [1–6], which have been first derived in the double scaling limit of the matrix model [1] and then developed in the form of the $\mathscr{W}$-algebra constraints [3,4]. Recently one of the authors (K.H.) [6] has found that the $\mathscr{W}$-algebra constraints are realized as the Ward identities of $\mathscr{W}_\infty$ symmetry [7] in the Liouville theory approach [5–13]. The scaling operators are then identified with the tachyon-like operators with discrete momenta. The non-linear terms come from the anomalous contributions from the boundaries of moduli space.

In this paper we investigate the non-critical NSR string, or 2D quantum supergravity. In this case the application of the continuum approach is extremely relevant because there is no definite matrix model to describe it. The BRST analysis of the physical spectrum was recently carried out [14]. It was then found that the ring structure and the associated $\mathscr{W}_\infty$ symmetry algebra are the same as those in the

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basonic case. We here study the Ward identities of $\mathscr{W}_\infty$ symmetry and derive the *usual $\mathscr{W}$-algebra* constraints, not super-$\mathscr{W}$ ones, as in the bosonic case.

We should refer to the work of Alvarez–Gaumé et al [15]. From the analysis of a kind of supersymmetrized matrix model they proposed that the super-Virasoro algebra constraints will be given in the case of the non-critical NSR string. In the continuum approach, however, we get a different result from their proposal. So we need further analysis to clarify the universality class of their model.

The paper is organized as follows. In Sect. 2 we summarize the BRST invariant states. The BRST charge of the super-Liouville theory coupled to the superconformal matter of central charge $\hat{c}_M (= \frac{2}{3} c_M) < 1$ is defined in the bosonized form. The picture changing isomorphism [16] is briefly explained. Then we define various discrete states which form the $\mathscr{W}_\infty$ algebra and the gravitationally dressed scaling operators, gravitational primaries and their descendants. Here we introduce the chiralities, which are classified by the momentum in the matter sector. Throughout this paper all calculations are carried out in positive chirality. In this case there exist $q - 1$ gravitational primaries. Their descendants are generated by one of the ring generators in the R-sector acting on them. For the negative chirality the roles of p and q are interchanged. In Sect. 3 we define the correlation functions of the interaction theory perturbed by the scaling operators such that (p, q)-*critical* theory is given [17]. Then we set up the Ward identities of the $\mathscr{W}_\infty$ symmetry. We first discuss the Ward identities which result in the Virasoro algebra constraints. Due to the nature of factorization, which is that the intermediate states become on-shell, the boundary of moduli space becomes dangerous. Here we evaluate the anomalous contributions coming from the boundaries of moduli space carefully, which give the non-linear terms in the Ward identities. Next we discuss the case resulting in the $\mathscr{W}$-algebra constraints. For the positive chirality we get $\mathscr{W}_q$ algebra. In the final section we argue the universality class of the non-critical NSR string. In our formalism the p-q duality is manifest. The universality of the theory is determined by the potential terms. Changing the potentials we can get various critical theories.

2. BRST Invariant States

2.1. Preliminary. In this section we summarize the BRST invariant states of the super-Liouville theory coupled to the super-conformal matter of central charge $\hat{c}_M = 1 - 2(p - q)^2/pq, \; p - q = 2\mathbf{Z}$. The BRST charge is given by

$$Q_{BRST} = \oint \frac{dz}{2\pi i} \left[c(z) \left(T(z) + \frac{1}{2} T^G(z) \right) - \frac{1}{2} \gamma(z) \left(G(z) + \frac{1}{2} G^G(z) \right) \right], \quad (2.1)$$

where $T = T^L + T^M$ and $G = G^L + G^M$ are the energy-momentum tensor and supercurrent for the Liouville-matter sector respectively,

$$T^L = -\frac{1}{2} \partial\phi\partial\phi + iQ^L \partial^2\phi - \frac{1}{2}\psi^L \partial\psi^L \,,$$

$$T^M = -\frac{1}{2} \partial\varphi\partial\varphi + iQ^M \partial^2\varphi - \frac{1}{2}\psi^M \partial\psi^M \,,$$

$$G^L = i\psi^L \partial\phi + 2Q^L \partial\psi^L \,,$$

$$G^M = i\psi^M \partial\varphi + 2Q^M \partial\psi^M \tag{2.2}$$

and

$$iQ^L = \frac{1}{2}(p+q)Q, \qquad Q^M = \frac{1}{2}(p-q)Q, \qquad Q = \frac{1}{\sqrt{pq}} . \tag{2.3}$$

The operators T^G and G^G are the energy-momentum tensor and the supercurrent in the ghost sector,

$$T^G = c\partial b + 2\partial cb - \frac{1}{2}\gamma\partial\beta - \frac{3}{2}\partial\gamma\beta ,$$

$$G^G = b\gamma - 3\partial c\beta - 2c\partial\beta . \tag{2.4}$$

In this paper we use the bosonized representations for the fermions $\psi^{L,M}$ and $\beta\gamma$-ghosts,

$$\psi^\pm(z) = \frac{1}{\sqrt{2}}(\psi^M(z) \pm i\psi^L(z)) = e^{\pm iv(z)} , \tag{2.5}$$

$$\beta(z) = e^{-u(z)}\partial\xi(z), \qquad \gamma(z) = e^{u(z)}\eta(z) , \tag{2.6}$$

where v and u are free bosons and ξ and η are components of a Fermi system with dimension 0 and 1 respectively.[1] These satisfy the following operator product expansions (OPE),

$$v(z)v(w) = u(z)u(w) = -\log(z-w) , \tag{2.7}$$

$$\xi(z)\eta(w) = \frac{1}{z-w} . \tag{2.8}$$

The BRST charge is then expressed in the form

$$Q_{BRST} = Q_I + Q_{II} + Q_{III} ,$$

$$Q_I = \oint \frac{dz}{2\pi i}c(T^B - b\partial c) ,$$

$$Q_{II} = -\frac{1}{2}\oint\frac{dz}{2\pi i}\eta[e^{u+iv}(i\partial\Omega^- + 2\beta^- i\partial v) + e^{u-iv}(i\partial\Omega^+ - 2\beta^+ i\partial v)] ,$$

$$Q_{III} = \frac{1}{4}\oint\frac{dz}{2\pi i}\eta\partial\eta e^{2u}b , \tag{2.9}$$

where $\Omega^\pm = \frac{1}{\sqrt{2}}(\varphi \pm i\phi)$ and $\beta^\pm = \frac{1}{\sqrt{2}}(Q^M \pm iQ^L)$. The operator T^B is the combined energy-momentum tensor of the fields, ϕ, φ, u, v, ξ and η:

$$T^B = -\frac{1}{2}\partial\phi\partial\phi + iQ^L\partial^2\phi - \frac{1}{2}\partial\varphi\partial\varphi + iQ^M\partial^2\varphi$$

$$- \frac{1}{2}\partial u\partial u + iQ^u\partial^2 u - \frac{1}{2}\partial v\partial v - \eta\partial\xi , \tag{2.10}$$

where $iQ^u = -1$.

Before introducing the concrete BRST invariant states, we briefly mention the picture changing isomorphism [16]. It is given by the action of the zero mode of the picture changing operator defined by

$$X_0 = \oint \frac{dz}{2\pi i}\frac{1}{z}X(z) , \tag{2.11}$$

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[1] The cocycle factor for bc-ghosts is omitted because we only consider the combined system of the holomorphic and the anti-holomorphic parts so that the phase is always canceled.

where

$$X(z) = -2\beta^+ \{Q_{BRST}, \xi(z)\} I_R^{-2}$$

$$= \sqrt{\frac{p}{2q}} \left[e^{u+iv} \left(i\partial\Omega^- - \sqrt{\frac{2q}{p}} i\partial v \right) + e^{u-iv} \left(i\partial\Omega^+ - \sqrt{\frac{2p}{q}} i\partial v \right) \right.$$

$$\left. -2c\partial\xi + \frac{1}{2} e^{2u} b\partial\eta + \frac{1}{2}\partial(e^{2u} b\eta) \right] (z) I_R^{-2} .$$

(2.12)

The cocycle factor I_R for the R-vacuum takes the following form:

$$I_R = e^{-\frac{\pi}{2} i(N_u - N_v)} ,$$

(2.13)

where $N_u = -\oint \frac{dz}{2\pi i} \partial u(z)$ and $N_v = i \oint \frac{dz}{2\pi i} \partial v(z)$. This cocycle factor is necessary for the definite statistics of the R-vacuum $e^{-\frac{1}{2} u(z) - \frac{i}{2} v(z)} I_R$ with respect to the BRST charge (2.9). Well-defined statistics of the operators in the R-sector is crucial for our analysis, because the generators of the $\mathcal{W}_\infty$ currents appear in the R-sector and the currents should have definite statistics with the BRST charge. The R-vacuum is bosonic with respect to the BRST charge if we take the above form of I_R. The operator X_0 increases the picture by one unit,

$$\mathcal{O}^{(a+1)}(z) = X_0 \mathcal{O}^{(a)}(z) ,$$

(2.14)

where $\mathcal{O}^{(a)}$ is a physical operator. The superscript a denotes the picture. Here we assign an integer to the NS-sector and a half-integer to the R-sector. There also exists the picture changing operator which decreases the picture by one unit,

$$Y_0 = \oint \frac{dz}{2\pi i} \frac{1}{z} Y(z), \qquad Y(z) = -4\beta^- c\partial\xi e^{-2u(z)} I_R^2 .$$

