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Beta Function and Anomaly of the Fermi Surface for a d = 1 System of Interacting Fermions in a Periodic Potential

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Abstract: We derive a perturbation theory, based on the renormalization group, for the Fermi surface of a one dimensional system of fermions in a periodic potential interacting via a short range, spin independent potential. The infrared problem is studied by writing the Schwinger functions in terms of running couplings. Their flow is described by a Beta function, whose existence and analyticity as a function of the running couplings is proved. If the fermions are spinless we prove that the Beta function is vanishing and the renormalization flow is bounded for any small interaction. If the fermions are spinning the Beta function is not vanishing but, if the conduction band is not filled or half filled and the interaction is repulsive, it is possible again to control the flow proving the partial asymptotic freedom of the theory. This is done showing that the Beta function is partially vanishing using the exact solution of the Mattis model, which is the spin analogue of the Luttinger model. In both these cases Schwinger functions are anomalous so that the system is a "Lutttinger liquid." Our results extend the work in [B.G.P.S], where neither spin nor periodic potential were considered; an explicit proof of some technical results used but not explicitly proved there is also provided.

1. Introduction and Statement of the Results

We study by renormalization group techniques the analyticity properties of the Beta function and the behaviour of the pair Schwinger function for momenta near the Fermi surface for a one dimensional system of n fermions moving in a common periodic field $-\partial_{\vec{x}}U(\vec{x})$ and interacting by a short range pair potential. We consider both spinless $\sigma = 0$ or spinning fermions $\sigma = \pm 1/2$. The recent interest about interacting electrons in a periodic potential [D.M., Sh.] motivates our study. The one dimensional hamiltonian is

$$H = T + \lambda V\,, \tag{1}$$

$$T = \sum_{\sigma} \int_{-L/2}^{L/2} d\vec{x}\, \psi^{+}_{\vec{x},\sigma} \left(-\frac{\partial^2}{2m} + U(\vec{x}) - \mu \right) \psi^{-}_{\vec{x},\sigma}\,,$$

$$V = \sum_{\sigma,\sigma'} \int_{-L/2}^{L/2} d\vec{x}d\vec{y}\, v(\vec{x}-\vec{y})(\psi^+_{\vec{x},\sigma}\psi^-_{\vec{x},\sigma})(\psi^+_{\vec{y},\sigma'},\psi^-_{\vec{y},\sigma'})\,,$$

where $\psi^{\pm}_{\vec{x},\sigma}$ are creation or annihilation field operators with spin σ on the Fock space of a fermion system confined in a box $(-L/2, L/2)$ with periodic boundary conditions and at zero temperature, $m > 0$ is the electron mass, μ is the chemical potential, $U(\vec{x}) = U(\vec{x}+a)$ is a C^∞-smooth periodic potential, which for simplicity will be assumed even $U(\vec{x}) = U(-\vec{x})$; a is the lattice spacing, $\lambda v(\vec{r})$ is the spin-independent, electron-electron, interaction, supposed to be even in $\vec{r}$, bounded C^∞ smooth and p_0^{-1} is the interaction range. Clearly we must have $L = Na, N$ integer, and we choose a system of units in which $\hbar = 1$.

It is well known (see for example [Ko.,T.P.]) that it is possible to find two functions $\varepsilon(\vec{k})$ and $\phi(\vec{k},\vec{x})$ defined for complex $\vec{k}$ and satisfying the equation:

$$-\phi(\vec{k},\vec{x})'' + U(\vec{x})\phi(\vec{k},\vec{x}) = \varepsilon(\vec{k})\phi(\vec{k},\vec{x})$$

with $\phi(\vec{k},\vec{x}+a) = e^{i\vec{k}a}\phi(\vec{k},\vec{x})$. Two such functions are holomorphic everywhere except on the vertical segments joining the point $\vec{k}_n = (n+1)\pi/a + ih_n$ with the point $\vec{k}_n^*$, and the point $-\vec{k}_n^*$ with the point $-\vec{k}_n$, where $h_n, n = 0,1,2,\dots$ is a sequence of real numbers such that $\lim_{n\to\infty} h_n = 0$. On the cuts the functions present a jump discontinuity and near the point $\vec{k}_n$ we have

$$\varepsilon(\vec{k}) = \varepsilon_n + c_n(\vec{k}-\vec{k}_n)^{1/2} + o_n[(\vec{k}-\vec{k}_n)^{1/2}]\,, \tag{2}$$

where c_n is a bounded constant, $\lim_{n\to\infty} 2ma^2(n+1)^{-2}\pi^{-2}\varepsilon_n = 1$, and

$$\phi(\vec{x},\vec{k}) = \frac{D(\vec{x})}{(\vec{k}-\vec{k}_n)^{1/4}}(1 + C(\vec{x})(\vec{k}-\vec{k}_n)^{1/2} + o_n[(\vec{k}-\vec{k}_n)^{1/2}]\,, \tag{3}$$

where $C(\vec{x})$ and $D(\vec{x})$ are holomorphic functions. The symmetry of $U(\vec{x})$ clearly implies analogous formulas for $\vec{k}_n^*, -\vec{k}_n$ and $-\vec{k}_n^*$. Finally it is possible to fix the mean of $U(\vec{x})$ so that $\varepsilon(0) = 0$.

For $\vec{k}$ real, $\phi(\vec{k},\vec{x})$ are called "Bloch waves" and $\varepsilon(\vec{k})$ is the "dispersion relation." The functions $\varepsilon_n(\vec{k})$ and $\phi_n(\vec{k},\vec{x})$, where n is the "band index," are ε and ϕ restricted to the segments $(n\pi/a,(n+1)\pi/a]$ and $[-(n+1)\pi/a,-n\pi/a)$. The periodic boundary conditions imply that $\vec{k} = \frac{2n\pi}{L}$. Physically one defines the Fermi momentum p_F so that the ground state energy of the hamiltonian Eq. (1) has the minimum at $n = \frac{2p_F L}{2\pi}$ when $\mu = \varepsilon(p_F)$, and the Fermi velocity v_0 is defined as the minimum energy increase by adding a particle to the ground state, divided by $\vec{k}_0 - p_F$, if $\vec{k}_0$ is the momentum of the particle added. We require that $p_F = \frac{2\pi}{L}(n_F + 1/2)$, where n_F is an integer (and this of course is possible if $p_F a/\pi$ is a positive rational number). It will be easy to see that our results remain valid also without this simplifying condition. The band "containing" the Fermi momentum (i.e. the band such that $\phi(p_F,\vec{x}) \equiv \phi_n(p_F,\vec{x})$) will be called the *conduction band.*

The creation or annihilation operators $\psi^{\pm}_{\vec{k},\sigma}$ of a Bloch wave are defined by

$$\psi^{\pm}_{x,\sigma} = e^{x_0 T}\psi^{\pm}_{\vec{x},\sigma}e^{-x_0 T} = \frac{1}{L}\sum_{\vec{k}} e^{\pm(\varepsilon(\vec{k})-\mu)x_0}\phi(\vec{k},\pm\vec{x})\psi^{\pm}_{\vec{k},\sigma}\,. \tag{4}$$

The "propagator" of a Bloch wave in a volume L with temperature β^{-1} is given by

$$g(x,y) = \delta_{\sigma,\sigma'}\frac{1}{L\beta}\sum_k \frac{e^{-ik_0(x_0-y_0)}}{-ik_0+E(\vec{k})}\phi(\vec{k},-\vec{x})\phi(\vec{k},\vec{y})$$
$$\equiv \delta_{\sigma,\sigma'}\frac{1}{L\beta}\sum_k e^{-ik_0(x_0-y_0)}\phi(\vec{k},-\vec{x})\phi(\vec{k},\vec{y})g(k)\,. \tag{5}$$

Here $x=(x_0,\vec{x}), k=(k_0,\vec{k}), e^{ik_0\beta}=-1, e^{i\vec{k}L}=1, E(\vec{k})=\varepsilon(\vec{k})-\mu$. It is important to note that in terms of the operators $\psi^{\pm}_{\vec{k},\sigma}$, the interaction V can be written

$$V=\sum_{\vec{n},\sigma,\sigma'}\left(\frac{1}{L}\right)^4\sum_{\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4}\lambda\hat{v}_n(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)\delta\left(\vec{k}_1+\vec{k}_2-\vec{k}_3-\vec{k}_4+\vec{n}\frac{2\pi}{a}\right)$$
$$\times\,\psi^+_{\vec{k}_1,\sigma}\psi^+_{\vec{k}_2,\sigma'}\psi^-_{\vec{k}_3,\sigma'}\psi^+_{\vec{k}_4,\sigma}\,, \tag{6}$$

if $\delta(\vec{k})=L\delta_{\vec{k},0}$ and $\delta_{i,j}$ is the Kronecker delta. The presence of the periodic potential has the effect that the sum of the "incoming " and "outgoing" momenta (i.e. $\vec{k}_1,\vec{k}_2$ and $\vec{k}_3,\vec{k}_4$ respectively) is not zero but it is equal to a vector of the reciprocal lattice $2\vec{n}\pi/a$, that is momentum is not conserved. The terms in which $\vec{n}\neq 0$ are called *Umklapp terms.*

If the fermions *are spinless and there is no periodic potential,* then one can prove rigorously, [B.G., B.G.M., B.G.P.S.] that the pair Schwinger function has an anomalous long distance behaviour. This essentially means that the occupation number at the Fermi surface is not discontinuous but is given by $n_{\vec{k}}-n_{p_F}\simeq ||\vec{k}|-p_F|^{2\eta(\lambda)}\mathrm{sign}(|\vec{k}|-p_F), \eta(\lambda)=O(\lambda^2)$ (*anomalous* Fermi surface). In the present paper we derive new, corresponding, results about one dimensional interacting Fermi systems in a periodic potential, both for spinning or spinless fermions.

If the fermions are spinless and the conduction band is not filled the Fermi surface is anomalous at small enough coupling; in fact we prove:

Theorem 1.1. *Given a C^∞-smooth pair potential $\lambda v(\vec{x}-\vec{y})$ with short range p_0^{-1}, a fermion mass m, a Fermi momentum p_F and a C^∞-smooth periodic even potential $U(\vec{x})$, and if $p_F\neq\frac{n\pi}{a}$, n integer, then there exists $\varepsilon>0$ such that one can define, for $|\lambda|<\varepsilon$, functions $p_F(\lambda),\eta(\lambda)$ analytic in λ and divisible by λ^2, such that:*

1. *the one dimensional spinless Fermi system with hamiltonian*

$$\sum_i^n\left(-\frac{\partial^2_{\vec{x}_i}}{2m}+U(\vec{x}_i)-\mu\right)+\lambda\sum_{i<j}v(\vec{x}_i-\vec{y}_j) \tag{7}$$

admits a ground state with a Euclidean pair Schwinger functions $S(k)$ verifying, for $|\vec{k}|-p_F^0$ and k_0 small, where $p_F^0=p_F+p_F(\lambda)$, the relation

$$S(k)=S_0(k)|p_0^{-1}k'|^{2\eta}+A_k(\lambda)|p_0^{-1}k'|^{-1+2\eta}\,,$$

where $|k'|=\sqrt{(|\vec{k}|-p_F^0)^2+v_0^{-2}k_0^2}, S_0(k)$ is the Schwinger function for the free gas with Fermi momentum p_F^0 and Fermi velocity v_0 and $|A_k(\lambda)|<C|\lambda|$, for a suitable constant C;

2. *$\eta(\lambda)$, the "anomalous exponent," has the expansion $\lambda^2 \hat{v}_0^2 + O(\lambda^3)$ with the coefficient $\hat{v}_0$ given by $\hat{v}_0 = \hat{v}_0(p_F, -p_F, -p_F, p_F)$.*

This result agrees with the considerations in [Sh.] based on an analogy with some solvable models, suggesting that the periodic potential in a spinless $d = 1$ interacting Fermi system has "no effect" unless $\lambda = O(1)$ (i.e. the results are not qualitatively different from those of the case of translation invariant spinless fermions in the continuum). If the fermions are spinning we have that, if the interaction is small and repulsive and the conduction band is neither filled nor half filled, the Fermi surface is anomalous; more precisely we prove:

Theorem 1.2. *Given a C^∞-smooth pair potential $\lambda v(\vec{x} - \vec{y})$ with short range p_0^{-1} and with $\hat{v}_0(-p_F, p_F, -p_F, p_F) > 0$, a fermion mass m, a Fermi momentum p_F and given a C^∞-smooth periodic even potential $U(\vec{x})$, and if $p_F \neq \frac{n\pi}{2a}$, n integer then there exists $\varepsilon > 0$ such that one can define functions $p_F(\lambda), \eta(\lambda)$ analytic in λ for $|\lambda - \varepsilon/2| < \varepsilon/2$, and divisible by λ^2, such that:*

1. *the one dimensional spinning Fermi system with hamiltonian*

$$\sum_i^n \left(-\frac{\partial_{\vec{x}_i}^2}{2m} + U(\vec{x}_i) - \mu \right) + \lambda \sum_{i<j} v(\vec{x}_i - \vec{y}_i) \tag{8}$$

admits a ground state with a Euclidean pair Schwinger function $S(k)$, verifying for $|\vec{k}| - p_F^0$ and k_0 small, where $p_F^0 = p_F + p_F(\lambda)$, the relation

$$S(k) = S_0(k)|p_0^{-1}k'|^{2\eta} + A_k(\lambda)|p_0^{-1}k'|^{-1+2\eta} ,$$

where $|k'| = \sqrt{(|\vec{k}| - p_F^0)^2 + v_0^{-2}k_0^2}, S_0(k)$ is the Schwinger function for the free gas with Fermi momentum p_F^0 and Fermi velocity v_0 and $|A_k(\lambda)| < C|\lambda|$, for a suitable constant C;

2. *$\eta(\lambda)$, the "anomalous exponent," has the expansion $\lambda^2 \hat{v}_0^2 + O(\lambda^3)$ with the coefficient $\hat{v}_0$ given by $\hat{v}_0 = \hat{v}_0(p_F, -p_F, -p_F, p_F)$.*

The above theorems show that the Fermi momentum and the Fermi velocity in the free system (i.e. $\lambda = 0$) and in the interacting system (i.e. $\lambda \neq 0$) are different: one usually says that the interaction *renormalizes* the values of the Fermi momentum and Fermi velocity. From a physical point of view it is more natural to fix the value of these quantities in the interacting theory so that we can replace the hamiltonian Eq. (1) by an hamiltonian containing two free parameters to be tuned so that the ground state has some fixed value of the Fermi momentum and Fermi velocity (we call them simply p_F and v_0). So we shall study

$$H = T + \lambda V + \alpha T + \nu N , \tag{9}$$

where $N = \sum_\sigma \int_{-L/2}^{L/2} d\vec{x}\, \psi_{\vec{x},\sigma}^+ \psi_{\vec{x},\sigma}^-$ and we prove, for instance in the spinning case, that there exists an $\varepsilon > 0$ so that it is possible to fix $\alpha(\lambda), \nu(\lambda), \eta(\lambda)$ analytic in $|\lambda - \varepsilon/2| < \varepsilon/2$ and divisible by λ^2 such that the Euclidean pair Schwinger function verifies, for $|\vec{k}| - p_F, k_0$ small, the relation

$$S(k) = S_0(k)|p_0^{-1}k'|^{2\eta} + A_k(\lambda)|p_0^{-1}k'|^{-1+2\eta} , \tag{10}$$

where $|k'| = \sqrt{(|\vec{k}| - p_F)^2 + v_0^{-2}k_0^2}$, $S_0(k)$ is the Schwinger function for the free gas with Fermi momentum p_F and Fermi velocity v_0, $|A_k(\lambda)| < C|\lambda|$, for a suitable constant C and the $\eta(\lambda)$, "anomalous exponent," has the expansion $\lambda^2\hat{v}_0^2 + O(\lambda^3)$ with the coefficient $\hat{v}_0$ given by $\hat{v}_0 = \hat{v}_0(p_F, -p_F, -p_F, p_F)$. This result is of course equivalent to Theorem 1.2, by a trivial application of a implicit function theorem. Analogous considerations can be made for the spinless case. From Eq. (10) we see that our choice of $\alpha(\lambda), v(\lambda)$ fix the value of the Fermi momentum to p_F; our choice will fix also the value of the Fermi velocity to v_0. This is not clear from Eq. (10): it is not, strictly speaking, proved here but it should be clear from the proofs that follow. We note that there are many ways different from Eq. (9) to introduce two free parameters in the hamiltonian (for instance one can have instead of αT a term like $\alpha\sum_\sigma \int_{-L/2}^{L/2} d\vec{x}\, \psi^+_{\vec{x},\sigma} U(x) \psi^-_{\vec{x},\sigma}$ or $\alpha\sum_\sigma \int_{-L/2}^{L/2} d\vec{x}\, \psi^+_{\vec{x},\sigma}(\frac{\partial_{\vec{x}}^2}{2m} - \mu)\psi^-_{\vec{x},\sigma}$), but it is easy to check that what follows does not depend on the particular choice Eq. (1). Finally in the cases not covered by the theorems we are not able to give rigorous results. The difficulties we met in such cases are discussed briefly in the last section.

