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The Strong Decay to Equilibrium for the Stochastic Dynamics of Unbounded Spin Systems on a Lattice

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Abstract: Using a method based on the application of hypercontractivity we prove the strong exponential decay to equilibrium for a stochastic dynamics of unbounded spin system on a lattice.

0. Introduction

In recent years essential progress has been made in understanding the ergodicity properties of the Markov semigroups P_t, $t \in \mathbb{R}^+$ defined on the space of continuous functions $\mathscr{C}(\Omega)$, with a configuration space $\Omega \equiv M^\Gamma$, M being a compact metric space and Γ a countable (infinite) set. An important method for the study of these properties was first introduced in [HS]. It involves three elements:

(i) a strong approximation property of the semigroup P_t, $t \in \mathbb{R}^+$ by the semigroups $P_t^{\Lambda,\omega}$ acting (essentially) on $\mathscr{C}(M^\Lambda)$, $\Lambda \subset \Gamma$ finite sets, and fixing a configuration $\omega \in \Omega$ outside Λ,

(ii) the finite volume ultracontractivity property of $P_t^{\Lambda,\omega}$, and

(iii) the uniform in volume Λ and boundary conditions ω hypercontractivity property of the semigroups $P_t^{\Lambda,\omega}$ on the spaces $L_p(E_\Lambda^\omega)$, $p \in (1,\infty)$, with E_Λ^ω being the corresponding invariant probability measures.

The first two properties have been well known for a long time for the situation of compact configuration space. Although the hypercontractivity property of a semigroup, or its equivalent property of corresponding invariant measure called the logarithmic Sobolev inequality (LS), was introduced almost twenty years ago, [G], for many years no nontrivial example involving an infinite dimensional configuration space was known. (For the trivial one corresponding to the Gaussian or some product measures see [G].) This was until a very nice Bakry–Emery criterion (B-E) for the logarithmic Sobolev inequality has been introduced in [BE], for a case of configuration space defined with a (finite dimensional) smooth, connected and compact Riemannian manifold M with positive Ricci curvature (or a case when the Ricci curvature is zero, but involving some special log-concave measures). Exploiting this criterion, Carlen and Stroock gave in [CS] the first nontrivial examples of probability measures (for some statistical mechanical spin system on a lattice) which satisfied the logarithmic Sobolev inequality (uniformly in volume and boundary conditions).

A general idea and technique to study the logarithmic Sobolev inequality by exploiting the associated Gibbs structure was introduced in [Z1–3] and developed later in [SZ1–3], (see also [MO, LY and SZ4]). This allowed us to consider the highly nontrivial situation where the Bakry–Emery criterion cannot work. That includes the case when M is a compact, smooth and connected Riemannian manifold with nonpositive Ricci curvature or the case (important for applications in statistical mechanics) when M is simply a finite set.

In the present paper we extend the ergodicity results to the other important case when M is a noncompact space. Besides that, the present paper is a necessary intermediate step towards a study of some other interesting questions concerning hypercontractive Markov semigroups and its applications in the field theory and statistical mechanics (as e.g. analyticity and particle structure, ergodicity properties for systems in continuum).

In the first section we construct a class of nontrivial semigroups P_t, $t \in \mathbb{R}^+$, as "a perturbation" of a Gaussian semigroup and we prove an analog of the strong approximation property appropriate in this case. Let us mention that a different construction based on the use of a cluster expansion, (and therefore for a very restricted type of perturbations), has been given in [Di]. (Let us note also that our construction is essentially independent of the Gaussian character of the free semigroup and can be easily carried out for more general cases when we have also multispin interactions of finite range with bounded derivative. However having some specific applications in mind and in order to simplify the notation we consider explicitly the case of local perturbations of the Gaussian semigroups.)

In the second section we study various ultracontractivity properties of the finite volume semigroups.

Next using the results of Sects. 1 and 2, we give in Sect. 3 a general strategy (extending that of [HS]) for proving the strong ergodicity result. We show that it is possible to apply our strategy even in the situation when the finite volume ultracontractivity fails. (Let us note that for the continuous space models, which we shall study in the future, one can have logarithmic Sobolev inequalities, but one cannot expect to have a local ultracontractivity. Therefore it is important to know that such an extended strategy can work.)

Section 4 is devoted to proving the logarithmic Sobolev inequalities for a (comprehensive) class of nontrivial examples with $\Omega = \mathbb{R}^{\mathbb{Z}}$, for which the B-E criterion does not work.

In Sect. 5 we extend these results to the higher dimensional lattices. In this case one has to introduce some restrictions, (because as is known in higher dimensions phase transition can occur), but still one can go beyond the B-E criterion. The results of Sects. 3–5 allow us to show a strong ergodicity result for the corresponding hypercontractive semigroups with pointwise exponential decay to equilibrium. (By this we have got an important extension of ergodicity results of [Di], where an ergodicity of the semigroups has been proven for a very restricted class of initial distributions. Let us mention here also the recent work [AKR], where the Logarithmic Sobolev inequalities, proven for convex interactions via the Bakry–Emery criterion, has been used to study the L_2-ergodicity of the corresponding semigroups.)

A summary and some discussion of interesting open questions are given in Sect. 6.

1. A Construction of the Stochastic Dynamics

We consider a lattice $\Gamma \equiv \mathbb{Z}^d$, $d \in \mathbb{N}$ with the usual Euclidean distance $d(\cdot\,,\cdot)$. Let $\mathfrak{F}$ denote the family of all finite subsets of Γ. For the purpose of discussion of the thermodynamic limit, we distinguish an increasing sequence $\mathfrak{F}_0 \equiv \{\Lambda_n \in \mathfrak{F}\}_{n\in\mathbb{N}}$ invading all the lattice Γ and called a countable exhaustion. We will assume that there is a number $d_0 \in (0, \infty)$ such that for any $n \in \mathbb{N}$ we have $d_0^{-1} \leqq \operatorname{diam} \Lambda_n/d(0, \partial\Lambda_n) \leqq d_0$.

Let $(\Omega, \Sigma) \equiv (\mathbb{R}, \mathscr{B}_{\mathbb{R}})^\Gamma$. By $\varphi_{\mathbf{i}}$, we denote the $\mathbf{i}^{\text{th}}$ coordinate function on (Ω, Σ), called the spin at site $\mathbf{i} \in \Gamma$. Given two configurations $\eta, \omega \in \Omega$ and a set $\Lambda \subset \Gamma$, we define a new configuration $\eta_\Lambda \bullet \omega_{\Lambda^\complement}$ by

$$(\eta_\Lambda \bullet \omega_{\Lambda^\complement})_{\mathbf{i}} \equiv \begin{cases} \eta_{\mathbf{i}} & \text{for } \mathbf{i} \in \Lambda \\ \omega_{\mathbf{i}} & \text{for } \mathbf{i} \in \Lambda^\complement \end{cases}.$$

For $\Lambda \subset \Gamma$ we define a σ-algebra Σ_Λ as the smallest σ-subalgebra of Σ with respect to which all the coordinate functions $\varphi_{\mathbf{i}}$, $\mathbf{i} \in \Lambda$ are measurable. A set of real bounded Σ_Λ-measurable functions will be denoted by $\mathfrak{A}_\Lambda$. The elements of the set $\mathfrak{A}_0 \equiv \bigcup_{\Lambda\in\mathfrak{F}} \mathfrak{A}_\Lambda$ are called the local functions. If $\Lambda \subset \mathbb{Z}^d$ is the smallest subset such that $f \in \mathfrak{A}_\Lambda$, for any set $\Lambda' \subset \mathbb{Z}^d$ we define $d(f, \Lambda') \equiv d(\Lambda, \Lambda')$. For many applications it is sufficient to restict ourselves to a smaller configuration space $(\mathscr{S}', \mathfrak{S})$, a subspace of (Ω, Σ) consisting of tempered sequences, i.e. configurations $\omega \equiv (\omega_{\mathbf{j}})_{\mathbf{j}\in\mathbb{Z}^d}$ satisfying a growth condition $|\omega_{\mathbf{j}}| \leqq C(1 + |\mathbf{j}|)^N$ with some positive constants C and N possibly dependent on ω. For a probability measure μ on (Ω, Σ) we set $\mu(f)$ to denote the corresponding expectation of a measurable function f and use the following notation for the two point truncated correlation function:

$$\mu(f, g) \equiv \mu fg - \mu f \mu g$$

of measurable functions f and g.

Let $\mu_{\mathbf{G}}$ denote a Gaussian probability measure on (Ω, Σ) with mean zero and a covariance $\mathbf{G}$. We assume that the inverse $\mathbf{G}^{-1}$ of the covariance $\mathbf{G}$ is of finite range, i.e. there is a positive number R such that

$$\mathbf{G}^{-1}_{\mathbf{ij}} \equiv 0 \quad \text{if } d(\mathbf{i}, \mathbf{j}) > R\,. \tag{1.1}$$

By $\mu_{\mathbf{G}}^{\partial\Lambda}$ we denote the Gaussian measure with the Dirichlet boundary condition on $\partial\Lambda$, i.e. the Gaussian measure with mean zero and covariance $\mathbf{G}_{\partial\Lambda}$ such that $(\mathbf{G}_{\partial\Lambda})_{\mathbf{ij}} = \mathbf{G}^{-1}_{\mathbf{ij}}$ if $\mathbf{i}, \mathbf{j} \in \Lambda$ or $\mathbf{i}, \mathbf{j} \in \Lambda^\complement$, and zero otherwise. Let $\hat{\mathscr{E}} \equiv \{\hat{E}_\Lambda\}_{\Lambda\in\mathfrak{F}}$, be the family of the regular conditional expectations $\hat{E}^\omega_\Lambda(F) \equiv E_{\mu_{\mathrm{G}}}(F|\Sigma_{\Lambda^\complement})(\omega)$ associated to the measure $\mu_{\mathbf{G}}$.

Let U be a semibounded function which can be represented as a sum of a function W with a bounded first and second derivative and a function V with nonnegative second derivative. For every $\Lambda \in \mathfrak{F}$ we define a local interaction energy by

$$U_\Lambda(\varphi) \equiv \sum_{\mathbf{i}\in\Lambda} U(\varphi_{\mathbf{i}})\,. \tag{1.2}$$

We introduce the family $\mathscr{E} \equiv \{E_\Lambda\}_{\Lambda \in \mathfrak{F}}$, called a local specification corresponding to the free measure μ_G and the local interaction U, by setting

$$E_\Lambda^\omega(f) \equiv \frac{\hat{E}_\Lambda^\omega(e^{-U_\Lambda} f)}{\hat{E}_\Lambda^\omega(e^{-U_\Lambda})} . \tag{1.3}$$

By $\mathscr{G}(\mathscr{E})$ we denote the set of Gibbs measures for $\mathscr{E}$, i.e. the set of all probability measures on $(\mathscr{S}', \mathfrak{S})$ satisfying the condition

$$\mu(E_\Lambda f) = \mu f . \tag{DLR}$$

By $\partial\mathscr{G}(\mathscr{E})$ we denote the set of all extremal Gibbs measures for the specification $\mathscr{E}$ (i.e. the set of these Gibbs measures which have no nontrivial convex linear representation in terms of other Gibbs measures for $\mathscr{E}$). Under our assumptions, it is known (see e.g. [BH-K]), that the set $\mathscr{G}(\mathscr{E})$ is nonempty.

For later purposes we need to introduce the gradient operator by setting $\nabla_\Lambda f \equiv (\nabla_{\mathbf{i}} f)_{\mathbf{i} \in \Lambda}$, $\Lambda \subset \Gamma$, where

$$\nabla_{\mathbf{i}} f(\omega) \equiv \partial f_{\mathbf{i}}(x|\omega)$$

with ∂ denoting the differentiation of a real function

$$\mathbb{R} \ni x \to f_{\mathbf{i}}(x|\omega) \equiv f(x \bullet_{\mathbf{i}} \omega) ,$$

the configuration $x \bullet_{\mathbf{i}} \omega \in \Omega$ being defined by declaring its $\mathbf{i}^{\text{th}}$ coordinate to be equal to $x \in \mathbb{R}$ and all the other coordinates coinciding with those of the configuration $\omega \in \Omega$, and it is assumed that $f_{\mathbf{i}}(\,\cdot\,|\omega)$ is differentiable for every $\mathbf{i} \in \Lambda$. We define also

$$|\nabla_\Lambda f|^2(\omega) \equiv \sum_{\mathbf{i} \in \Lambda} |\nabla_{\mathbf{i}} f|^2(\omega) .$$

If $\Lambda = \Gamma$, we omit the reference to the set Λ in the above notation. Let $\mathscr{C}^{(n)}(\Omega)$, $n \in \mathbb{N}$ denote the set of all functions f for which for any $\mathbf{i} \in \Gamma$ we have $f_{\mathbf{i}}(\,\cdot\,|\omega) \in \mathscr{C}^{(n)}(\mathbb{R})$, with $\mathscr{C}^{(n)}(\mathbb{R})$ being the set of functions for which n derivatives exist and are bounded. In the space $\mathscr{C}(\Omega) \equiv \mathscr{C}^{(0)}(\Omega)$ we will use the supremum norm denoted later on by $\|\cdot\|_u$. In $\mathscr{C}^{(1)}$ we introduce a seminorm

$$|||f||| \equiv \sum_{\mathbf{k} \in \Gamma} \|\nabla_{\mathbf{k}} f\|_u .$$

We say that a probability measure μ on $(\mathscr{S}', \mathfrak{S})$ satisfies a logarithmic Sobolev inequality (LS) with a coefficient $c \in (0, \infty)$ if for some $q \in [1, \infty)$ (and therefore, by general arguments [G], for all $q \in [1, \infty)$) we have

$$\mu f^q \log f \leqq \frac{2}{q} c\mu |\nabla f^{\frac{q}{2}}|^2 + \mu f^q \log(\mu f^q)^{\frac{1}{q}} \tag{LS}$$

for any positive function f for which the right-hand side is finite.

Using the gradient operator we define for all local functions $f \in \mathscr{C}^2(\Omega)$ the following operators:

$$\mathscr{L}_\Lambda f \equiv \sum_{\mathbf{i} \in \Lambda} \mathscr{L}_{\mathbf{i}} f , \tag{1.4}$$

where

$$\mathscr{L}_{\mathbf{i}} f \equiv \nabla_{\mathbf{i}}^2 f - \beta_{\mathbf{i}} \nabla_{\mathbf{i}} f \tag{1.5}$$

with a coefficient $\beta_{\mathbf{i}}$, frequently called a diffusion coefficient, given by

$$\beta_{\mathbf{i}} \equiv (\mathbf{C}^{-1}\varphi)_{\mathbf{i}} + \nabla_{\mathbf{i}} U(\varphi_{\mathbf{i}}) \tag{1.6}$$

with $\mathbf{C}^{-1}$ being an inverse of some covariance matrix specified later. In the case when $\Lambda = \Gamma$ and $U \equiv 0$, it is known that the corresponding operator extends to a generator $\mathscr{L}_{\mathbf{C}}$ of a Markov semigroup $P_t^{\mathbf{C}} \equiv e^{t\mathscr{L}_{\mathbf{C}}}$, (called a Gaussian semigroup), and we have

$$P_t^{\mathbf{C}} f(\omega) = \int \mu_{\mathbf{C}}(d\varphi) f((1 - \tau_{2t}^{\mathbf{C}})^{\frac{1}{2}} \varphi + \tau_t^{\mathbf{C}} \omega) \tag{1.7}$$

with

$$\tau_t^{\mathbf{C}} \equiv e^{-t\mathbf{C}^{-1}} . \tag{1.8}$$

We will show also how to construct the semigroup for the infinite lattice system corresponding to a nontrivial local interaction. For this, let us first note that, for every $\Lambda \in \mathfrak{F}$ and $\omega \in \Omega$, we have well defined the following (essentially finite dimensional) semigroup

$$P_t^{\Lambda,\omega} f(\eta) \equiv e^{t\mathscr{L}_{\Lambda,\partial\Lambda}} f(\eta_\Lambda \bullet \omega_{\Lambda^{\complement}}) \tag{1.9}$$

with the generator $\mathscr{L}_{\Lambda,\partial\Lambda}$ defined by (1.4)–(1.6) with the covariance $\mathbf{C} = \mathbf{G}_{\partial\Lambda}$. Using this we define a tensor product semigroup on $\mathscr{C}(\Omega)$

$$P_t^{(k)} f(\eta) \equiv \lim_{\mathfrak{F}_0} P_t^{\Lambda_{n+1}\setminus\Lambda_n,\omega} \cdots P_t^{\Lambda_{k+1}\setminus\Lambda_k,\bullet} f(\eta) , \tag{1.10}$$

where $\lim_{\mathfrak{F}_0}$ means the limit (in the uniform norm) as $n \to \infty$ with $\Lambda_l \in \mathfrak{F}_0$, $l = k, \dots, n+1$. Clearly the right-hand side is independent of $\omega \in \Omega$ and as one can see its generator is given by (1.4)–(1.6) with the suitably chosen covariance function $\mathbf{C}$ (having the Dirichlet boundary conditions on $\bigcup_{n \geqq k} \partial\Lambda_n$). We will show that the limit $P_t f = \lim_{k\to\infty} P_t^{(k)} f$ exists for every local function f and in fact extends to a Markov semigroup for the infinite lattice system. We will need the following fact.