(2.15)

This operation is the inverse of the X_0, that is

$$X_0 Y_0 = Y_0 X_0 = 1 + \{Q_{BRST}, \varepsilon\}$$

(2.16)

for some operator ε. Therefore the operators $\mathcal{O}^{(a)}$ and $X_0 Y_0 \mathcal{O}^{(a)}$ are BRST-equivalent,

$$X_0 Y_0 \mathcal{O}^{(a)}(z) = \mathcal{O}^{(a)}(z) + \{Q_{BRST}, \varepsilon\mathcal{O}^{(a)}(z)\} .$$

(2.17)

In the following, since all operators which are related by the picture changing are identified in the correlation function, we often omit the superscript describing the picture except when we define the normalization of the operators.

2.2. $\mathcal{W}_\infty$ Symmetry Currents. The BRST cohomology of the non-critical NSR string [14] has been calculated as in the bosonic case [13]. The non-trivial states called "discrete states" arise at the special value of momenta parametrized by $r, s \in \mathbf{Z}$,

$$ip^L = \alpha_{r,s} = iQ^L + \frac{1}{\sqrt{2}}(r\beta_+ - s\beta_-) ,$$

(2.18)

$$p^M = \beta_{r,s} = Q^M + \frac{1}{\sqrt{2}}(r\beta_+ + s\beta_-) .$$

(2.19)

The states with $r - s \in 2\mathbf{Z}$ appear in the NS-sector, while those with $r - s \in 2\mathbf{Z} + 1$ are in the R-sector. Here we only consider the states with $r, s < 0$. In this case there are the discrete operators of ghost number zero $B_{r,s}$, which provide a ring of operators,

$$B_{r,s}(z)B_{r',s'}(w) = B_{r+r'+1,s+s'+1}(w) \,. \tag{2.20}$$

It is clear that the entire ring is generated by two elements, $x = B_{-1,-2}$ and $y = B_{-2,-1}$ in the R-sector such that

$$B_{r,s} = x^{-s-1}y^{-r-1}, \quad r, s \in \mathbf{Z}_- \,. \tag{2.21}$$

The partners of $B_{r,s}$ at the ghost number one, $\Psi_{r,s}$ give the spin-1 currents $R_{r,s} = b_{-1}\Psi_{r,s}$, which satisfy the $\mathcal{W}_\infty$ algebra. Here we normalize them such as

$$R_{r,s}(z)R_{r',s'}(w) = \frac{1}{z-w}\frac{1}{2}(rs' - r's)R_{r+r'+1,s+s'+1}(w) \,. \tag{2.22}$$

Combining $R_{r,s}$ and $\bar{B}_{r,s}$ we can construct the symmetry currents

$$W_{r,s}(z,\bar{z}) = R_{r,s}(z)\bar{B}_{r,s}(\bar{z}), \quad r, s \in \mathbf{Z}_- \,, \tag{2.23}$$

which satisfy

$$\partial_{\bar{z}}W_{r,s}(z,\bar{z}) = \{\bar{Q}_{BRST}, [\bar{b}_{-1}, W_{r,s}(z,\bar{z})]\} \,. \tag{2.24}$$

In the following section we need the explicit forms of the discrete operators. We define the ring elements x and y in $-\frac{1}{2}$ picture as

$$x^{(-1/2)} = B_{-1,-2}^{(-1/2)} = \left[e^{-\frac{1}{2}u - \frac{i}{2}v} + \sqrt{\frac{2q}{p}}e^{-\frac{3}{2}u + \frac{i}{2}v}\partial\xi c \right] e^{-\frac{1}{2}q\mathcal{Q}\phi + \frac{i}{2}q\mathcal{Q}\varphi}I_R \,, \tag{2.25}$$

$$y^{(-1/2)} = B_{-2,-1}^{(-1/2)} = \left[\sqrt{\frac{2q}{p}}e^{-\frac{3}{2}u - \frac{i}{2}v}\partial\xi c - \frac{q}{p}e^{-\frac{1}{2}u + \frac{i}{2}v} \right] e^{-\frac{1}{2}p\mathcal{Q}\phi - \frac{i}{2}p\mathcal{Q}\varphi}I_R \,. \tag{2.26}$$

The ring elements in other pictures are defined by the picture changing operators acting on them. For example the ring element $x^{(1/2)} = X_0 x^{(-1/2)}$ is given by

$$x^{(1/2)} = \left[\frac{1}{2}\left\{ cb + \sqrt{\frac{p}{q}}(\partial\phi + i\partial\varphi) \right\} e^{\frac{1}{2}u + \frac{i}{2}v} - \sqrt{\frac{p}{2q}}c\partial\xi e^{-\frac{1}{2}u - \frac{i}{2}v} \right.$$
$$\left. + \frac{1}{2}\sqrt{\frac{p}{2q}}b\eta e^{\frac{3}{2}u - \frac{i}{2}v} + \frac{p}{2q}e^{\frac{1}{2}u - \frac{3}{2}iv} \right] e^{-\frac{1}{2}q\mathcal{Q}\phi + \frac{i}{2}q\mathcal{Q}\varphi}I_R^{-1} \tag{2.27}$$

and $y^{(1/2)} = X_0 y^{(-1/2)}$ is

$$y^{(1/2)} = \left[\frac{1}{2}\left\{ cb + \sqrt{\frac{q}{p}}(\partial\phi - i\partial\varphi) \right\} e^{\frac{1}{2}u - \frac{i}{2}v} + \sqrt{\frac{q}{2p}}c\partial\xi e^{-\frac{1}{2}u + \frac{i}{2}v} \right.$$
$$\left. - \frac{1}{2}\sqrt{\frac{q}{2p}}b\eta e^{\frac{3}{2}u + \frac{i}{2}v} + \frac{q}{2p}e^{\frac{1}{2}u + \frac{3}{2}iv} \right] e^{-\frac{1}{2}p\mathcal{Q}\phi - \frac{i}{2}p\mathcal{Q}\varphi}I_R^{-1} \,. \tag{2.28}$$

These are useful when we calculate the OPE's. The explicit forms of the operators $R_{r,s}$ are given on occasions we need them.

2.3. Gravitational Primaries and Descendants. Let us consider the gravitationally dressed primary fields inside and outside the minimal Kac table in superconformal field theory. The theory is classified into two chiralities by the choice of the momentum of matter sector.[2] Throughout this paper we only consider the positive chirality. The negative chirality is given by the interchange of p and q. It is then convenient to change the signs of fields, φ, v, ξ and η simultaneously to keep the BRST charge invariant under the interchange.

The gravitationally dressed primary fields $O_j(z, \bar{z}) = O_j(\bar{z})O_j(z)$ are defined in the NS-sector by

$$O_j^{(-1)}(z) = c(z)e^{-u(z)}e^{\alpha_j \phi(z) + i\beta_j \varphi(z)}I_R^2 \,, \tag{2.29}$$

where $j \in \mathcal{N}$ and those in the R-sector are

$$O_j^{(-1/2)}(z) = c(z)e^{-\frac{1}{2}u(z) + i\frac{1}{2}v(z)}e^{\alpha_j \phi(z) + i\beta_j \varphi(z)}I_R \,, \tag{2.30}$$

where $j \in \mathcal{R}$. The sets $\mathcal{N}$ and $\mathcal{R}$ are defined by

$$\mathcal{N} = \{j | j = 1, 2, 3, \ldots \quad (j \not\equiv q \bmod q)\} \,, \tag{2.31}$$

$$\mathcal{R} = \{j | j = \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \ldots \quad (j \not\equiv \tfrac{1}{2}q \bmod q) \quad \text{for } q \in 2\mathbf{Z} + 1$$

$$\text{and } j = 1, 2, 3, \ldots \quad (j \not\equiv \tfrac{1}{2}q \bmod q) \quad \text{for } q \in 2\mathbf{Z}\} \,. \tag{2.32}$$

The Liouville and the matter momenta are defined by

$$\alpha_j = \left(\frac{p+q}{2} - j\right)Q, \qquad \beta_j = \left(\frac{p-q}{2} + j\right)Q \,. \tag{2.33}$$

These correspond to the momenta with $r = 0$ and $s = -2j/q$ in the definition of (2.18–19). The expressions in other pictures are summarized in the Appendix. Note that for the negative chirality β_j changes into β_{-j}, while α_j does not.