Technical comments. The proof of the theorems is based on the renormalization group and follows the ideas in [B.G., B.G.M., B.G.P.S] for a system of spinless fermions with no periodic potential. However our proof contains new technical results necessary to treat our model. The first difficulty one meets in studying fermions in a periodic potential is that the estimates on a Bloch wave propagator are not trivial as in the case of plane waves and requires a careful analysis, for the presence of the non-analyticity points in the dispersion relation and in the Bloch waves themselves. Due to this fact and contrary to the case studied in [B.G., B.G.M., B.G.P.S] we are not able to find a definition of quasi-particles such that the free quasi-particle propagator "at scale h" decays exponentially with the distance; we have only a power law decay (see below, or [B.G.P.S.], for the precise notion of quasi-particles and of scales). In [B.G.P.S.] the convergence of the anomalous Beta function, as a power series in the running coupling constants (redefined precisely below, for completeness), was proved by using in an essential way that the fixed scale quasi-particle propagators have an exponential decay; extending the proof to propagators with a weaker decay, i.e. power-law, is non-trivial and requires some new technical results (see App. 2, 3).

The use, for spinning fermions, of the localization introduced in [B.G., B.G.M., B.G.P.S] leads to local terms containing irrelevant operators (this was the reason for the apparent difficulties found by [B.G.] in the spinning case). From our definition of localization we have in the spinning case six relevant running couplings when the band is not half-filled, and seven when it is half-filled; the Umklapp scattering is relevant only at half-filling. In the spinless case there are four running couplings like in the $U(\vec{x}) = 0$ case and Umklapp is never relevant, not even in the half-filled case (by Pauli's exclusion principle).

In the spinless case the exact solution of the Luttinger model, [M.L.], is the key to the proof that the Beta function is vanishing, the renormalization flow is bounded and that the Fermi surface is anomalous. Schwinger functions and running couplings are analytic for $|\lambda| < \varepsilon$. However there is a major difference between the spinless and spinning case, namely in the spinning case Schwinger functions are not analytic in λ around $\lambda = 0$ and this is a manifestation of the fact that the analysis of the renormalization group flow is substantially different from the spinless case: the beta function is not vanishing, not even to the second order. We show nevertheless that, if the conduction band is not half filled and the interaction is replusive, it is possible to control the flow, but this can be done only giving up analyticity (but retaining almost surely Borel summability). This is done proving that the Beta function is partially vanishing using the exact solution of the Mattis model [M.]. Our proof is conceptually similar to the proof in [B.G.P.S.] for the vanishing of the Beta function in the spinless case and provides the proof of some technical results stated and used in [B.G.P.S.] for this proof but not explicitly proved there.

Another noticeable difference is that our discussion is consistently performed at finite volume and temperature β^{-1}, taking care in getting bounds which are uniform in $L, \beta \to \infty$ (in [B.G.P.S.] the theory is developed directly for $\beta = \infty, L = \infty$ and the uniformity in β, L is not really discussed). As a byproduct we extend and derive explicitly the expression of the running couplings at the cut off scale [B.G.P.S.] Eq. (7.10).

2. Multiscale Decomposition and Effective Potential

We consider a Grassmann algebra, whose elements $\psi^s_{k,\sigma}$ verify $\{\psi^s_{k,\sigma}, \psi^{s'}_{k',\sigma'}\} = 0$. The *Euclidean fields* are, if $\varepsilon = \pm$,

$$\psi^\varepsilon_{x,\sigma} = \frac{1}{\beta L} \sum_k e^{i\varepsilon k_0 x_0} \phi(\vec{k}, \varepsilon \vec{x}) \psi^\varepsilon_{k,\sigma} ,$$

where $e^{ik_0\beta} = -1, e^{i\vec{k}L} = 1$ (see [B.G.P.S.] or [B.G.]). Although β and L are kept finite we will write $\int \frac{dk}{(2\pi)^2}$ instead of $\frac{1}{\beta L}\sum_k$ to make the notation more clear. A "functional integration" is defined on the monomials by the Wick rule

$$\int P(d\psi) \psi^+_{x_1,\sigma_1} \dots \psi^+_{x_n,\sigma_n} \psi^-_{y_1,\sigma'_1} \dots \psi^-_{y_n,\sigma'_n} = \sum_{\pi \in P_n} (-1)^\pi \prod_i g(x_i, y_{\pi(i)}) \delta_{\sigma_i, \sigma'_{\pi(i)}} ,$$

where P_n is the set of all the permutation of n elements and $(-1)^\pi$ is the parity of the permutation. The above integration rule is extended to a more complicated expression by linearity. We call this rule for associating numbers to grassmannian monomials, and by linear extension, to grassmannian polynomials a *grassmannian integration with propagator* $g(x, y)$ or a *measure* with propagator $g(x, y)$. For our purposes the case in which $g(x, y)$ is given by Eq. (5) will be of interest.

All the properties of the Gibbs state generated by the hamiltonian Eq. (1) at temperature $\beta^{-1} = 0$ can be deduced from the functional V_{eff} defined by

$$e^{-V_{eff}(\varphi)} = \frac{1}{\mathcal{N}} \int P(d\psi) e^{-\overline{V}(\psi + \varphi)} , \tag{11}$$

where $\mathcal{N} = \int P(d\psi) e^{-\overline{V}(\psi)}$ is a normalization constant (so that $V_{eff}(0) = 0$) and $\overline{V} = \lambda V + \alpha T + \nu N$ with, if $x_i = (x_{0,i}, \vec{x}_i)$,

$$V = \sum_{\sigma,\sigma'} \int_{\Lambda \times \Lambda} dx_1 dx_2 v(\vec{x}_1 - \vec{x}_2) \delta(x_{0,1} - x_{0,2}) \psi^+_{x_1,\sigma} \psi^+_{x_2,\sigma'} \psi^-_{x_2,\sigma'} \psi^-_{x_1,\sigma} ,$$

$$T = \sum_\sigma \int_\Lambda dx \psi^+_{x,\sigma} \left(-\frac{\partial^2_{\vec{x}}}{2m} + U(\vec{x}) - \mu \right) \psi^-_{x,\sigma} \quad N = \sum_\sigma \int_\Lambda dx \psi^+_{x,\sigma} \psi^-_{x,\sigma} ,$$

and $\Lambda = (-\beta/2, \beta/2) \times (-L/2, L/2)$. *The Schwinger functions* are defined by

$$S(x_1, \sigma_1, \ldots, x_n, \sigma_n, y_1, \sigma_1', \ldots, y_n, \sigma_n') = \frac{1}{\mathcal{N}} \int P(d\psi) e^{-\bar{V}(\psi)} \psi^+_{x_1,\sigma_1} \ldots \psi^+_{x_n,\sigma_n} \psi^-_{y_1,\sigma_1'} \ldots \psi^n_{y_n,\sigma_n'} \,. \tag{12}$$

We decompose the measure into a product of two independent measures, i.e. $P(d\psi) = P(d\psi^{u.v.}) \cdot P(d\psi^{i.r.})$. The grassmannian integral in Eq. (11) can be rewritten as

$$e^{-V_{eff}(\varphi)} = \frac{\mathcal{N}_0}{\mathcal{N}} \int P(d\psi_{i.r.}) e^{-V^0(\psi_{i.r.}+\varphi)} \,, \tag{13}$$

$$e^{-V^0(\psi_{i.r.}+\varphi)} = \frac{1}{\mathcal{N}_0} \int P(d\psi_{u.v.}) e^{-\bar{V}(\psi_{i.r.}+\psi_{u.v.}+\varphi)} \,, \tag{14}$$

where $\psi^{u.v.}, \psi^{i.r.}, \phi$ are anticommuting grassmannian fields, $\mathcal{N}_0 = \int P(d\psi_{u.v.}) e^{-\bar{V}(\psi_{u.v.})}$ and $P(d\psi^{u.v.}), P(d\psi^{i.r.})$ denote respectively the grassmannian integrations with vanishing cross propagator and with propagators $g_{u.v.}, g_{i.r.}$ given by

$$g_{u.v}(x, y) = \int \frac{dk}{(2\pi)^2} \frac{e^{-ik_0(x_0-y_0)}}{-ik_0 + E(\vec{k})} \phi(\vec{k}, -\vec{x}) \phi(\vec{k}, \vec{y}) h(k_0^2 + E(\vec{k}^2)) \,, \tag{15}$$

$$g_{i.r}(x, y) = \int \frac{dk}{(2\pi)^2} \frac{e^{-ik_0(x_0-y_0)}}{-ik_0 + E(\vec{k})} \phi(\vec{k}, -\vec{x}) \phi(\vec{k}, \vec{y}) (1 - h(k_0^2 + E(\vec{k})^2)) \,,$$

where $h(t)$ is a C^∞ function in its argument t and it is identically 1 if $t > \mu^2$, hence $h(k_0^2 + E(\vec{k})^2)$ is identically 1 if $\vec{k}$ is above the first band ($|\vec{k}| > \pi/a$) and it is also identically 1 for $k_0, \vec{k}$ near (0,0): this property shows that the integral in $g_{u.v}(x, y)$ involves only $(k_0, \vec{k})$'s which are "far" from the Fermi surface $k_0 = 0, E(\vec{k}) = 0$, thus justifing the u.v. name.

It is possible to check that (see App. 1)

$$g_{u.v.}(x, y) = G(x - y) + R(x, y) \,,$$
$$G(x) = H(\vec{x} - \vec{y}) H(x_0 - y_0) \theta(x_0 - y_0) e^{(x_0 - y_0)\mu} \left(\frac{m}{2\pi(x_0 - y_0)} \right)^{1/2} e^{-\frac{m(\vec{x}-\vec{y})^2}{2(x_0-y_0)}} \,, \tag{16}$$

where $R(x, y) \leqq \frac{C_N}{1+(p_0|x-y|)^N}, |x|^2 = x_0^2 v_0^2 + \vec{x}^2$ for all N, and $H(t)$ is a smooth function of compact support such that $H(t) = 1$, if $|t| \leqq 1$ and $H(t) = 0$ if $|t| \geqq \gamma \geqq 1$.

The Eq. (16) shows that $g_{u.v.}$ has the same properties of the ultraviolet propagator with $U(\vec{x}) = 0$, so that we can repeat the proofs leading to the theory of the ultraviolet problem for the $U(\vec{x}) = 0$ case in [B.G.P.S.], (it is trivial to include the presence of the spin in developing the proof). In [B.G.P.S.] as well as in our case the following statement is rather easy and the difficult part does not concern the ultraviolet problem but only the infrared one:

Theorem 2.1. *There exists* $\varepsilon > 0$ *such that* V^0, *see Eq* (14), *can be written for* $|z| \leqq \varepsilon, z = (\lambda, \alpha, \nu)$ *and if* $\psi \equiv \psi^{i.r.}$, *in the following way as a sum of a term* *linear in the couplings*:

$$V^0(\psi) = \sum_{\sigma,\sigma'} \lambda \int_\Lambda dxdy\, v(x-y)\psi^+_{x,\sigma}\psi^+_{y,\sigma'}\psi^-_{y,\sigma'}\psi^-_{x,\sigma} + 2\lambda \int dxdy\, v(x-y)R(x,y)\psi^+_{x,\sigma}\psi^-_{y,\sigma}$$
$$+ \sum_\sigma \int_\Lambda dx(\nu - 2\lambda K(x))\psi^+_{x,\sigma}\psi^-_{x,\sigma} + \alpha \sum_\sigma \int_\Lambda dx \psi^+_{x,\sigma}\left(-\frac{\Delta}{2m} + U(x) - \mu\right)\psi^-_{x,\sigma} \tag{17}$$

with $K(x) = \int_\Lambda v(\vec{x}-\vec{y})R(y,y)dy$ *and of a remainder of order at least 2 in the couplings* λ, α, ν,

$$\sum_\sigma \int_\Lambda dx \psi^+_{x,\sigma}\left(-\frac{\Delta}{2m} + U(x) - \mu\right)\psi^-_{y,\sigma} W_2(x,y;z) \tag{18}$$
$$+ \sum_{\sigma_1,\dots,\sigma_n} \sum_{\substack{n,n_1,n_2\\ n_1+n_2=n}} \int dx_1 \dots dx_{2n} \psi^+_{x_1} \dots \psi^+_{x_{n_1}} \psi^-_{x_{n_1+1}} \psi^-_{x_{2n-n_2}}$$
$$\times \left(-\frac{\Delta}{2m} + U(x_{2n-n_2}) - \mu\right)\psi^-_{x_{2n-n_2+1}} \cdots \left(-\frac{\Delta}{2m} + U(x_{2n}) - \mu\right)$$
$$\times \psi^-_{x_{2n}} W_{n_1,n_2}(z, x_1, \dots, x_{2n}),$$

where the kernels W_{n_1,n_2} *are products of suitable delta functions times bounded functions analytic in* z *if* $|z| \leqq \varepsilon$ *and, if* $d(x_1 \dots, x_n)$ *is the length of the shortest tree connecting the points ("tree distance" or "graph distance"), the following bounds hold*:

$$\int dx_1 \dots dx_{2n} |W_{n_1,n_2}(z, x_1, \dots)|(1 + d(x_1, \dots, x_n))^N < c(N)\Lambda |z|^{max(2,n-1)},$$

while W_2 *satisfies the weaker bound*

$$|\int dxdy \phi(\vec{k}_1, \vec{x})\phi(-\vec{k}_2, \vec{y})e^{-ik_0(x_0-y_0)}W_2(x,y;z)| \leqq D|z|^2$$

for $|\vec{k}_1|, |\vec{k}_2| \leqq \pi/a$ *and* $|z| \leqq \varepsilon$.

A more interesting and difficult problem is the analysis of the "infrared" integration Eq. (13): We decompose the grassmannian integration $P(d\psi)$ into a product of independent grassmannian integrations, that is $P(d\psi_{i.r.}) = \prod_{h=-\infty}^0 P(d\psi^h)$. This can be done by setting $g_{i.r.}(k) = \sum_{h=-\infty}^0 g^h(k)$ and by writing $\psi_{i.r.} = \sum_h \psi^{(h)}$, with ψ^h being a family of grassmannian fields with vanishing "cross propagator" (i.e. independent) and with propagator $\int \psi^h_{k_1,\sigma_1}\psi^h_{k_2,\sigma_2}P(d\psi^h) = \delta_{\sigma_1,\sigma_2}\delta(k_1 - k_2)g^h(k_1)$:

$$g^h(k) = \frac{f(\gamma^{-2h+2}(k_0^2 + E(\vec{k})^2))}{-ik_0 + E_0(\vec{k})}, \tag{19}$$

where $f(x) = h(x)(1 - h(\frac{x}{\gamma^2}))$ is a C^∞ function with compact support and $\gamma > 1$. However such decomposition is not suitable for a renormalization group analysis, because $g^h(k)$ have no good scaling properties (as the system has two intrinsic scale lengths, i.e. a and p_F). In order to overcome this difficulty we introduce new grassmannian fields $\psi^h_{k,\vec{\omega},\sigma}$, called *quasi-particles* field operators, with propagators

$g_{\vec\omega}^{(h)}(k)$ and vanishing cross propagators, such that

$$\psi^h_{k,\sigma} = \sum_{\vec\omega=\pm 1} \psi^h_{k,\vec\omega,\sigma}, \quad \psi^h_{k,\sigma} = \sum_{\vec\omega=\pm 1} \psi^h_{k,\vec\omega,\sigma}, \quad g^h(k) = \sum_{\vec\omega=\pm 1} g^h_{\vec\omega}(k)\,.$$

Although there are infinitely many ways to represent $g^h(k)$ in this form, there is at least one such that $g^h_{\vec\omega}(k)$ has good scaling property. Define

$$g^h(k) = \sum_{\vec\omega=\pm 1} \theta(\vec\omega\vec k)\frac{f(\gamma^{-2h+2}(k_0^2 + E(\vec k)^2))}{-ik_0 + E(\vec k)}\,,$$

where θ is the step function. If $k = k' + (0, \vec\omega p_F)$, i.e. $\vec k'$ is the momentum measured from the Fermi surface and is restricted to values of the form $\frac{2\pi}{L}(n + 1/2)$, where n is an integer, it is possible to prove that:

Lemma 2.1. *If v_0 is different from zero then the quasi-particle propagator can be written as*

$$\gamma^{-h} g^h_{\vec\omega}(\gamma^{-h}k') = \gamma^{-h}\bar g^h_{\vec\omega}(\gamma^{-h}k') + C_h(\gamma^{-h}k')\,,$$

where

$$\bar g^h_{\vec\omega}(\gamma^{-h}k') = \frac{f(\gamma^2([\gamma^{-h}k_0]^2 + ([\gamma^{-h}\vec k']v_0)^2)}{-i[\gamma^{-h}v_0k_0] + \vec\omega[\gamma^{-h}\vec k']} v_0\,,$$

and $C_h(t)$ is C^∞ with support contained in $(-\pi/a, \pi/a)$ and such that $|C_h(t)| \leqq M$, where M does not depend on h, T, L.

Note that $v_0 = 0$ only if $p_F = 0, \pi/a$, as we have supposed that $|p_F| \leqq \pi/a$.