Lemma 1.1. *For $f \in \mathscr{C}^1(\Omega)$ and any stochastic dynamics $\bar{P}_t$ defined above, we have with $f_t \equiv \bar{P}_t f$ and any $\mathbf{k} \in \Gamma$ the following inequality*:

$$\|\nabla_{\mathbf{k}} f_t\|_u \leqq \|\nabla_{\mathbf{k}} f\|_u + \frac{\bar{C}}{(2R+1)^d} \sum_{d(\mathbf{j},\mathbf{k}) \leqq R} \int_0^t ds \|\nabla_{\mathbf{j}} f_s\|_u \tag{1.11}$$

with a constant $\bar{C} \in (0,\infty)$ independent of $\mathbf{k} \in \Lambda$ and the function f. If additionally $f \in \mathfrak{A}_\Lambda$, $\Lambda \in \mathfrak{F}$, this implies

$$\sum_{d(k,\Lambda) \geqq NR} \|\nabla_{\mathbf{k}} f_t\|_u \leqq \varepsilon(t,N) |||f||| \tag{1.12}$$

with

$$\varepsilon(t,N) \equiv e^{\bar{C}t} - \sum_{n=0}^{N} \frac{(\bar{C}t)^n}{n!} \leqq \left(\frac{\bar{C}te}{N}\right)^N e^{\bar{C}t} . \tag{1.13}$$

Proof. Let $f \in \mathscr{C}^1(\Omega)$ and let $f_t \equiv \bar{P}_t f \equiv e^{t\bar{\mathscr{L}}} f$. Then, using the fact that $\bar{\mathscr{L}}$ is a Markov generator, we have

$$\frac{d}{dt}(\nabla_{\mathbf{k}} f_t)^2 = 2\nabla_{\mathbf{k}} f_t \nabla_{\mathbf{k}} \bar{\mathscr{L}} f_t \leqq \bar{\mathscr{L}}(\nabla_{\mathbf{k}} f_t)^2 + 2\nabla_{\mathbf{k}} f_t [\nabla_{\mathbf{k}}, \bar{\mathscr{L}}] f_t . \tag{1.14}$$

Since $\bar{\mathscr{L}} \equiv \sum_{\mathbf{j}} \bar{\mathscr{L}}_{\mathbf{j}}$ and

$$[\nabla_{\mathbf{k}}, \bar{\mathscr{L}}_{\mathbf{j}}] f_t = -\mathbf{C}_{\mathbf{kj}}^{-1} \nabla_{\mathbf{j}} f_t - \delta_{\mathbf{kj}} \nabla_{\mathbf{k}}^2 U(\varphi_{\mathbf{k}}) \nabla_{\mathbf{k}} f_t$$
$$= -\mathbf{C}_{\mathbf{kj}}^{-1} \nabla_{\mathbf{j}} f_t - \delta_{\mathbf{kj}} \nabla_{\mathbf{k}}^2 W(\varphi_{\mathbf{k}}) \nabla_{\mathbf{k}} f_t - \delta_{\mathbf{kj}} \nabla_{\mathbf{k}}^2 V(\varphi_{\mathbf{k}}) \nabla_{\mathbf{k}} f_t\,, \tag{1.15}$$

using our assumptions that $\nabla_{\mathbf{k}}^2 V \geqq 0$ and that the covariance $\mathbf{C}$ is of the finite range, we get

$$\frac{d}{dt}(\nabla_{\mathbf{k}} f_t)^2 \leqq \bar{\mathscr{L}}(\nabla_{\mathbf{k}} f_t)^2 + 2 \sum_{d(\mathbf{j},\mathbf{k}) \leqq R} |\nabla_{\mathbf{k}} f_t \mathbf{C}_{\mathbf{kj}}^{-1} \nabla_{\mathbf{j}} f_t| + 2\|\nabla_{\mathbf{k}}^2 W(\varphi_{\mathbf{k}})\|_u (\nabla_{\mathbf{k}} f_t)^2\,. \tag{1.16}$$

This implies that

$$\frac{d}{ds}\bar{P}_{t-s}(\nabla_{\mathbf{k}} f_s)^2 \leqq 2 \sum_{d(\mathbf{j},\mathbf{k}) \leqq R} \bar{P}_{t-s}|\nabla_{\mathbf{k}} f_s \mathbf{C}_{\mathbf{kj}}^{-1} \nabla_{\mathbf{j}} f_s| + 2\|\nabla_{\mathbf{k}}^2 W(\varphi_{\mathbf{k}})\|_u \bar{P}_{t-s}(\nabla_{\mathbf{k}} f_s)^2\,. \tag{1.17}$$

Integrating the inequality (1.17) with respect to $s \in [0, t]$ and taking the supremum with respect to the configuration $\omega \in \Omega$ we obtain

$$\begin{aligned}
\|\nabla_{\mathbf{k}} f_t\|_u^2 &\leqq \|\bar{P}_t(\nabla_{\mathbf{k}} f)^2\|_u + 2 \sum_{d(\mathbf{j},\mathbf{k}) \leqq R} |\mathbf{C}_{\mathbf{kj}}^{-1}| \int_0^t ds \|\bar{P}_{t-s}|\nabla_{\mathbf{k}} f_s \nabla_{\mathbf{j}} f_s|\,\|_u \\
&\quad + 2\|\nabla_{\mathbf{k}}^2 W(\varphi_{\mathbf{k}})\|_u \int_0^t ds \|\bar{P}_{t-s}(\nabla_{\mathbf{k}} f_s)^2\|_u \\
&\leqq \|\nabla_{\mathbf{k}} f\|_u^2 + 2 \sum_{d(\mathbf{j},\mathbf{k}) \leqq R} |\mathbf{C}_{\mathbf{kj}}^{-1}| \int_0^t ds \|\nabla_{\mathbf{k}} f_s\|_u \|\nabla_{\mathbf{j}} f_s\|_u \\
&\quad + 2 \int_0^t ds \|\nabla_{\mathbf{k}}^2 W(\varphi_{\mathbf{k}})\|_u \|\nabla_{\mathbf{k}} f_s\|_u^2\,.
\end{aligned} \tag{1.18}$$

Hence we get

$$\|\nabla_{\mathbf{k}} f_t\| \leqq \|\nabla_{\mathbf{k}} f\| + \frac{\bar{C}}{(2R+1)^d} \sum_{d(\mathbf{j},\mathbf{k}) \leqq R} \int_0^t ds \|\nabla_{\mathbf{j}} f_s\| \tag{1.19}$$

with the constant

$$\bar{C} \leqq 2(\mathbf{C}_{\mathbf{kk}} + \|\nabla_{\mathbf{k}}^2 W\|_u)(2R+1)^d\,. \tag{1.20}$$

This ends the proof of the first part of the lemma. The inequality (1.12) follows from (1.11) by simple general arguments, which do not depend on the semigroup but only the inequality (1.11), (see e.g. [SZ2], 1.8 Lemma). □

Lemma 1.2. *There is a constant $B \in (0, \infty)$ such that*

$$\bar{P}_t|\varphi_{\mathbf{j}}|(\omega) \leqq B(1 + \omega_{\mathbf{j}}^2)^{\frac{1}{2}} e^{\alpha t} \tag{1.21}$$

with

$$\alpha \equiv \sup_{\mathbf{j}} \sum_{\mathbf{k}} |\mathbf{C}_{\mathbf{jk}}^{-1}|\,. \tag{1.22}$$

Proof. To prove (1.21) we first observe that

$$\bar{P}_t(1+\varphi_{\mathbf{j}}^2)^{\frac{1}{2}}(\omega) = (1+\omega_{\mathbf{j}}^2)^{\frac{1}{2}} + \int_0^t ds \bar{P}_s \bar{\mathscr{L}}(1+\varphi_{\mathbf{j}}^2)^{\frac{1}{2}}(\omega) . \tag{1.23}$$

Now using our assumption about the function U, it is not difficult to see that for any $\mathbf{j} \in \Gamma$ we have

$$\bar{P}_t(1+\varphi_{\mathbf{j}}^2)^{\frac{1}{2}} \leqq (1+\omega_{\mathbf{j}}^2)^{\frac{1}{2}} + D + \sum_{d(\mathbf{k},\mathbf{j}) \leqq R} |\mathbf{C}_{\mathbf{jk}}^{-1}| \int_0^t ds \bar{P}_s(1+\varphi_{\mathbf{k}}^2)^{\frac{1}{2}} \tag{1.24}$$

with the constant $D \leqq (1 + \|W'\|_u + |\inf_x(xV'(x)/(1+x^2)^{\frac{1}{2}})|)$. Iterating the inequality (1.24) we obtain

$$\bar{P}_t(1+\varphi_{\mathbf{j}}^2)^{\frac{1}{2}} \leqq B(1+\omega_{\mathbf{j}}^2)^{\frac{1}{2}} \exp(\alpha t) \tag{1.25}$$

with some constant $B \in (0,\infty)$ dependent only on the constant D and

$$\alpha = \sup_{\mathbf{j}} \sum_{d(\mathbf{k},\mathbf{j}) \leqq R} |\mathbf{C}_{\mathbf{jk}}^{-1}| . \tag{1.26}$$

The inequalities (1.25) and (1.26) clearly imply (1.21) and (1.22). This ends the proof of the lemma. □

Using the above lemmata we get the following useful estimate.

Lemma 1.3. *For any $\Lambda_n, \Lambda_{n+1} \in \mathfrak{F}_0$, and any local function $f \in \mathfrak{A}_{\Lambda_0}$, $\Lambda_0 \subset \Lambda_n$ we have*

$$|P_t^{\Lambda_{n+1}} f - P_t^{\Lambda_n} f|(\omega) \leqq \bar{B} \sum_{d(\mathbf{k},\partial\Lambda_n) \leqq R} (1+\omega_{\mathbf{k}}^2)^{\frac{1}{2}} e^{\alpha t} \varepsilon\left(t, \left[\frac{d(f,\partial\Lambda_n)}{R}\right]\right) |||f||| \tag{1.27}$$

with some constant $\bar{B} \in (0,\infty)$ independent of Λ_n, Λ_{n+1}, the function f, $\omega \in \Omega$ and $t \in \mathbb{R}^+$; $[\frac{d(f,\partial\Lambda_n)}{R}]$ denotes the corresponding biggest least integer.

Proof. To get (1.27) we note that for two Markov semigroups $P_{t,i} \equiv e^{t\mathscr{L}_i}$ $(i = 1,2)$, setting $f_{t,i} \equiv P_{t,i} f$, we have

$$\frac{d}{dt}(f_{t,1} - f_{t,2}) = \mathscr{L}_1(f_{t,1} - f_{t,2}) + (\mathscr{L}_1 - \mathscr{L}_2) f_{t,2} . \tag{1.28}$$

Hence

$$(f_{t,1} - f_{t,2}) = \int_0^t ds\, P_{(t-s),1}(\mathscr{L}_1 - \mathscr{L}_2) f_{s,2} . \tag{1.29}$$

Using this for the semigroups $P_t^{\Lambda_{n+1}}$ and $P_t^{\Lambda_n}$, respectively, we get

$$\begin{aligned} |P_t^{\Lambda_{n+1}} f - P_t^{\Lambda_n} f|(\omega) &= \left| \int_0^t ds P_{(t-s)}^{\Lambda_{n+1}} \sum_{\substack{\mathbf{j} \in \Lambda_{n+1} \\ \mathbf{k} \in \Lambda_n}} \mathbf{C}_{\mathbf{jk}}^{-1} \varphi_{\mathbf{k}} \nabla_{\mathbf{k}} P_s^{\Lambda_n} f(\omega) \right| \\ &\leqq \sum_{\substack{\mathbf{j} \in \Lambda_{n+1} \\ \mathbf{k} \in \Lambda_n}} |\mathbf{C}_{\mathbf{jk}}^{-1}| \int_0^t ds P_{(t-s)}^{\Lambda_{n+1}} (1+\varphi_{\mathbf{k}}^2)^{\frac{1}{2}} \|\nabla_{\mathbf{k}} P_s^{\Lambda_n} f\|_u . \end{aligned} \tag{1.30}$$

From (1.30) and Lemma 1.1 used together with Lemma 1.2, we obtain

$$|P_t^{\Lambda_{n+1}} f - P_t^{\Lambda_n} f|(\omega) \leqq (2R+1)^d \max_{\mathbf{jl}} |\mathbf{C}_{\mathbf{jl}}^{-1}| \frac{B}{\alpha} \sum_{d(\mathbf{k},\partial\Lambda_n)\leqq R} (1+\omega_{\mathbf{k}}^2)^{\frac{1}{2}} e^{\alpha t}$$

$$\times \varepsilon\left(t, \left[\frac{d(f,\partial\Lambda_n)}{R}\right]\right) |||f||| \,. \tag{1.31}$$

This implies (1.27) and ends the proof of Lemma 1.3. □

From the last lemma we get

Proposition 1.4. *For every* $f \in \mathfrak{A} \cap \mathscr{C}^1(\mathscr{S}')$ *the following limit exists*:

$$P_t f \equiv \lim_{\mathfrak{F}_0} P_t^\Lambda f \,, \tag{1.32}$$

and the family $\{P_t, t \in \mathbb{R}^+\}$ *extends to the Markov semigroup on* $\mathscr{C}(\mathscr{S}')$. *Moreover for any constant* $A \in (0,\infty)$ *we have the following* **property of finite speed of propagation of interaction**: *for every function* $f \in \mathfrak{A}_0 \cap \mathscr{C}^1(\mathscr{S}')$,

$$|P_t f - P_t^\Lambda f|(\omega) \leqq D\left(\sum_{d(\mathbf{k},\Lambda\complement)\leqq R} e^{-A d(0,\mathbf{k})}(1+\omega_{\mathbf{k}}^2)^{\frac{1}{2}}\right) |||f||| e^{-At} \tag{1.33}$$

with some constant D *dependent only on the smallest set* Λ_0 *such that* $f \in \mathfrak{A}_{\Lambda_0}$, *provided that*

$$d(f,\partial\Lambda) \geqq Ct \tag{1.34}$$

with some sufficiently large constant $C \in (0,\infty)$ *dependent only on the choice of* A.