As in the bosonic case the scaling operators can be classified into two sets, the gravitational primaries and their descendants. Here we define the following $q - 1$ scaling operators as the gravitational primaries,

$$O_k^{NS}(k = 1, 2, \ldots, \tfrac{1}{2}(q-1)), \qquad O_k^R(k = \tfrac{1}{2}, \tfrac{3}{2}, \ldots, \tfrac{1}{2}q - 1) \tag{2.34}$$

for $q \in 2\mathbf{Z} + 1$ and

$$O_k^{NS}(k = 1, 2, \ldots, \tfrac{1}{2}q), \qquad O_k^R(k = 1, 2, \ldots, \tfrac{1}{2}q - 1) \tag{2.35}$$

for $q = 2\mathbf{Z}$. Then all other scaling operators, gravitational descendants, can be obtained by the ring element x acting on the primaries repeatedly,

$$O_{\frac{n}{2}q+k}(z, \bar{z}) = \sigma_n(O_k)(z, \bar{z}) \propto (x\bar{x})^n O_k(z, \bar{z}) \,, \tag{2.36}$$

where we omit the superscript distinguishing the sectors, which is easily recovered by noting that the ring element x is in the R-sector. Here the action of the ring

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[2] Here we correct the misleading argument of the last paragraph in Sect. 2 of ref. [6] (Nucl. Phys. B413 (1994) 278).

element x on the scaling operators are easily calculated as

$$x(z)O_j^{NS}(w) = -i\frac{j}{q}O_{j+\frac{1}{2}q}^R(w),$$ (2.37)

$$x(z)O_j^R(w) = iO_{j+\frac{1}{2}q}^{NS}(w)$$ (2.38)

for the holomorphic part. The coefficients for the anti-holomorphic part are their complex conjugate.

Finally we introduce the normalized scaling operators defined by

$$\hat{O}_j^{NS}(z,\bar{z}) = \Lambda_{NS}(j)O_j^{NS}(z,\bar{z}), \qquad \Lambda_{NS}(j) = \frac{\Gamma(\frac{j}{q})}{\Gamma(-\frac{j}{q})},$$ (2.39)

$$\hat{O}_j^R(z,\bar{z}) = \Lambda_R(j)O_j^R(z,\bar{z}), \qquad \Lambda_R(j) = i\frac{j}{q}\frac{\Gamma(\frac{1}{2}+\frac{j}{q})}{\Gamma(\frac{1}{2}-\frac{j}{q})}.$$ (2.40)

The role of the normalization factors will be clear in the next section. Here note that $\Lambda_{NS}(j)$ $(j > 0)$ vanishes at $j = q$ mod q and $\Lambda_R(j)$ $(j > 0)$ does at $j = \frac{1}{2}q$ mod q.

3. Ward Identities of $\mathscr{W}_\infty$ Symmetry

3.1. Correlation Functions. Let us define the correlation functions of the non-critical NSR string [17]. We consider the interaction theory defined by the action

$$S = S_0(p,q) + \mu\hat{O}_1^{NS} - t\hat{O}_{\frac{1}{2}(p+q)}^{NS},$$ (3.1)

where $S_0(p,q)$ is the kinetic term with the background charges (2.3) and

$$\hat{O}_j = \int d^2z b_{-1}\bar{b}_{-1}\hat{O}_j(z,\bar{z}).$$ (3.2)

The potential term $\hat{O}_1^{NS}$ is the normalized dressed operator with the lowest dimensional matter field in the NS-sector and $\hat{O}_{(p+q)/2}^{NS}$ is nothing but the screening charge for the matter sector. After integrating over the zero modes of the Liouville and the matter fields the correlation function of the scaling operators is expressed as the free field one:

$$\left\langle\!\!\left\langle \prod_{j\in S}\hat{O}_j \right\rangle\!\!\right\rangle_g = \left(-\lambda\frac{Q}{\pi}\right)^{-\frac{\chi}{2}}\mu^s\frac{\Gamma(-s)}{\alpha_1}\frac{t^n}{n!}\int d\xi_0 d\bar{\xi}_0$$

$$\times \left\langle \prod_{j\in S}\hat{O}_j(\hat{O}_1^{NS})^s(\hat{O}_{\frac{1}{2}(p+q)}^{NS})^n(\bar{X}_0 X_0)^m\bar{\xi}_0\xi_0 \right\rangle_g,$$ (3.3)

where g is the genus, $\chi = 2 - 2g$ and

$$s = \frac{1}{p+q-2}\left[(p+q)\chi - \sum_{j\in S}(p+q-2j)\right],\qquad(3.4)$$

$$n = \frac{1}{p+q-2}\left[-2\chi + \sum_{j\in S}(2-2j)\right].\qquad(3.5)$$

The Γ-function comes from the zero mode integral of ϕ. The zero mode integral of φ (compactified in the finite interval) gives the Kronecker delta which guarantees the momentum neutrality of the matter sector. If s and n are integers, the correlation functions can be calculated. However s and n are not integer in general so that, according to the argument of [10, 17], we define them by analytic continuation in s and n, where $n!$ is defined by $\Gamma(n+1)$. The insertion of the operators X_0 is to ensure the neutrality of the picture. If we assign the picture -1 to all the NS-operators and $-\frac{1}{2}$ to the R-operators, we then get $m = -\frac{1}{2}N_R$[3] because of the relation $N_{NS} + N_R + s + n = \chi$, where N_{NS} and N_R are the numbers of the NS- and the R-states in S respectively. The integral of the zero mode of ξ, which is essentially unity, is introduced to ensure the picture changing isomorphism. The spin structures are simply summed over all possibilities [11].

In the following we calculate the Ward identities of the $\mathcal{W}_\infty$ symmetry,

$$\int d^2z\partial_{\bar z}\left\langle\!\!\left\langle W_{-k,-n-k}(z,\bar z)\prod_{j\in S}\hat O_j\right\rangle\!\!\right\rangle_g = 0,\quad (k=1,\ldots,q-1; n\in\mathbf{Z}_{\geqq 1-k}).\quad(3.6)$$

We will see that these are expressed as the $\mathcal{W}_n^{(k+1)}$ constraints as in the bosonic case. The equations for $k=1$ are the Virasoro constraints and others are the $\mathcal{W}$-algebra constraints.

3.2. Virasoro Constraints.

We first discuss the Ward identities for the symmetry current $W_{-1,-n-1}(z,\bar z) = R_{-1,-n-1}(z)\bar B_{-1,-n-1}(\bar z)$. The explicit form of the field $R_{-1,-n-1}(z)$ is given as follows:

$$R^{(a)}_{-1,-2m-1}(z) = m!J_0^-(b_{-1}O^{(a)}_{0,-2m-2})(z),\qquad(3.7)$$

$$R^{(a)}_{-1,-2m-2}(z) = -i(m+1)!J_0^-(b_{-1}O^{(a)}_{0,-2m-3})(z),\qquad(3.8)$$

where the current for even $n = 2m$ is in the NS-sector and that for odd $n = 2m+1$ is in the R-sector. The zero mode of $SU(2)$ current $J^-(z)$ is defined by

$$J_0^- = \oint\frac{dz}{2\pi i}J^-(z),\qquad(3.9)$$

where

$$J^-(z) = \frac{2q}{p}(b_{-1}O^{(0)}_{-2,0})(z) = \left(e^{-iv(z)} + \frac{q}{p}e^{iv(z)}\right)e^{-\Phi(z)},\qquad(3.10)$$

$$\Phi(z) = \frac{1}{2}(p-q)Q\phi(z) + \frac{i}{2}(p+q)Q\varphi(z).\qquad(3.11)$$

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[3] This corresponds to inserting the operator $(\bar Y_0 Y_0)^{N_R/2}$ instead of X_0.

The operator $O_{0,s}^{(a)}(z)$ is defined at the -1 and $-\frac{1}{2}$ picture as

$$O_{0,s}^{(-1/2-\kappa)}(z) = c(z)e^{(-1/2-\kappa)u(z)+i(1/2-\kappa)v(z)}e^{\alpha_{0,s}\phi(z)+i\beta_{0,s}\varphi(z)}I_R^{1+2\kappa} , \qquad (3.12)$$

where $\kappa = 0$ for the R-sector and $\frac{1}{2}$ for the NS-sector. The field $B_{-1,-n-1}$ is defined in Eq. (2.21).