We define the "position space" quasi-particle fields as

$$\hat\psi^{h,\pm}_{x,\sigma} = \int \frac{dk}{(2\pi)^2} e^{\pm i(k_0 t + \vec k\vec x)}\psi^h_{k,\sigma} \quad \hat\psi^{h,\pm}_{x,\sigma} = \sum_{\vec\omega=\pm 1} e^{\pm i\vec\omega p_F \vec x}\hat\psi^{h,\pm}_{x,\vec\omega,\sigma}\,,$$

$$\hat\psi^{h,\pm}_{x,\vec\omega,\sigma} = \int \frac{dk'}{(2\pi)^2} e^{\pm i(k_0 t + \vec k' \vec x)}\psi^h_{k'+\vec\omega p_F,\vec\omega,\sigma} \quad \hat g^h_{\vec\omega}(x) = \int \frac{dk'}{(2\pi)^2} e^{-i(k_0 t + \vec k' \vec x)} g^h_{\vec\omega}(k')\,. \tag{20}$$

Note that, at variance with the work in the translation invariant case, [B.G.], the fields $\sum_{h,\vec\omega} e^{\pm i\vec\omega p_F \vec x}\hat\psi^{h,\pm}_{x,\sigma,\vec\omega}$ cannot be identified with the fields $\psi^{i.r.}_{x,\sigma}$ above: this is an important difference with respect to [B.G.]. The relation between $\hat\psi^{h,\pm}_{x,\sigma,\vec\omega}$ and $\psi^{i.r.\pm}_{x,\sigma}$ is more complicated as it is given by

$$\psi^{i.r.\pm}_{x,\sigma} = \int dk\phi(\vec k, \pm\vec x)e^{\pm ik_0x_0} \int dx' e^{\mp ikx'} \sum_{h,\vec\omega} e^{\pm i\vec\omega p_F \vec x'}\hat\psi^{h,\pm}_{x',\vec\omega,\sigma}\,.$$

The natural definition, if we wanted to operate in analogy with [B.G.], would be introducing the field

$$\psi^{h,\pm}_{x,\vec\omega,\sigma} = \int \frac{dk'}{(2\pi)^2} e^{\mp ik_0x_0}\frac{\phi(\vec k' + \vec\omega p_F, \pm x)}{\phi(\vec\omega p_F, \pm\vec x)}\psi^{h,\pm}_{k'+\vec\omega p_F,\vec\omega,\sigma}$$

with propagator $g^h_{\vec\omega}(x,y)$ and to set $\psi^{i.r.,\pm}_{x,\sigma} = \sum_{\vec\omega,h}\phi(\vec\omega p_F, \pm\vec x)\psi^{h,\pm}_{x,\vec\omega,\sigma}$. Such definition would be "the same" as in [B.G.], with of course plane waves replaced by Block waves, but the definition of "localization" would become very cumbersome, so we prefer not to use these fields.

It is not difficult to check that:

Lemma 2.2. *For $N > 1$ we have*:

$$\hat{g}^h_{\vec{\omega}}(x) < \gamma^h \frac{C_N(a, p_F)}{1 + p_0^{-N}[\gamma^{2h}(\vec{x})^2_\pi + v_0^2\gamma^{2h}(x_0)^2_\pi)]^{N/2}} ,$$

$$\hat{C}_h(x) < \gamma^{2h} \frac{C_N(a, p_F)}{1 + p_0^{-N}[\gamma^{2h}(\vec{x})^2_\pi + \gamma^{2h}v_0^2(x_0)^2_\pi)]^{N/2}} ,$$

where $\hat{C}_h(x)$ is the Fourier transform of $C_h(\gamma^{-h}\vec{k})$, $(\vec{x})_\pi = \frac{\sin \vec{x}\pi/L}{\pi/L}$ and $(x_0)_\pi = \frac{\sin \pi/\beta}{\pi/\beta}$. Moreover

$$\hat{g}^h_{\vec{\omega}}(x) \leqq \gamma^h \frac{C_N(a, p_F)}{1 + \gamma^{hN}|p_0 x|^N} \quad \hat{C}_h(x) \leqq \gamma^{2h} \frac{C_N(a, p_F)}{1 + \gamma^{hN}|p_0 x|^N} \; for \; |\vec{x}| \leqq L/2, \, |x_o| \leqq \beta/2$$

with $|x|^2 = v_0^2 x_0^2 + \vec{x}^2$.

One could hope that, by using an analytic (rather than C^∞) cut off function h to realise the decomposition Eqs. (15),(19), and by making a shift in the integral of the infrared propagator $g^h_{\vec{\omega}}(x, y)$ or $\hat{g}^h_{\vec{\omega}}(x)$ to a complex line with imaginary part $\gamma^h \bar{h}$, $\gamma^h \bar{h} + h_n$, following a path similar to that of Appendix 1, one would obtain that $\hat{g}^h_{\vec{\omega}}(x)$ decays exponentially for large distance. However this does not happen: essentially because one cannot use in this case the symmetry properties used in Appendix 1 and because of the accumulation of the non-analyticity points on the real $\vec{k}$ line. We strongly suspect that there is no way to realize a multiscale decomposition for our problem such that the quasi-particles have propagator with exponential decay in the x-variables. In any event we were not able to find it: so that we preferred a compact support cut-off. In this way the analysis of the perturbative expansion is clearer, as the distinction between ultraviolet and infrared term is sharper.

3. The Effective Potential in the Infrared Region

In this section we set $v_0 = 1$ for simplicity and we begin the analysis of the infrared problem, which consists in the study of the possibility to give a rigorous meaning to $V_{\mathit{eff}}(\phi)$ defined by the functional integration Eq. (13). We start by studying the functional integral $\int P(d\psi_{i.r})e^{-V^0(\psi_{i.r})}$, which is the normalization constant in Eq. (9). We can represent $V^0(\psi)$, see Eq. (17), (18), in terms of quasi-particles fields $\psi^\varepsilon_{\vec{k},\vec{\omega},\sigma}$, where $\varepsilon = \pm 1$, so that $V^0(\psi)$ is given by a sum of terms like

$$V^0_m(\psi) = \int \frac{dk_1'}{(2\pi)^2} \cdots \frac{dk_n'}{(2\pi)^2} f^m(k_1', \ldots, k_m'; \underline{\vec{\omega}})\delta\left(\sum_{i=1}^m (k_i' + \vec{\omega}_i p_F)\varepsilon_i + \frac{2n\pi}{a} \right) \prod_{i=1}^m \psi^{\varepsilon_i}_{k_i' + \vec{\omega}_i p_F, \vec{\omega}_i, \sigma_i} . \tag{21}$$

We can isolate the relevant part of $V^0(\psi)$ by introducing a *localization operator* $\mathscr{L}$ on the Fermi surface acting on $V^0(\psi)$ as follows; $\mathscr{L} V^0_m(\psi) = 0$ for $m > 4$, while (see discussion after Eq. (30), (31) for a motivation of the *localization* name given to $\mathscr{L}$) a "natural" definition for $\mathscr{L}$ if $m = 2, 4$ should be to computing $f^m(k_1',\dots,k_m';\underline{\vec\omega})$ and (if $m=2$) its derivative at the Fermi surface, i.e. for $k_1'=,\dots,=k_m'=0$. However k' cannot assume the value 0 as $k'=k-(0,\vec\omega p_F)$ has the form $k'=(2\pi/L(n_1+1/2),2n_2\pi/\beta+1/2)$, with n_1,n_2 integer, for the antiperiodic boundary temporal conditions and the definition of $p_F=2\pi/L(n_F+1/2)$, and this takes to the complicated formulae below:

$$\begin{aligned}
&\mathscr{L}\int\prod_{i=1}^{4}dk_i' f^4(k_1',k_2',k_3',k_4';\underline{\vec\omega})\delta(k_1'+k_2'-k_3'-k_4'\\
&\qquad+\left(\vec\omega_1+\vec\omega_2-\vec\omega_3-\vec\omega_4)p_F+\frac{2n\pi}{a}\right)\\
&\qquad\psi^+_{k_1'+\vec\omega_1 p_F,\vec\omega_1,\sigma}\psi^+_{k_2'+\vec\omega_2 p_F,\vec\omega_2,\sigma'}\psi^-_{k_3'+\vec\omega_3 p_F,\vec\omega_3,\sigma'}\psi^-_{k_4'+\vec\omega_4 p_F,\vec\omega_4,\sigma}\\
&\qquad=\delta_{(\vec\omega_1+\vec\omega_2-\vec\omega_3-\vec\omega_4)p_F+n\frac{2\lambda}{a},0}f^{4,L}(\underline{\vec\omega})\int\prod_{i-1}^{4}dk_i'\delta(k_1'+k_2'-k_3'-k_4')\\
&\qquad\times\psi^+_{k_1'+\vec\omega_1 p_F,\vec\omega_1,\sigma}\psi^+_{k_2'+\vec\omega_2 p_F,\vec\omega_2,\sigma'}\psi^-_{k_3'+\vec\omega_3 p_F,\vec\omega_3,\sigma'}\psi^-_{k_4'+\vec\omega_4 p_F,\vec\omega_4,\sigma}\,,
\end{aligned}\tag{22}$$

where $\delta_{a,b}$ is the Kronecher delta equal to 1 if $a=b$ and zero otherwise;

$$\begin{aligned}
&\mathscr{L}\int dk_1'dk_2'\delta(k_1'-k_2'+(\vec\omega_1-\vec\omega_2)p_F+2n\pi/a)f^2(k_1',k_2';\underline{\vec\omega})\\
&\quad\times\psi^+_{k_1'+\vec\omega_1 p_F,\vec\omega_1,\sigma}\psi^-_{k_2'+\vec\omega_2 p_F,\vec\omega_2,\sigma}=\\
&\quad\delta_{(\vec\omega_1-\vec\omega_2)p_F+2n\pi/a,0}\int dk_1'dk_2'\delta(k_1'-k_2')\left[f^{2,L_a}(\underline{\vec\omega})+E(\vec K_1'+\vec\omega_1 p_F)\right.\\
&\quad\left.\times\,\vec\omega_1 f^{2,L_b}(\underline{\vec\omega})+k_1^0 f^{2,L_c}(\underline{\vec\omega})\right]\psi^+_{k_1'+\vec\omega_1 p_F,\vec\omega_1,\sigma}\psi^-_{k_2'+\vec\omega_2 p_F,\vec\omega_2,\sigma}\,,
\end{aligned}\tag{23}$$

where if $\tilde\partial_{\vec k}f(\vec k)$, the discrete derivative, is defined for instance as $\frac{f(\vec k+2\pi/L)-f(\vec k)}{2\pi/L}$, and an analogous definition is set for k_0, and:

$$\begin{aligned}
f^{4,L}_{h_{v_0}}(\underline{\vec\omega})&=\frac14\sum_{i,j=1}^{2}f^4_{h_{v_0}}\left[\left((-1)^i\frac{\pi}{\beta},(-1)^j\frac{\pi}{L}\right),\left((-1)^i\frac{\pi}{\beta},(-1)^j\frac{\pi}{L}\right),\right.\\
&\qquad\left.\left((-1)^i\frac{\pi}{\beta},(-1)^j\frac{\pi}{L}\right),\left((-1)^i\frac{\pi}{\beta},(-1)^j\frac{\pi}{L}\right);\underline{\vec\omega}\right]\,,\\
f^{2,L_a}_{h_{v_0}}(\underline{\vec\omega})&=\frac14\sum_{i,j=1}^{2}f^2_{h_{v_0}}\left((-1)^i\frac{\pi}{\beta},(-1)^j\frac{\pi}{L};\underline{\vec\omega}\right)\,,\\
f^{2,L_b}_{h_{v_0}}(\underline{\vec\omega})&=\frac12\sum_{i=1}^{2}\tilde\partial_{\vec k_1'}f^2_{h_i}\left((-1)^i\frac{\pi}{\beta},-\frac{\pi}{L};\underline{\vec\omega}\right)\,,\\
f^{2,L_c}_{h_{v_0}}(\underline{\vec\omega})&=\frac12\sum_{i=1}^{2}\tilde\partial_{k_1^0}f^2_{h_{v_0}}\left(-\frac{\pi}{\beta},(-1)^i\frac{\pi}{L};\underline{\vec\omega}\right)\,.
\end{aligned}$$

In Eq. (22) the Kroneker δ can be satisfied only by $n=0$ and $\vec\omega_1=\vec\omega_4=-\vec\omega_2=-\vec\omega_3$, $\vec\omega_1=\vec\omega_3=-\vec\omega_2=-\vec\omega_4$ or $\vec\omega_1=\vec\omega_2=\vec\omega_3=\vec\omega_4$ unless $p_F=\pi/2a$, *i.e.* the conduction band is half filled, in which case the Kroneker δ can also be satisfied also by $n=1$ and $\vec\omega_1=\vec\omega_2=-\vec\omega_3=-\vec\omega_4$, *i.e.* umklapp is relevant only if the conduction band is half filled. In Eq. (23) we must have $n=0$ and $\vec\omega_1=\vec\omega_2$.

The relevant part of $V^0(\psi)$ in the spinning case is then

$$\mathscr{L}V^0 = v_0 F_v + \alpha_0 F_\alpha + \zeta_0 F_\zeta + g_{1,0}F_1 + g_{2,0}F_2 + g_{3,0}\delta_{p_F,\pi/2a}F_3 + g_{4,0}F_4 \,, \tag{24}$$

$$F_v = \sum_{\vec\omega,\sigma} \int dk_1 dk_2 \psi^+_{k_1'+\vec\omega p_F,\vec\omega,\sigma}\psi^-_{k_2'+\vec\omega p_F,\vec\omega,\sigma}\delta(k_1'-k_2')\,,$$

$$F_\alpha = \sum_{\vec\omega,\sigma} \int dk_1 dk_2 E(\vec{k}_1' + \vec\omega p_F)\psi^+_{k_1'+\vec\omega p_F,\vec\omega,\sigma}\psi^-_{k_2'+\vec\omega p_F,\vec\omega,\sigma}\delta(k_1'-k_2')\,,$$

$$F_\zeta = \sum_{\vec\omega,\sigma} \int dk_1 dk_2 - ik_1^0\psi^+_{k_1'+\vec\omega p_F,\vec\omega,\sigma}\psi^-_{k_2'+\vec\omega p_F,\vec\omega,\sigma}\delta(k_1'-k_2')\,,$$

$$F_1 = \sum_{\vec\omega,\sigma,\sigma'} \int \prod_{i=1}^4 dk_i \psi^+_{k_1'+\vec\omega p_F,\vec\omega,\sigma}\psi^+_{k_2'-\vec\omega p_F,-\vec\omega,\sigma'}\psi^-_{k_3'+\vec\omega p_F,\vec\omega,\sigma'}\psi^-_{k_4'-\vec\omega p_F,-\vec\omega,\sigma}\delta\left(\sum_i \varepsilon_i k_i'\right)\,,$$

$$F_2 = \sum_{\vec\omega,\sigma,\sigma'} \int \prod_{i=1}^4 dk_i' \psi^+_{k_1'+\vec\omega p_F,\vec\omega,\sigma}\psi^+_{k_2'-\vec\omega p_F,-\vec\omega,\sigma'}\psi^-_{k_3'-\vec\omega p_F,-\vec\omega,\sigma'}\psi^-_{k_4'+\vec\omega p_F,\vec\omega,\sigma}\delta\left(\sum_i \varepsilon_i k_i'\right)\,,$$

$$F_3 = \delta_{p_F,\pi/2a}\sum_{\vec\omega,\sigma,\sigma'} \int \prod_{i=1}^4 dk_i' \psi^+_{k_1'+\vec\omega p_F,\vec\omega,\sigma}\psi^+_{k_2'+\vec\omega p_F,\vec\omega,\sigma'}\psi^-_{k_3'-\vec\omega p_F,-\vec\omega,\sigma'}\psi^-_{k_4'-\vec\omega p_F,-\vec\omega,\sigma}$$
$$\delta\left(\sum_i \varepsilon_i k_i'\right)\,,$$

$$F_4 = \sum_{\vec\omega,\sigma,\sigma'} \int \prod_{i=1}^4 dk_i' \psi^+_{k_1'+\vec\omega p_F,\vec\omega,\sigma}\psi^+_{k_2'+\vec\omega p_F,\vec\omega,-\sigma}\psi^-_{k_3'+\vec\omega p_F,\vec\omega,-\sigma}\psi^-_{k_4'+\vec\omega p_F,\vec\omega,\sigma}\delta\left(\sum_i \varepsilon_i k_i'\right)\,,$$

where $\varepsilon_1 = \varepsilon_2 = -\varepsilon_3 = -\varepsilon_4 = 1$. We note that the relevant part of $V^0(\psi)$ is similar to the phenomenological hamiltonian introduced in [S.] via heuristic considerations. Note, moreover, that there are no terms renormalizing the periodic potential in the relevant part of $V^0(\psi)$. In the spinless case $\sigma = 0$ and $\mathscr{L}V_h = v_0 F_v + \alpha_0 F_\alpha + \zeta_0 F_\zeta + \lambda_0 F$, where

$$F = \sum_{\vec\omega} \int \prod_{i=1}^4 dk_i' \psi^+_{k_1'+\vec\omega p_F,\vec\omega}\psi^+_{k_2'-\vec\omega p_F,-\vec\omega}\psi_{k_3'-\vec\omega p_F,-\vec\omega}\psi_{k_4'+\vec\omega p_F,\vec\omega}\delta\left(\sum_i \varepsilon_i k_i'\right)\,.$$

Therefore, in the spinless case, $g_{2,h} = g_{1,h} = \lambda_h$ and $F_3 = F_4 = 0$ (because of the anticommuting property of the grassmannian variables).