Proof. Proposition 1.4 follows from Lemma 1.3 by choosing the constant C in (1.34) sufficiently large, so that

$$\left(\log\left(\frac{\bar{C}te}{[d(f,\partial\Lambda)/R]}\right) + \bar{C} + \alpha\right) \leqq -2A \,, \tag{1.35}$$

(where $\bar{C}$ is given in Lemma 1.1, see (1.13)). □

2. Some Properties of the Stochastic Dynamics

For $\Lambda \in \mathfrak{F}$, let μ_Λ denote a Gibbs measure on $(\mathscr{S}', \mathfrak{S})$ corresponding to the interaction U and the free measure $\mu_{\mathbf{G}}^{\partial\Lambda} \equiv \mu_{\mathbf{G}_{\partial\Lambda}}$ with Dirichlet boundary conditions on $\partial\Lambda$. Let $\mathscr{L}_\Lambda$ and $\mathscr{L}_0^{\partial\Lambda}$ be the Markov generators defined by (1.4)–(1.6) with the covariance $\mathbf{G}_{\partial\Lambda}$ and additionally with zero local interaction, respectively. Let P_t^Λ and $P_{t,0}^{\partial\Lambda}$ denote the corresponding semigroups. From now on we assume also that there is a constant $m_0 \in (0,\infty)$ such that for any $\Lambda \in \mathfrak{F}$ we have $\mathbf{G}_{\partial\Lambda}^{-1} \geqq m_0^2 I$ in the sense of quadratic forms. We have

Proposition 2.1. *For any* $f \in \mathscr{C}(\mathscr{S}') \cap \mathfrak{A}_\Lambda$, *we have*

$$P_t^\Lambda f(\eta) = \frac{1}{Z_\Lambda} e^{\frac{1}{2}U_\Lambda(\eta)} \hat{E}_\eta^{\partial\Lambda}\left(e^{\int_0^t ds \mathscr{V}_\Lambda(\varphi_s)} e^{-\frac{1}{2}U_\Lambda(\varphi_t)} f(\varphi_t)\right) \tag{2.1}$$

with $\hat{E}_\eta^{\partial\Lambda}$ *being the path-space measure associated to the semigroup* $P_{t,0}^{\partial\Lambda}$ *and the initial point* $\eta \in \mathscr{S}'$, *and* $Z_\Lambda \equiv \mu_{\mathbf{G}}^{\partial\Lambda} e^{-U_\Lambda}$, *and we have set*

$$\mathscr{V}_\Lambda(\varphi) \equiv -\frac{1}{2}\sum_{\mathbf{i}\in\Lambda}\left(\frac{1}{2}(\nabla_{\mathbf{i}}U)^2(\varphi_{\mathbf{i}}) - \nabla_{\mathbf{i}}^2 U(\varphi_{\mathbf{i}}) + (\mathbf{G}_{\partial\Lambda}^{-1}\varphi)_{\mathbf{i}}\nabla_{\mathbf{i}}U(\varphi_{\mathbf{i}})\right) . \tag{2.2}$$

Moreover there is a constant $T_0 \in (0,\infty)$ *such that for any* $T \geqq T_0$ *the semigroup is weakly-ultracontractive in the sense that for any* $f \in \mathscr{C}(\mathscr{S}') \cap \mathfrak{A}_\Lambda$ *we have*

$$|P_T^\Lambda f(\eta)| \leqq I_\Lambda(\eta)(\mu_\Lambda(f(\varphi))^2)^{\frac{1}{2}} \tag{2.3}$$

with

$$\begin{aligned} I_\Lambda(\eta) \equiv{}& \frac{1}{Z_\Lambda^{\frac{1}{2}}} e^{\frac{1}{2}U_\Lambda(\eta)}\left(\hat{E}_\eta^{\partial\Lambda} e^{p\int_0^T ds\mathscr{V}_\Lambda(\varphi_s)}\right)^{\frac{1}{p}} \\ &\times\left(\mu_{\mathbf{G}}^{\partial\Lambda}\left(\frac{d\mu_{\mathbf{G}}^{\partial\Lambda}((1-\tau_{2T})^{-\frac{1}{2}}(\varphi-\tau_T\eta))}{d\mu_{\mathbf{G}}^{\partial\Lambda}(\varphi)}\right)^r\right)^{\frac{1}{qr}} \end{aligned} \tag{2.4}$$

defined with arbitrary $q \in (1,2)$, $\frac{1}{q}+\frac{1}{p}=1$ *and* $\frac{1}{r}+\frac{q}{2}=1$, *and* $\tau_t \equiv e^{-t\mathbf{G}_{\partial\Lambda}^{-1}}$.

Proof. The representation of the semigroup given by (2.1)–(2.2) follows by standard arguments by use of the Feynman–Kac formula. To prove the second part of the proposition we first observe that by the Hölder inequality with positive p,q such that $\frac{1}{q}+\frac{1}{p}=1$ and $q\in(1,2)$, we get

$$|P_T^\Lambda f(\eta)| \leqq \frac{1}{Z_\Lambda} e^{\frac{1}{2}U_\Lambda(\eta)}\left(\hat{E}_\eta^{\partial\Lambda} e^{p\int_0^T ds\mathscr{V}_\Lambda(\varphi_s)}\right)^{\frac{1}{p}}(\hat{E}_\eta^{\partial\Lambda} e^{-\frac{q}{2}U_\Lambda(\varphi_T)}|f(\varphi_T)|^q)^{\frac{1}{q}} . \tag{2.5}$$

Since

$$\hat{E}_\eta^{\partial\Lambda} e^{-\frac{q}{2}U_\Lambda(\varphi_T)}|f(\varphi_T)|^q = \mu_{\mathbf{G}}^{\partial\Lambda}(e^{-\frac{q}{2}U_\Lambda((1-\tau_{2T})^{\frac{1}{2}}\varphi+\tau_T\eta)}|f((1-\tau_{2T})^{\frac{1}{2}}\varphi+\tau_T\eta)|^q) , \tag{2.6}$$

changing the integration variables

$$\varphi_\Lambda \to \varphi'_\Lambda \equiv (1-\tau_{2T})^{\frac{1}{2}}\varphi_\Lambda + \tau_T\eta_\Lambda , \tag{2.7}$$

we obtain

$$\hat{E}_\eta^{\partial\Lambda} e^{-\frac{q}{2}U_\Lambda(\varphi_T)}|f(\varphi_T)|^q = \int \mu_{\mathbf{G}}^{\partial\Lambda}(d\varphi')\frac{d\mu_{\mathbf{G}}^{\partial\Lambda}(\varphi)}{d\mu_{\mathbf{G}}^{\partial\Lambda}(\varphi')} \cdot e^{-\frac{q}{2}U_\Lambda(\varphi')}|f(\varphi')|^q \tag{2.8}$$

with

$$\begin{aligned} \frac{d\mu_{\mathbf{G}}^{\partial\Lambda}(\varphi)}{d\mu_{\mathbf{G}}^{\partial\Lambda}(\varphi')} ={}& \det{}_\Lambda\{(1-\tau_{2T})^{-\frac{1}{2}}\} \\ &\times\exp\left\{-\frac{1}{2}(\varphi-\tau_T\eta),\mathbf{G}_{\partial\Lambda}^{-1}(1-\tau_{2T})^{-1}(\varphi-\tau_T\eta)) - (\varphi,\mathbf{G}_{\partial\Lambda}^{-1}\varphi)\right\} . \end{aligned} \tag{2.9}$$

Hence using the Hölder inequality with $\frac{1}{r} + \frac{q}{2} = 1$ we get

$$\hat{E}_\eta^{\partial\Lambda} e^{-\frac{q}{2}U_\Lambda(\varphi_T)} |f(\varphi_T)|^q \leqq \left(\mu_{\mathbf{G}}^{\partial\Lambda} \left(\frac{d\mu_{\mathbf{G}}^{\partial\Lambda}((1-\tau_{2T})^{-\frac{1}{2}}(\varphi - \tau_T\eta))}{d\mu_{\mathbf{G}}^{\partial\Lambda}(\varphi)} \right)^r \right)^{\frac{1}{r}}$$

$$\times (\mu_{\mathbf{G}}^{\partial\Lambda} e^{-U_\Lambda(\varphi)} |f(\varphi)|^2)^{\frac{q}{2}} . \tag{2.10}$$

If $T \geqq T_0$ for some sufficiently large $T_0 \equiv T_0(r) \in (0,\infty)$, the first term on the right-hand side of (2.10) is finite. Combining (2.5)–(2.10) we obtain

$$|P_T^\Lambda f(\eta)| \leqq I_\Lambda(\eta)(\mu_\Lambda |f(\varphi)|^2)^{\frac{1}{2}} \tag{2.11}$$

with

$$I_\Lambda(\eta) \equiv Z_\Lambda^{-\frac{1}{2}} e^{\frac{1}{2}U_\Lambda(\eta)} \left(\hat{E}_\eta^{\partial\Lambda} e^{p\int_0^T ds \mathscr{V}_\Lambda(\varphi_s)} \right)^{\frac{1}{p}}$$

$$\times \left(\mu_{\mathbf{G}}^{\partial\Lambda} \left(\frac{d\mu_{\mathbf{G}}^{\partial\Lambda}((1-\tau_{2T})^{-\frac{1}{2}}(\varphi - \tau_T\eta))}{d\mu_{\mathbf{G}}^{\partial\Lambda}(\varphi)} \right)^r \right)^{\frac{1}{qr}} . \tag{2.12}$$

This ends the proof of Proposition 2.1. □

For later purposes let us note the following lemma describing the growth of the functional I_Λ.

Lemma 2.2. *Suppose the local interaction is polynomially bounded at infinity, i.e. there are constants $u_0 \in (0,\infty)$ and $K \in \mathbb{N}$ such that*

$$|U(x)| \leqq u_0(1+|x|)^K . \tag{2.13}$$

Then for any $\eta \in \mathscr{S}'$ there are constants $N \equiv N(\eta) \in \mathbb{N}$ and $u \equiv u(\eta) \in (0,\infty)$ such that for every $\Lambda \in \mathfrak{F}_0$ we have

$$I_\Lambda(\eta) \leqq \exp\{u|\Lambda|^N\} . \tag{2.14}$$

Proof. Using our assumption about the local interaction U and Jensen's inequality we get

$$Z_\Lambda^{-\frac{1}{2}} \leqq e^{u_1|\Lambda|} \tag{2.15}$$

with some constant $u_1 \in (0,\infty)$. Using the same assumption about U together with the fact that we consider the finite range strictly positive operator $\mathbf{G}_{\partial\Lambda}^{-1}$, from the formula (2.2) we have

$$\mathscr{V}_\Lambda \leqq v_1|\Lambda| \tag{2.16}$$

with some other constant $v_1 \in (0,\infty)$, whence one can easily show that

$$\frac{1}{Z_\Lambda^{\frac{1}{2}}} \left(\hat{E}_\eta^{\partial\Lambda} e^{p\int_0^T ds \mathscr{V}_\Lambda(\varphi_s)} \right)^{\frac{1}{p}} \leqq \exp(C_1|\Lambda|) , \tag{2.18}$$

with the constant $C_1 = u_1 + Tv_1$. Since for any $\eta \in \mathscr{S}'$ there is an $a \equiv a(\eta) \in (0,\infty)$ and an $n \equiv n(\eta) \in \mathbb{N}$ such that

$$|\eta_{\mathbf{i}}| \leqq a(1 + d(0,\mathbf{i}))^n , \tag{2.19}$$

using our growth restriction on the interaction U, we have

$$|U_\Lambda(\eta)| \leqq \sum_{\mathbf{i}\in\Lambda} u_0(1+|\eta_{\mathbf{i}}|)^K \leqq b_1(\eta)|\Lambda|^N \tag{2.20}$$

with some constant $b_1(\eta) \in (0,\infty)$ and $N = 1 + \frac{nK}{d}$. This gives us the bound on the second factor from the right-hand side of (2.12). Finally to bound the last factor from the right-hand side of (2.12), we note that choosing $T \geqq T_0$ with T_0 satisfying

$$r \leqq \varepsilon(e^{2m^2T_0} - 1) \tag{2.21}$$

with some $\varepsilon \in (0,1)$, one can easily see that the corresponding Gaussian integral in this factor exists and we have

$$\left(\mu_{\mathbf{G}}^{\partial\Lambda}\left(\frac{d\mu_{\mathbf{G}}^{\partial\Lambda}((1-\tau_{2T})^{-\frac{1}{2}}(\varphi-\tau_T\eta))}{d\mu_{\mathbf{G}}^{\partial\Lambda}(\varphi)}\right)^r\right)^{\frac{1}{qr}} \leqq \exp\left\{C_2|\Lambda| + C_3\sum_{\mathbf{i}\in\Lambda}\eta_{\mathbf{i}}^2\right\} \tag{2.22}$$

with some constants $C_2, C_3 \in (0,\infty)$ independent of $\Lambda \in \mathfrak{F}_0$ and $\eta \in \mathcal{S}'$. Because of (2.19), we can bound it as follows:

$$\left(\mu_{\mathbf{G}}^{\partial\Lambda}\left(\frac{d\mu_{\mathbf{G}}^{\partial\Lambda}((1-\tau_{2T})^{-\frac{1}{2}}(\varphi-\tau_T\eta))}{d\mu_{\mathbf{G}}^{\partial\Lambda}(\varphi)}\right)^r\right)^{\frac{1}{qr}} \leqq \exp\{b_2(\eta)|\Lambda|^{(1+\frac{2n}{d})}\} \tag{2.23}$$

with some constant $b_2(\eta) \in (0,\infty)$ dependent only on η. Combining (2.15)–(2.23) we arrive at the bound (2.14) with a constant $u \equiv C_1 + b_1(\eta) + b_2(\eta)$ and $N \equiv 1 + \frac{n}{d}\max(K,2)$. This ends the proof of Lemma 2.2. □

Remark. Let us remark that, as easily follows from the definition of I_Λ in (2.4) and the considerations in the above proof, there is $q_0 \in (0,\infty)$ such that for any $q \geqq q_0$ we have

$$\mu_{\Lambda'} I_\Lambda^{\frac{1}{q}} \leqq e^{D|\Lambda|}$$

with some constant $D \in (0,\infty)$ for any $\Lambda, \Lambda' \in \mathfrak{F}$, $\Lambda \subset \Lambda'$.

Proposition 2.1 together with Lemma 2.2 give us a very simple and useful devise, which has a chance to be true not only on the lattice but also in the continuum. The estimate on Lemma 2.2 contains unfortunately a drawback, namely the restriction on the growth of the local interaction. One can overcome it by restricting the space of configurations (which is reasonable since for fast growing local interactions the infinite volume measure lives on the much smaller space of very slowly growing sequences). However in the case of lattice spin systems one can use also the following stronger tool, which on the other hand requires the growth of the local interaction to be sufficiently fast.

Proposition 2.3. *Suppose the local interaction satisfies the following growth condition:*

$$U(x) \geqq |x|^{2+\delta} \tag{2.24}$$

with some $\delta > 0$, for all sufficiently large $x \in \mathbb{R}$. Then there is $T \in (0,\infty)$ such that

$$\|P_T^\Lambda f\|_u \leqq e^{M_0|\Lambda|}\mu_\Lambda|f| \tag{2.25}$$

with a constant $M_0 \in (0,\infty)$ independent of $f \in \mathfrak{A}_\Lambda \cap L_2(\mu_\Lambda)$.

Proof. Proposition 2.3 is a simple consequence of the general beautiful ultracontractivity estimates of [DaSi] plus some careful arguments tracing the volume dependence of interesting constants. We start by proving the following lemma:

Lemma 2.4. *Suppose for any* $p \in (2, \infty)$ *we have*

$$\mu_\Lambda f^p \log f \leqq c_\Lambda(p)\mu_\Lambda(f^{(p-1)}(-\mathscr{L}_{\Lambda,\partial\Lambda} f)) + \Gamma_\Lambda(p)\mu_\Lambda f^p + \mu_\Lambda f^p \log(\mu_\Lambda f^p) \tag{2.26}$$

for any positive function $f \in \mathfrak{A}_\Lambda$ *for which the expectations on the right-hand side are finite, with the constants* $c_\Lambda(p)$ *and* $\Gamma_\Lambda(p)$ *such that*

$$T \equiv \int_2^\infty c_\Lambda(p)\frac{dp}{p} < \infty \tag{2.27}$$

and

$$M_\Lambda \equiv \int_2^\infty \Gamma_\Lambda(p)\frac{dp}{p} < \infty\,. \tag{2.28}$$

Then we have the following ultracontractivity estimate:

$$\|P_T^\Lambda f\|_u \leqq e^{M_\Lambda}(\mu_\Lambda f^2)^{\frac{1}{2}}\,. \tag{2.29}$$

In particular if the local interaction U satisfies the growth condition (2.24), *we have*

$$c_\Lambda(p) \equiv c(p) \equiv (\log p)^{-2} \tag{2.30}$$

and

$$\Gamma_\Lambda(p) \equiv C\frac{(\log p)^{2\beta}}{p}|\Lambda|\,, \tag{2.31}$$

with some $\beta \varepsilon (0, \infty)$, *and thus there is a constant* $M_0 \in (0, \infty)$ *such that*

$$M_\Lambda \leqq \frac{1}{2}M_0|\Lambda|\,. \tag{2.32}$$

Using (2.29) and (2.32) together with the duality arguments, we get the desired ultracontractivity estimate (2.25). This ends the proof of Proposition 2.3. □

Remark. Let us also mention that the growth condition (2.24) is not the optimal one. Actually for the ultracontractivity to be true it suffices that the interaction grows at infinity slightly faster than $x^2(\log x)^2$, see [DaSi].