Let us calculate OPE between the current and the scaling operator. We first evaluate the NS-current $\times$ NS-operator, that is

$$W_{-1,-2m-1}^{(0)}(z,\bar{z})O_k^{(-1)}(w,\bar{w})$$

$$= \frac{1}{z-w}(-1)^m \prod_{l=0}^{m}\left(\frac{k}{q}+l\right)\prod_{l=0}^{m-1}\left(\frac{k}{q}+l\right)O_{mq+k}^{(-1)}(w,\bar{w})$$

$$= \frac{1}{z-w}\frac{k}{q}\Lambda_{NS}^{-1}(k)\Lambda_{NS}(mq+k)O_{mq+k}^{(-1)}(w,\bar{w}) . \qquad (3.13)$$

The Λ-factors are renormalized into the scaling operators so that we obtain

$$W_{-1,-2m-1}(z,\bar{z})\hat{O}_k^{NS}(w,\bar{w}) = \frac{1}{z-w}\frac{k}{q}\hat{O}_{mq+k}^{NS}(w,\bar{w}) . \qquad (3.14)$$

Here we omit the superscripts describing the picture. The other cases are also calculated easily and we get the following results:

$$W_{-1,-2m-1}(z,\bar{z})O_k^R(w,\bar{w})$$

$$= \frac{1}{z-w}\frac{k}{q}\Lambda_R^{-1}(k)\Lambda_R(mq+k)O_{mq+k}^R(w,\bar{w}) , \qquad (3.15)$$

$$W_{-1,-2m-2}(z,\bar{z})O_k^{NS}(w,\bar{w})$$

$$= \frac{1}{z-w}\frac{k}{q}\Lambda_{NS}^{-1}(k)\Lambda_R(mq+k+\tfrac{1}{2}q)O_{mq+k+\frac{1}{2}q}^R(w,\bar{w}) , \qquad (3.16)$$

$$W_{-1,-2m-2}(z,\bar{z})O_k^R(w,\bar{w})$$

$$= \frac{1}{z-w}\frac{k}{q}\Lambda_R^{-1}(k)\Lambda_{NS}(mq+k+\tfrac{1}{2}q)O_{mq+k+\frac{1}{2}q}^{NS}(w,\bar{w}) , \qquad (3.17)$$

The Λ-factors are also renormalized into the scaling operators. As a result these OPE's can be summarized in the single equation

$$W_{-1,-n-1}(z,\bar{z})\hat{O}_k(w,\bar{w}) = \frac{1}{z-w}\frac{k}{q}\hat{O}_{\frac{n}{2}q+k}(w,\bar{w}) , \qquad (3.18)$$

where we omit the superscript distinguishing the NS- and the R-sector which can be easily recovered.

From the OPE calculated above we get the following expression:

$$0 = \int d^2z \partial_{\bar{z}} \left\langle\!\!\left\langle W_{-1,-n-1}(z,\bar{z}) \prod_{j \in S} \hat{O}_j \right\rangle\!\!\right\rangle_g$$

$$= \pi \frac{p+q}{2q} t \left\langle\!\!\left\langle \hat{O}_{\frac{n}{2}q+\frac{1}{2}(p+q)} \prod_{j \in S} \hat{O}_j \right\rangle\!\!\right\rangle_g - \pi \frac{1}{q} \mu \left\langle\!\!\left\langle \hat{O}_{\frac{n}{2}q+1} \prod_{j \in S} \hat{O}_j \right\rangle\!\!\right\rangle_g$$

$$+ \pi \sum_{k \in S} \frac{k}{q} \left\langle\!\!\left\langle \hat{O}_{\frac{n}{2}q+k} \prod_{\substack{j \in S \\ (j \neq k)}} \hat{O}_j \right\rangle\!\!\right\rangle_g + \int d^2z \left\langle\!\!\left\langle \partial_{\bar{z}} W_{-1,-n-1}(z,\bar{z}) \prod_{j \in S} \hat{O}_j \right\rangle\!\!\right\rangle_g .$$

$$(3.19)$$

The first and the second correlators of the r.h.s. come from the OPE with the potentials $\hat{O}_{(p+q)/2}^{NS}$ and $\hat{O}_1^{NS}$ respectively. Usually the last correlator would vanish because the divergence of the current is the BRST trivial. However, as discussed in the previous paper [6], the boundary of moduli space is now dangerous and the last correlator gives anomalous contributions.

The anomalous contributions from the boundaries of the moduli space are calculated by inserting the complete set at the intermediate line as

$$- \lambda \frac{Q}{\pi} \sum_{N=0}^{\infty} \int_{-\infty}^{\infty} \frac{dh}{2\pi} \left\langle\!\!\left\langle F_1 \int_{|z| \leq 1} d^2z \{\bar{Q}_{BRST}, [\bar{b}_{-1}, W_{-1,-n-1}(z,\bar{z})]\} \right.\right.$$

$$\times D \left\{ \sum_{k \in \mathcal{N}} | -h, \beta_{-k}, N; -1 \rangle\!\rangle \langle\!\langle h, \beta_k, N; -1 | \right.$$

$$\left.\left.\left. + \sum_{k \in \mathcal{R}} | -h, \beta_{-k}, N; -3/2 \rangle\!\rangle \langle\!\langle h, \beta_k, N; -1/2 | \right\} F_2 \right\rangle\!\!\right\rangle ,$$

$$(3.20)$$

where the operators in S are divided into the sets F_1 and F_2. The integer N stands for the oscillator level of the states. The zero level states are defined by

$$|h, \beta_k; b\rangle = \bar{c}(\bar{0})c(0)e^{bu(0,\bar{0})+i(b+1)v(0,\bar{0})}e^{i(h+Q^L)\phi(0,\bar{0})+i\beta_k\varphi(0,\bar{0})}|0\rangle ,$$

$$(3.21)$$

where $b = -\frac{1}{2}, -1$ and $-\frac{3}{2}$.[4] The state $|h, \beta_k, N; b\rangle$ is the eigenstate of the Hamiltonian $H = L_0 + \bar{L}_0$ with the eigenvalue $h^2 + k^2 Q^2 + 2N$, which is normalized as

$$\langle\!\langle h, \beta_k, N; -1 | h', \beta_{k'}, N'; -1 \rangle\!\rangle_{g=0} = \frac{-\pi}{\lambda Q} 2\pi\delta(h+h')\delta_{k+k',0}\delta_{N,N'} ,$$

$$(3.22)$$

$$\langle\!\langle h, \beta_k, N; -1/2 | h', \beta_{k'}, N'; -3/2 \rangle\!\rangle_{g=0} = \frac{-\pi}{\lambda Q} 2\pi\delta(h+h')\delta_{k+k',0}\delta_{N,N'} .$$

$$(3.23)$$

---

[4] In the following we describe $\bar{z} = 0$ as $\bar{0}$.

The zero mode integral of the Liouville field now produces the δ-function. D is the propagator defined by

$$D = \int\limits_{e^{-\tau} \leq |z| \leq 1} \frac{d^2 z}{|z|^2} z^{L_0} \bar{z}^{\bar{L}_0} = 2\pi \left(\frac{1}{H} - \lim_{\tau \to \infty} \frac{1}{H} e^{-\tau H} \right) . \tag{3.24}$$

The parameter τ is introduced as a regulator. The last term stands for the boundary of moduli space.

Since the BRST charge commutes with the Hamiltonian, there is no contribution from $1/H$ term in the propagator. The boundary term is non-trivial, which gives the non-vanishing quantities at the limit $\tau \to \infty$. We first claculate in the case that the intermediate state is in the NS-sector,

$$\lim_{\tau \to \infty} \lambda \frac{Q}{\pi} \sum_{k \in \mathscr{N}} \int\limits_{-\infty}^{\infty} \frac{dh}{2\pi} \int\limits_{e^{-\tau} \leq |z| \leq 1} d^2 z \left\langle\!\!\left\langle F_1 [\bar{b}_{-1}, W^{(a)}_{-1,-n-1}(z,\bar{z})] \right.\right.$$

$$\left.\left. \times \bar{Q}_{BRST} \frac{2\pi}{H} e^{-\tau H} | -h, \beta_{-k}; -1 \rangle\!\rangle \langle\!\langle h, \beta_k; -1 | F_2 \right\rangle\!\!\right\rangle$$

$$= \lim_{\tau \to \infty} \lambda \frac{Q}{\pi} \sum_{k \in \mathscr{N}} \int\limits_{-\infty}^{\infty} \frac{dh}{2\pi} \int\limits_{e^{-\tau} \leq |z| \leq 1} d^2 z \langle\!\langle F_1 [\bar{b}_{-1}, W^{(a)}_{-1,-n-1}(z,\bar{z})]$$

$$\times \pi \bar{\partial} \bar{c}(\bar{0}) | -h, \beta_{-k}; -1 \rangle\!\rangle e^{-\tau(h^2 + k^2 Q^2)} \langle\!\langle h, \beta_k; -1 | F_2 \rangle\!\rangle , \tag{3.25}$$