The most natural definition for the *effective potential* on scale γ^{-k}, $k < 0$ for the infrared problem (but not the correct one, as it will appear clear in the following) would be:

$$e^{-V^h(\psi^{\leqq h})} = \frac{\mathscr{N}^0}{\mathscr{N}}\int P(d\psi^{h+1})\dots P(d\psi^0)e^{-V^0(\psi^{\leqq 0})}\,, \tag{25}$$

where $\psi^{\leqq k} = \sum_{h=-\infty}^k \psi^h$. The inductive evaluation of Eq. (25) is made by writing at each step $V^h(\psi^{\leqq h}) = \mathscr{L}V^h(\psi^{\leqq h}) + \mathscr{R}V^h(\psi^{\leqq h})$, where $\mathscr{L}V^h$ is given by an equation like Eq. (24) with $(v_0, \alpha_0, \zeta_0, g_{1,0}, g_{2,0}, g_{3,0}, g_{4,0})$ replaced by $(\gamma^h v_h, \alpha_h, \zeta_h, g_{1,h}, g_{2,h}, g_{3,h}, g_{4,h}) \equiv \vec{v}_h$ plus an additive constant $t_h|\Lambda|$, $|\Lambda| = L\beta$, *i.e.* the vacuum contribution. The quantities $\gamma^h v_h, \alpha_h, \zeta_h$ are $\vec\omega$ independent (by the "rotation invariance" of

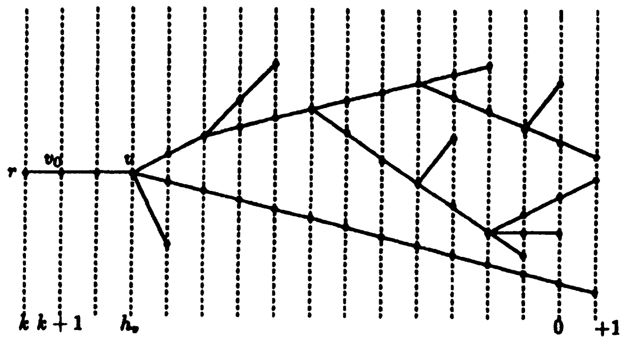

Fig. 1.

the theory, i.e. by the invariance under the transformation $\vec{x} \to -\vec{x}$). An essential role in this analysis will be played by the tree expansion.

We call τ_n the set of all the *labeled trees with n end points* $\tau \in \tau_n$ that can be constructed as follows (see also the picture). Draw on the (x, y) plane vertical lines at $x = k, k+1, \ldots, 0, 1$. Let τ (*the root*) be a point on the line $x = k$. Starting from τ draw an horizontal line leading to a point v_0 on the line $x = k_{v_0} = k+1$. Choose $s_{v_o} \geqq 0$ and draw s_{v_0} lines starting from v_0 leading to s_v points $v_1, \ldots v_{s_v}$ on the line $x = k_{v_i} = k_v + 1$, *i.e.* the lines cannot go back. Do the same thing starting with the points v_i and go on recursively. A point v is called an *end point* if $s_v = 0$, *i.e.* if there is no line starting from this point. Moreover a point v is a *trivial vertex* if $s_v = 1$ and a *non-trivial vertex* if $s_v \geqq 2$. Finally if $h_v = 1$, then v is necessarily an end point. Clearly this process ends when all the reached points are end points. A *cluster* v with frequency h_v is the set of the end-points reachable from the vertex v with frequency h_v; and the tree provides an organisation of the endpoints into a hierarchy of clusters. Each non-trivial or trivial vertex bears a label $\mathscr{R}$ except v_0 (see the picture) on which can bear either a label $\mathscr{R}$ or a lable $\mathscr{L}$. To each tree we associate a term $V^{(k)}(\tau, \psi^{(\leqq k)})$ defined recursively as follows. If τ has only one end-point with frequency $k+1$ then $V^{(k)}(\tau, \psi^{(\leqq k)})$ is equal to one of the terms of Eq. (24) with $\vec{v}_k$ instead of $\vec{v}_0$ or, only if $k = 0$, one of the monomial in $\mathscr{R}V^0$. We attach a label to each endpoint of the tree to distinguish among these possibilities. Otherwise

$$V^{(k)}(\tau, \psi^{(\leqq k)}) = \mathscr{O}\frac{1}{s_v!}\mathscr{E}^T_{k+1}\left[V^{(k+1)}(\tau^1, \psi^{(\leqq k+1)}), \ldots)\right], \tag{26}$$

where $\mathscr{O}$ is $\mathscr{L}$ or $\mathscr{R}$ if the vertex v bears an $\mathscr{L}$ or $\mathscr{R}$ label, $n \geqq 2$, $\tau^1 \ldots \tau^{s_v}$ are the subtrees starting from v and the symbols $\mathscr{E}^T_h$ denote the truncated expectations with respect to a measure with covariance $g^{(h)}$. We have that $\mathscr{O}$ is equal to $\mathscr{L}$ only if $v = v_0$ and the tree contributes to the local part of the effective potential. We also associate to each field a labels, f, $f = 1, \ldots, n_\tau$, where n_τ is the number of the fields associated with all the endpoints of the tree. To every field with label f corresponds a momentum $k(f)$ and the indices $\vec{\omega}(f), \sigma(f), \varepsilon(f) = \pm 1$ and, also, the index $s(f) = 0, 1, 2$ allowing us to distinguish the three possibilities $\psi^{\varepsilon(f)}_{k(f),\vec{\omega}(f),\sigma(f)}, E(\vec{k})\psi^{\varepsilon(f)}_{k(f),\vec{\omega}(f),\sigma(f)}, -ik_0\psi^{\varepsilon(f)}_{k(f),\vec{\omega}(f),\sigma(f)}$. We call I_{v_0} the set of f labels.

It is possible to check that the effective potential Eq. (25) can be written as

$$V^{(k)}(\psi^{(\leqq k)}) = \sum_{n=1}^{\infty} \sum_{\tau\in\tau_n} V^{(k)}(\tau, \psi^{(\leqq k)})\,.$$

From Eq. (26) we see that each set of running coupling constants $\vec{v}_h$ is determined once that a set $\vec{v}_0$ is given from the relation $\vec{v}_{h-1} = \vec{v}_h + \beta_h(\vec{v}_h, \vec{v}_{h+1}, \dots \vec{v}_0)$, where β, called *beta function*, is a sum over all the trees contributing to the relevant part of the effective potential. We define

$$V^{(k)}(\tau, \psi^{(\leqq k)}) = \int dk_{v_0} \sum_{P_{v_0}} V^{(k)}(\tau, P_{v_0}, k_{v_0})\tilde{\psi}^{(\leqq k)}(P_{v_0})\delta\left(\sum_{f\in P_{v_0}} \varepsilon(f)k(f) + 2n\pi/a\right)\,,$$

where k_{v_0} is the set of all the momentum variables of the n_τ fields, P_{v_0} is a non-empty subset of I_{v_0}, $|P_{v_0}|$ are the number of elements of this subset, $\sum_{P_{\iota_0}}$ is the sum over such subsets and $\tilde{\psi}^{(\leqq k)}(P_{v_0}) = \prod_{f\in P_{\iota_0}} \psi^{\varepsilon(f),(\leqq k)}_{k(f),\vec{\omega}(f),\sigma(f),s(f)}$.

If in Eq. (26) we expanded the expectations by Wick's theorem, we could represent the r.h.s. as a sum of *Feynaman graphs* (see [B.G., B.G.P.S.]). A Feynman graph is constructed by symbolising the fields associated with every end-point of the tree as oriented *half lines* emerging from that point and enclosing the end-points belonging to the cluster v together with their half lines into a ideal box. We pair, i.e. *contract*, the half lines in *internal lines* (all but the *external lines* $\tilde{\psi}^{(\leqq k)}(P_{v_0})$) and we associate to each of them a *propagator* g^{h_v}, if the line is contained in the ideal box containing the cluster v and not in any one with higher frequency. Every graph contributes to the effective potential with a term of the form $\int dk^{P_{v_0}} W_{h_{v_0}}(k^{P_{\iota_0}})\delta(\sum_{f\in P_{\iota_0}} \varepsilon(f)k(f) + 2n\pi/a)\tilde{\psi}^{(\leqq k)}(P_{v_0})$, where $k^{P_{\iota_0}}$ is the set of the variables $k(f)$ with $f \in P_{v_0}$ and W, called *value of the graph*, is the product of the propagators of the graph and of the running couplings or the kernels in Theorem 2.1 associated to the end points, integrated over the momenta of the internal lines.

Furthermore, if G_τ is the set of all Feynman graphs associated with τ, given $g \in G_\tau$, it is natural to associate a *subgraph* g_v to the vertex v enclosing into an ideal box the cluster v and cutting into half lines the lines connecting points in the v cluster with points outside from it. Each g_v is of the form $\int dk^{P_\iota} W_{h_v}(k^{P_v})\delta(\sum_{f\in P_\iota} \varepsilon(f)k(f) + 2n\pi/a)\tilde{\psi}^{(\leqq h_\iota - 1)}(P_v)$ where $\tilde{\psi}^{(\leqq h_v-1)}(P_v)$ are the half lines emerging from v before contraction and P_v is defined as P_{v_0}. On all this term the $\mathscr{R}$ operation acts, if $v \neq v_0$, while if $v \equiv v_0$ the operation $\mathscr{L}$ or $\mathscr{R}$ acts, depending on whether it contributes to the relevant or to the irrelevant part of the effective potential. It is convenient to write $W(k)$ as a function of $k' = k - \vec{\omega} p_F$ introduced in the preceding section defining $W_{h_\iota}(k'^{P_v} + \vec{\omega}^{P_v} p_F) = \mathscr{f}_{h_v}(k'^{P_v}; \vec{\omega}^{P_v})$.

We call *scaling dimension* $D(P_{v_0}) = -2 + \sum_{A\in P_{\iota_0}}(1/2 + \chi_A)$, where $\chi_A = 0$ if $A = \psi^h_{k+\vec{\omega}p_F,\vec{\omega},\sigma}, \chi_A = 1$ if $A = E(k)\psi^h_{k'+\vec{\omega}p_F,\vec{\omega},\sigma}$ or $A = -ik_0\psi^h_{k'+\vec{\omega}p_F,\vec{\omega},\sigma}$. The *size* of a generic graph associated with a monomial $\tilde{\psi}^{\leqq h}(P_{v_0})$ with value W^h is defined by

$$\|W^h\| = \sup_{k_2^{ex},\dots k_n^{ex}} \gamma^{hD(P_{v_0})} d_h(k_1^{ex})\dots d_h(k_n^{ex})W^h(k_1^{ex},\dots k_n^{ex})\,, \tag{27}$$

where $d_n(k)$ is the characteristic function of the support of $h(\gamma^{-2h+2}(k_0^2 + E(\vec{k})^2))$.

In order to motivate our definition of localization suppose for a moment that $\mathscr{R} = I$, where I is the identity operator; by a standard calculation it is possible to prove that the size, Eq. (27), of a Feynman graph is bounded by $\|W\| < c^n \varepsilon^n \prod_v \gamma^{-(h_v - h_{v'})D(P_v)}$, where v' is the vertex preceding v in the tree ordering. To obtain an estimate of the perturbative contribution of order n to the effective potential, we must sum over trees. In order to have an estimate uniform in β, L it is necessary that $D(P_v) > 0$ for all P_v. But we have that $D(P_v) = -1$ if $|P_v| = 2$ and $\sum_{A \in P_v} \chi_A = 0$, while $D(P_v) = 0$ if $|P_v| = 4$ and $\sum_{A \in P_v} \chi_A = 0$ or $|P_v| = 2$ and $\sum_{A \in P_v} \chi_A = 1$. Like in [B.G.] one could define as "relevant part" of the effective potential the sum of its local quadratic and quartic parts in the fields. However such definitions would still contain irrelevant terms. This can be easily understood by remarking that for h suitably small the contributions to the effective potential V^h having forms:

$$\int \prod_{i=1}^{4} dk'_i f^4_{h_v}(k'_1, k'_2, k'_3, k'_4; \underline{\vec{\omega}}) \delta\left(k'_1 + k'_2 - k'_3 - k'_4 + (\vec{\omega}_1 + \vec{\omega}_2 - \vec{\omega}_3 - \vec{\omega}_4) p_F + \frac{2n\pi}{a} \right)$$

$$\psi^+_{k'_1 + \vec{\omega}_1 p_F, \vec{\omega}_1, \sigma} \psi^+_{k'_2 + \vec{\omega}_2 p_F, \vec{\omega}_2, \sigma'} \psi^-_{k'_3 + \vec{\omega}_3 p_F, \vec{\omega}_3, \sigma} \psi^-_{k'_4 + \vec{\omega}_4 p_F, \vec{\omega}_4, \sigma}$$

$$\int dk'_1 dk'_2 \delta(k'_1 - k'_2 + (\vec{\omega}_1 - \vec{\omega}_2) p_F + 2n\pi/a) f^2(k'_1, k'_2; \underline{\vec{\omega}}) \psi^+_{k'_1 + \vec{\omega}_1 p_F, \vec{\omega}_1, \sigma} \psi^-_{k'_2 + \vec{\omega}_2 p_F, \vec{\omega}_2, \sigma}$$

are vanishing unless $(\vec{\omega}_1 + \vec{\omega}_2 - \vec{\omega}_3 - \vec{\omega}_4) p_F + \frac{2n\pi}{a} = 0$ in the first case and $(\vec{\omega}_1 - \vec{\omega}_2) p_F + 2n\pi/a = 0$ in the second as the delta's in the above equations cannot be satisfied for the support properties of the propagator (see Eq. (19) and relative discussion).

It is possible to check by a standard calculation that the size of the generic Feynman graph contributing to the effective potential defined above is bounded by

$$\|W\| < C^m \varepsilon^n \prod_v \gamma^{-(h_v - h_{v'})(D(P_v) + z_v)}, \tag{28}$$

where $D(P_v) + z_v > 0$ (the $\mathscr{R}$ operation was defined in order to make true such an inequality), $\varepsilon = \max |\vec{v}_h|$ and c is a suitable constant independent from n. By repeating the estimates in [G.] it is easy to see that Eq. (28) implies that $|V^{(k)}(\tau, \psi^{\leqq k})| \leqq \varepsilon^n c^n n!$. We shall use (and prove) an equation stronger than Eq. (28), hence we do not dicuss its proof in more detail.

It is convenient to see the effect of $\mathscr{L}$ when $V^k(\psi^{\leqq k})$ is written as an integral over the coordinates. Writing

$$\int dk'^{P_v} \delta\left(\sum_{f \in P_v} \varepsilon(f) k(f) + \varepsilon(f) \vec{\omega}(f) p_F + 2n\pi/a \right) f_{h_v}(k'^{P_v}; \vec{\omega}^{P_v}) \tilde{\psi}^{(\leqq h_v - 1)}(P_v)$$

$$= \int dx^{P_v} W(x^{P_v}) \hat{\psi}^{(\leqq h_v - 1)}(P_v),$$

where $\hat{\psi}^{(\leqq k)}(P_v) = \prod_{f \in P_{v_0}} \hat{\psi}^{\varepsilon(f),(\leqq k)}_{x(f), \vec{\omega}(f), \sigma(f), s(f)}$, $\vec{\omega}^{P_v}$ is the set of quasi-particle variables $\vec{\omega}(f)$ such that $f \in P_v$ and $W(x^{P_v})$ is a not traslation invariant function unless $\sum_{f\ in P_v} \varepsilon(f) \vec{\omega}(f) p_F + 2n\pi/a = 0$ in which case it is translation invariant (see the delta-function in the above equation). By Eq. (22), (23) we see, by performing a Fourier transform, that the $\mathscr{L}$-operation acts in the following ways:

$$\mathscr{L}\int_\Lambda W_{h_v}(x_1,\ldots x_n;\underline{\vec\omega})\hat\psi^+_{x_1,\vec\omega_1,\sigma_1}\ldots\hat\psi^-_{x_n,\vec\omega_n,\sigma_n}dx_1\ldots dx_n=0\quad n>4\,, \tag{29}$$

$$\mathscr{L}\int_\Lambda W_{h_v}(x_1-x_4,\,x_2-x_4,\,x_3-x_4;\underline{\vec\omega})\hat\psi^+_{x_1,\vec\omega_1,\sigma}\hat\psi^+_{x_2,\vec\omega_2,\sigma'}\hat\psi^-_{x_3,\vec\omega_3,\sigma'}\hat\psi^-_{x_4,\vec\omega_4,\sigma}\prod_{i=1}^4 dx_i=$$
$$\delta_{(\vec\omega_1+\vec\omega_2-\vec\omega_3-\vec\omega_4)p_F+2n\pi/a,0}\int_\Lambda dt_1dt_2dt_3W_{h_v}(t_1,\,t_2,\,t_3;\underline{\vec\omega})s_{L,\beta}(t_1,\,t_2,\,t_3)$$
$$\times\int_\Lambda\delta(x_1-x_2)\delta(x_2-x_3)\delta(x_3-x_4)\hat\psi^+_{x_1,\vec\omega_1,\sigma},\hat\psi^+_{x_2,\vec\omega_2,\sigma'}\hat\psi^-_{x_3,\vec\omega_3,\sigma'}\hat\psi^-_{x_4,\vec\omega_4,\sigma}\prod_{i=1}^4 dx_i\,, \tag{30}$$

where $s_{L,\beta}(t_1,t_2,t_3)=\frac{1}{4}\sum_{j,k=1}^2 e^{i\frac{\pi}{L}(-1)^j(\vec t_1+\vec t_2-\vec t_3)+i\frac{\pi}{\beta}(-1)^k(t_1^0+t_2^0-t_2^0)}$ and is present only because we are studying the sytem with finite volume and with temperature different from zero; moreover

$$\mathscr{L}\int_\Lambda dx_1dx_2\hat\psi^+_{x_1,\vec\omega_1,\sigma}\hat\psi^-_{x_2,\vec\omega_2,\sigma}W_{h_v}(x_1-x_2;\underline{\vec\omega})=\delta_{\vec\omega_1,\vec\omega_2}\int_\Lambda dtW_{h_v}(t;\vec\omega)\cos\left(\frac{\pi}{L}\vec t\right)\cos\left(\frac{\pi}{\beta}t_0\right)$$
$$\times\int_\Lambda dx_1dx_2\delta(x_1-x_2)\hat\psi^+_{x_1,\vec\omega,\sigma}\hat\psi^-_{x_2,\vec\omega,\sigma}$$
$$+\int_\Lambda dtW_{h_v}(t;\vec\omega)\frac{\beta}{\pi}\sin\left(t_0\frac{\pi}{\beta}\right)\cos\left(\frac{\pi}{L}\vec t\right)\int_\Lambda dx_1dx_2\partial_{x_2^0}\delta(x_1-x_2)\hat\psi^+_{x_1,\vec\omega,\sigma}\hat\psi^-_{x_2,\vec\omega,\sigma} \tag{31}$$
$$+\int_\Lambda dtW_{h_v}(t;\vec\omega)\frac{L}{\pi}\sin\left(\vec t\frac{\pi}{L}\right)\cos\left(\frac{\pi}{\beta}t_0\right)\int_\Lambda dx_1dx_2\overline{\partial}_{\vec x_2,\vec\omega}\delta(x_1-x_2)\hat\psi^+_{x_1,\vec\omega,\sigma}\hat\psi^-_{x_2,\vec\omega,\sigma}\,,$$

where the kernels W are antiperiodic functions both in the time and space variables with period β and $L, t=(t^0,\vec t), \overline{\partial}_{\vec x,\vec\omega}$, the *covariant derivative*, is defined by

$$\overline{\partial}_{\vec x,\vec\omega}f(\vec x)=i\sum_{\vec k}\vec\omega E(\vec k+\vec\omega p_F)\hat f(\vec k)\,.$$