Proof of Lemma 2.4. The implication of (2.29) from (2.26) assuming (2.27)–(2.28) is a general result which one can find in [G] and [DaSi]. To verify the estimate (2.32) of interest to us, we use the following arguments. Let $\rho_{\mathbf{i}}$ be a probability measure on $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$ defined as follows:

$$\rho_{\mathbf{i}}(dx) \equiv \frac{1}{Z}e^{-\frac{1}{2}(\mathbf{G}_{\partial\Lambda}^{-1})_{\mathbf{ii}}x^2 - U(x)}dx\,. \tag{2.33}$$

Then using a result of [DaSi] we have for any $\varepsilon \in (0, \infty)$ the following inequality:

$$\rho_{\mathbf{i}} f^2 \log f \leqq \varepsilon\rho_{\mathbf{i}}|\nabla_{\mathbf{i}} f|^2 + \gamma_0(\varepsilon)\rho_{\mathbf{i}} f^2 + \rho_{\mathbf{i}} f^2 \log(\rho_{\mathbf{i}} f^2)^{\frac{1}{2}} \tag{2.34}$$

true with some function $\gamma_0(\varepsilon)$ diverging at zero at most as $\varepsilon^{-\beta}$, with $\beta \equiv 1 + \frac{2}{\delta}$, for any positive function f for which the right-hand side is finite. By simple inductive arguments one can see ([Z4]) that the inequality (2.34) implies the following inequality involving the product measure $\rho_\Lambda \equiv \bigotimes_{\mathbf{i}\in\Lambda} \rho_{\mathbf{i}}$:

$$\rho_\Lambda f^2 \log f \leqq \varepsilon \rho_\Lambda |\nabla_\Lambda f|^2 + |\Lambda| \gamma_0(\varepsilon) \rho_\Lambda f^2 + \rho_\Lambda f^2 \log(\rho_\Lambda f^2)^{\frac{1}{2}} . \tag{2.35}$$

Now setting e^{-A_Λ} to be the density of the measure μ_Λ with respect to the measure ρ_Λ, we have

$$\mu_\Lambda f^2 \log f = \rho_\Lambda (e^{-\frac{1}{2}A_\Lambda} f)^2 \log(e^{-\frac{1}{2}A_\Lambda} f) + \rho_\Lambda \left((e^{-\frac{1}{2}A_\Lambda} f)^2 \frac{1}{2} A_\Lambda \right) . \tag{2.36}$$

Hence using the inequality (2.35) we get

$$\begin{aligned}\mu_\Lambda f^2 \log f \leqq\ & \varepsilon \rho_\Lambda |\nabla_\Lambda (e^{-\frac{1}{2}A_\Lambda} f)|^2 + \rho_\Lambda \left((e^{-\frac{1}{2}A_\Lambda} f)^2 \frac{1}{2} A_\Lambda \right) \\ & + |\Lambda| \gamma_0(\varepsilon) \mu_\Lambda f^2 + \mu_\Lambda f^2 \log(\mu_\Lambda f^2)^{\frac{1}{2}} ,\end{aligned} \tag{2.37}$$

which implies

$$\begin{aligned}\mu_\Lambda f^2 \log f \leqq\ & 2\varepsilon \mu_\Lambda |\nabla_\Lambda f|^2 + \mu_\Lambda \left(f^2 \left(\frac{1}{2} A_\Lambda + 2\varepsilon |\nabla_\Lambda A_\Lambda|^2 \right) \right) \\ & + |\Lambda| \gamma_0(\varepsilon) \mu_\Lambda f^2 + \mu_\Lambda f^2 \log(\mu_\Lambda f^2)^{\frac{1}{2}} .\end{aligned} \tag{2.38}$$

Since A_Λ is a quadratic form in $\varphi_{\mathbf{k}}$, $\mathbf{k} \in \Lambda$, using our assumption about the interaction U, it is easy to see that we have the following inequality for any $\varepsilon \in (0, \infty)$:

$$\frac{1}{2} A_\Lambda + 2\varepsilon |\nabla_\Lambda A_\Lambda|^2 \leqq \varepsilon (|\nabla_\Lambda \log \Psi_\Lambda|^2 - \Delta_\Lambda \log \Psi_\Lambda) + a_0 |\Lambda| (1 + \varepsilon^{-\beta}) \tag{2.39}$$

with some constant $a_0 \in (0, \infty)$, $\beta \equiv 1 + \frac{2}{\delta}$, and Ψ_Λ denoting the density of the measure μ_Λ with respect to Lebesgue measure. This inequality clearly implies that we have

$$\mu_\Lambda f^2 \log f \leqq \varepsilon \mu_\Lambda |\nabla_\Lambda f|^2 + |\Lambda| \gamma_1(\varepsilon) \mu_\Lambda f^2 + \mu_\Lambda f^2 \log(\mu_\Lambda f^2)^{\frac{1}{2}} \tag{2.40}$$

with a function $\gamma_1(\varepsilon)$ diverging at zero not faster than $\varepsilon^{-\beta}$. For $p \geqq 2$, substituting $f^{\frac{p}{2}}$ instead of f into (2.40), after simple transformations we arrive at the following inequality:

$$\begin{aligned}\mu_\Lambda f^p \log f \leqq\ & \frac{p}{2(p-1)} \varepsilon \mu_\Lambda (f^{p-1}(-\mathscr{L} f)) \\ & + |\Lambda| \frac{2}{p} \gamma_1(\varepsilon) \mu_\Lambda f^p + \mu_\Lambda f^p \log(\mu_\Lambda f^p)^{\frac{1}{p}} .\end{aligned} \tag{2.41}$$

Hence, choosing $\varepsilon = \frac{2(p-1)}{p} (\log p)^{-2}$ we arrive at the inequality

$$\mu_\Lambda f^p \log f \leqq c(p) \mu_\Lambda (f^{p-1}(-\mathscr{L} f)) + \Gamma_\Lambda(p) \mu_\Lambda f^p + \mu_\Lambda f^p \log(\mu_\Lambda f^p)^{\frac{1}{p}} \tag{2.42}$$

with

$$c(p) \equiv (\log p)^{-2} \tag{2.43}$$

and

$$\Gamma_\Lambda(p) \equiv C\frac{(\log p)^{2\beta}}{p}|\Lambda| \tag{2.44}$$

with some constant $C \in (0,\infty)$ for all $p \in (2,\infty)$. From this the rest of Lemma 2.4 easily follows. □

3. Ergodicity of the Stochastic Dynamics of Infinite Spin Systems

Let μ, respectively μ_Λ, $\Lambda \in \mathfrak{F}_0$, be a probability measure, respectively a sequence of probability measures, on $(\mathscr{S}', \mathfrak{S})$. Let P_t and P_t^Λ, $\Lambda \in \mathfrak{F}_0$, be the Markov semigroup preserving μ and μ_Λ, $\Lambda \in \mathfrak{F}_0$, respectively, and satisfying the following condition

C.1 (Exponential approximation property). *For any $A \in (0,\infty)$ we have*

$$|P_t f(\eta) - P_t^\Lambda f(\eta)| \leqq D\left(\sum_{d(\mathbf{k},\Lambda\complement)\leqq R} e^{-Ad(0,\mathbf{k})}(1+\eta_\mathbf{k}^2)\right)|||f|||e^{-At} \tag{3.1}$$

with a constant $D \in (0,\infty)$ dependent only on the smallest set Λ_0 such that $f \in \mathfrak{A}_{\Lambda_0}$, provided that

$$d(f,\partial\Lambda) \geqq Ct \tag{3.2}$$

with some sufficiently large constant $C \in (0,\infty)$ dependent only on A.

We assume also the following

C.2 (Finite volume weak-ultracontractivity). *There is a positive function $I_\Lambda(\eta)$ satisfying*

$$\lim_{\mathfrak{F}_0} \exp(-\varepsilon|\Lambda|^{\frac{1}{d}})\log I_\Lambda(\eta) = 0 \tag{3.3}$$

for any $\varepsilon \in (0,\infty)$, and such that for every function $F \in \mathfrak{A}_\Lambda \cap L_2(\mu_\Lambda)$ we have

$$|P_T^\Lambda F|(\eta) \leqq I_\Lambda(\eta)(\mu_\Lambda F^2)^{\frac{1}{2}} \tag{3.4}$$

for some $T \in (0,\infty)$ independent of $\Lambda \in \mathfrak{F}$, $\eta \in \mathscr{S}'$ and the function F.

In this section we prove first the following result

Theorem 3.1. *Suppose that the conditions* C.1 *and* C.2 *are satisfied and that*

– there is a constant $B \in (0,\infty)$ such that for any $\Lambda \in \mathfrak{F}_0$ and $\mathbf{k} \in \Lambda$, we have

$$|\mu_\Lambda \varphi_\mathbf{k}| \leqq B \tag{3.5}$$

and

– there is a constant $c \in (0,\infty)$ such that for any $\Lambda \in \mathfrak{F}_0$ we have the following logarithmic Sobolev inequality:

$$\mu_\Lambda f^2 \log f \leqq c\mu_\Lambda|\nabla_\Lambda f|^2 + \mu_\Lambda f^2 \log(\mu_\Lambda f^2)^{\frac{1}{2}} \tag{3.6}$$

for every function f for which the right-hand side is finite. Then the limit

$$\mu \equiv \lim_{\mathfrak{F}_0} \mu_\Lambda \tag{3.7}$$

exists. If additionally the condition (3.3) *is satisfied with* $I_\Lambda(\eta)$ *replaced by its integral with the measure* μ*, then for any local function* $f \in \mathfrak{A}_0 \cap \mathscr{C}^1$ *we have*

$$|P_t f(\eta) - \mu f| \leqq C_m(\eta) \left\{ \sup_\Lambda (\mu_\Lambda (f - \mu_\Lambda f)^2)^{\frac{1}{2}} + |||f||| \right\} e^{-mt} \tag{3.8}$$

for any $m \in (0, \inf_\Lambda gap_2(\mathscr{L}_{\Lambda,\partial\Lambda}))$*, with* $gap_2(\mathscr{L}_{\Lambda,\partial\Lambda})$ *being the spectral gap of the selfadjoint operator* $-\mathscr{L}_{\Lambda,\partial\Lambda}$ *in* $L_2(\mu_\Lambda)$*, and with a constant* $C_m(\eta) \in (0, \infty)$ *dependent only on* $\eta \in \mathscr{S}'$ *and m.*

Remarks. As we have observed in Sect. 2, in our situation the condition (3.3) is satisfied with $I_\Lambda(\eta)$ replaced by its integral with the measures $\mu_{\Lambda'}$, uniformly in $\Lambda' \in \mathfrak{F}$, and thus also with the measure μ.

Let us note also that (3.6) implies $gap_2(\mathscr{L}_{\Lambda,\partial\Lambda}) \geqq \frac{1}{c}$, see [Rot, Si1].

Proof. First of all let us mention that, as easily and directly follows from our assumption (3.6), for any $\mathbf{k} \in \Gamma$ and $a \in \mathbb{R}$, we have

$$\mu_\Lambda e^{a\varphi_\mathbf{k}} \leqq e^{\frac{1}{2}a^2 + Ba} . \tag{3.9}$$

Thus by standard arguments the sequence $\{\mu_\Lambda\}_{\Lambda \in \mathfrak{F}_0}$ is compact in the weak topology in the space of probability measures on $(\mathscr{S}', \mathfrak{S})$. Let μ denote its accumulation point. Then clearly μ satisfies (3.5) and (3.6), and thus also (3.9), with the same constants.

Let us consider now a function $f \in \mathfrak{A}_0 \cap \mathscr{C}^1$. Then for any $\Lambda \in \mathfrak{F}_0$, we have

$$|P_t f(\eta) - \mu f| \leqq |P_t f(\eta) - P_t^\Lambda f(\eta)| + |P_t^\Lambda f(\eta) - \mu_\Lambda f| + |\mu_\Lambda f - \mu f| . \tag{3.10}$$

The first term on the right-hand side of (3.10) can be estimated with the use of the exponential approximation property C.1, i.e. assuming we have (3.2) satisfied, this term is bounded as in (3.1). For the second term on the right-hand side of (3.10) we have with any $q \in (1, \infty)$,

$$|P_t^\Lambda(f(\eta) - \mu_\Lambda f)| = (|P_t^\Lambda(f(\eta) - \mu_\Lambda f)|^q)^{\frac{1}{q}} \leqq (P_T^\Lambda |P_{t-T}^\Lambda (f - \mu_\Lambda f)|^q(\eta))^{\frac{1}{q}} , \tag{3.11}$$

where we have used the Hölder inequality for P_T^Λ. Now applying the weak-ultracontractivity property C.2, we get

$$(P_T^\Lambda |P_{t-T}^\Lambda (f - \mu_\Lambda f)|^q(\eta)|)^{\frac{1}{q}} \leqq I_\Lambda(\eta)^{\frac{1}{q}} (\mu_\Lambda |P_{t-T}^\Lambda (f - \mu_\Lambda f)|^{2q})^{\frac{1}{2q}} . \tag{3.12}$$

Since by our assumption the measures μ_Λ satisfy logarithmic Sobolev inequalities with the same coefficient $c \in (0, \infty)$, with some $\delta \in (0, 1)$, we have the following hypercontractivity estimate:

$$(\mu_\Lambda |P_{t-T}^\Lambda (f - \mu_\Lambda f)|^{2q})^{\frac{1}{2q}} \leqq (\mu_\Lambda |P_{(1-\delta)t}^\Lambda (f - \mu_\Lambda f)|^2)^{\frac{1}{2}} \tag{3.13}$$

assuming

$$2q = 1 + e^{\frac{2}{c}(\delta t - T)} . \tag{3.14}$$

Using the fact that the logarithmic Sobolev inequality (3.6) implies that the self-adjoint operator $-\mathscr{L}_{\Lambda,\partial\Lambda}$ in $L_2(\mu_\Lambda)$ has a spectral gap $gap_2(\mathscr{L}_{\Lambda,\partial\Lambda}) \geqq c^{-1}$ at the bottom of its spectrum (see [Rot, Si2]), we get

$$(\mu_\Lambda|P^\Lambda_{t-T}(f-\mu_\Lambda f)|^{2q})^{\frac{1}{2q}} \leqq e^{-(1-\delta)gap_2(\mathscr{L}_{\Lambda,\partial\Lambda})^t}(\mu_\Lambda(f-\mu_\Lambda f)^2)^{\frac{1}{2}}\,. \tag{3.15}$$

Combining (3.11)–(3.15) we obtain the following bound on the middle term on the right-hand side of (3.10):

$$|P^\Lambda_t(f(\eta)-\mu_\Lambda f)| \leqq I_\Lambda(\eta)^{\frac{1}{q}}(\mu_\Lambda(f-\mu_\Lambda f)^2)^{\frac{1}{2}}e^{-(1-\delta)gap_2(\mathscr{L}_{\Lambda,\partial\Lambda})^t}\,. \tag{3.16}$$

To estimate the last term on the right-hand side of (3.10) let us note that

$$|\mu_\Lambda f-\mu f| \leqq \mu|P^\Lambda_t f(\omega)-\mu_\Lambda f| + \mu|P_t f(\omega)-P^\Lambda_t f(\omega)| \tag{3.17}$$

with $\omega \in \mathscr{S}'$ denoting the integration variable with respect to the measure μ. Applying the same arguments as in (3.11)–(3.16) we get

$$\mu|P^\Lambda_t f-\mu_\Lambda f| \leqq \mu(I_\Lambda^{\frac{1}{q}})(\mu_\Lambda(f-\mu_\Lambda f)^2)^{\frac{1}{2}}e^{-(1-\delta)gap_2(\mathscr{L}_{\Lambda,\partial\Lambda})^t}\,. \tag{3.18}$$

(Let us remark that frequently one has an independent estimate on the last term on the right-hand side of (3.10) from the construction of the measure μ. Then one can use this for the estimate of quantity which interests us.) The second term from the right-hand side of (3.17) can be easily estimated with the use of the exponential approximation property and (3.9) for the measure μ. Now the final ergodicity estimate (3.8) follows from our considerations by choosing the sequence of sets $\Lambda \equiv \Lambda(t) \to \mathbb{Z}^d$ so that the condition (3.2) is satisfied. □

As for the ergodicity properties of the infinite volume semigroup, it is not very natural to assume something about the ergodicity of finite volume semigroups, therefore we would like also to present the following result. (For a corresponding result for the case of compact single spin space M and the discussion of its relevance for applications in statistical mechanics see [SZ4].)