where we omit $N \neq 0$ states because these states vanish exponentially as $e^{-2N\tau}$ at $\tau \to \infty$. Noting the following OPE,

$$[\bar{b}_{-1}, W^{(a)}_{-1,-n-1}(z,\bar{z})] \bar{\partial} \bar{c}(\bar{0}) | -h, \beta_{-k}; -1 \rangle$$

$$= A^{NS}_n(h) | -h + i\frac{n}{2}qQ, \beta_{\frac{n}{2}q-k}; a-1 \rangle |z|^{2\{ -\frac{n}{2} -1 + i\frac{n}{2}qQ(-h+Q^L) + \frac{n}{2}qQ\beta_{-k} \}} , \tag{3.26}$$

and changing the variable to $z = e^{-\tau x + i\theta}$, where $0 < x < 1$ and $0 < \theta < 2\pi$, the above expression becomes

$$\lim_{\tau \to \infty} \lambda \frac{Q}{\pi} \sum_{k \in \mathscr{N}} \int\limits_{-\infty}^{\infty} \frac{dh}{2\pi} \int\limits_0^1 2\pi \tau dx \, \pi A^{NS}_n(h)$$

$$\times \langle\!\langle F_1 | -h + i\frac{n}{2}qQ, \beta_{\frac{n}{2}q-k}; a-1 \rangle\!\rangle \langle\!\langle h, \beta_k; -1 | F_2 \rangle\!\rangle$$

$$\times \exp[-\tau\{h^2 + k^2 Q^2 - 2x(i\frac{n}{2}qQh + \frac{n}{2}kqQ^2)\}] . \tag{3.27}$$

The coefficient A^{NS}_n is calculated after the integrations are carried out. Since the exponential term is highly peaked in the limit $\tau \to \infty$, the saddle point estimation becomes exact. The saddle point of the h integral is $h = i\frac{n}{2}qQx$, so that (3.27) becomes

$$\lim_{\tau \to \infty} \lambda Q\tau \sqrt{\frac{2\pi}{2\tau}} \sum_{k \in \mathscr{N}} \int\limits_0^1 dx \, A^{NS}_n(i\frac{n}{2}qQx) \exp(-\tau Q^2(\frac{n}{2}qx - k)^2)$$

$$\times \langle\!\langle F_1 | -i\frac{n}{2}qQ(x-1), \beta_{\frac{n}{2}q-k}; a-1 \rangle\!\rangle \langle\!\langle i\frac{n}{2}qQx, \beta_k; -1 | F_2 \rangle\!\rangle . \tag{3.28}$$

The x integral is also evaluated at the saddle point

$$x = \frac{2k}{nq} .$$

(3.29)

If the saddle points are located within the interval $0 < x = 2k/nq < 1$, the integral gives the non-vanishing contributions. Thus the sum of the integer k is restricted within $0 < k < \frac{n}{2}q$ and we get

$$\pi \frac{\lambda}{q} \sum_{\substack{k \in \mathcal{N} \\ (0<k<\frac{n}{2}q)}} \frac{2}{n} \tilde{A}_n^{NS}(ikQ) \langle\langle F_1 O_{\frac{n}{2}q-k}^{(a-1)} \rangle\rangle \langle\langle O_k^{(-1)} F_2 \rangle\rangle ,$$

(3.30)

where $\tilde{A}_n^{NS}$ is the coefficient such that the intermediate states are normalized in the form $O_j^{(a)}$.[5] In the same way we can calculate the anomalous contribution in the case that the intermediate states are in the R-sector. The result is

$$\pi \frac{\lambda}{q} \sum_{\substack{k \in \mathcal{R} \\ (0<k<\frac{n}{2}q)}} \frac{2}{n} \tilde{A}_n^{R}(ikQ) \langle\langle F_1 O_{\frac{n}{2}q-k}^{(a-3/2)} \rangle\rangle \langle\langle O_k^{(-1/2)} F_2 \rangle\rangle ,$$

(3.31)

Let us calculate the coefficients $\tilde{A}_n^{NS}(ikQ)$ and $\tilde{A}_n^{R}(ikQ)$. We need to evaluate the following OPE:

$$R_{-1,-n-1}^{(a)}(z)[\bar{b}_{-1}, \bar{B}_{-1,-n-1}^{(a)}(\bar{z})] \bar{\partial}\bar{c}(\bar{0})| - h, \beta_{-k}; -3/2 + \kappa \rangle|_{h=ikQ}$$

$$= \frac{1}{|z|^2} \tilde{A}_n^{NS,R}(ikQ) O_{\frac{n}{2}q-k}^{(a-3/2+\kappa)}(0,\bar{0})|0\rangle ,$$

(3.32)

where $\kappa = 0(R)$ and $\frac{1}{2}(NS)$. The holomorphic part is easily evaluated by using the explicit form of $R_{-1,-n-1}$. To evaluate the anti-holomorphic part it is necessary to calculate the following OPE's:

$$[\bar{b}_{-1}, \bar{x}^{(1/2)}(\bar{z})] \bar{\partial}\bar{c}(\bar{0})| - ikQ, \beta_{-k}; -3/2 + \kappa \rangle$$

$$= \left(\frac{1}{2}\bar{b}(\bar{z}) e^{\frac{1}{2}u(\bar{z})+i\frac{1}{2}v(\bar{z})} - \sqrt{\frac{p}{2q}} \bar{\partial}\bar{\xi}(\bar{z}) e^{-\frac{1}{2}u(\bar{z})-i\frac{1}{2}v(\bar{z})} \right) e^{-\frac{1}{2}qQ\phi(\bar{z})+\frac{1}{2}qQ\varphi(\bar{z})} \bar{I}_R^{-1}$$

$$\times \bar{\partial}\bar{c}(\bar{0})| - ikQ, \beta_{-k}; -3/2 + \kappa \rangle$$

$$= \frac{i}{2} \frac{1}{\bar{z}} O_{\frac{1}{2}q-k}^{(-1+\kappa)}(\bar{0})|0\rangle$$

(3.33)

and

$$\bar{x}^{(1/2)}(\bar{z}) \bar{\partial}\bar{c}(\bar{0})| - ikQ, \beta_{-k}; -3/2 + \kappa \rangle$$

$$= i \left(-\frac{k}{q} + \frac{1}{2} \right) \bar{\partial}\bar{c}(\bar{0}) O_{\frac{1}{2}q-k}^{(-1+\kappa)}(\bar{0})|0\rangle .$$

(3.34)

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[5] Note that, for instance, $O_j^{(-3/2)}(z) = -\frac{q}{j} c(z) e^{-\frac{3}{2}u(z)-\frac{1}{2}v(z)+\alpha_j\phi(z)+\beta_j\varphi(z)} I_R^3$.

In general, noting $\bar{B}_{-1,-n-1} = \bar{x}^n$ and $[\bar{b}_{-1}, \bar{x}^n] = [\bar{b}_{-1}, \bar{x}]\bar{x}^{n-1} + \cdots + \bar{x}^{n-1}[\bar{b}_{-1}, \bar{x}]$, we get the following results:

$$\tilde{A}_{2m}^{NS}(ikQ) = m\Lambda_{NS}(k)\Lambda_{NS}(mq - k) , \tag{3.35}$$

$$\tilde{A}_{2m+1}^{NS}(ikQ) = (m + \tfrac{1}{2})\Lambda_{NS}(k)\Lambda_R(mq - k + \tfrac{1}{2}q) , \tag{3.36}$$

for the case of the NS-intermediate states and

$$\tilde{A}_{2m}^{R}(ikQ) = m\Lambda_R(k)\Lambda_R(mq - k) , \tag{3.37}$$

$$\tilde{A}_{2m+1}^{R}(ikQ) = (m + \tfrac{1}{2})\Lambda_R(k)\Lambda_{NS}(mq - k + \tfrac{1}{2}q) \tag{3.38}$$

for the case of the R-intermediate states. These Λ-factors are renormalized in the scaling operators and we get the simple expression,

$$\int d^2z \left\langle\!\!\!\left\langle \bar{\partial}W_{-1,-2m-1}(z,\bar{z}) \prod_{j\in S} \hat{O}_j \right\rangle\!\!\!\right\rangle_g$$

$$= \frac{1}{2!}\pi\frac{\lambda}{q} \sum_{\substack{k\in\mathcal{N}\\(0<k<mq)}} \left[\left\langle\!\!\!\left\langle \hat{O}_{mq-k}^{NS}\hat{O}_k^{NS} \prod_{j\in S} \hat{O}_j \right\rangle\!\!\!\right\rangle_{g-1} \right.$$

$$\left. + \sum_{\substack{S=X\cup Y\\g=g_1+g_2}} \left\langle\!\!\!\left\langle \hat{O}_{mq-k}^{NS} \prod_{j\in X} \hat{O}_j \right\rangle\!\!\!\right\rangle_{g_1} \left\langle\!\!\!\left\langle \hat{O}_k^{NS} \prod_{j\in Y} \hat{O}_j \right\rangle\!\!\!\right\rangle_{g_2} \right]$$