If $\delta(x_i-x_j)$ are integrated away in the r.h.s. of Eq. (29), (30),(31) we see that the action of $\mathscr{L}$ has the effect that the monomials in the fields are changed into *local* expressions. This is the main reason for which we introduced the fields $\hat\psi^\sigma_{x,\vec\omega,\sigma}$ rather than working with $\psi^\sigma_{x,\vec\omega,\sigma}$ Eq. (4). The non-trivial action of $\mathscr{R}$ on the terms with four external lines is

$$\int_\Lambda\mathscr{R}W_{h_v}(x_1-x_4,\,x_2-x_4,\,x_3-x_4;\underline{\vec\omega})\hat\psi^+_{x_1,\vec\omega_1,\sigma}\hat\psi^+_{x_2,\vec\omega_2,\sigma'}\hat\psi^-_{x_3,\vec\omega_3,\sigma'}\hat\psi^-_{x_4,\vec\omega_4,\sigma}\prod_{i=1}^4 dx_i= \tag{32}$$
$$\int_\Lambda\prod_{i=1}^4 dx_i\hat\psi^+_{x_1,\vec\omega_1,\sigma}\hat\psi^+_{x_2,\vec\omega_2,\sigma'}\hat\psi^-_{x_3,\vec\omega_3,\sigma'}\hat\psi^-_{x_4,\vec\omega_4,\sigma}\big\{W_{h_v}(x_1-x_4,x_2-x_4,x_3-x_4;\underline{\vec\omega})$$
$$-\delta(x_1-x_2)\delta(x_2-x_3)\delta(x_3-x_4)\int_\Lambda dt_1dt_2dt_3s_{L,\beta}(t_1,\,t_2,\,t_3)W_{h_v}(t_1,\,t_2,\,t_3;\underline{\vec\omega})\big\}\,.$$

In the estimates of the following section it is convenient integrating $\delta's$ in Eq. (32) obtaining a different, equivalent form:

$$
\begin{aligned}
&\int \mathscr{R}W(x_1 - x_4, x_2 - x_4, x_3 - x_4; \underline{\vec{\omega}})\hat{\psi}^{+}_{x_1,\vec{\omega}_1,\sigma}\hat{\psi}^{+}_{x_2,\vec{\omega}_2,\sigma'}\hat{\psi}^{-}_{x_3,\vec{\omega}_3,\sigma'}\hat{\psi}^{-}_{x_4,\vec{\omega}_4,\sigma}\prod_{i=1}^{4} dx_i \qquad (33)\\
&\quad = \int \prod_{i=1}^{4} dx_i W(x_1 - x_4, x_2 - x_4, x_3 - x_4; \underline{\vec{\omega}})(1 - s_{L,\beta}(\vec{x}_1 - \vec{x}_4, x_2 - x_4, x_3 - x_4))\\
&\quad \times \hat{\psi}^{+}_{x_1,\vec{\omega}_1,\sigma}\hat{\psi}^{+}_{x_1,\vec{\omega}_2,\sigma'}\hat{\psi}^{-}_{x_1,\vec{\omega}_3,\sigma'}\hat{\psi}^{-}_{x_1,\vec{\omega}_4,\sigma} +\\
&\quad \int \prod_{i=1}^{4} dx_i W(x_1 - x_4, x_2 - x_4, x_3 - x_4; \underline{\vec{\omega}}) \cdot \Big\{\hat{\psi}^{+}_{x_1,\vec{\omega}_2,\sigma}\hat{\psi}^{+}_{x_2,\vec{\omega}_2,\sigma'}\hat{\psi}^{-}_{x_3,\vec{\omega}_3,\sigma'}\hat{\psi}^{-}_{x_4,\vec{\omega}_4,\sigma}\\
&\quad - 1/2\sum_{i=1}^{2}\hat{\psi}^{+}_{x_i,\vec{\omega}_1,\sigma}\hat{\psi}^{+}_{x_i,\vec{\omega}_2,\sigma'}\hat{\psi}^{-}_{x_i,\vec{\omega}_3,\sigma'}\hat{\psi}^{-}_{x_i,\vec{\omega}_4,\sigma}\Big\},
\end{aligned}
$$

where $s_{L,\beta}(t_1, t_2, t_3)$ is, once again, present only because one wants to distinguish carefully the $L, \beta < \infty$ from the $L, \beta = \infty$ case. Equation (33) shows that there are two ways in which the renormalization acts on a subgraph with four half lines connecting points in the cluster v to points outside it. One way renormalization affects the graphical analysis is that one of the half-lines does not represnt a ψ-field as we can write:

$$
\begin{aligned}
&(\hat{\psi}^{+}_{x_1,\vec{\omega}_1}D^{+}_{x_{2,1},\vec{\omega}_1}\hat{\psi}^{-}_{x_3,\vec{\omega}_3}\hat{\psi}^{-}_{x_4,\vec{\omega}_4} + \hat{\psi}^{+}_{x_1,\vec{\omega}_1}\hat{\psi}^{+}_{x_2,\vec{\omega}_2}D^{-}_{x_{3,1},\vec{\omega}_3} + \hat{\psi}^{+}_{x_1,\vec{\omega}_1}\hat{\psi}^{+}_{x_2,\vec{\omega}_2}\hat{\psi}^{-}_{x_3,\vec{\omega}_3}D^{-}_{x_{4,1},\vec{\omega}_4} \qquad (34)\\
&+ D^{+}_{x_{1,2},\vec{\omega}_1}\hat{\psi}^{+}_{x_1,\vec{\omega}_1}\hat{\psi}^{-}_{x_3,\vec{\omega}_3}\hat{\psi}^{-}_{x_4,\vec{\omega}_4} + \hat{\psi}^{+}_{x_1,\vec{\omega}_1}\hat{\psi}^{+}_{x_2,\vec{\omega}_2}D^{-}_{x_{3,2},\vec{\omega}_3}\hat{\psi}^{-}_{x_4,\vec{\omega}_4} + \hat{\psi}^{+}_{x_1,\vec{\omega}_1}\hat{\psi}^{+}_{x_2,\vec{\omega}_2}\hat{\psi}^{-}_{x_3,\vec{\omega}_3}D^{-}_{x_{4,2},\vec{\omega}_4},
\end{aligned}
$$

where

$$
D^{\varepsilon}_{x_{j,i},\vec{\omega}} = \hat{\psi}^{\varepsilon}_{x_i,\vec{\omega}} - \hat{\psi}^{\varepsilon}_{x_j,\vec{\omega}} = (x_i - x_j)\int_0^1 du\,\partial\hat{\psi}^{\varepsilon}_{x'_{j,i}(u),\vec{\omega}}, \qquad (35)
$$

$$
x'_{j,i}(u) = ux_j + (1-u)x_i, \quad \partial = (\partial_t, \partial_{\vec{\omega}}).
$$

$x'_{j,i}(u)$ are called *interpolated points*. It is easy to check that the effect of this substitution (*i.e.* a D-line instead of a ψ-line) is that in the estimate of a generic Feynman graph, in which the line representing D-field has an end in the cluster v and the other in cluster the v', there is an extra factor $\gamma^{-(h_v - h_{v'})}$ with respect to the case in which $\mathscr{R} = 1$.

The other way in which $\mathscr{R}$ can act, from Eq. (33) is that the kernel W is substituted by a kernel $W(1-s)$ and this produces, in the estimate, at least an extra factor $\gamma^{-(h-h^*)}$ with respect to the not renormalized case, with $h^* = \min(h_L, h_\beta)$, where h_L is such that if $h < h_L$ $g^h_{\vec{\omega}} \equiv 0$ for $\vec{k}$ of the form $2n\pi/L$, n integer. Of course $\gamma^{-h_L} = const\ L$. We define h_β in the same way. Of course if $h \leqq h^*$ and $V^h(\phi) = -\log\int P(d\psi^{h+1})e^{-V^{h+1}(\psi^{h+1}+\phi)}$, then

$$
V^h(\phi) = V^{h-1}(\phi) \qquad (36)
$$

that is the effective potential *stops flowing*. Equation (36) is the analogue of Lemma 1 in [B.G.P.S.] whose proof (not explicitly written there) is not so trivial for the exponential decay of the propagators in momentum space. Since there are no non-trivial vertex with $h' \leqq h^*$, then we have that $\gamma^{-(h_v - h^*)} \leqq \gamma^{-(h_v - h_{v'})}$. Similar considerations can be made on the terms with two external lines.

4. Analyticity of the Anomalous Effective Potential

In order to see if the flow of the relevant running coupling $\vec{v}_h$ (see lines following Eq. (25)) is bounded we write then the equations for the $\vec{v}_h$ up to the second order:

$$
\begin{aligned}
g_{1,h-1} &= g_{1,h} - 2\beta g_{1,h}^2 + O(\gamma^h)\,,\\
g_{2,h-1} &= g_{2,h} - \beta g_{1,h}^2 + O(\gamma^h)\,,\\
g_{4,h-1} &= g_{4,h} + O(\gamma^h)\,,\\
\alpha_{h-1} &= \alpha_h + \tilde\beta_1 g_{1,h}^2 + \tilde\beta_2 g_{2,h}^2 + O(\gamma^h)\,,\\
\zeta^{h-1} &= \zeta^h + \tilde\beta_1 g_{1,h}^2 + \tilde\beta_2 g_{2,h}^2 + O(\gamma^h)\,,\\
v^{h-1} &= \gamma v^h + O(\gamma^h)\,,
\end{aligned}
\tag{37}
$$

where $\beta, \tilde\beta_1, \tilde\beta_2 > 0$ and the equation for t_h (see lines following Eq. (25)) is not written for simplicity as $\vec{v}_h$ does not depend on t_h. We have supposed that $p_F \neq \pi/2a$ postponing the discussion of the case $p_F = \pi/2a$ to the last section. These equations are qualitatively similar to the equation founded by Solyom [S.] for his phenomenological hamiltonian. If $g_{1,0} < 0$ and the corrections $O(\gamma^h)$ are neglected, then $g_{1,h}$ grows so that the second order truncation has no meaning. If $g_{1,0} \geqq 0$ things looks different. It is easy to check by using the general methods of stability theory that, for v_0 small enough:

$$
g_{1,h} \simeq \frac{g_{1,0}}{1-2\beta h g_{1,0}} \to 0\,, \quad g_{4,h} = g_{4,0}\,,
$$

$$
g_{2,h} \to g_{2,\infty} \simeq g_{2,0} - \beta \sum_{h=0}^{-\infty} \frac{g_{1,0}^2}{(1-2\beta h g_{1,0})^2} = g_{2,0} + O(g_{1,0}^2)\,. \tag{38}
$$

However also in this case the flow is unbounded. In fact we have that $\alpha_h = \alpha_0 + \sum_{h=-\infty}^{0}(\tilde\beta_1 g_{1,h}^2 + \tilde\beta_2 g_{2,h}^2)$ and similarly for ζ_h; so that in any case $\alpha_h, \zeta_h \to \infty$ because $g_{2,h}, g_{4,h}$ do not go to zero. Note that even if the third order contribution to the Beta function makes that $g_{2,h}, g_{4,h}$ tend to zero, this would happen very slowly, *i.e.* not faster than $1/\sqrt{-h}$.

This suggests that we try a new and more general scaling approach, including the one described in the preceding section, the *anomalous scaling*. Given a sequence of positive numbers Z_h, with $Z_0 = 1$ we can write, for $h \leqq 0$,

$$
\int P_{Z_{h+1}}(d\psi^{(\leqq h)}) e^{-\bar V^{(h)}(\sqrt{Z_{h+1}}\psi^{(\leqq h)})} = \int \tilde P_{Z_h}(d\psi^{(h)}) P_{Z_h}(d\psi^{(<h)}) e^{-V^{(h)}(\sqrt{Z_h}\psi^{(\leqq h)})}\,, \tag{39}
$$

where $P_{Z_h}(d\psi^{(h)})$ denotes the Grassmannian integration with propagator $\frac{g^h}{Z_h}$ and $\tilde P_{Z_h}(d\psi^{(h)})$ the one with propagator $\frac{\tilde g^h}{Z_h}$ where, if we call $C_h(k) = \sum_{k=-\infty}^{h} f(\gamma^{-2k+2}(k_0^2 + E(\vec k)^2))$, $\tilde g^{(h)}(k)$ is given by $g^{(h)}(k) + \tau^{(h)}(k)$ with:

$$
\tau^{(h)}(k) = \frac{C_h(k)(1-C_h(k))}{-ik_0 + E(\vec k)} \frac{z_h}{1 + z_h C_h(k)}\,.
$$

$\tilde{V}^h(\psi)$ is determined from $V^{h+1}(\psi)$ using the following relation, for $h < 0$:

$$e^{-\tilde{V}^{(h)}(\sqrt{Z_{h+1}}\psi^{(\leqq h)})} = \int \tilde{P}_{Z_{h+1}}(d\psi^{(h+1)}) e^{-V^{(h+1)}\left[\sqrt{Z_{h+1}}(\psi^{(h+1)}+\psi^{(\leqq h)})\right]} .$$

Note that $\tilde{V}^0(\psi) = V^0(\psi)$ and that the relevant part of $\tilde{V}^h(\psi)$ (we write in it also the constant part of $\tilde{V}^h(\psi^{\leqq h})$) is given by

$$\mathscr{L}\tilde{V}^h(\sqrt{Z_{h+1}}\psi^{\leqq h}) = Z_{h+1}\gamma^h n_h F_v + Z_{h+1}a_h F_\alpha + Z_{h+1}z_h F_\zeta + Z_{h+1}^2 \sum_{i=1}^4 \tilde{g}_i^h F_i + t_h|\Lambda| ,$$

where $|\Lambda| = L\beta$. The sequence of Z_h is chosen so that the relevant part of $V^h(\psi)$ does not contain the term proportional to F_ζ, i.e.

$$\mathscr{L}V^h(\sqrt{Z_h}\psi^{\leqq h}) = Z_h\gamma^h v_h F_v + Z_h\delta_h F_\alpha + Z_h^2 \sum_{i=1}^4 g_i^h F_i + \gamma^{2h}\theta_h|\Lambda| , \qquad (40)$$

and this is achieved by taking $Z_h = Z_{h+1}(1 + z_h)$. Clearly we have $v_h = \frac{Z_{h+1}}{Z_h} n_h$, $g_h = \left(\frac{Z_{h+1}}{Z_n}\right)^2 \tilde{g}^h$, $\delta_h = \frac{Z_{h+1}}{Z_h}(a_h - z_h)$, $\gamma^{2h}\theta_h = t_h + t_h'$, where $t_h' = \int \frac{dk}{(2\pi)^2} log\left(1 + \frac{Z_h - Z_{h+1}}{Z_{h+1}} C_h(k)\right)$. V^h is called *anomalous effective potential.*

We can write $V^{(k)}(\hat{\psi}^{(\leqq k)}) = \sum_{n=1}^{\infty}\sum_{\tau\in\tau_n} \tilde{V}^{(k)}(\tau, \hat{\psi}^{(\leqq k)})$ with

$$\tilde{V}^{(k)}(\tau, Z_k^{\frac{1}{2}}\psi^{(\leqq k)}) = \mathcal{O}\frac{1}{s_{v_0}!} E_{k+1}^T[V^{k+1)}(\tau^1, Z_{k+1}^{\frac{1}{2}}\psi^{(\leqq k+1)}), \ldots)] , \qquad (41)$$

where $n \geqq 2, \tau^1 \ldots \tau^{s_{v_0}}$ are the subtrees starting from v_0 (the first vertex above the root), the symbols E_h, E_h^T denote the expectations with respect to a grassmannian integration with propagator $Z_h^{-1}\tilde{g}^{(h)}$ and $\mathcal{O}$ is equal to $\mathscr{L}^*$, if the tree contributes to the local part of the potential, or $\mathscr{R}$, if it contributes to the irrelevant part, where $V^h(\psi) = \mathscr{L}^*\tilde{V}^{(h)}(\frac{Z_{h+1}}{Z_h}\psi) + \mathscr{R}\tilde{V}^{(h)}(\frac{Z_{h+1}}{Z_h}\psi)$ and $\mathscr{L}^*\tilde{V}^{(h)}$ differs from $\mathscr{L}\tilde{V}^{(h)}$ only because it does not contain the addend proportional to $-ik_0\psi^+_{k,\sigma,\vec{\omega}}\psi^-_{k,\sigma,\vec{\omega}}$.