Theorem 3.2. *Suppose that the conditions* C.1 *and* C.2 *are satisfied and that*

– *there is* $r \in (1,\infty)$ *such that*

$$\lim_{\mathfrak{F}_0} \exp(-\varepsilon|\Lambda|^{\frac{1}{d}})\log\left\|\frac{d\mu_\Lambda}{d\mu_{|\mathfrak{S}_\Lambda}}\right\|_{L_r(\mu)} = 0 \tag{3.19}$$

for every $\varepsilon \in (0,\infty)$, *and*

– *the measure* μ *satisfies the logarithmic Sobolev inequality with a coefficient* $c \in (0,\infty)$, *i.e. we have*

$$\mu f^2 \log f \leqq c\mu|\nabla f|^2 + \mu f^2 \log(\mu f^2)^{\frac{1}{2}} \tag{3.20}$$

for any function f *for which the right-hand side is finite. Then for any local function* $f \in \mathfrak{A}_{\Lambda_0} \cap \mathscr{C}^1$, $\Lambda_0 \in \mathfrak{F}$, *we have*

$$|P_t f(\eta)-\mu f| \leqq C_m(\eta)\{(\mu(f-\mu f)^2)^{\frac{1}{2}} + |||f|||\}e^{-mt} \tag{3.21}$$

for any $m \in (0, gap_2(\mathscr{L}))$, *with a constant* $C_m(\eta) \in (0,\infty)$ *dependent only on* $\eta \in \mathscr{S}'$, *the set* $\Lambda_0 \in \mathfrak{F}$ *and the choice of* m.

Proof. Let $f \in \mathfrak{A}_{\Lambda_0} \cap \mathscr{C}^1$, $\Lambda_0 \in \mathfrak{F}$. Then for any $\Lambda \in \mathfrak{F}$, $\Lambda_0 \subset \Lambda$, we have

$$|P_t f(\eta) - \mu f| \leqq |P_t^\Lambda f(\eta) - \mu f| + |P_t f(\eta) - P_t^\Lambda f(\eta)| \,. \tag{3.22}$$

The second term on the right-hand side of (3.22) can be estimated with the use of the exponential approximation property C.1. Assuming we have (3.2) satisfied, the second term on the right-hand side of (3.22) is bounded as in (3.1).

For the first term on the right-hand side of (3.22) we have with any $q \in (1, \infty)$,

$$|P_t^\Lambda(f(\eta) - \mu f)| = (|P_t^\Lambda(f(\eta) - \mu f)|^q)^{\frac{1}{q}} \leqq (P_T^\Lambda |P_{t-T}^\Lambda(f - \mu f)(\eta)|^q)^{\frac{1}{q}} \,, \tag{3.23}$$

where we have used the Hölder inequality for P_T^Λ. Now applying the weak-ultracontractivity property C.2, with the same $r \in (1, \infty)$ as in our first assumption (3.19), we obtain

$$\begin{aligned} (P_T^\Lambda |P_t^\Lambda(f - \mu f)(\eta)|^q)^{\frac{1}{q}} &\leqq I_\Lambda(\eta)^{(\frac{1}{q})}(\mu_\Lambda |P_{t-T}^\Lambda(f - \mu f)|^{2q})^{\frac{1}{2q}} \\ &\leqq I_\Lambda(\eta)^{\frac{1}{q}} \left\| \frac{d\mu_\Lambda}{d\mu_{|\mathfrak{S}_\Lambda}} \right\|_{L_r(\mu)}^{\frac{1}{q}} (\mu |P_{t-T}^\Lambda(f - \mu f)|^{2sq})^{\frac{1}{2sq}} \,, \end{aligned} \tag{3.24}$$

where $s^{-1} + r^{-1} = 1$. Hence we get

$$\begin{aligned} (P_T^\Lambda |P_{t-T}^\Lambda(f - \mu f)(\eta)|^q)^{\frac{1}{q}} &\leqq I_\Lambda(\eta)^{\frac{1}{q}} \left\| \frac{d\mu_\Lambda}{d\mu_{|\mathfrak{S}_\Lambda}} \right\|_{L_r(\mu)}^{\frac{1}{q}} (\mu |P_{t-T}(f - \mu f)|^{2sq})^{\frac{1}{2sq}} \\ &+ I_\Lambda(\eta)^{\frac{1}{q}} \left\| \frac{d\mu_\Lambda}{d\mu_{|\mathfrak{S}_\Lambda}} \right\|_{L_r(\mu)}^{\frac{1}{q}} (\mu |P_{t-T}^\Lambda f - P_{t-T} f)|^{2sq})^{\frac{1}{2sq}} \,. \end{aligned} \tag{3.25}$$

If the measure μ satisfies the logarithmic Sobolev inequality, then we have, with any $\delta \in (0, 1)$ and

$$2sq = 1 + e^{\frac{2}{c}(\delta t - T)} \,, \tag{3.26}$$

the following hypercontractivity estimate:

$$\begin{aligned} (\mu |P_{t-T}(f - \mu f)|^{2sq})^{\frac{1}{2sq}} &\leqq (\mu |P_{(1-\delta)t}^\Lambda(f - \mu f)|^2)^{\frac{1}{2}} \\ &\leqq (\mu(f - \mu f)^2)^{\frac{1}{2}} e^{-gap_2(\mathscr{L})(1-\delta)t} \,, \end{aligned} \tag{3.27}$$

where in the last step we have used the fact that the logarithmic Sobolev inequality implies also the spectral gap $gap_2(\mathscr{L}) \geqq c^{-1}$ at the bottom of the spectrum of the operator $(-\mathscr{L})$ in $L_2(\mu)$. To estimate the second term on the right-hand side of (3.25) we use the approximation property to get

$$(\mu |P_{t-T}^\Lambda f - P_{t-T} f)|^{2sq})^{\frac{1}{2sq}} \leqq C_1 \sum_{\mathbf{k} \in \Lambda \complement} e^{-Ad(0,\mathbf{k})}(1 + (\mu |\omega_{\mathbf{k}}|^{2sq})^{\frac{1}{2sq}}) |||f||| e^{-At} \tag{3.28}$$

with some positive constant C_1 independent of Λ, t and f. Since μ satisfies the logarithmic Sobolev inequality, it has also satisfy the exponential bound (3.9). Using this we have

$$(\mu|\omega_{\mathbf{k}}|^{2sq})^{\frac{1}{2sq}} \leqq C_2(sq)^{\frac{1}{2}} \tag{3.29}$$

with some positive constant C_2 independent of $\mathbf{k} \in \Gamma$ and sq. From (3.28) and (3.29) we see that if the conditions (3.2) and (3.26) are satisfied, we obtain

$$(\mu|P^{\Lambda}_{t-T}f - P_{t-T}f)|^{2sq})^{\frac{1}{2sq}} \leqq C_3 \sum_{\mathbf{k}\in\Lambda\complement} e^{-Ad(0,\mathbf{k})}(sq)^{\frac{1}{2}}|||f|||e^{-At} \leqq C_4|||f|||e^{-At} \,. \tag{3.30}$$

Choosing a sequence of $\Lambda \equiv \Lambda(t)$ so that $d(f,\partial\Lambda) = C[t]$ with a large positive constant C, we see also that the factors $I_{\Lambda}(\eta)^{\frac{1}{q}}$ and $\|\frac{d\mu_{\Lambda}}{d\mu_{|\mathfrak{S}_{\Lambda}}}\|^{\frac{1}{q}}_{L_l(\mu)}$ converge to one. Thus combining all our considerations we arrive at the estimate (3.21). This ends the proof of the theorem. □

4. Logarithmic Sobolev Inequality for Gibbs Measures: An Example on $\Gamma = \mathbb{Z}$

In this section we give a first nontrivial example of a spin system for which the logarithmic Sobolev inequality is true for the corresponding Gibbs measure in infinite volume as well as for finite volume Gibbs measures uniformly in volume. Here we take advantage of one dimensionality of the lattice to have almost "for free" the exponential decay of correlations in the system. (A higher dimensional situation in discussed in the next section.) Let $\mathscr{E} = \{E^{\omega}_{\Lambda}\}_{\Lambda\in\mathfrak{F},\omega\in\mathscr{S}'}$ be a local specification defined by (1.3), corresponding to a free Gaussian measure $\mu_{\mathbf{G}}$, with a strictly positive inverse covariance $\mathbf{G}^{-1}$ of a finite range $R > 0$, and a local interaction given by a real function $U \equiv V + W$ defined with a nonnegative convex function V and a bounded function W having the first and the second derivative bounded. We show the following result

Theorem 4.1. *Let $\Gamma = \mathbb{Z}$. There is a constant $c \in (0,\infty)$ such that for any $\Lambda \subseteq \mathbb{Z}^d$ we have*

$$\mu_{\Lambda}f\log f \leqq 2c\mu_{\Lambda}|\nabla_{\Lambda}f^{\frac{1}{2}}|^2 + \mu_{\Lambda}f\log\mu_{\Lambda}f \tag{4.1}$$

for all nonnegative functions f for which the right-hand side is finite. This implies that also the unique Gibbs measure μ for the local specification $\mathscr{E} = \{E^{\omega}_{\Lambda}\}_{\Lambda\in\mathfrak{F},\omega\in\mathscr{S}'}$ satisfies the logarithmic Sobolev inequality with the same coefficient c.

Proof. The basic idea of the proof is similar to that given in [Z2] for one dimensional spin systems with a compact spin space, however now some technicalities are much more involved. We prove the Logarithmic Sobolev inequality (4.1) only for the case of infinite volume Gibbs measure μ on $S'(\mathbb{Z})$; the proof for the case of arbitrary volume Λ is similar. First of all for a large $L \in \mathbb{N}$ (to be chosen later) we define the sets Γ_i, $i = 0,1$ as follows

$$\Gamma_i \equiv \bigcup_{\mathbf{k}\in\mathbb{Z}+\frac{i}{2}} ([0,2(L+R)] + \mathbf{k}(2L+4R))\,. \tag{4.2}$$

As one can see from this definition, each of the sets Γ_i consists of intervals of length $2(L+R)$ separated by the distance $2R$. Moreover we have

$$\Gamma_1 = \Gamma_0 + (L+2R)\,, \tag{4.3}$$

and thus $\Gamma = \Gamma_0 \cup \Gamma_1$. Now we introduce the following regular conditional expectations associated to any Gibbs measure for the local specification $\mathscr{E}$

$$E_{\Gamma_i} = \prod_{\mathbf{k} \in \mathbb{Z} + \frac{i}{2}} E^{\omega}_{\Lambda_{\mathbf{k}}} \tag{4.4}$$

with $\Lambda_{\mathbf{k}} \equiv [0, 2(L+R)] + \mathbf{k}(L+2R)$, $\mathbf{k} \in \mathbb{Z} + \frac{i}{2}$, and we set

$$\mathscr{P} \equiv E_{\Gamma_1} E_{\Gamma_0} \,. \tag{4.5}$$

We will show that the following lemma is true.

Lemma 4.2. *There is a constant $\bar{c} \in (0, \infty)$ such that*

$$\mathscr{P} f \log f \leqq 2\bar{c}\mathscr{P}|\nabla f^{\frac{1}{2}}|^2 + \mathscr{P} f \log \mathscr{P} f \tag{4.6}$$

for any nonnegative function f for which the right-hand side is finite. Moreover there is a constant $B \in (0, \infty)$ such that for any differentiable function f we have

$$|\nabla (\mathscr{P} f)^{\frac{1}{2}}|^2 \leqq B\mathscr{P}|\nabla f^{\frac{1}{2}}|^2 \,, \tag{4.7}$$

and additionally there is an $L \in \mathbb{N}$ such that with some constant $\lambda \in (0,1)$, we have

$$|\nabla (\mathscr{P} f)^{\frac{1}{2}}|^2 \leqq \lambda \mathscr{P}|\nabla f^{\frac{1}{2}}|^2 \tag{4.8}$$

for any nonnegative differentiable function $f \in \mathfrak{A}(\Gamma_0 \backslash \Gamma_1)$.

The proof of this lemma is much more complicated than in the case of spin systems with compact spin space and will be given later. Now assuming Lemma 4.2 we proceed as follows. Let $f \in \mathfrak{A}_{\Lambda_0}$, for some $\Lambda_0 \in \mathfrak{F}$, be a positive and differentiable function. Then from our assumption it is not difficult to see that also any function f_n, $n \in \mathbb{Z}^+$ defined for $n = 0$ is equal to f and for any $n \in \mathbb{N}$ by

$$f_n \equiv \mathscr{P}^n f \,,$$

has the same properties. Moreover for $n \in \mathbb{N}$ we have $f_n \in \mathfrak{A}(\Gamma_0 \backslash \Gamma_1)$. Now using the first part of Lemma 4.2, for any $n \in \mathbb{N}$ and $k = 1, \ldots, n$ we have

$$\begin{aligned}
\mathscr{P}^{n-k+1} f_{k-1} \log f_{k-1} &= \mathscr{P}^{(n-k)}(\mathscr{P} f_{k-1} \log f_{k-1}) \\
&\leqq \mathscr{P}^{(n-k)}\left(2\bar{c}\mathscr{P}|\nabla f_{k-1}^{\frac{1}{2}}|^2 + f_k \log f_k\right) \\
&= 2\bar{c}\mathscr{P}^{(n-k+1)}|\nabla f_{k-1}^{\frac{1}{2}}|^2 + \mathscr{P}^{(n-k)}(f_k \log f_k) \,.
\end{aligned} \tag{4.9}$$

Hence we get

$$\mathscr{P}^n f \log f \leqq \frac{\bar{c}}{2} \sum_{k=1}^{n} \mathscr{P}^{(n-k+1)}|\nabla f_{k-1}^{\frac{1}{2}}|^2 + \mathscr{P}^n f \log \mathscr{P}^n f \,. \tag{4.10}$$

Using the second part of Lemma 4.2 we have

$$|\nabla f_1^{\frac{1}{2}}|^2 \leqq B\mathscr{P}|\nabla f^{\frac{1}{2}}|^2 \,, \tag{4.11a}$$

and for $k \geqq 2$

$$|\nabla f_{k-1}^{\frac{1}{2}}|^2 \leqq B\lambda^{k-2}\mathscr{P}^{k-1}|\nabla f^{\frac{1}{2}}|^2 . \tag{4.11b}$$

From this it is not difficult to see that for any $n \in \mathbb{N}$ we have

$$\mathscr{P}^n f \log f \leqq 2c\mathscr{P}^n|\nabla f^{\frac{1}{2}}|^2 + \mathscr{P}^n f \log \mathscr{P}^n f \tag{4.12}$$

with a constant $c \in (\bar{c}, \bar{c}(1+2B(1-\lambda)^{-1}))$. Now to finish the proof, it is sufficient to show that in $L_1(\mu)$ we have

$$\lim_{n\to\infty} \mathscr{P}^n f = \mu f \tag{4.13}$$

for any local function $f \in \mathscr{C}^1$. To see this we utilize an idea of [Z3] as follows. For any $m, n \in \mathbb{N}$ we have

$$\mu|\mathscr{P}^{m+n}f(\omega) - \mathscr{P}^n f(\omega)| \leqq \mu\mathscr{P}^m|\mathscr{P}^n f(\tilde{\omega}) - \mathscr{P}^n f(\omega)| \tag{4.14}$$

with $\tilde{\omega}$ being the integration variable with respect to $\mathscr{P}^m$. By our construction we have $\mathscr{P}^n f \in \mathfrak{A}_{\Lambda_n}$ where $\Lambda_n \equiv \{\mathbf{i} : d(\mathbf{i}, \Lambda_0) \leqq n(2L+3R)\}$. Let $\{\mathbf{i}_k : k = 1, \ldots, |\Lambda_n|\}$ be an enumeration of elements of Λ_n. For any two configuration $\omega, \tilde{\omega} \in \mathscr{S}'$, we define the interpolating configurations $\omega^{\mathbf{i}_l}$ by setting

$$(\omega^{\mathbf{i}_l})_{\mathbf{i}_k} = \begin{cases} \omega_{\mathbf{i}_k} & \text{for } k \leqq l \\ \tilde{\omega}_{\mathbf{i}_k} & \text{for } k > l \end{cases} .$$