$$+ \frac{1}{2!}\pi\frac{\lambda}{q} \sum_{\substack{k\in\mathcal{R}\\(0<k<mq)}} \left[\left\langle\!\!\!\left\langle \hat{O}_{mq-k}^{R}\hat{O}_k^{R} \prod_{j\in S} \hat{O}_j \right\rangle\!\!\!\right\rangle_{g-1} \right.$$

$$\left. + \sum_{\substack{S=X\cup Y\\g=g_1+g_2}} \left\langle\!\!\!\left\langle \hat{O}_{mq-k}^{R} \prod_{j\in X} \hat{O}_j \right\rangle\!\!\!\right\rangle_{g_1} \left\langle\!\!\!\left\langle \hat{O}_k^{R} \prod_{j\in Y} \hat{O}_j \right\rangle\!\!\!\right\rangle_{g_2} \right] \tag{3.39}$$

for the current in the NS-sector and

$$\int d^2z \left\langle\!\!\!\left\langle \bar{\partial}W_{-1,-2m-2}(z,\bar{z}) \prod_{j\in S} \hat{O}_j \right\rangle\!\!\!\right\rangle_g$$

$$= \frac{1}{2!}2\pi\frac{\lambda}{q} \sum_{\substack{k\in\mathcal{R}\\(0<k<mq+\frac{1}{2}q)}} \left[\left\langle\!\!\!\left\langle \hat{O}_{mq-k+\frac{1}{2}q}^{NS}\hat{O}_k^{R} \prod_{j\in S} \hat{O}_j \right\rangle\!\!\!\right\rangle_{g-1} \right.$$

$$\left. + \sum_{\substack{S=X\cup Y\\g=g_1+g_2}} \left\langle\!\!\!\left\langle \hat{O}_{mq-k+\frac{1}{2}q}^{NS} \prod_{j\in X} \hat{O}_j \right\rangle\!\!\!\right\rangle_{g_1} \left\langle\!\!\!\left\langle \hat{O}_k^{R} \prod_{j\in Y} \hat{O}_j \right\rangle\!\!\!\right\rangle_{g_2} \right] \tag{3.40}$$

for the current in the R-sector. The first and the third terms of the r.h.s. in (3.39) and the first in (3.40) are variants of the boundaries (3.30) and (3.31). The factor

1/2! corrects for double counting. The factor 2 in (3.40) comes from the currents in the R-sector boundaries (3.30) and (3.31) gives the same contributions.

Combining the results (3.19) and (3.39–40) we complete the calculation of the Ward identities for $W_{-1,-n-1}$. The final result can be summarized in the simple form as in the Virasoro constraint of the bosonic case,

$$\mathscr{L}_n \tau \Big|_{\substack{x_1^{NS}=-\mu \\ x_{(p+q)/2}^{NS}=t \\ \text{other } x_j^{NS,R}=0}} = 0, \quad \tau = e^{Z(x_j^{NS}, x_j^R)}, \tag{3.41}$$

where $\mathscr{L}_n$ is defined by

$$\mathscr{L}_{2m} = \sum_{\substack{k,l \in \mathscr{N} \\ -k+l=mq}} \frac{k}{q} x_k^{NS} \partial_l^{NS} + \sum_{\substack{k,l \in \mathscr{R} \\ -k+l=mq}} \frac{k}{q} x_k^R \partial_l^R$$

$$+ \frac{1}{2}\frac{\lambda}{q} \sum_{\substack{k,l \in \mathscr{N} \\ k+l=mq}} \partial_k^{NS} \partial_l^{NS} + \frac{1}{2}\frac{\lambda}{q} \sum_{\substack{k,l \in \mathscr{R} \\ k+l=mq}} \partial_k^R \partial_l^R \tag{3.42}$$

for even $n = 2m$ and

$$\mathscr{L}_{2m+1} = \sum_{\substack{k \in \mathscr{N}, l \in \mathscr{R} \\ -k+l=(m+\frac{1}{2})q}} \frac{k}{q} x_k^{NS} \partial_l^R + \sum_{\substack{k \in \mathscr{R}, l \in \mathscr{N} \\ -k+l=(m+\frac{1}{2})q}} \frac{k}{q} x_k^R \partial_l^{NS}$$

$$+ \frac{\lambda}{q} \sum_{\substack{k \in \mathscr{N}, l \in \mathscr{R} \\ k+l=(m+\frac{1}{2})q}} \partial_k^{NS} \partial_l^R \tag{3.43}$$

for odd $n = 2m + 1$. The partition function $Z(x_j^{NS}, x_j^R)$ is defined by the action

$$S = S_0(p,q) - \sum_{j \in \mathscr{N}} x_j^{NS} \hat{O}_j^{NS} - \sum_{j \in \mathscr{R}} x_j^R \hat{O}_j^R. \tag{3.44}$$

Note that the Virasoro generators for even n, which belong to the NS-sector, form the Virasoro sub-algebra by themselves.

Until now we describe the sector explicitly. It is, however, distinguished by the sets $\mathscr{N}$ and $\mathscr{R}$ so that the superscripts on x_j and ∂_j can be removed and the Virasoro generators are expressed in the single form as

$$\mathscr{L}_n = \sum_{\substack{k \in \mathscr{N} \\ -k+l=\frac{n}{2}q}} \frac{k}{q} x_k \partial_l + \frac{1}{2}\frac{\lambda}{q} \sum_{\substack{k \in \mathscr{N} \\ k+l=\frac{n}{2}q}} \partial_k \partial_l$$

$$+ \sum_{\substack{k \in \mathscr{R} \\ -k+l=\frac{n}{2}q}} \frac{k}{q} x_k \partial_l + \frac{1}{2}\frac{\lambda}{q} \sum_{\substack{k \in \mathscr{R} \\ k+l=\frac{n}{2}q}} \partial_k \partial_l. \tag{3.45}$$

The sector of ∂_l is easily recovered by noting that the Virasoro generators $\mathscr{L}_n$ for $n =$ even (odd) belongs to the NS (R)-sector.

3.3. $\mathscr{W}$-Algebra Constraints. In this subsection we discuss the Ward identities for the symmetry currents which derive the $\mathscr{W}$-algebra constraints. We first consider the current $W_{-2,-n-2}$. For $n = -1$ the explicit form of the current $W_{-2,-1}(z,\bar{z}) = R_{-2,-1}(z)\bar{B}_{-2,-1}(\bar{z})$ is defined by $\bar{B}_{-2,-1}(\bar{z}) = \bar{y}(\bar{z})$ and

$$R^{(a)}_{-2,-1}(z) = -4iJ_0^+(b_{-1}O^{(a)}_{-3,0})(z)\,, \tag{3.46}$$

where

$$J_0^+ = \oint \frac{dz}{2\pi i}J^+(z), \qquad J^+(z) = \left(\mathrm{e}^{-iv(z)} + \frac{q}{p}\mathrm{e}^{iv(z)}\right)\mathrm{e}^{\Phi(z)} \tag{3.47}$$

and

$$O^{(-1/2)}_{-3,0}(z) = c(z)\mathrm{e}^{-\frac{1}{2}u(z)-\frac{i}{2}v(z)+\alpha_{-3,0}\phi(z)+i\beta_{-3,0}\varphi(z)}I_R\,. \tag{3.48}$$

The current for general n is defined by using the $\mathscr{W}_\infty$ algebra (2.22) as

$$W_{-2,-n-2}(z,\bar{z}) = \frac{-2}{2n+3}[Q_{-1,-n-2}, W_{-2,-1}(z,\bar{z})]\,, \tag{3.49}$$

where $Q_{r,s} = \oint \frac{dz}{2\pi i}W_{r,s}(z,\bar{z})$.