We write the effective potential as an integral in position space, i.e. we write:

$$V^{(k)}(\tau, \sqrt{Z_k}\psi^{(\leqq k)}) = \int dx_{v_0} \sum_{P_{v_0}} (Z_k)^{\frac{1}{2}|P_{v_0}|} V^{(k)}(\tau, P_{v_0}, x_{v_0})\psi^{(\leqq k)}(P_{v_0})$$

where x_{v_0} is the set of all the coordinate variables. We define also the kernels

$$W^{(k)}(\tau, P_{v_0}, x^{P_{v_0}}) = \int_\Lambda d(x_{v_0}\backslash x^{P_{v_0}}) V^{(k)}(\tau, P_{v_0}, x_{v_0}) ,$$

so that

$$V^{(k)}(\tau, Z_k^{\frac{1}{2}}\psi^{(\leqq k)}) = \sum_{P_{v_0}} \int_\Lambda dx^{P_{v_0}} W^{(k)}(\tau, P_{v_0}, x^{P_{v_0}}) Z_k^{\frac{1}{2}|P_{v_0}|} \tilde{\psi}^{(\leqq k)}(P_{v_0}) .$$

Here $x^{P_{v_0}}$ is the set of points on which the monomial $\tilde{\psi}^{(\leqq k)}(P_{v_0})$ depends (recall that there can be more than one point for each field). Let us define: $\vec{v}_h = (g_h^\alpha, \delta_h, v_h)$

and $\varepsilon_k = \max_{i,h\geqq k}|v_{i,h}|$ and let us formulate the following theorem:

Theorem 4.1. *There exist a constant* $\bar{\varepsilon} > 0$, *such that, if* $\varepsilon_k \leqq \bar{\varepsilon}$ *and* $\sup_{k<h'<h} |\frac{Z_h}{Z_{h'}}| \leqq 1 + c_2\bar{\varepsilon}^2, c_2 > 0$ *then, for every N*:

$$\int_\Lambda d(x^{(P_{v_0})})(1+\gamma^k d(P_{v_0}))^N \sum_{\tau\in\tau_n} |W(\tau, P_{v_0}, x^{P_{v_0}})| \leqq \Lambda\gamma^{-kD(P_{v_0})}(C_N\varepsilon_k)^n ,$$

where $d(P_{v_0})$ *is the length of the shortest tree graph connecting the set of points* $x^{(P_{v_0})}, C_N$ *is a constant and* $D(P_{v_0})$ *is the scaling dimension of the monomial* $\psi^{\leqq k}(P_{v_0})$.

Proof. Let be $\mathscr{E}^T_{h_v}(\tilde{\psi}(P_1),\ldots,\tilde{\psi}(P_k))$ the truncated expectation with propagator $\tilde{g}^{(h_v)}$ of n fields. By using a well known expansion of truncated expectation in terms of interpolating parameters $s_t, t = 1,\ldots,k-1$ [Le., B.G.P.S.], we can write:

$$\mathscr{E}^T_{h_v}(\tilde{\psi}(P_1),\ldots,\tilde{\psi}(P_k)) = \sum_{T_v}\prod_{l\in T_v} \tilde{g}^{(h_v)}(x_l - y_l)\int dP_{T_v}(s)\, det\, G^{T_v}(s) , \tag{42}$$

where T_v is an *anchored tree graph* between the clusters of space vertices from which the fields labeled with $P_1,\ldots,P_k$ emerge: this means that T_v is a set of lines connecting two points in different clusters, which become a tree graph if one identifies all the points in the same cluster; we call $T = \cup_v T_v$ and, if $l \in T_v, x_l, y_l$ are the end-points of the line and are such that $x_l \equiv x_{i,j}$ or $y_l \equiv x_{i',j'}$, where $x_{i,j}$ is the coordinate of the i^{th} field of the j^{th} monomial $\tilde{\psi}(P_j).G^{T_v}(s)$ is a $(n-k+1)\times(n-k+1)$ matrix whose elements are $G^{T_v}_{iji'j'} = S_{jj'}\tilde{g}^{(h_v)}(x_{i',j'} - x_{i,j})$ with $x_{i',j'} - x_{i,j}$ not belonging to $T_v, S_{jj'} = \prod_{t=j}^{j'-1} s_t$ and $dP_{T_v}(s)$ is a normalised measure which depends on s_t and T_v.

From Eq. (41) and App. 2 we have that

$$V^{(k)}(\tau, P_{v_0}, x_{v_0}) = \sum_{\{P_v\}}\prod_{v\ not\ e.p.}\left(\frac{Z_{h_v}}{Z_{h_v-1}}\right)^{\frac{1}{2}|P_v|}\frac{1}{s_v!}\tilde{\mathscr{E}}^T_{h_v}(\tilde{\psi}(P_{v^1}/Q_{v^1}),\ldots,$$
$$\tilde{\psi}(P_{v^{s_v}}/Q_{v^{s_v}}))\prod_{v\ not\ e.p.}\vec{v}_{h_v} , \tag{43}$$

where $\prod_{v\ not\ e.p.}\vec{v}_{h_v}$ is the product of the running couplings associated to the end points of the tree τ (which are n if $\tau\in\tau_n$), P_v is a not empty subset of I_v, the field labels reachable from v, v_i are the s_v vertices immediately following $v, \sum_{P_v}$ represents the sum over all the compatible choices of the subsets P_v such that $Q_v \subset P_v$, $P_v = \cup Q_{v_i}$ and $\tilde{\mathscr{E}}^T_{h_v}$ obeys to an equation like Eq. (42) in which $\tilde{g}^{(h_v)}(x_l - y_l)$ is replaced by $|\xi_l - h_l|^{z_l}\tilde{g}^{(h_v)}(\xi_l - \eta_l)$, where z_l is a positive integer $z_l \leqq 2$ and such that $\sum_{l\in T_v} z_l \leqq 2$. The end points of the lines in T_v ξ_l, η_l can be simple points $x_{j,i}$ or interpolated points $x'_{j,i}$ (see Eq. (35)), i.e

$$x'_{j,i} = \sum_{i'}\varepsilon_{i',j}(\underline{u})x_{i',j} ,\quad \sum_{i'}\varepsilon_{i',j}(\underline{u}) = 1 ,\quad \varepsilon_{i',j}(0) = \delta_{i',i} ,$$

and $G^{T_v}_{iji'j'} = S_{jj'}\tilde{g}^{(h_v)}(x'_{i',j'} - x'_{i,j})$ with $x'_{i',j'} - x'_{i,j}$ not belonging to T_v.

Estimating the determinant in Eq. (42) by the Grahm-Hadamard (see for instance [B.G.P.S.]) inequality it follows:

$$|V^{(k)}(\tau,P_{v_0},x_{v_0})| \leqq \bar{\varepsilon}^n \sum_{\{P_v\}} \prod_{v\ not\ e.p.} \left(\frac{Z_{h_v}}{Z_{h_v-1}}\right)^{\frac{1}{2}|P_v|} C^{|Q_v|-|P_v|} .. J(\tau,P_{v_0},x_{v_0}) \cdot \gamma^{\frac{h_v}{2}\sum_{j=0}^{2}(2j+1)\sum_i(|P^j_{v^i}|-|Q^j_{v^i}|)}, \tag{44}$$

where P^j_v denotes the subset of P_v correspondent to field with a derivative index of order j, Q^j_v is defined analogously and:

$$J(\tau,P_{v_0},x_{v_0}) = \left(\prod_{v\ not\ e.p} \frac{1}{s_v!}\right) \int \prod_{v\ not\ e.p} \sum_{T_v} \prod_{l\in T_v} d\underline{u}|\xi_l-\eta_l|^{z_l}\tilde{g}^{(h_l)}(\gamma^{h_v}(\xi_l-\eta_l)) . \tag{45}$$

The interpolated points and the terms $|\xi_l-\eta_l|$ are produced by the renormalization, see Eq. (35). In [B.G.P.S.] it is found an equation very similar to Eq. (44), except that in the equation analogous to Eq. (45) the terms $|\xi_l-\eta_l|$ produced by the renormalization would not coincide with the argument of the propagator belonging to the anchored tree T_v and z_l would be bounded by $2n$. However the proof given in [B.G.P.S.] for the boundedness of the integral over the coordinates of the analogue of Eq. (44) would not apply here as it requires exponential decay for the propagator. The elimination of the latter condition is the main technical innovation that we develop in this section.

We can perform in Eq. (45) the change of variables $\xi_l-\eta_l \to y_l$ realised from the following linear system $y = A_T(\underline{u})x$, where $A_T(\underline{u})$ is a square matrix $n\times n$ whose elements are functions of the interpolated parameters u, so that we can write:

$$\begin{aligned}&\int_\Lambda dx_{v_0}(1+\gamma^k d(P_{v_0}))^N J(\tau,P_{v_0},x_{v_0}) \\ &= \sum_T \left(\prod_{v\ not\ e.p} \frac{1}{s_v!}\right) \int d\underline{u}|\det A_T(u)^{-1}| \prod_{l\in T} \int dy_l(1+\gamma^k d_y(P_{v_0}))^N |y_l|^{z_l}\tilde{g}^{(h_v)}(\gamma^{h_v}y_l) ,\end{aligned} \tag{46}$$

where $d_y(P_{v_0})$ is the length of the shortest tree graph expressed in terms of y-variables. In Appendix 3 it is proved that $|\det A_T(u)| = 1$ so:

$$\int_\Lambda dx_{v_0}(1+\gamma^k d(P_{v_0}))^N J(\tau,P_{v_0},x_{v_0}) \leqq |\Lambda| C^{2n} \prod_v C^{\sum_i|P_{v^i}|-|Q_{v^i}|}\gamma^{-2h_v(s_v-1)-z_v h_v} . \tag{47}$$

Then Eq. (44), (46) imply that

$$\gamma^{kD(P_{v_0})} \sum_{\tau\in\tau_n} \int_\Lambda dx_{v_0}(1+\gamma^k d(P_{v_0}))^N |V^{(k)}(\tau,P_{v_0},x_{v_0})| \leqq (CC'\varepsilon_k)^n , \tag{48}$$

so that the theorem is proved. We remark that, without the result of Appendix 2, one would obtain Eq. (48) with C^n replaced by C^n_n, where C_n would be a function of n and C_n could only be bounded by $n!$ or worse.

5. The Flow of Renormalization Group

The Beta function can be written as $\beta_h(\vec{v}_h,\ldots,\vec{v}_0) = \overline{\beta}_h(\vec{v}_h \ldots \vec{v}_0) + \gamma^h \hat{\beta}_h(\vec{v}_h,\ldots,\vec{v}_0)$. In the spinless case $v_{i,h} = \{\nu_h, \delta_h, Z_h, \lambda_h\}$ and the function $\overline{\beta}_h(\vec{v}_h \ldots \vec{v}_0)$ is the same as in the $U(\vec{x}) = 0$ case. So repeating the arguments in [B.G.M., B.G.P.S] it is possible to prove that the flow is bounded and anomalous so that Theorem 1.1 holds.

The spinning case is more involved. The equations for the running coupling can be written explicitly in the following way:

$$
\begin{aligned}
g_{1,h-1} &= \frac{Z_h^2}{Z_{h-1}^2}[g_{1,h} + g_{1,h}^2(-2\beta + B_1(g_{\geqq h}, \delta_{\geqq h}, \nu_{\geqq h}) + \gamma^h R_1(g_{\geqq h}, \delta_{\geqq h}, \nu_{\geqq h})] ,\\
g_{2,h-1} &= \frac{Z_h^2}{Z_{h-1}^2}[g_{2,h} + g_{2,h}B_2(g_{2,\geqq h}, g_{4,\geqq h}, \delta_{\geqq h}, \nu_{\geqq h})\\
&\quad + g_{1,h}^2(-\beta + B_3(g_{\geqq h}, \delta_{\geqq h}, \nu_{\geqq h})) + \gamma^h R_2(g_{\geqq h}, \delta_{\geqq h}, \nu_{\geqq h})] ,\\
g_{4,h-1} &= \frac{Z_h^2}{Z_{h-1}^2}[(g_{4,h} + B_4(g_{2,\geqq h}, g_{4,\geqq h}, \delta_{\geqq h}, \nu_{\geqq h})\\
&\quad + g_{1,h}^2 B_5(g_{\geqq h}, \delta_{\geqq h}, \nu_{\geqq h}) + \gamma^h R_3(g_{\geqq h}, \delta_{\geqq h}, \nu_{\geqq h})] ,
\end{aligned}
$$

$$
\begin{aligned}
\delta_{h-1} &= \frac{Z_h}{Z_{h-1}}[\delta_h + \delta_h g_h^2 B_6(g_{\geqq h}) + \nu_h^2 B_7(g_{\geqq h}, \delta_{\geqq h}, \nu_{\geqq h})\\
&\quad + \gamma^h R_4(g_{\geqq h}, \delta_{\geqq h}, \nu_{\geqq h})] , \qquad (49)\\
\nu_{h-1} &= \gamma \frac{Z_h}{Z_{h-1}}[\nu^h + \nu_h g_h^2 B_8(g_{\geqq h}, \delta_{\geqq h}, \nu_{\geqq h}) + \gamma^h R_5(g_{\geqq h}, \delta_{\geqq h}, \nu_{\geqq h})] ,\\
1 &= \frac{Z_h}{Z_{h-1}}[1 + g_{h,1}^2 \tilde{\beta}_1 + g_{h,2}^2 \tilde{\beta}_2 + g_h^2 B_9(g_{\geqq h}) + \delta_h g_h^2 B_{10}(g_{\geqq h}, \delta_{\geqq h})\\
&\quad + g_h^2 \nu_h^2 B_{11}(g_{\geqq h}, \delta_{\geqq h}, \nu_{\geqq h}) + \gamma^h R_6(g_{\geqq h}, \delta_{\geqq h}, \nu_{\geqq h})] ,
\end{aligned}
$$

where by g_h we mean generically the quartic running coupling, i.e. one of $g_{h,1}, g_{h,2}, g_{h,4}$. We know from Sect. (4) that B_i and R_i are expressed by a power series in $\vec{v}_h$ converging to an analytic function if the running coupling $\vec{v}_h$ are such that $|\vec{v}_h| \leqq \varepsilon$. In writing Eq. (49) we make explicit the lowest order contribution in the running couplings to the power series B_i and we use some symmetry considerations.

Using the last relation in Eq. (49) to eliminate $\frac{Z_h}{Z_{h-1}}$, the fact that $\gamma > 1$ and the implicit function theorem it is possible to prove that the above equations are equivalent to:

$$\mu_{h-1} = \mu_h + \hat{G}_\mu^h(g_{1,h}, \mu_h; \ldots; g_{1,0}, \mu_0; \nu_h) + \gamma^h \hat{R}_\mu^h(g_{1,h}, \mu_h; \ldots; g_{1,0}, \mu_0; \nu_h) , \qquad (50)$$

$$g_{1,h-1} = g_{1,h} + \hat{G}_1^h(g_{1,h}, \mu_h; \ldots; g_{1,0}, \mu_0; \nu_h) + \gamma^h \hat{R}_1^h(g_{1,h}, \mu_h; \ldots; g_{1,0}, \mu_0; \nu_h) , \qquad (51)$$

$$\nu_{h-1} = \gamma \nu_h + \hat{G}_\nu^h(g_{1,h}, \mu_h; \ldots; g_{1,0}, \mu_0; \nu_h) + \gamma^h \hat{R}_\nu^h(g_{1,h}, \mu_h; \ldots; g_{1,0}, \mu_0; \nu_h) , \qquad (52)$$

where $\mu_h = g_{2,h}, g_{4,h}, \delta_h$ and $\hat{G}^h, \hat{R}^h$, are analytic for $|g_{1,h'}| < \varepsilon, |\mu_{h'}| < \varepsilon$, if $h' \geqq h$ and $|\nu_h| < \varepsilon$.

Eq. (52), (49), given any sequence of $g_{1,h}, \mu_h$ with $|g_{1,h}|, |\mu_h| < \varepsilon$, imply that there is a unique ν_0, analytic in $g_{1,h}, \mu_h$ for $|g_{1,h}|, |\mu_h| < \varepsilon$, such that $|\nu_h| < \varepsilon$ and ν_h converges to 0 for $h \to -\infty$ at the rate $O(\gamma^h)$. The proof of the existence of ν_0 is essentially a version of the unstable manifold theorem. The equation for ν_0 is:

$$\nu_0 - \sum_{i=-\infty}^{0} \gamma^i [\hat{G}_\nu^i(g_{1,i}, \mu_i; \ldots; g_{1,0}, \mu_0; \nu_i) + \gamma^i \hat{R}_\nu^i(g_{1,i}, \mu_i; \ldots; g_{1,0}, \mu_0; \nu_i)] = 0 . \quad (53)$$

By Theorem 2.1 this value ν_0 is obtained, given α, λ, by a unique choice of ν.