Using this notation we have

$$|\mathscr{P}^n f(\tilde{\omega}) - \mathscr{P}^n f(\omega)| \leqq \sum_{l=1}^{|\Lambda_n|-1} |\mathscr{P}^n f(\omega^{\mathbf{i}_{l+1}}) - \mathscr{P}^n f(\omega^{\mathbf{i}_l})| . \tag{4.15}$$

Since

$$|\mathscr{P}^n f(\omega^{\mathbf{i}_{l+1}}) - \mathscr{P}^n f(\omega^{\mathbf{i}_l})| \leqq \left| \int_{\omega_{\mathbf{i}_l}}^{\tilde{\omega}_{\mathbf{i}_l}} dx \nabla_{\mathbf{i}_l}\mathscr{P}^n f(x \bullet_{\mathbf{i}_l} \omega^{\mathbf{i}_l}) \right| \leqq (|\omega_{\mathbf{i}_l}| + |\tilde{\omega}_{\mathbf{i}_l}|)\|\nabla_{\mathbf{i}}\mathscr{P}^n f\|_u , \tag{4.16}$$

we get

$$\begin{aligned} \mu|\mathscr{P}^{m+n}f(\omega) - \mathscr{P}^n f(\omega)| &\leqq \sum_{\mathbf{i}\in\Lambda_n} 2\mu|\omega_{\mathbf{i}}| \cdot \|\nabla_{\mathbf{i}}\mathscr{P}^n f\|_u \\ &\leqq 4\|f^{\frac{1}{2}}\| \left(\sum_{\mathbf{i}\in\Lambda_n} \mu|\omega_{\mathbf{i}}| \cdot \||\nabla_{\mathbf{i}}(\mathscr{P}^n f)^{\frac{1}{2}}|\|_u \right) . \end{aligned} \tag{4.17}$$

Since under our assumptions (see e.g. [BH-K]) there is a constant $a \in (0, \infty)$ such that

$$\mu|\omega_{\mathbf{i}}| \leqq a \tag{4.18}$$

for any $\mathbf{i} \in \Gamma$, using the second part of the Lemma 4.2 and the fact that $|\Lambda_n| \leqq |\Lambda_0| + n(2L+3R)$, we finally obtain

$$\mu|\mathscr{P}^{m+n}f(\omega) - \mathscr{P}^n f(\omega)| \leqq 4a\|f^{\frac{1}{2}}\| \cdot \||f^{\frac{1}{2}}|\|B\lambda^{\frac{n-1}{2}}(|\Lambda_0| + n(2L+3R)) . \tag{4.19}$$

Using the fact that $\lambda \in (0,1)$, we see that the right-hand side of (4.19) converges to zero. This together with the fact that $\mathscr{P}^n$ satisfies DLR equation with the measure μ imply (4.13) and ends the proof of Theorem 4.1 (assuming Lemma 4.2) for positive local functions $f \in \mathscr{C}^1$. From this Theorem 4.1 follows by general arguments [G]. $\square$

Proof of Lemma 4.2. To prove Lemma 4.2 we first note that we have

Lemma 4.3. *For every $\Lambda \in \mathfrak{F}$ there is $c_0 \in (0,\infty)$ such that for any $\omega \in \mathscr{S}'$ we have*

$$E_\Lambda^\omega f \log f \leqq 2c_0 E_\Lambda^\omega |\nabla f^{\frac{1}{2}}|^2 + E_\Lambda^\omega f \log E_\Lambda^\omega f \tag{4.20}$$

for any positive differentiable function $f \in \mathfrak{A}_0$.

Proof of Lemma 4.3. To prove the lemma we observe that the measure E_Λ^ω has uniformly bounded from above and below density with respect to the measure

$$\tilde{E}_\Lambda^\omega F \equiv \frac{\bar{E}_\Lambda^\omega(e^{-V_\Lambda}F)}{\bar{E}_\Lambda^\omega(e^{-V_\Lambda})} \tag{4.21}$$

with $V_\Lambda \equiv \sum_{\mathbf{i}\in\Lambda} V(\varphi_\mathbf{i})$ and $\bar{E}_\Lambda^\omega$ being the conditional expectation associated to the Gaussian measure. Since by our assumption V_Λ is a convex function, the Bakry–Emery criterion, [BE], implies that the measure $\tilde{E}_\Lambda^\omega$ satisfies logarithmic Sobolev inequality with a coefficient independent of $\omega \in \Omega$ (and in fact of $\Lambda \subset \Gamma$). Using this and the general arguments of [HS] (Lemma 5.1), the lemma follows. $\square$

From Lemma 4.3, by the product property of logarithmic Sobolev inequality [G], we see that both E_{Γ_i}, $i = 0,1$ satisfy LS with the same coefficient c_0. Therefore we have

$$\begin{aligned}\mathscr{P}f \log f &= E_{\Gamma_1}(E_{\Gamma_0} f \log f) \leqq E_{\Gamma_1}(2c_0 E_{\Gamma_0}|\nabla_{\Gamma_0} f^{\frac{1}{2}}|^2 + E_{\Gamma_0} f \log E_{\Gamma_0} f)\\ &\leqq 2c_0(\mathscr{P}|\nabla_{\Gamma_0} f^{\frac{1}{2}}|^2 + E_{\Gamma_1}|\nabla_{\Gamma_1}(E_{\Gamma_0} f)^{\frac{1}{2}}|^2) + \mathscr{P}f \log \mathscr{P}f\,. \end{aligned} \tag{4.22}$$

To bound the second term on the right-hand side of (4.22) we will need the following fact:

Lemma 4.4. *Let $\Lambda = [0, 2(L+R)]$. For any positive and differentiable $f \in \mathfrak{A}_{\Lambda_0}$, $\Lambda_0 \subseteq \Lambda$ and any $\mathbf{j} \in \partial_R \Lambda$, we have*

$$|\nabla_\mathbf{j}(E_\Lambda f)^{\frac{1}{2}}|^2 \leqq 2E_\Lambda |\nabla_\mathbf{j} f^{\frac{1}{2}}|^2 + C|\Lambda| e^{-Md(f,\partial\Lambda)} E_\Lambda |\nabla_\Lambda f^{\frac{1}{2}}|^2 \tag{4.23}$$

with some constants $C, M \in (0,\infty)$ independent of the function f. For $\mathbf{j} \in \Lambda$, by the definition of local specification, the left-hand side of (4.23) *vanishes, whereas if $d(\mathbf{j},\Lambda) > R$ we can omit the second term from the right-hand side.*

The lemma will be proven later. Now let us note that for every $\mathbf{j} \in \Gamma_1$ we have either $\mathbf{j} \in \Gamma_0$, in which case $\nabla_\mathbf{j}(E_{\Gamma_0} f)^{\frac{1}{2}} = 0$, or there is a unique set $\Lambda_\mathbf{j} \subset \Gamma_0$ such that $\mathbf{j} \in \partial_R \Lambda_\mathbf{j}$. In the second case using Lemma 4.4 we get

$$\begin{aligned}|\nabla_\mathbf{j}(E_{\Gamma_0} f)^{\frac{1}{2}}|^2 &\leqq E_{\Gamma_0\setminus\Lambda_\mathbf{j}} |\nabla_\mathbf{j}(E_{\Lambda_\mathbf{j}} f)^{\frac{1}{2}}|^2\\ &\leqq 2E_{\Gamma_0}|\nabla_\mathbf{j} f^{\frac{1}{2}}|^2 + C|\Lambda_\mathbf{j}| e^{-M\,d(f,\partial\Lambda_\mathbf{j})} E_{\Gamma_0}|\nabla_{\Lambda_\mathbf{j}} f^{\frac{1}{2}}|^2\,. \end{aligned} \tag{4.24}$$

Summing this inequality over $\mathbf{j} \in \Gamma_1 \backslash \Gamma_0$ and integrating with the measure E_{Γ_1}, we obtain

$$E_{\Gamma_1} |\nabla_{\Gamma_1} (E_{\Gamma_0} f)^{\frac{1}{2}}|^2 \leqq (2 + 2RC) \mathscr{P} |\nabla f^{\frac{1}{2}}|^2 . \tag{4.25}$$

This together with (4.22) gives (4.6) with a constant $\bar{c} = c_0(3 + 2RC)$. To get (4.8) we observe that for $\mathbf{j} \in \Gamma_1 \complement$ we have

$$|\nabla_{\mathbf{j}} (E_{\Gamma_1} E_{\Gamma_0} f)^{\frac{1}{2}}|^2 \leqq E_{\Gamma_1 \backslash \Lambda_{\mathbf{j}}} |\nabla_{\mathbf{j}} (E_{\Lambda_{\mathbf{j}}} E_{\Gamma_0} f)^{\frac{1}{2}}|^2 \leqq 2RC |\Lambda_{\mathbf{j}}| e^{-ML} E_{\Gamma_1} |\nabla_{\Lambda_{\mathbf{j}}} (E_{\Gamma_0} f)^{\frac{1}{2}}|^2 , \tag{4.26}$$

where we have used Lemma 4.4 together with the fact that $(E_{\Gamma_0} f)^{\frac{1}{2}} \in \mathfrak{A}_{\Gamma_1 \backslash \Gamma_0}$. To estimate (4.26) we use (4.24) together with our present assumption that $f \in \mathfrak{A}_{\Gamma_0 \backslash \Gamma_1}$, to obtain

$$|\nabla_{\mathbf{j}} (E_{\Gamma_1} E_{\Gamma_0} f)^{\frac{1}{2}}|^2 \leqq (2RC)^2 (2(L + R))^2 e^{-2ML} \mathscr{P} |\nabla f^{\frac{1}{2}}|^2 . \tag{4.27}$$

If $L \in \mathbb{N}$ is sufficiently large, we get (4.8). This ends the proof of Lemma 4.2. □

Proof of Lemma 4.4. Let f be a positive differentiable function and let $\mathbf{j} \in \partial_R \Lambda$. In order to prove the inequality of interest to us, we observe that

$$\nabla_{\mathbf{j}} (E_\Lambda f)^{\frac{1}{2}} = 2 (E_\Lambda f)^{-\frac{1}{2}} \nabla_{\mathbf{j}} E_\Lambda f . \tag{4.28}$$

Therefore it is sufficient for us to find a suitable bound on the second factor on the right side of (4.28). For this we first use the definition of our local specification to get

$$\nabla_{\mathbf{j}} E_\Lambda f = E_\Lambda \nabla_{\mathbf{j}} f - \sum_{\substack{\mathbf{k} \in \Lambda \\ d(\mathbf{k},\mathbf{j}) \leqq R}} G_{\mathbf{jk}}^{-1} E_\Lambda (f, \varphi_{\mathbf{k}}) . \tag{4.29}$$

To bound the first term on the right-hand side of (4.29) we use the formula $\nabla_{\mathbf{j}} f = 2 f^{\frac{1}{2}} \nabla_{\mathbf{j}} f^{\frac{1}{2}}$ and the Hölder inequality to get

$$|E_\Lambda \nabla_{\mathbf{j}} f| \leqq 2 (E_\Lambda f)^{\frac{1}{2}} (E_\Lambda |\nabla_{\mathbf{j}} f^{\frac{1}{2}}|^2)^{\frac{1}{2}} . \tag{4.30}$$

To discuss the second term from the right-hand side of (4.29) we use the following representation of the two point truncated correlation function

$$E_\Lambda (f, \varphi_{\mathbf{k}}) = \frac{1}{2} E_\Lambda \otimes \tilde{E}_\Lambda (f - \tilde{f})(\varphi_{\mathbf{k}} - \tilde{\varphi}_{\mathbf{k}}) , \tag{4.31}$$

where $\tilde{E}_\Lambda$ denotes the isomorphic copy of the measure E_Λ and for a function F we have set $\tilde{F} \equiv F(\tilde{\varphi})$, with $\tilde{\varphi}$ denoting the integration variables with respect to the measure $\tilde{E}_\Lambda$. Introducing new integration variables q and p by

$$\varphi_{\mathbf{i}} = \frac{1}{\sqrt{2}} (q_{\mathbf{i}} + p_{\mathbf{i}}) , \tag{4.32}$$

$$\tilde{\varphi}_{\mathbf{i}} = \frac{1}{\sqrt{2}} (q_{\mathbf{i}} - p_{\mathbf{i}}) ,$$

we get the following representation of (4.31):

$$E_\Lambda (f, \varphi_{\mathbf{k}}) = \frac{1}{2} E_\Lambda \otimes \tilde{E}_\Lambda (E_{q,\Lambda} (f - \tilde{f})(\varphi_{\mathbf{k}} - \tilde{\varphi}_{\mathbf{k}})) , \tag{4.33}$$

where $E_{q,\Lambda}$ denotes the conditional expectation given a fixed configuration q_Λ associated to the measure $E_\Lambda \otimes \tilde{E}_\Lambda$ via (4.32). It is easy to see that we have

$$E_{q,\Lambda}(F(\varphi,\tilde{\varphi})) = \frac{\mu_{G,p}^{\partial\Lambda}(e^{-\mathscr{V}_\Lambda(p|q)}F(\varphi(q,p),\tilde{\varphi}(q,p)))}{\mu_{G,p}^{\partial\Lambda}(e^{-\mathscr{V}_\Lambda(p|q)})} \tag{4.34}$$

with $\mu_{G,p}^{\partial\Lambda}$ denoting the corresponding Gaussian measure for p variables and the (conditional) local action $\mathscr{V}_\Lambda(p|q)$ is given by the following formula:

$$\mathscr{V}_\Lambda(p|q) \equiv \sum_{\mathbf{i}\in\Lambda} \mathscr{V}(p_\mathbf{i}|q_\mathbf{i}) \equiv \sum_{\mathbf{i}\in\Lambda}\left(U\left(\frac{1}{\sqrt{2}}(q_\mathbf{i}+p_\mathbf{i})\right)+U\left(\frac{1}{\sqrt{2}}(q_\mathbf{i}-p_\mathbf{i})\right)\right) . \tag{4.35}$$

Let us mention that, as follows from the last formula, $\mathscr{V}(p_\mathbf{i}|q_\mathbf{i})$ is symmetric with respect to the change $p_\mathbf{i} \to -p_\mathbf{i}$. Since $f \in \mathfrak{A}_{\Lambda_0}$, we have

$$E_{q,\Lambda}(f-\tilde{f})(\varphi_\mathbf{k}-\tilde{\varphi}_\mathbf{k}) = \sqrt{2}E_{q,\Lambda}((f-\tilde{f})p_\mathbf{k}) = \sqrt{2}E_{q,\Lambda}((f-\tilde{f})E_{q,\Lambda\setminus\Lambda_0}p_\mathbf{k}) \,, \tag{4.36}$$

where $E_{q,\Lambda\setminus\Lambda_0}$ is the conditional expectation with respect to the variables $p_\mathbf{i}$, $\mathbf{i}\in\Lambda_0$, associated to the measure $E_{q,\Lambda}$. Now using this together with the algebraic formula $f-\tilde{f} = (f^{\frac{1}{2}}+\tilde{f}^{\frac{1}{2}})(f^{\frac{1}{2}}-\tilde{f}^{\frac{1}{2}})$ and the Hölder inequality we get

$$|E_{q,\Lambda}(f-\tilde{f})p_\mathbf{k}| \leqq (E_{q,\Lambda}(f^{\frac{1}{2}}+\tilde{f}^{\frac{1}{2}})^2)^{\frac{1}{2}}(E_{q,\Lambda}(f^{\frac{1}{2}}-\tilde{f}^{\frac{1}{2}})^2(E_{q,\Lambda\setminus\Lambda_0}p_\mathbf{k})^2)^{\frac{1}{2}} . \tag{4.37}$$

To estimate the second factor from the right-hand side of (4.37) let us note that we have

$$E_{q,\Lambda\setminus\Lambda_0}^{\mathbf{p}}p_\mathbf{k} = \sum_{\mathbf{i}\in\Lambda_0}\int_0^{p_\mathbf{i}} dx \sum_{\substack{\mathbf{i}\in\Lambda_0\complement \\ d(\mathbf{i},\mathbf{l})\leqq R}} G_{\mathbf{il}}^{-1}E_{q,\Lambda\setminus\Lambda_0}^{\mathbf{p}^\mathbf{i}}(p_\mathbf{l},p_\mathbf{k}) \,, \tag{4.38}$$

where $\{\mathbf{p}^\mathbf{i}\}$, $\mathbf{i}\in\Lambda_0$ is an interpolating sequence defined by

$$(p^\mathbf{i})_\mathbf{j} \equiv \begin{cases} 0 & \text{for } \mathbf{j}<\mathbf{i} \\ x & \text{for } \mathbf{j}=\mathbf{i} \\ p_\mathbf{i} & \text{for } \mathbf{j}>\mathbf{i} \end{cases} . \tag{4.39}$$

Hence

$$|E_{q,\Lambda\setminus\Lambda_0}^{\mathbf{p}}p_\mathbf{k}| \leqq \sum_{\mathbf{i}\in\Lambda_0}|p_\mathbf{i}| \sum_{\substack{\mathbf{i}\in\Lambda_0\complement \\ d(\mathbf{i},\mathbf{l})\leqq R}} |G_{\mathbf{il}}^{-1}| \cdot \|E_{q,\Lambda\setminus\Lambda_0}(p_\mathbf{l}p_\mathbf{k})\|_u . \tag{4.40}$$

To estimate the right-hand side of (4.40) we use the following lemma proven later.