Let us calculate the OPE between the current and scaling operators. We first calculate the $W_{-2,-1}$ current. It is convenient to use the field $R_{-2,-1}$ in the $\frac{1}{2}$-picture given by

$$R^{(1/2)}_{-2,-1}(z) = -4i\oint_{C_0} \frac{dz'}{2\pi i}\left(\mathrm{e}^{-iv(z'+z)} + \frac{q}{p}\mathrm{e}^{iv(z'+z)}\right)\mathrm{e}^{\Phi(z'+z)}$$

$$\times \left[-\frac{p}{2q}\mathrm{e}^{\frac{1}{2}u(z)-\frac{3}{2}iv(z)} - \left(i\partial v(z) - \sqrt{\frac{p}{2q}}i\partial\Omega^-(z)\right)\mathrm{e}^{\frac{1}{2}u(z)+\frac{i}{2}v(z)}\right]$$

$$\times \mathrm{e}^{-\Phi(z)-\frac{1}{2}pQ\phi(z)-\frac{1}{2}pQ\varphi(z)}I_R^{-1}\,. \tag{3.50}$$

We need to calculate the perturbed OPE. The OPE between $R^{(1/2)}_{-2,-1}$ and the scaling operators is, for example, calculated as

$$R^{(1/2)}_{-2,-1}(z)O^{(-1)}_k(0)V^{(-1)}_l(w) = -2i\left[\frac{k}{q}\frac{1}{z^2} + \frac{l}{q}\frac{1}{(z-w)^2} - \frac{k+l}{q}\frac{1}{z(z-w)}\right]$$

$$\times z^{1-\frac{k}{q}}(z-w)^{1-\frac{l}{q}}w^{-2+\frac{k+l}{q}}\left(-\frac{k+l}{q}+\frac{1}{2}\right)O^{(-3/2)}_{k+l-\frac{1}{2}q}(0)\,, \tag{3.51}$$

where $V_l(w) = (b_{-1}O_l)(w)$. Using the expression for y (2.28), the OPE for the anti-holomorphic part is calculated as

$$\bar{B}^{(1/2)}_{-2,-1}(\bar{z})O^{(-1)}_k(\bar{0})V^{(-1)}_l(\bar{w})$$

$$= \frac{1}{\bar{z}-\bar{w}}\bar{z}^{1-\frac{k}{q}}(\bar{z}-\bar{w})^{1-\frac{l}{q}}\bar{w}^{-2+\frac{k+l}{q}}$$

$$\times \left[-\frac{1}{2}\bar{c}\mathrm{e}^{-\frac{3}{2}u-\frac{1}{2}v} + \sqrt{\frac{q}{2p}}\bar{c}\bar{\partial}\bar{c}\bar{\partial}\bar{\xi}\mathrm{e}^{-\frac{5}{2}u+\frac{1}{2}v}\right]\mathrm{e}^{\alpha_{k+l-q/2}\phi(\bar{0})+i\beta_{k+l-q/2}\varphi(\bar{0})}\bar{I}_R^3$$

$$= \frac{1}{2}\left(\frac{k+l}{q}-1\right)\frac{1}{\bar{z}-\bar{w}}\bar{z}^{1-\frac{k}{q}}(\bar{z}-\bar{w})^{1-\frac{l}{q}}\bar{w}^{-2+\frac{k+l}{q}}O^{(-3/2)}_{k+l-\frac{1}{2}q}(\bar{0})\,. \tag{3.52}$$

Combining the holomorphic and anti-holomorphic parts and integrating over w we get

$$W^{(1/2)}_{-2,-1}(z,\bar z)O^{(-1)}_k(0,\bar 0)\int d^2wV^{(-1)}_l(w,\bar w)$$

$$= \frac{1}{z}i\left(\frac{k+l}{2}-\frac{1}{2}\right)\left(\frac{k+l}{2}-1\right)\left[\frac{k}{q}I_1+\frac{l}{q}I_{-1}-\frac{k+l}{q}I_0\right]O^{(-3/2)}_{k+l-\frac{1}{2}q}(0,\bar 0)$$

$$= \frac{1}{z}2\pi\frac{k}{q}\frac{l}{q}\Lambda^{-1}_{NS}(k)\Lambda^{-1}_{NS}(l)\Lambda_R(k+l-\tfrac{1}{2}q)O^{(-3/2)}_{k+l-\frac{1}{2}q}(0,\bar 0)\,, \tag{3.53}$$

where the integrals I_n $(n=0,\pm1)$ are defined by

$$I_n = \int d^2y|y|^{2(-2+\frac{k+l}{q})}|1-y|^{-\frac{2l}{q}}(1-y)^n$$

$$= \pi\frac{\Gamma(\frac{k+l}{q}-1)\Gamma(1-\frac{k}{q})\Gamma(1-\frac{l}{q}+n)}{\Gamma(2-\frac{k+l}{q})\Gamma(\frac{k}{q}+n)\Gamma(\frac{l}{q})}\,. \tag{3.54}$$

The Λ-factors are renormalized in the scaling operators. The other cases are also calculated in the same way. Furthermore, using the $\mathcal{W}_\infty$ algebra (3.49) we obtain the following OPE:

$$W_{-2,-n-2}(z,\bar z)\hat O_k(0,\bar 0)\int d^2w\hat V_l(w,\bar w) = \frac{1}{z}\pi2!\frac{k}{q}\frac{l}{q}\hat O_{\frac{n}{2}q+k+l}(0,\bar 0)\,. \tag{3.55}$$

The superscripts distinguishing the sectors are omitted again.

The non-linear terms are calculated as in the Virasoro case. We need to evaluate the following boundary contribution:

$$\lim_{\tau\to\infty}\lambda\frac{Q}{\pi}\int_{-\infty}^{\infty}\frac{dh}{2\pi}\left\langle\!\!\left\langle F'_1\left\{\int_{e^{-\tau}\leq|z|\leq1}d^2z\bar\partial W_{-2,-n-2}(z,\bar z)\int_{|w|\leq|z|}d^2w\hat V_l(w,\bar w)\right.\right.\right.$$

$$\left.+\int_{e^{-\tau}\leq|z|\leq1}d^2z\hat V_l(z,\bar z)\int_{|w|\leq|z|}d^2w\bar\partial W_{-2,-n-2}(w,\bar w)\right\}$$

$$\times\frac{2\pi}{H}e^{-\tau H}\left\{\sum_{k\in\mathcal{N}}|-h,\beta_{-k},N;-1\rangle\!\rangle\langle\!\langle h,\beta_k,N;-1|\right.$$

$$\left.\left.\left.+\sum_{k\in\mathcal{R}}|-h,\beta_{-k},N;-3/2\rangle\!\rangle\langle\!\langle h,\beta_k,N;-1/2|\right\}F'_2\right\rangle\!\!\right\rangle$$

$$= \pi^22!\frac{\lambda}{q}\sum_{\substack{k\in\mathcal{N}\\(0<k<\frac{n}{2}q+l)}}\frac{l}{q}\langle\!\langle F'_1\hat O_{\frac{n}{2}q+l-k}\rangle\!\rangle\langle\!\langle\hat O_kF'_2\rangle\!\rangle$$

$$+\pi^22!\frac{\lambda}{q}\sum_{\substack{k\in\mathcal{R}\\(0<k<\frac{n}{2}q+l)}}\frac{l}{q}\langle\!\langle F'_1\hat O_{\frac{n}{2}q+l-k}\rangle\!\rangle\langle\!\langle\hat O_kF'_2\rangle\!\rangle\,, \tag{3.56}$$

where the primes of F_1 and F_2 stand for the exclusion of the operator $\hat{O}_l$ in S. This is an application of the methods developed in the calculations of (3.20) and (3.55). The integrals of h and z are also evaluated by using the saddle point method.

As a variant of the boundary contribution (3.56), there exists the following one:

$$
\pi^2 2! \frac{\lambda^2}{q^2} \left\{ \sum_{\substack{k \in \mathcal{N}, l \in \mathcal{N} \\ k+l+r=\frac{n}{2}q}} + \sum_{\substack{k \in \mathcal{R}, l \in \mathcal{N} \\ k+l+r=\frac{n}{2}q}} + \sum_{\substack{k \in \mathcal{N}, l \in \mathcal{R} \\ k+l+r=\frac{n}{2}q}} + \sum_{\substack{k \in \mathcal{R}, l \in \mathcal{R} \\ k+l+r=\frac{n}{2}q}} \right\} \langle\!\langle F_1 \hat{O}_r \rangle\!\rangle \langle\!\langle \hat{O}_k F_2 \rangle\!\rangle \langle\!\langle \hat{O}_l F_3 \rangle\!\rangle \,.
$$

$$(3.57)$$

This is obtained by replacing the scaling operator V_l in the expression (3.56) with the factorization formula $-(\lambda Q/\pi h_l) V_{-l}^{(-1)} \langle\!\langle O_l^{(-1)} F_3 \rangle\!\rangle$ for $l \in \mathcal{N}$ and $-(\lambda Q/\pi h_l)$ $(l/q)^2 V_{-l}^{(-3/2)} \langle\!\langle O_l^{(-1/2)} F_3 \rangle\!\rangle$ for $l \in \mathcal{R}$, where $1/h_l = \int dh(h^2 + l^2 Q^2)^{-1} = \pi/lQ$. This is based on the argument that in the case of the non-critical string the intermediate states become on-shell [6].