By a similar argument it is possible to choose δ_0 (and this corresponds to fixing α) such that $\delta_h \to 0$ for $h \to -\infty$: this choice corresponds to requiring that the Fermi velocity is fixed to 1 (see [B.G.M]) but we can avoid the work of checking the latter statement here because this choice, contrary to the choice of ν_0, is not essential to control the flow of $g_{1,h}, \mu_h$. With the above choice of ν_0, δ_0 we have that

$$\begin{aligned} \mu_{h-1} &= \mu_h + \hat{G}_\mu^h(g_{1,h}, \mu_h; \ldots; g_{1,0}, \mu_0; 0) + \gamma^h \hat{R}_\mu^h(g_{1,h}, \mu_h; \ldots; g_{1,0}, \mu_0; \nu_h) , \\ g_{1,h-1} &= g_{1,h} + \hat{G}_1^h(g_{1,h}, \mu_h; \ldots; g_{1,0}, \mu_0; 0) + \gamma^h \hat{R}_1^h(g_{1,h}, \mu_h; \ldots; g_{1,0}, \mu_0; \nu_h) , \\ \nu_{h-1} &= \gamma \nu_h + \hat{G}_\nu^h(g_{1,h}, \mu_h; \ldots; g_{1,0}, \mu_0; 0) + \gamma^h \hat{R}_\nu^h(g_{1,h}, \mu_h; \ldots; g_{1,0}, \mu_0; \nu_h) . \end{aligned} \quad (54)$$

Remembering that $\hat{G}_1^h(g_{1,h}, \mu_h; \ldots; g_{1,0}, \mu_0; 0) = g_{1,h}^2(-2\beta + \overline{G}_1^h(g_{1,h}, \mu_h; \ldots; g_{1,0}, \mu_0; 0)$, with $\overline{G}_1^h$ analytic as a function of its argument, it is clear that, for any sequence of $|\mu_h| < \varepsilon$ and any complex $g_{1,0}$, such that $|g_{1,0} - \varepsilon/2| < \varepsilon/2$, then $|g_{1,h} - \varepsilon/2| \leqq \varepsilon/2$ and $g_{1,h} \to 0$ for $h \to -\infty$ as $O(\frac{1}{|h|})$.

Remark. Chosen ν as above and λ complex such that $|\lambda - \tilde{\varepsilon}/2| < \tilde{\varepsilon}/2$, where $\tilde{\varepsilon}$ is chosen so that $|\nu_0|, |\mu_0| < \varepsilon$ and $|g_{1,0} - \varepsilon/2| \leqq \varepsilon/2$, if it happens that $|\mu_{h'}| < \varepsilon$ for $h' \geqq h$, then $\vec{v}_{h'}$ for $h' \geqq h - 1$ is analytic as a function of λ in $|\lambda - \tilde{\varepsilon}/2| < \tilde{\varepsilon}/2$ and $\nu_h, g_{1,h} \to 0$ for $h \to \infty$.

We want to show that $|\mu_h| < \varepsilon$ for all h if $|\lambda - \frac{\tilde{\varepsilon}}{2}| < \frac{\tilde{\varepsilon}}{2}$. We define the function $\lim_{h \to -\infty} \lim_{T,L \to \infty} \hat{G}_i^h(v; \ldots; v; 0) = G_i(v)$, where $i = \mu, \nu, 1$ and $v = g_1, \mu$. We prove that:

$$G_\mu(0, \mu) = 0 . \quad (55)$$

Equation (55) can be proved by comparing the beta function of our system with the Beta function of the Mattis model. The Mattis model describes two spinning fermions with linear dispersion relation. The Hamiltonian is:

$$\begin{aligned} T_0' + H_I' =& \sum_{\vec{\omega},\sigma} \int d\vec{x} : \psi^+_{\vec{x},\vec{\omega},\sigma}(i\vec{\omega}\vec{\partial})\psi^-_{\vec{x},\vec{\omega},\sigma} : + \alpha \sum_{\vec{\omega},\sigma} \int d\vec{x} : \psi^+_{\vec{x},\vec{\omega},\sigma}(i\vec{\omega}\vec{\partial})\psi^-_{\vec{x},\vec{\omega},\sigma} : \\ &+ \sum_{\vec{\omega},\sigma,\sigma'} \int d\vec{x} d\vec{y} \lambda g_4(\vec{x} - \vec{y}) : (\psi^+_{\vec{x},\vec{\omega},\sigma}\psi^-_{\vec{x},\vec{\omega},\sigma}) :: (\psi^+_{\vec{y},\vec{\omega},\sigma'}\psi^-_{\vec{y},\vec{\omega},\sigma'}) : \\ &+ \sum_{\vec{\omega},\sigma,\sigma'} \int d\vec{x} d\vec{y} \lambda g_2(\vec{x} - \vec{y}) : (\psi^+_{\vec{x},\vec{\omega},\sigma}\psi^-_{\vec{x},\vec{\omega},\sigma}) :: (\psi^+_{\vec{y},-\vec{\omega},\sigma'}\psi^-_{\vec{y},-\vec{\omega},\sigma'}) : , \end{aligned} \quad (56)$$

where :: denotes the Wick ordering respect to the free vacuum and $\psi^\pm_{\vec{x},\vec{\omega},s}$ are creation or annihilation operators of $\vec{\omega}$-fermions. Note that, contrary to the model with hamiltonian Eq. (1), the Mattis model hamiltonian is written directly in terms of quasi-particles. We can introduce a family of Grassmannian variables $\psi^\varepsilon_{\vec{x},\vec{\omega},\sigma}$ and study the

Mattis model by a renormalization group analysis similar to the one discussed so far. The ultraviolet part of the theory is rather delicate, due to the linear dispersion relation of the propagator of the Mattis model, but it has been discussed in [G.Sc.] (only for the Luttinger model, but the same considerations trivially hold also for the Mattis model).

The discussion of the infrared part is made by repeating the arguments in Sect. 2 – Sect. 4 with $U(\vec{x}) = 0$. The infrared integration is written as $\prod_{\varepsilon,\sigma,\vec{\omega}=\pm 1} \prod_{h=-\infty}^{0} P(d\psi_{k,\vec{\omega},\sigma}^{\varepsilon,h})$, where the covariance of $\psi_{k,\vec{\omega},\sigma}^{\varepsilon,h}$ is

$$g_{\vec{\omega},k',M}^{\geqq h} = \frac{\sum_{h'=h}^{0} f(\gamma^{-2h'}(k_0^2 + \vec{k}'^2)}{-ik_0 + \vec{\omega}\vec{k}'} \,. \tag{57}$$

The effective potential is given by Eq. (24) with $g_{1,h} = g_{3,h} = 0$ and $v_h = 0$ by the symmetry of the interaction and for the parity of the Mattis model propagator.

The Beta function is given by $\mu_{h-1} = \mu_h + B_{\mu,M}^h(\mu_h;\ldots;\mu_0) + \gamma^h \hat{R}_\mu^h(\mu_h;\ldots;\underline{\mu}_0)$. The crucial point is that

$$B_{\mu,M}^h(\mu_h;\ldots;\mu_0) = G_\mu^h(\mu_h,0;\ldots;\mu_0,0;0) \,.$$

This essentially follows from the fact that the propagators $g_{\vec{\omega}}^h(x)$ of our model differ from the Mattis one only for terms of order γ^h (see Lemma 2.1) and from the fact that in our model $v_h = O(\gamma^h)$. The analysis in the preceding section shows that the Beta function is analytic as a function of its argument in a circle with radius independent from β, L (see Th. 4.1); this implies that the limit of the Beta function as $\beta \to \infty$ is an analytic function of its argument in the same domain. We call from now on μ_h^L the running coupling in the theory with finite volume (but not temporal cut-off) and $\lim_{L\to\infty} \mu_h^L = \mu_h$. If $L_h = p_0^{-1}\gamma^{-h}$ the following lemma, analogous to Lemma 2 in [B.G.P.S.], holds:

Lemma 5.1. *If $\mu_{h'}$ is defined and $|\mu_{h'}| \leqq \varepsilon_1 \leqq \varepsilon/2$, for $h' \geqq h$, then also $\mu_{h'}^{L_h}$ is defined for $h' \geqq h$ and*

$$|\mu_{h'}^{L_h} - \mu_{h'}| \leqq c_1 \varepsilon_1^{3/2} \gamma^{h-h'} \quad h' \geqq h \tag{58}$$

for some constant c_1.

Proof. We proceed inductively noting that Eq. (58) holds for $h' = 0$ and proving that Eq. (58) holds for $h' - 1$ if it holds for the couplings with frequency greater than or equal to h'. We write

$$\mu_{h'-1}^{L_h} - \mu_{h'-1} = \mu_{h'}^{L_h} - \mu_{h'} + \beta_{\mu,M}^{h',L_h}(\mu_{h'}^{L_h},\ldots,\mu_0^{L_h}) - \beta_{\mu,M}^{h'}(\mu_{h'},\ldots,\mu_0) \,, \tag{59}$$

where $\beta_{\mu,M}^{h',L_h}$ and $\beta_{\mu,M}^{h'}$ are the Beta functions for the Mattis model with finite or infinite volume. It is convenient to write the second difference in Eq. (59) as

$$[\beta_{\mu,M}^{h',L_h}(\mu_{h'}^{L_h},\ldots,\mu_0^{L_h}) - \beta_{\mu,M}^{h',L_h}(\mu_{h'},\ldots,\mu_0)] + [\beta_{\mu,M}^{h',L_h}(\mu_{h'},\ldots,\mu_0) - \beta_{\mu,M}^{h'}(\mu_{h'},\ldots,\mu_0)] \,. \tag{60}$$

The first term can be bounded, proceeding as in Sect. 4 and using the inductive assumption, by $c_2 \varepsilon_1^{5/2} \gamma^{h-h'}$. In order to estimate the second difference in Eq. (60)

we remember that $\beta^{h'}_{\mu,M}(\mu_{h'},\ldots,\mu_0)$ can be written, by Eq. (42), (43), as a sum over trees τ; to each tree is associated a product of terms which can be written as the integral of products of propagators whose arguments $\xi_l - \eta_l$ form an anchored tree graph T times a determinant; all except one of the points belonging to the anchored tree graph are integrated. We perform a change of variables $\xi_l - \eta_l = y_l$ as in Eq. (46) and we write

$$\beta^{h'}_{\mu,M}(\mu_{h'},\ldots,\mu_0) = \hat{\beta}^{h'}_{\mu,M}(\mu_{h'},\ldots,\mu_0) + \tilde{\beta}^{h'}_{\mu,M}(\mu_{h'},\ldots,\mu_0)\,,$$

where $\hat{\beta}^{h'}_{\mu,M}$ contains terms integrated in $|\vec{y}_l| \leqq L/2$ for each y_l belonging to the spanning tree. All the terms contained in $\tilde{\beta}^{h'}_{\mu,M}$ contain at least an integral with domain $|\vec{y}_l| > L/2$ so that, proceeding as in Sect. 4 and remembering Eq. (46), it follows that $\tilde{\beta}^{h'}_{\mu,M}$ is bounded by $c_4\varepsilon_1^2\gamma^{h-h'}$.

It remains to bound $\beta^{h',L_h}_{\mu,M}(\mu_{h'},\ldots,\mu_0) - \hat{\beta}^{h'}_{\mu,M}(\mu_{h'},\ldots,\mu_0)$; this can made by noting that this term can be written as a sum over trees similar to that one in Sect. 4 for $\beta^{h',L}$, with the integrals over the arguments of the spanning tree T with domain $|\vec{y}_l| \leqq L/2$: the only difference is that at least one of the propagators of the spanning tree $g^{h'',L}(y_l), h'' > h'$ is replaced by $g^{h''}(y_l) - g^{h'',L_h}(y_l)$, or at least one of the determinants is replaced by $\det G^T - \det G^{T,L_h}$. Noting that the number of possible substitutions is bounded by C^n, if $\tau \in \tau_n$, and that

$$|g^{h'}(y) - g^{h',L_h}(y)| \leqq \frac{\gamma^{-h'}}{L_h}\frac{C_N}{1+\gamma^{h'N}|y|^N}\,, |\vec{y}| \leqq \frac{L}{2}\,;$$

$$|\det G^T - \det G^{T,L_h}| \leqq \frac{C^{|P|}}{L_h}\gamma^{h|P|/2} \tag{61}$$

if $|P|$ is the number of the fields in G and proceeding as in Sect. 4 we find that this term is bounded by $c_3\varepsilon_1^2\gamma^{h-h'}$. Finally we can write

$$\begin{aligned}|\mu^{L_h}_{h'-1} - \mu_{h'-1}| &\leqq c_1\varepsilon_1^{3/2}\gamma^{h-h'} + c_2\varepsilon_1^{5/2}\gamma^{h-h'} + c_3\varepsilon_1^2\gamma^{h-h'} + c_4\varepsilon_1^2\gamma^{h-h'}\\ &\leqq c_1\varepsilon_1^{3/2}\gamma^{h-h'+1}\,,\end{aligned}$$

where the first term comes from the first difference in Eq. (59) and the last three from the second difference in Eq. (60). The above inequality is always verified if ε_1 is chosen suitable small.

Let us remark that the effective potential at scale h can be also obtained by a one step integration by the relation

$$e^{V^h(\psi^{\leqq h})} = \frac{\mathcal{N}_0}{\mathcal{N}}\int P(d\psi^{>h})e^{-V^0(\psi^{\leqq 0})}\,.$$

$V^h(\psi^{\leqq h})$ is given by a series over *one step* Feynman graphs similar to those ones of Sect. 4 except that to each internal line is associated the propagator $g^{>h}_{\vec{\omega}}(k) = \sum_{k=h+1}^{0} g^h_{\vec{\omega}}(k)$ and to each vertex one of the terms in V^0 Eq. (17), (18). The expansion is well defined if $|\lambda| \leqq \frac{const}{L}$, which is not $O(1)$, and hence *very* *small* as $L \to \infty$. By the definition of the localization in Sect. (4) it follows that

$$Z_h^L \gamma^h \nu_h = \frac{1}{4}\sum_{j=1}^{2} V_2^h\left(0,(-1)^j\frac{\pi}{L};\vec\omega\right) = 0 \ ; \ Z_h^L - 1 = \frac{i\partial}{\partial k_0}\sum_{i=1}^{2} V_2^h\left(k_0,(-1)^i\frac{\pi}{L};\vec\omega\right)|_{k_0=0} \ , \tag{62}$$

$$Z_h^L - 1 + Z_h^L\delta_h^L = \frac{\vec\omega L}{2\pi}\left(V_2^h\left(0,\frac{\pi}{L};\vec\omega\right) - V_2^h\left(0,-\frac{\pi}{L};\vec\omega\right)\right) \ ,$$

$$(Z_h^L)^2 g_{2,h}^L = \frac{1}{2}\sum_k V_4^h\left(\left(0,(-1)^k\frac{\pi}{L}\right);\left(0,(-1)^k\frac{\pi}{L}\right);\left(0,(-1)\frac{\pi}{L}\right);\left(0,(-1)^k\frac{\pi}{L}\right);\underline{\vec\omega}_1\right) ,$$

$$(Z_h^L)^2 g_{4,h}^L = \frac{1}{2}\sum_k V_4^h\left(\left(0,(-1)^k\frac{\pi}{L}\right);\left(0,(-1)^k\frac{\pi}{L}\right);\left(0,(-1)\frac{\pi}{L}\right);\left(0,(-1)^k\frac{\pi}{L}\right);\underline{\vec\omega}_2\right) ,$$

where $V_2^h(k';\vec\omega)$ is given by the sum over all the one step graphs with two external lines with momentum $\vec{k'} + \vec\omega p_F$ and quasi-particle index $\vec\omega$ and $V_4^h(k_1',k_2',k_3',k_4';\underline{\vec\omega}_i)$ is given by the sum over the one step Feynman graphs with four external lines with momentum $k_i' + \vec\omega_i p_F$ and quasi-particle index $\vec\omega_i$, $i = 1,2,3,4$ and $\underline{\vec\omega}_1 = (1,-1,-1,1),, \underline{\vec\omega}_2 = (1,1,1,1)$. Of course the one step expansion for $V_2(k';\vec\omega)^h$ and $V_{4,i}^h(k_1',k_2',k_3',k_4';\underline{\vec\omega})$ *is convergent only if* $|\lambda| \leq \frac{const}{L}$. If $L \equiv L_{h-n}$ and $n > 2$ it follows from the compact support properties of $g_{\vec\omega}^{\geq h}(k)$ that $g_{\vec\omega}^{\geq h}(k) \equiv 0$ in a small domain around the point $(\pi/L, 0)$ so that the one step graphs contributing to $\mu_h^{L_{h-n}}$ or $Z_h^{L_{h-n}}$ are only the *irreducible* ones, defined as graphs which cannot be splitted into two parts by cutting a single internal line. We call $\sum^h(k;\vec\omega)$ and $\Gamma_i^h(k_1,k_2,k_3,k_4;\underline{\vec\omega})$ the sum over the irreducible one step graphs contributing to $V_2^h(k';\vec\omega)$ and $V_{4,i}^h(k_1',k_2',k_3',k_4';\underline{\vec\omega})$. Then Eq. (62) can be written, if $L \equiv L_{h-n}$ and $n > 2$, replacing V_2^h and V_4^h by $\sum^h$ and Γ^h.