Lemma 4.5. *There are constants $C_1, M \in (0,\infty)$ such that we have*

$$\|E_{q,\Lambda\setminus\Lambda_0}(p_\mathbf{l}p_\mathbf{k})\|_u \leqq C_1 e^{-\frac{1}{2}Md(\mathbf{k},\mathbf{l})} . \tag{4.41}$$

Using (4.38)–(4.41), we get

$$(E_{q,\Lambda}(f^{\frac{1}{2}}-\tilde{f}^{\frac{1}{2}})^2(E_{q,\Lambda\setminus\Lambda_0}p_{\mathbf{k}})^2)^{\frac{1}{2}} \leqq C_2 e^{-\frac{1}{2}Md(f,\partial\Lambda)} \times \left(E_{q,\Lambda}(f^{\frac{1}{2}}-\tilde{f}^{\frac{1}{2}})^2 \sum_{\mathbf{i}\in\Lambda_0} p_{\mathbf{i}}^2\right)^{\frac{1}{2}} \tag{4.42}$$

with a constant $C_2 \in (0,\infty)$ independent of Λ, q and the function f. Now let $\Phi_{q,\Lambda} \equiv \Phi_{q,\Lambda}(p)$ denote the density of the measure $E_{q,\Lambda}$ with respect to the Lebesgue measure dp_Λ. Then it is clear from our assumptions about the local interaction that there are constants, $a, b \in (0,\infty)$ such that, for any $\Lambda \in \mathfrak{F}$ and configuration q, we have

$$\sum_{\mathbf{i}\in\Lambda_0} p_{\mathbf{i}}^2 \leqq a(|\nabla \log \Phi_{q,\Lambda}|^2 - \Delta \log \Phi_{q,\Lambda}) + b|\Lambda| . \tag{4.43}$$

Hence it is easy to see (by similar arguments as used in Sect. 2 in the proof of Lemma 2.4) that

$$E_{q,\Lambda}\left((f^{\frac{1}{2}}-\tilde{f}^{\frac{1}{2}})^2 \sum_{\mathbf{i}\in\Lambda_0} p_{\mathbf{i}}^2\right) \leqq aE_{q,\Lambda}(|\nabla_{\Lambda,p} f^{\frac{1}{2}}|^2 + |\nabla_{\Lambda,p}\tilde{f}^{\frac{1}{2}}|^2) + b|\Lambda| E_{\Lambda,q}(f^{\frac{1}{2}}-\tilde{f}^{\frac{1}{2}})^2 , \tag{4.44}$$

where $\nabla_{\Lambda,p}$ denotes the gradient with respect to the variables $p_{\mathbf{i}}$, $\mathbf{i} \in \Lambda$. To estimate the second term on the right-hand side of (4.44), we use the following lemma, which will be proven at the end.

Lemma 4.6. *There is a constant $m_0 \in (0,\infty)$ such that for any $\Lambda \in \mathfrak{F}$, $q \in \mathscr{S}'$ and all differentiable functions $f \in \mathfrak{A}_{\Lambda_0}$ we have*

$$E_{\Lambda,q}(f^{\frac{1}{2}}-\tilde{f}^{\frac{1}{2}})^2 \leqq m_0^{-1} E_{q,\Lambda}|\nabla_{\Lambda,p}(f^{\frac{1}{2}}-\tilde{f}^{\frac{1}{2}})|^2 . \tag{4.45}$$

Using Lemma 4.6 we get

$$\left(E_{q,\Lambda}\left((f^{\frac{1}{2}}-\tilde{f}^{\frac{1}{2}})^2 \sum_{\mathbf{i}\in\Lambda_0} p_{\mathbf{i}}^2\right)\right)^{\frac{1}{2}} \leqq (2a + bm_0^{-1}|\Lambda|)E_{q,\Lambda}\left|\nabla_{\Lambda,p}(f^{\frac{1}{2}}-\tilde{f}^{\frac{1}{2}})\right|^2 . \tag{4.46}$$

Using this together with (4.42), (4.37) and (4.36), (4.33), we get

$$|E_\Lambda(f,\varphi_{\mathbf{k}})| \leqq C_3|\Lambda|^{\frac{1}{2}} e^{-\frac{1}{2}Md(f,\partial\Lambda)} E_\Lambda \otimes \tilde{E}_\Lambda \times ((E_{q,\Lambda}(f^{\frac{1}{2}}+\tilde{f}^{\frac{1}{2}})^2)^{\frac{1}{2}}(E_{q,\Lambda}|\nabla_{\Lambda,p}(f^{\frac{1}{2}}-\tilde{f}^{\frac{1}{2}})|^2)^{\frac{1}{2}}) \tag{4.47}$$

with some constant $C_3 \in (0,\infty)$ independent of Λ, q and the function f. Hence by simple arguments involving Hölder inequality and the definition of the conditional expectation, we arrive at the following inequality:

$$|E_\Lambda(f,\varphi_{\mathbf{k}})| \leqq 2(E_\Lambda f)^{\frac{1}{2}}(C_4|\Lambda|^{\frac{1}{2}} e^{-\frac{1}{2}Md(f,\partial\Lambda)}(E_\Lambda|\nabla_\Lambda f^{\frac{1}{2}}|^2)^{\frac{1}{2}}) \tag{4.48}$$

with some constant $C_4 \in (0,\infty)$ independent of $\Lambda \in \mathfrak{F}$, $\omega \in \mathscr{S}'$ and the function $f \in \mathfrak{A}_{\Lambda_0} \cap \mathscr{C}^1$. Combining (4.48) with (4.29)–(4.30), and using (4.28), we get

$$|\nabla_{\mathbf{j}}(E_\Lambda f)^{\frac{1}{2}}| \leqq (E_\Lambda|\nabla_{\mathbf{j}} f^{\frac{1}{2}}|^2)^{\frac{1}{2}} + C_4|\Lambda|^{\frac{1}{2}} e^{-\frac{1}{2}Md(f,\partial\Lambda)}(E_\Lambda|\nabla_\Lambda f^{\frac{1}{2}}|^2)^{\frac{1}{2}} \tag{4.49}$$

from which Lemma 4.4 easily follows. □

Proof of Lemma 4.5. Let Λ_n, $n = 1, \ldots, N$ be a sequence of intervals of lengths equal to the range of the interaction R, such that $\mathbf{k} \in \Lambda_1$ and $\mathbf{l} \in \Lambda_N$ for some $N \leqq \left[\frac{d(\mathbf{k},\mathbf{l})}{R}\right] + 1$. Let

$$\varepsilon_{n,n+1} \equiv -\sum_{\substack{\mathbf{i} \in \Lambda_n \\ \mathbf{j} \in \Lambda_{n+1}}} \varphi_{\mathbf{i}} G^{-1}_{\mathbf{ij}} \varphi_{\mathbf{j}} \,. \tag{4.50}$$

Using the symmetry $p \to -p$ of the conditional local interaction $\mathscr{V}_\Lambda(\,\cdot\,|q)$ and the fact that G^{-1} has the finite range R, it is not difficult to see that we have

$$E_{q,\Lambda}(p_{\mathbf{k}} p_{\mathbf{l}}) = E_{q,\Lambda}\left(\prod_{n=1}^{N} \mathrm{th}(\varepsilon_{n,n+1}(p)) p_{\mathbf{k}} p_{\mathbf{l}}\right) \,. \tag{4.51}$$

Let us define the following partition of unity χ^{σ_n}, $\sigma_n \in \{-1,+1\}$ by setting:

$$\chi^{\sigma_n=+1}(p) \equiv \chi(\forall \mathbf{j} \in \Lambda_n,\, |p_{\mathbf{j}}| \leqq H)\,, \tag{4.52}$$

and let

$$\chi^{\sigma_n=-1}(p) \equiv 1 - \chi^{\sigma_n=+1}(p) = \chi(\exists \mathbf{j} \in \Lambda_n,\, |p_{\mathbf{j}}| > H)\,. \tag{4.53}$$

Using this notation we have

$$E_{q,\Lambda}(p_{\mathbf{k}} p_{\mathbf{l}}) = \sum_{\sigma_n,\, n=1,\ldots,N} E_{q,\Lambda}\left(\prod_{n=1}^{N} (\chi^{\sigma_n}(p)\mathrm{th}(\varepsilon_{n,n+1}(p))) p_{\mathbf{k}} p_{\mathbf{l}}\right) \,. \tag{4.54}$$

We divide the sum on the right-hand side of (4.54) into two parts. The first, denoted by S_I, will contain all terms for which the number of n's satisfying $\sigma_n = +1$ and $\sigma_{n+1} = +1$ is bigger than δN, for some small constant $\delta \in (0, \frac{1}{2})$. Since in the case when $\sigma_n = +1$ and $\sigma_{n+1} = +1$ we have

$$\|\mathrm{th}(\varepsilon_{n,n+1})\| \leqq e^{-M_0} \tag{4.55}$$

for some $M_0 \in (0, \infty)$, using an easy to prove fact (see e.g. [BH-K]) that

$$\sup_{q,\Lambda} (E_{q,\Lambda}(p_{\mathbf{j}}^2))^{\frac{1}{2}} \leqq C \tag{4.56}$$

with some constant $C \in (0, \infty)$, we see that the first sum has the following bound:

$$|S_I| \leqq C^2 e^{-M_0 \delta N} \leqq D_1 e^{-\frac{M_0 \delta}{R} d(\mathbf{k},\mathbf{l})} \tag{4.57}$$

with some constant $D_1 \in (0, \infty)$ independent of Λ and $\mathbf{k}, \mathbf{l} \in \Lambda$. To bound the second part of the sum from the right-hand side of (4.54), denoted later by S_{II}, we estimate each term as follows:

$$\left|E_{q,\Lambda}\left(\prod_{n=1}^{N} (\chi^{\sigma_n}\, \mathrm{th}(\varepsilon_{n,n+1}(p))) p_{\mathbf{k}} p_{\mathbf{l}}\right)\right| \leqq \left(E_{q,\Lambda}\left(\prod_{n=1}^{N} \chi^{\sigma_n}(p)\right)\right)^{\frac{1}{2}} (E_{q,\Lambda}(p_{\mathbf{k}}^2 p_{\mathbf{l}}^2))^{\frac{1}{2}} \,. \tag{4.58}$$

Now it is not difficult to see (e.g. by the same arguments as in [BeH-K]) that the second factor on the right-hand side of (4.58) is bounded by a constant $D_2 \in (0, \infty)$ independent of q, Λ, σ and $\mathbf{k}, \mathbf{l} \in \Lambda$. On the other hand using the fact that the local conditional interaction is fast increasing at infinity, we get

$$E_{q,\Lambda}\left(\prod_{n=1}^{N}\chi^{\sigma_n}(p)\right) \leqq e^{-M(H)(\frac{1}{2}-\delta)N} \tag{4.59}$$

with a constant $M(H) \in (0,\infty)$ growing to infinity with H and independent of q and Λ. This is because for every configuration of σ's in the sum S_{II} we have to have at least $\frac{1}{2}N$ factors $\chi(|p_{\mathbf{i}}| > H)$. Taking into account that the number of terms in this sum does not exceed 2^{NR}, for any fixed $\delta < \frac{1}{2}$ we can choose $H \in (0,\infty)$ sufficiently large so that $[M(H)(\frac{1}{2}-\delta) - \log(2R)] > M_0(\frac{1}{2}-\delta)$. Then we get the following estimate

$$|S_{II}| \leqq D_2 e^{-(\frac{1}{2}-\delta)\frac{M_0}{R}d(\mathbf{k},\mathbf{l})} \tag{4.60}$$

with some constant $D_2 \in (0,\infty)$ independent of q, Λ and $\mathbf{k}, \mathbf{l} \in \Lambda$. Combining (4.57) and (4.60) we get the inequality (4.41) with $M = \min(2\delta, 1-2\delta)M_0/R$. This ends the proof of Lemma 4.5. □

Proof of Lemma 4.6. Let $F \in \mathfrak{A}_{\Lambda_0} \cap \mathscr{C}^1$ with some $\Lambda_0 \subset \Lambda \in \mathfrak{F}$. We need to prove that

$$E_{q,\Lambda}(F(\varphi(q,p)) - F(\tilde{\varphi}(q,p)))^2 \leqq m_0^{-1} E_{q,\Lambda}|\nabla_p(F(\varphi(q,p)) - F(\tilde{\varphi}(q,p)))|^2 \tag{4.61}$$

with some constant $m_0 \in (0,\infty)$ independent of q and Λ, and ∇_p denoting the gradient with respect to the variables p. To see (4.61), we note that denoting by

$$\rho_\Lambda(p_{\Lambda_0}) \equiv \mu_G^{\partial\{\Lambda\setminus\Lambda_0\}}\left(\exp\left\{-\mathscr{V}_{\Lambda\setminus\Lambda_0}(p|q) - \sum_{\substack{\mathbf{j}\in\Lambda\setminus\Lambda_0, \mathbf{i}\in\Lambda_0 \\ d(\mathbf{i},\mathbf{j})\leqq R}} p_{\mathbf{j}} G_{\mathbf{ij}}^{-1} p_{\mathbf{i}}\right\}\right), \tag{4.62}$$

for any function $g \in \mathfrak{A}_{\Lambda_0}$ we have

$$E_{q,\Lambda}(g^2) = \frac{\mu_G^{\partial\Lambda_0}(e^{-\mathscr{V}_{\Lambda_0}(p|q)}\rho_{\Lambda,\Lambda_0} g^2)}{\mu_G^{\partial\Lambda_0}(e^{-\mathscr{V}_{\Lambda_0}}\rho_{\Lambda,\Lambda_0})}. \tag{4.63}$$

Let us note that the local conditional interaction $\mathscr{V}_{\Lambda_0}(p|q)$ can be represented in the following form:

$$\mathscr{V}_{\Lambda_0}(p|q) = \sum_{\mathbf{i}\in\Lambda_0}(\mathscr{V}^c(p_{\mathbf{i}}|q_{\mathbf{i}}) + \mathscr{V}^{nc}(p_{\mathbf{i}}|q_{\mathbf{i}})) \equiv \mathscr{V}_{\Lambda_0}^c(p|q) + \mathscr{V}_{\Lambda_0}^{nc}(p|q) \tag{4.64}$$

with $\mathscr{V}^c(\,\cdot\,|q_{\mathbf{i}})$, respectively $\mathscr{V}^{nc}(\,\cdot\,|q_{\mathbf{i}})$, being convex, respectively not necessarily convex but satisfying

$$\sup_{q_{\mathbf{i}}}\|\mathscr{V}^{nc}(\,\cdot\,|q_{\mathbf{i}})\|_u \leqq v_1 \tag{4.65}$$

with some constant $v_1 \in (0,\infty)$. Using this we see that

$$E_{q,\Lambda}(g^2) \leqq e^{2v_1|\Lambda_0|}\frac{\mu_G^{\partial\Lambda_0}(e^{-\mathscr{V}_{\Lambda_0}^c(p|q)}\rho_{\Lambda,\Lambda_0} g^2)}{\mu_G^{\partial\Lambda_0}(e^{-\mathscr{V}_{\Lambda_0}^c}\rho_{\Lambda,\Lambda_0})}. \tag{4.66}$$