Combining the contributions (3.55–57) and their variants and taking into account the factor $1/2!$ for (3.56) and $1/3!$ for (3.57) to avoid the overcounting, it is found that the Ward identity of the current $W_{-2,-n-2}$ gives the $\mathscr{W}_n^{(3)}$ constraint,

$$
\mathscr{W}_n^{(3)} \tau \Big|_{\substack{x_1^{NS}=-\mu \\ x_{(p+q)/2}^{NS}=t \\ \text{other } x_j^{NS,R}=0}} = 0 \,,
$$

$$(3.58)$$

where τ function is defined in (3.41) and

$$
\mathscr{W}_n^{(3)} = \sum_{\substack{l \in \mathcal{N}, k \in \mathcal{N} \\ -l-k+r=\frac{n}{2}q}} \frac{l}{q}\frac{k}{q} x_l x_k \partial_r + \frac{\lambda}{q} \sum_{\substack{l \in \mathcal{N}, k \in \mathcal{N} \\ -l+k+r=\frac{n}{2}q}} \frac{l}{q} x_l \partial_k \partial_r + \frac{1}{3}\frac{\lambda^2}{q^2} \sum_{\substack{l \in \mathcal{N}, k \in \mathcal{N} \\ l+k+r=\frac{n}{2}q}} \partial_l \partial_k \partial_r
$$

$$
+ \sum_{\substack{l \in \mathcal{R}, k \in \mathcal{N} \\ -l-k+r=\frac{n}{2}q}} \frac{l}{q}\frac{k}{q} x_l x_k \partial_r + \frac{\lambda}{q} \sum_{\substack{l \in \mathcal{R}, k \in \mathcal{N} \\ -l+k+r=\frac{n}{2}q}} \frac{l}{q} x_l \partial_k \partial_r + \frac{1}{3}\frac{\lambda^2}{q^2} \sum_{\substack{l \in \mathcal{R}, k \in \mathcal{N} \\ l+k+r=\frac{n}{2}q}} \partial_l \partial_k \partial_r
$$

$$
+ \sum_{\substack{l \in \mathcal{N}, k \in \mathcal{R} \\ -l-k+r=\frac{n}{2}q}} \frac{l}{q}\frac{k}{q} x_l x_k \partial_r + \frac{\lambda}{q} \sum_{\substack{l \in \mathcal{N}, k \in \mathcal{R} \\ -l+k+r=\frac{n}{2}q}} \frac{l}{q} x_l \partial_k \partial_r + \frac{1}{3}\frac{\lambda^2}{q^2} \sum_{\substack{l \in \mathcal{N}, k \in \mathcal{R} \\ l+k+r=\frac{n}{2}q}} \partial_l \partial_k \partial_r
$$

$$
+ \sum_{\substack{l \in \mathcal{R}, k \in \mathcal{R} \\ -l-k+r=\frac{n}{2}q}} \frac{l}{q}\frac{k}{q} x_l x_k \partial_r + \frac{\lambda}{q} \sum_{\substack{l \in \mathcal{R}, k \in \mathcal{R} \\ -l+k+r=\frac{n}{2}q}} \frac{l}{q} x_l \partial_k \partial_r + \frac{1}{3}\frac{\lambda^2}{q^2} \sum_{\substack{l \in \mathcal{R}, k \in \mathcal{R} \\ l+k+r=\frac{n}{2}q}} \partial_l \partial_k \partial_r \,.
$$

$$(3.59)$$

Applying the $\mathscr{W}_\infty$ algebra recursively, we easily get the following perturbed OPE:

$$
W_{-k,-n-k}(z,\bar{z}) \hat{O}_{l_1}(0,\bar{0}) \int \hat{V}_{l_2} \cdots \int \hat{V}_{l_k} = \frac{1}{z} \pi^{k-1} k! \prod_{j=1}^{k} \frac{l_i}{q} \hat{O}_{\frac{n}{2}q+l_1+\cdots+l_k}(0,\bar{0}) \,. \quad (3.60)
$$

This OPE corresponds to the single derivative term of the $\mathscr{W}_n^{(k+1)}$ constraint

$$\mathscr{W}_n^{(k+1)} = \sum_{-l_1-\cdots-l_k+r=\frac{n}{2}q} \frac{l_1}{q} \cdots \frac{l_k}{q} x_{l_1} \cdots x_{l_k} \partial_r + \cdots . \qquad (3.61)$$

We can also calculate the boundary corresponding to the two derivative term. The terms with more derivatives are calculated as variants of the two derivative term.

4. Summary and Discussion

In this paper we investigated the Ward identities of the $\mathscr{W}_\infty$ symmetry in the super-Liouville theory coupled to the super-conformal matter of central charge $\hat{c}_M = 1 - 2(p-q)^2/pq$, $p - q \in 2\mathbf{Z}$. Throughout this paper we discussed the case of the positive chirality defined by the momenta (2.33) and normalization factors (2.39–40). Then we found that the Ward identities of the $\mathscr{W}_\infty$ symmetry currents $W_{-k,-n-k}$ are expressed in the form of the $\mathscr{W}_n^{(k+1)}$ constraints which form the *usual* $\mathscr{W}_q$ algebra as in the bosonic case. The Ward identities of $k \geqq q$ will be redundant. The similar reduction from the $\mathscr{W}_\infty$ to the $\mathscr{W}_q$ algebra appears in the matrix model approach [18]. For the negative chirality the roles of p and q are interchanged and we get the $\mathscr{W}_p$ algebra constraints.

The universality is determined by the potential terms. The (p,q)-*critical* theory is given by choosing the first scale operator $\hat{O}_1^{NS}$ and the $\frac{1}{2}(p+q)^{\text{th}}$ scaling operator $\hat{O}_{(p+q)/2}^{NS}$ that is one of the screening charge operator for the matter sector. If we choose the potential $\hat{O}_l^{NS}$ instead of $\hat{O}_{(p+q)/2}^{NS}$, we get the lth *critical* theory which is subject to the $\mathscr{W}_q$ algebra constraints with $x_1^{NS} = -\mu$, $x_l^{NS} = t$ and other $x_j^{NS,R} = 0$ in (3.41) and (3.58). The μ and t dependences are then given as follows:

$$s = \frac{1}{l-1} \left[l\chi - \sum_{j \in S} (l-j) \right] , \qquad (4.1)$$

$$n = \frac{1}{l-1} \left[-\chi + \sum_{j \in S} (1-j) \right] \qquad (4.2)$$

instead of (3.4) and (3.5). Thus we can not classify the universality only from the background charges (2.3). Really, for the positive chirality, we get the same model from the (p',q) model defined by the action

$$S = S_0(p',q) + \mu \hat{O}_1^{NS}(p',q) - t \hat{O}_l^{NS}(p',q) , \qquad (4.3)$$

where $p' \neq p$ ($p' - q \in 2\mathbf{Z}$). It is easily seen by noting that all OPE coefficients of the boundary calculations, or the Λ-factors are independent of p. If we set $l = \frac{1}{2}(p+q)$ we get the (p,q)-critical theory from the (p',q) theory.

The NS-sector is closed in itself. If we consider only the NS-sector and choose the potential of $l = p+q$ in (4.3) instead of $l = \frac{1}{2}(p+q)$,[6] we can not distinguish the universality of the model from that of the (p,q)-critical theory of the non-critical bosonic string [17].

---

[6] Here $p' - q \in 2\mathbf{Z}$, but it is not necessary to be $p - q \in 2\mathbf{Z}$.

Appendix: Note on the Scaling Operators

Here we summarize the expressions of the scaling operators with the pictures other than $-\frac{1}{2}$ and -1. The picture changing isomorphism indicates that the scaling operator in the R-sector is expressed at the picture $-\frac{3}{2}$ as

$$O_j^{(-3/2)}(z) = -\frac{q}{j}c(z)e^{-\frac{3}{2}u(z)-\frac{i}{2}v(z)+\alpha_j\phi(z)+\beta_j\varphi(z)}I_R^3$$

for the holomorphic part. Using (2.17), this is also expressed as

$$O_j^{(-3/2)}(z) = -4\sqrt{\frac{q}{2p}}c\partial c\partial\xi(z)e^{-\frac{5}{2}u(z)+\frac{i}{2}v(z)+\alpha_j\phi(z)+\beta_j\varphi(z)}I_R^3 \ .$$

In the NS-sector we get, for instance, the following expressions,

$$O_j^{(0)}(z) = -\left[\left(\frac{j}{q}-\frac{1}{2}\right)e^{iv(z)} + \frac{p}{2q}e^{-iv(z)}\right]c(z)e^{\alpha_j\phi(z)+\beta_j\varphi(z)}$$

$$-\frac{1}{2}\sqrt{\frac{p}{2q}}\eta(z)e^{u(z)+\alpha_j\phi(z)+\beta_j\varphi(z)} \ ,$$

$$O_j^{(-2)}(z) = -\frac{q}{j}\left(e^{-iv(z)} + \frac{q}{p}e^{iv(z)}\right)c(z)e^{-2u(z)+\alpha_j\phi(z)+i\beta_j\varphi(z)}I_R^4 \ ,$$

and so on.

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