Additionally we observe, that as follows from Lemma 4.5, we have

$$\begin{aligned}
&\sum_{\mathbf{i},\mathbf{j}\in\Lambda_0\cap\partial_R\Lambda\setminus\Lambda_0} p_{\mathbf{i}}p_{\mathbf{j}}\frac{\partial^2\rho_{\Lambda,\Lambda_0}}{\partial p_{\mathbf{i}}\partial p_{\mathbf{j}}}\\
&\quad=\sum_{\mathbf{i},\mathbf{j}\in\Lambda_0\cap\partial_R\Lambda\setminus\Lambda_0} p_{\mathbf{i}}p_{\mathbf{j}}\left\langle\sum_{\mathbf{k}\in\Lambda\setminus\Lambda_0}G^{-1}_{\mathbf{ik}}p_{\mathbf{k}},\sum_{\mathbf{k}\in\Lambda\setminus\Lambda_0}G^{-1}_{\mathbf{jk}}p_{\mathbf{k}}\right\rangle_{\Lambda\setminus\Lambda_0,p_{\Lambda_0}}\\
&\quad\leqq C\sum_{\mathbf{i}\in\Lambda_0\cap\partial_R\Lambda\setminus\Lambda_0} p_{\mathbf{i}}^2 \qquad (4.67)
\end{aligned}$$

with some constant $C\in(0,\infty)$ independent of Λ, Λ_0 and p_{Λ_0}; we have used $\langle\,\cdot\,,\,\cdot\,\rangle_{\Lambda\setminus\Lambda_0,p_{\Lambda_0}}$ to denote the corresponding covariance of the conditional expectation given p_{Λ_0} associated to the measure $E_{q,\Lambda}$. Using this, we see that there is a convex, even with respect to p, function $\bar{\mathscr{V}}^{c}_{\Lambda_0}(p|q)$ and a constant $v_2\in(0,\infty)$, such that we have

$$e^{-2v_2|\Lambda_0|}\frac{\mu_G^{\partial\Lambda_0}(e^{-\bar{\mathscr{V}}^c_{\Lambda_0}(p|q)}g^2)}{\mu_G^{\partial\Lambda_0}(e^{-\bar{\mathscr{V}}^c_{\Lambda_0}(p|q)})}\leqq E_{q,\Lambda}(g^2)\leqq e^{2v_2|\Lambda_0|}\frac{\mu_G^{\partial\Lambda_0}(e^{-\bar{\mathscr{V}}^c_{\Lambda_0}(p|q)}g^2)}{\mu_G^{\partial\Lambda_0}(e^{-\bar{\mathscr{V}}^c_{\Lambda_0}(p|q)})} \qquad (4.68)$$

for any function $g\in\mathfrak{A}_{\Lambda_0}$. If the function g is odd, by standard arguments one shows that there is a constant $\bar{m}_0\in(0,\infty)$, such that

$$\frac{\mu_G^{\partial\Lambda_0}(e^{-\bar{\mathscr{V}}^c_{\Lambda_0}(p|q)}g^2)}{\mu_G^{\partial\Lambda_0}(e^{-\bar{\mathscr{V}}^c_{\Lambda_0}(p|q)})}\leqq \bar{m}_0^{-1}\frac{\mu_G^{\partial\Lambda_0}(e^{-\bar{\mathscr{V}}^c_{\Lambda_0}(p|q)}|\nabla_p g|^2)}{\mu_G^{\partial\Lambda_0}(e^{-\bar{\mathscr{V}}^c_{\Lambda_0}(p|q)})}\,. \qquad (4.69)$$

This together with (4.68) implies (4.61) and ends the proof of the lemma. □

Having completed the proof of Theorem 4.1 we can now combine it with the results of Sects. 1–3 and easily get the following corollary.

Theorem 4.2. *Let P_t, $t\in\mathbb{R}^+$ be a semigroup on $\mathscr{C}(\mathscr{S}'(\mathbb{Z}))$ corresponding to the free Gaussian measure μ_G (with mean zero and with strictly positive inverse of covariance $\mathbf{G}^{-1}$ of a finite range R), and a local interaction $U\equiv V+W$ as in Theorem 4.1. Let μ be an invariant Gibbs measure for P_t. Then for any $f\in\mathfrak{A}_{\Lambda_0}\cap\mathscr{C}^1$, $\Lambda_0\in\mathfrak{F}$, we have*

$$|P_tf(\eta)-\mu f|\leqq C_\delta(\eta)((\mu(f-\mu f)^2)^{\frac{1}{2}}+|||f|||)e^{-mt} \qquad (4.70)$$

for any $m\in(0,(1-\delta)gap_2\mathscr{L})$, with a constant $C_\delta(\eta)\in(0,\infty)$ dependent only on $\eta\in\mathscr{S}'$, $\delta\in(0,1)$ and $\Lambda_0\in\mathfrak{F}$.

5. Logarithmic Sobolev Inequality for Gibbs Measures: The Higher Dimensional Examples

In this section we would like to show that also on the higher dimensional lattice one can give examples where our strategy allows to prove the Logarithmic Sobolev inequality in more general situations than admitted by the Γ_2 criterion of [BE]. A careful reader has surely noticed that the basic ingredients we used in the previous section were: the uniform (in volume and external conditions) cluster property of conditional measures and certain operator forms bounds. The later are essentially independent of the dimension of the lattice, but the uniform cluster property is a more delicate matter and on a higher dimensional lattice it does not come "for free" as in one dimension. It should be however clear for a reader that in general the following result is true.

Theorem 5.1. *Suppose there are constants $C, M \in (0,\infty)$ such that for any sufficiently large cube $\Lambda_0 \subset \mathbb{Z}^d$ and any $\omega \in \Omega$ we have*

$$|E^{\omega}_{\Lambda_0}(\varphi_{\mathrm{i}}, \varphi_{\mathrm{j}})| \leqq Ce^{-Md(\mathrm{i},\mathrm{j})} . \tag{5.1}$$

Then there is a constant $c \in (0,\infty)$ such that for any cube $\Lambda \in \mathbb{Z}^d$ we have

$$\mu_\Lambda f \log f \leqq 2c\mu_\Lambda |\nabla_\Lambda f^{\frac{1}{2}}|^2 + \mu_\Lambda f \log \mu_\Lambda f \tag{5.2}$$

for all nonnegative functions f for which the right-hand side is finite. This implies that also the unique Gibbs measure μ for the local specification $\mathscr{E} = \{E^{\omega}_{\Lambda}\}_{\Lambda\in\mathfrak{F},\omega\in\mathscr{S}'}$ satisfies the logarithmic Sobolev inequality with the same coefficient c.

After a comprehensive description of the one dimensional case in Sect. 4 it should be easy for a reader to use the geometrical considerations of [SZ3] to reproduce the details of the proof of Theorem 5.1. Therefore we would like to restrict ourselves to description of a large class of models satisfying the assumption of the uniform cluster property. Our main goal will be to show that this class contains examples where the local interactions can have arbitrarily large negative second derivative, for which the Γ_2 form of the Bakry–Emery criterion [BE] cannot be positive and therefore one cannot get the logarithmic Sobolev inequality using the arguments of [BE]. To achieve our goal we note that if the function U, used to define the local interaction, differs from a convex function only on the sets of small probability then one can get an estimate (5.1) by the following special version of cluster expansion; (for the general principles of cluster expansion, see e.g. [MM, Br]). First of all we observe that using (4.33) and (4.34), we have

$$|E^{\omega}_{\Lambda}(\varphi_{\mathrm{i}}, \varphi_{\mathrm{j}})| \leqq \sup_q |E_{q,\Lambda}(p_{\mathrm{i}}\, p_{\mathrm{j}})| , \tag{5.3}$$

where

$$E_{q,\Lambda}(F(p)) = \frac{\mu^{\partial\Lambda}_{G,p}(e^{-\mathscr{V}_\Lambda(p|q)}F(p))}{\mu^{\partial\Lambda}_{G,p}(e^{-\mathscr{V}_\Lambda(p|q)})} \tag{5.4}$$

with $\mu^{\partial\Lambda}_{G,p}$ denoting the corresponding Gaussian measure for p variables and the (conditional) local action $\mathscr{V}_\Lambda(p|q)$ is given by the following formula:

$$\mathscr{V}_\Lambda(p|q) \equiv \sum_{\mathrm{i}\in\Lambda} \mathscr{V}(p_{\mathrm{i}}|q_{\mathrm{i}}) \equiv \sum_{\mathrm{i}\in\Lambda} \left(U\left(\frac{1}{\sqrt{2}}(q_{\mathrm{i}} + p_{\mathrm{i}})\right) + U\left(\frac{1}{\sqrt{2}}(q_{\mathrm{i}} - p_{\mathrm{i}})\right)\right) . \tag{5.5}$$

The conditional local interaction has the following symmetry useful for us later:

$$\mathscr{V}_\Lambda(p|q) = \mathscr{V}_\Lambda(-p|q)\,. \tag{5.6}$$

We will like to apply a cluster expansion to the probability measures given by (5.4). For this we define a family of covariances $G(\mathbf{s})$, $\mathbf{s} \equiv \{s_j \in [0,1]\}_{\mathbf{j}\in\Lambda}$, by setting

$$G^{-1}(\mathbf{s})_{\mathbf{ii}} \equiv G^{-1}_{\mathbf{ii}}\,,$$

$$G^{-1}(\mathbf{s})_{\mathbf{ij}} \equiv= s_{\mathbf{i}} G^{-1}_{\mathbf{ij}} s_{\mathbf{j}} \quad \text{for } \mathbf{i} \neq \mathbf{j}\,. \tag{5.7}$$

Let $\mu_{G(\mathbf{s}),p}$ be the corresponding (interpolating) Gaussian measure. Substituting this measure into the formula (5.4) we define a probability measure $E_{q,\mathbf{s},\Lambda}$. Let us define non-normalized expectation by

$$\bar{E}_{q,\mathbf{s},\Lambda} \equiv \mu^{\partial\Lambda}_{G(\mathbf{s}),p}(e^{-\mathscr{V}_\Lambda(p|q)}) \cdot E_{q,\mathbf{s},\Lambda}\,. \tag{5.8}$$

The cluster expansion of interest to us is generated by the successive application of the fundamental theorem of calculus in the following form:

if $s_l = 1$

$$\bar{E}_{q,\mathbf{s},\Lambda} p_{\mathbf{i}} p_{\mathbf{j}} = (\bar{E}_{q,\mathbf{s},\Lambda} p_{\mathbf{i}} p_{\mathbf{j}})|_{s_l=0} + \int_0^1 d\tilde{s}_l \frac{d}{d\tilde{s}_l} \bar{E}_{q,\tilde{\mathbf{s}},\Lambda} p_{\mathbf{i}} p_{\mathbf{j}}\,, \tag{5.9}$$

where $\tilde{\mathbf{s}}_{\mathbf{k}} = s_{\mathbf{k}}$ for $\mathbf{k} \neq \mathbf{l}$ and otherwise to the integration variable. Applying (5.9) successively and taking into the account the symmetry property (5.6), one gets the following representation for the quantity of interest to us:

$$E_{q,\mathbf{s},\Lambda} p_{\mathbf{i}} p_{\mathbf{j}} = \sum_{\substack{X \subset \Lambda \\ X \ni \mathbf{i},\mathbf{j}}} \frac{Z_{\Lambda\setminus X}}{Z_\Lambda} \int_{0 \leqq s_X \leqq 1} ds_X \frac{\partial}{\partial s_X} \bar{E}_{q,\mathbf{s},\Lambda} p_{\mathbf{i}} p_{\mathbf{j}}\,, \tag{5.10}$$

where the summation is running only on the connected sets X containing $\mathbf{i}$ and $\mathbf{j}$, the integration is over $s_{\mathbf{k}} \in [0,1]$, $\mathbf{k} \in X$ and $\frac{\partial}{\partial s_X} \equiv \prod_{\mathbf{k}\in X} \frac{\partial}{\partial s_{\mathbf{k}}}$. By standard arguments one gets the following general result:

Proposition 5.2. *There is a constant* $\lambda \in (0,1)$ *such that if*

$$\sup_{s_X} \left| \frac{\partial}{\partial s_X} \bar{E}_{q,\mathbf{s},\Lambda} p_{\mathbf{i}} p_{\mathbf{j}} \right| \leqq \lambda^{|X|}\,, \tag{5.11}$$

then also

$$\frac{Z_{\Lambda\setminus X}}{Z_\Lambda} \leqq e^{B|X|} \tag{5.12}$$

with some constant $B \ll |\log\lambda|$ *and the cluster expansion* (5.10) *converges uniformly in* $\Lambda \subset \mathbb{Z}^d$ *and the estimate* (5.1) *is true.*

Let us mention that clearly the inequality (5.11) is true if for every $\mathbf{i} \in \mathbb{Z}^d$, we have $G^{-1}_{\mathbf{ii}} \geqq g_0$, with some sufficiently large constant $g_0 \in (0,\infty)$ and the local interaction is given by a convex function U. Now to finish the construction of the example of interest to us it is sufficient to observe that if the cluster expansion is convergent uniformly in Λ and q for a given local interaction given by a function U, then it is also convergent for any interaction given by $U + \delta W$, provided that $\|\delta W\|_u < \varepsilon$ with some $\varepsilon \in (0,\infty)$ sufficiently small. This condition obviously allows us to take a function having an arbitrary second derivative and therefore one can violate the positivity of the Γ_2 form from the Bakry–Emery criterion.

Remark. Let us note that the Erice cluster expansion of [GJS] could be not suitable in general for the above arguments, since it depends not only on the supremum norm of the perturbation. On the other hand this expansion converges also in the situations of some polynomial interactions where the above given (crude) expansion would be divergent. One could however hope that by careful modification of the arguments (based on the multiscale analysis), it should also be possible to get Logarithmic Sobolev inequalities for the (unique) infinite volume Gibbs measure.

Finally let us mention that by the results of Sect. 3 in all the above considered models we have the following strong ergodicity result.

Theorem 5.2. *For any model on* $\mathbb{Z}^d$ *satisfying the conditions of Theorem* 5.1, *the corresponding semigroup is strongly ergodic in the sense of Theorems* 3.1 *and* 3.2.

6. Conclusions

In the present work we have shown that an extension of a general strategy for proving strong ergodicity properties of the Markov semigroups to the case of noncompact configuration space is possible. In particular we have shown that the general idea of proving LS for the corresponding Gibbs measures works also in the present setting, although some technical details require more work than in the case of compact configuration space. By this we provide an important class of nontrivial situations, where LS is true and which remains beyond the applicability region of the Bakry–Emery criterion. We have given a comprehensive characterization of the models in one dimension. In higher dimensions the situation is more complicated and although we have constructed a class of nontrivial examples for which the LS is true but the Γ_2 criterion fails, there are still interesting cases to study. For example let us mention the $\lambda : P(\varphi) :_{2,\delta}$ lattice models of euclidean field theory with a small coupling constant $\lambda \in (0, \infty)$, for which one could expect LS, but for which it seems to be impossible to prove the uniform cluster property (as in Theorem 5.1) by the cluster expansion, which would be uniform also in the lattice spacing. It would be very interesting to study such lattice systems.

Although we know already quite a lot about the lattice systems, we still know very little about the continuum systems. From what we have done in the present work it seems to be clear that the logarithmic Sobolev inequality is also true for the unique Gibbs measure of a one dimensional continuum system corresponding to the free Gaussian measure μ_0, with mean zero and a covariance $(-d^2/dx^2 + m_0^2)^{-1}$, $m_0 > 0$, and the local interaction as considered before. This provides the first nontrivial example which is not in the B-E class.

A more intriguing situation is in two dimensions for the measures describing the models of euclidean field theory, [Si], with the weak polynomial interactions. (As remarked in [Z3] for the model with the exponential interaction one can use the B-E criterion together with some approximation procedure to get LS.) For the $\lambda : P(\varphi) :_2$ already the finite volume LS is nontrivial, although one can give heuristic arguments that it should be true. The problem for the infinite volume measures seems to be at the moment much more complicated.

After we have constructed a number of examples of hypercontractive semigroups it would be very interesting to study their kernels further. Besides purely mathematical interests, we believe it could provide us with a better understanding of the analyticity properties and the particle structure of the corresponding theories.

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