By definition $\sum^h(k;\vec\omega)$ and $\Gamma_i^h(k_1,k_2,k_3,k_4;\underline{\vec\omega})$ are simply related to the two point or truncated four point Schwinger function with infrared cut-off at scale h:

$$S^{>h,L}(k',\vec\omega) = \frac{1}{-ik_0 + \vec\omega\vec{k'} + \sum^h(k')} \ , \tag{63}$$

$$S_4^{>h,T,L}(k_1',\vec\omega_1,+,\sigma;k_2',\vec\omega_2,+,\sigma';k_3',\vec\omega_3,-,\sigma';k_4',\vec\omega_4,-,\sigma) =$$
$$S^{>h,L}(k_1',\vec\omega_1)S^{>h,L}(k_2',\vec\omega_2)S^{>h,L}(k_3',\vec\omega_3)S^{>h,L}(k_4',\vec\omega_4)\Gamma^h(k_1',k_2',k_3',k_4';\underline{\vec\omega}) \ . \tag{64}$$

The above equations can be proven as an identity between graph at any order.

Equations (62), (63), (64) by substitution and some algebra implies that if $L \equiv L_{h-n}$ and $n > 2$:

$$\frac{L}{\pi}\frac{1}{Z_h^L(1+\delta_h^L)} = S^{>h,L}(0,\pi/L;\vec\omega) \ ; \ \frac{L^2}{\pi^2}\frac{1}{Z_h^L(1+\delta_h^L)^2} = \frac{\partial S^{>h,L}(0,\pi/L;\vec\omega)}{\partial k_0} \ , \tag{65}$$

$$\left(\frac{L}{\pi(1+\delta_h^L)}\right)^4 \frac{g_{2,h}^L}{(Z_h^L)^2} = \sum_{i=1}^{2} S^{>h,T,L}(0,(-1)^i\pi/L;0,(-1)^i\pi/L;0,$$
$$(-1)^i\pi/L;0,(-1)^i\pi/L;\underline{\vec\omega}_1) \ , \tag{66}$$

$$\left(\frac{L}{\pi(1+\delta_h^L)}\right)^4 \frac{g_{4,h}^L}{(Z_h^L)^2} = \sum_{i=1}^{2} S^{>h,T,L}(0,(-1)^i\pi/L;0,(-1)^i\pi/L;0,$$
$$(-1)^i\pi/L;0,(-1)^i\pi/L;\underline{\vec\omega}_2) \ ,$$

Remark. The above identity is proved by a one step integration, so is proved in the region $|\lambda| \leqq const/L$, where the one step expansion for the effective potential and the Schwinger function is analytic. However the series for the effective potential and Schwinger functions at scale h are analytic in λ if the running coupling constants $\mu_{h'}, h' > h$ are such that $|\mu_{h'}| < \varepsilon$, see Sect. 4 and below, so that in this case Eq. (65), (66) holds also in the domain $|\mu_{h'}| < \varepsilon$.

μ^L_∞, Z^L_∞, if $S^{>-\infty,L} \equiv S^L$ are given by Eq. (65), if $S^{>-\infty,T,L} \equiv S^{T,L}$ replaces $S^{>h,T,L}$.

Using the exact solution of the Mattis model [M.] and the explicit evaluation of its Schwinger function [Ma1] it is possible to prove the following lemma (analogous to Lemma 3 of [B.G.P.S.] but not explicitly proved there):

Lemma 5.2. *In the Mattis model there exists an ε such that, if $|\lambda| \leqq \varepsilon$, then μ^L_∞ is bounded and $|\mu^L_\infty| \leqq const\lambda$ uniformly in L; moreover $Z^L_\infty = A_L(\lambda)L^{2\eta(\lambda)}$ with $A_L(\lambda)$ and $\eta(\lambda)$ bounded in λ for $|\lambda| \leqq \varepsilon$ and $O(\lambda^2)$.*

Proof. If $\varepsilon_\rho(p) = \mathrm{sech}\phi(\vec{p})_\rho = \sqrt{(1+\alpha+\frac{\lambda\hat{g}_4(\vec{p})}{\pi})^2 + (\frac{\lambda\hat{g}_2(\vec{p})}{\pi})^2}$ in [Ma1] it is shown that

$$S^L(x,\vec{\omega}) = \overline{S}^L_0(x,\vec{\omega})e^{-Q^L_\rho(x)},$$

$$\overline{S}^L_0(x,\vec{\omega}) = \frac{\theta(x_0)}{L}\frac{e^{-\frac{c}{L}(x_0+\imath\vec{x})}}{1-e^{-\frac{2\pi}{L}(x_0+\imath\vec{x})}} - \frac{\theta(-x_0)}{L}\frac{e^{-\frac{c}{L}(|x_0|-\imath\vec{x})}}{1-e^{-\frac{2\pi}{L}(|x_0|+\imath\vec{x})}} = e^{-\frac{c}{L}|x_0|}\tilde{S}(x;\vec{\omega}), \tag{67}$$

$$\begin{aligned} Q^L_\rho = &\frac{\pi}{L}\sum_{\vec{p}>0}\frac{s_\rho(\vec{p})^2}{\vec{p}}(1-e^{-\vec{p}|t|\varepsilon_\rho(\vec{p})}\cos\vec{p}\vec{x}) - \frac{\pi}{L}\sum_{\vec{p}>0}\frac{\cos\vec{p}\vec{x}}{\vec{p}}(e^{-\vec{p}|t|} - e^{-\vec{p}|t|\varepsilon_\rho(\vec{p})}) \\ &- \frac{\pi}{L}\sum_{\vec{p}>0}\frac{\sin\vec{p}\vec{x}}{\vec{p}}(e^{-\vec{p}|t|} - e^{-\vec{p}|t|\varepsilon_\rho(\vec{p})}), \end{aligned} \tag{68}$$

where $c = \pi + \lambda g_4(0) > 0$ which is the solubility condition of the Mattis model, $S_\rho(p) = \sinh\phi(p)_\rho$ and $\hat{g}_i(\vec{p})$ is the Fourier transform of $g_\imath(\vec{r})$. In the limit $L \to \infty$ the asymptotic behaviour of the two point Schwinger function is [Ma1]:

$$\frac{1}{i\vec{\omega}\vec{x} + \varepsilon(o)t}\frac{1}{(\vec{x}^2+\varepsilon(0)^2t^2)^\eta}(A(\phi) + A_1(\phi,x)), \tag{69}$$

where $A_1(\phi,x)$ is bounded near $\hat{g}_2(\vec{p}) = \hat{g}_4(\vec{p}) = 0$ and $\eta = \frac{1}{2}(1 - \frac{\lambda\hat{g}_2(0)^2}{(2\pi(1+\alpha)+\lambda\hat{g}_4(0))^2})^{-1/2} - 1/2$. $\varepsilon(0)$ is the Fermi velocity. It is possible to choose α as an analytic function of λ so that $\varepsilon(0) = 1$.

From Eq. (65), (66) it is easy to see that $Z^L_\infty = L^{2\eta}\frac{I^L_2}{(I^L_1)^2}$ $\delta^L_\infty = -1 + \frac{I^L_1}{\pi I^L_2}$, where

$$S^L(0,\pi/L;1) = \frac{L}{L^{2\eta}}\int_0^\infty dx_0 \int_{-\frac{1}{2}}^{\frac{1}{2}} d\vec{x}e^{-cx_0}(L^{1+2\eta}\tilde{S}^L(Lx)e^{-Q^L_\rho(xL)})e^{i\pi\vec{x}} = \frac{L}{L^{2\eta}}I^L_1, \tag{70}$$

$$\partial_{k_0}S^L(0,\pi/L;\vec{\omega}) = \frac{L^2}{L^{2\eta}}\int_0^\infty dx_0 \int_{-\frac{1}{2}}^{\frac{1}{2}} d\vec{x}e^{-cx_0}x_0(L^{1+2\eta}\tilde{S}^L(Lx)e^{-Q^L_\rho(xL)})e^{i\pi\vec{x}} = \frac{L^2}{L^{2\eta}}I^L_2.$$

From Eq. (69) it follows that I_1^L, I_2^L has a limit for $L \to \infty$.

We study now the four points Schwinger function. We limit ourselves to $S^L(x_1,+,\sigma;x_2,+,\sigma';x_3,-,\sigma';x_4,-,\sigma)$ but similar considerations could be made also for $S^L(x_1,+,\sigma;x_2,+,-\sigma;x_3,-,-\sigma;x_4,-,\sigma)$. In [Ma1] it is shown that

$$S^{T,L}(x_1,+,\sigma;x_2,+,\sigma';x_3,-,\sigma';x_4,-,\sigma) = S^L(x_1-x_4;1)S^L(x_2-x_3;-1)(e^A-1)\,,$$

where $A = F(x_1-x_3)+F(x_2-x_4)-F(x_1-x_2)-F(x_3-x_4)$ and $F(x) = \frac{2\pi}{L}\sum_{\vec{p}} \frac{s_\rho(\vec{p})c_\rho(\vec{p})}{\vec{p}}(1-e^{-\vec{p}|x_0|\varepsilon_\rho(\vec{p})}\cos\vec{p}\vec{x})$ with $c_\rho(p)\tau\cosh\phi(\vec{p})_c$. Performing the change of variables $u = x_1-x_4, v = x_2-x_3, z = x_1-x_3$ we can write

$$\begin{aligned}\frac{g_{2,\infty}^L (I_2^L)^2 L^4}{L^{4\eta}} &= \frac{L^4}{L^{4\eta}}\int_0^\infty du_0 \int_{-1}^1 d\vec{u}\, e^{-cu_0}(L^{1+2\eta}\tilde{S}^L(Lu)e^{-Q_\rho^L(uL)} \\ &\times \int_0^\infty dv_0 \int_{-1}^1 d\vec{v}\, e^{-cv_0}(L^{1+2\eta}\tilde{S}^L(Lv)e^{-Q_\rho^L(vL)}) \\ &\times \left.\int_0^\infty dz_0 \int_{-1}^1 d\vec{z}(\cos\pi\vec{u}\cos\pi\vec{v} - \sin\pi\vec{u}\sin\pi\vec{v})\,(e^{A_\rho(uL,vL,zL)}-1)\right\}\,.\end{aligned} \tag{71}$$

It is easy to see that the expression between curly brackets of the r.h.s. of Eq. (71) has a limit for $L \to \infty$. This is made by using the asymptotic expressions of $S^L(x,\vec{\omega})$, Eq. (69), and of $F(x)$ (i.e. $s(0)c(0)\log|x_0^2+\varepsilon_\rho^2(0)\vec{x}^2|$) and by dividing the integration domain of the integral in the r.h.s. of Eq. (71) in several regions:

$$D_1 = \{u_0,v_0,z_0 \leqq 1\},\quad D_2 = \{u_0,v_0 \geqq 1, z_0 \leqq 1\},\quad D_3 = \{u_0,v_0 \leqq 1, z_0 \geqq 1\}\,,$$

$$D_4 = \{u_0 \leqq 1,\ z_0,v_0 \geqq 1\},\quad D_5 = \{v_0 \leqq 1, z_0,u_0 \geqq 1\},\quad D_6 = \{u_0,v_0,z_0 \geqq 1\}\,.$$

In each region the infrared divergences are integrable. In D_2 the integration over u_0,v_0 is controlled by the exponential factors; in D_3 one has to use that $(e^A-1) \simeq_{z_0\to\infty} \frac{f(u_0,v_0,\vec{u},\vec{v},\vec{z})}{z_0^2}$, where $f(u_0,v_0,\vec{u},\vec{v},\vec{z})$ is a polynomial of second order in the variables $u_0,v_0,\vec{u},\vec{v},\vec{z}$; in D_4 it is convenient to split the integration domain in two regions, one with $|z_0-v_0| \leqq 1$ and the other with $|z_0-v_0| \geqq 1$. In the first of these regions one can perform a change of variables $u_0,v_0,z_0 \to y_0 = z_0-v_0, u_0, v_0$ and the integral is of course divergence free as $y_0, u_o \leqq 1$, while v_0 is controlled by the exponential term; in the second region one can use that $(e^A-1) \simeq_{x_0\to\infty} \frac{f_1(u_0,v_0,\vec{u},\vec{v},\vec{z})}{(z_0-v_0)^2} + \frac{f_2(u_0,v_0,\vec{u},\vec{v},\vec{z})}{z_0(z_0-v_0)}$. Similar considerations hold for D_5 and D_6.

Suppose now that μ_h, although start arbitrary small, can reach $O(\varepsilon/2)$ at h_0; then v_{h_0} is "close" to $v_{h_0}^{L_{h_0}}$, by the considerations above Eq. (58); but, by the consideration at the end of Sect. 3, $v_{h_0}^{L_{h_0}} = v_{-\infty}^{L_{h_0}}$, so we can conclude that $v_{-\infty}^{L_{h_0}} = O(\varepsilon/2)$. But this is in contradiction with the fact that, by the exact solution $v_{-\infty}^{L_{h_0}} = O(\lambda)$ (Lemma 5.1); this essentially proves that:

Lemma 5.3. *There exists an ε such that, if $|\lambda| \leqq \varepsilon$ then for any $h \leqq 0$ μ_h is analytic in λ and $|\mu_h| < const \cdot \lambda$ uniformly in L; moreover $Z_h = A_L(\lambda)\gamma^{-h2\eta(\lambda)}$ with $A_L(\lambda), \eta(\lambda)$ analytic in λ in $|\lambda| < \varepsilon$ and $O(\lambda^2)$.*

Proof. Let us suppose that, given $\varepsilon_1 \leqq \varepsilon/2$, there exists $h_0 > -\infty$ such that

$$|\mu_h| \leqq \varepsilon_1/2 < |\mu_{h_0}| < \varepsilon_1$$

for $h \geqq h_0$. We start with a small μ, say $|\mu| \leqq \varepsilon_1/4$ and, from Lemma 5.1 we have that it is possible to fix $n > 2$ so that, if $h' \geqq h_0$ $|\mu_{h'} - \mu_{h'}^{L_{h_0-n}}| \leqq c_1\varepsilon^{3/2}\gamma^{-n} \leqq \varepsilon_1/8$ (for instance). Note that, if $|\mu_{h'}| \leqq \varepsilon_1 \leqq \varepsilon, h' \geqq h$ the bounds in Sect. 4 imply that

$$|\mu_{h'}^L - \mu_{h'+1}^L| \leqq \varepsilon_1^2 \,,$$

and this equation imply that $|\mu_{h_0}^{L_{h_0-n}} - \mu_{h_0-n}^{L_{h_0-n}}| \leqq 2b\varepsilon_1^2 n$ and the factor 2 takes into account the small growth of $\mu_{h'}^{L_{h_0-n}}$ for $h < h_0$. But by Eq. (36) it holds that $\mu_{h_0-n}^{L_{h_0-n}} = \mu_{\infty}^{L_{h_0-n}}$ and, $\mu_{\infty}^{L_{h_0-n}}$, defined by Eq. (65), (66) with $h = -\infty$, is close to μ by $c_2\varepsilon_1^2$ for Lemma 5.2 so that $|\mu_{h_0} - \mu| \leqq \varepsilon_1/8 + 2b\varepsilon_1^2 n + c_2\varepsilon_1^2$.

We note finally that the Beta function has an essentially short memory as the number of scales h' above h at which one must know $\vec{v}_h$ in order to compute $\vec{v}_{h-1}$ is essentially finite; in fact (see [B.G.P.S.])

$$\hat{B}_{\mu,M}^h(\mu_h;\ldots;\mu_0) = \hat{B}_{\mu,M}^h(\mu_h;\ldots;\mu_h) + \sum_{k=h+1}^{0} D^{h,k}(\mu_h;\ldots;\mu_0)$$

with $|D^{h,k}| < K\varepsilon^2\gamma^{-1/2(k-h)}$, if $|g_{1,h}|, |\mu_h| < \varepsilon$ and K is a constant. But a dynamical system of the form $\mu_{h-1} = \mu_h + B(\mu_h)$ with B vanishing at least to the second order cannot have trajectories bounded by a constant unless $B \equiv 0$ (see [B.G.P.S.]). This argument implies Eq. (55).

We return then to the study of Eq. (54): the vanishing of the Beta function Eq. (55) implies that, fixed v, δ as the above analytic functions in λ, if $|\lambda - \tilde{\varepsilon}/2| < \tilde{\varepsilon}/2$, then $|v_{i,h} - v_{i,0}| \leqq c\varepsilon^2$ and:

$$g_{1,h} \to_{h\to-\infty} 0 \quad g_{2,h} \to_{h\to-\infty} g_{2,\infty} \quad g_{4,h} \to_{h\to-\infty} g_{4,\infty} \quad \delta_h \to_{h\to-\infty} 0$$

$$\frac{Z_h}{Z_{h-1}} \to_{h\to-\infty} \gamma^{-2\eta} \,, \tag{72}$$

where $g_{2,\infty}, g_{4,\infty}, \eta$ are analytic functions in λ for $|\lambda - \tilde{\varepsilon}/2| < \tilde{\varepsilon}/2$. From Eq. (49) it is easy to see that $g_{2,\infty}, g_{4,\infty}$ are $O(\varepsilon)$ and $\eta = O(\varepsilon^2)$ as $\eta = cg_{2,\infty}^2 + O(\varepsilon^2), c > 0$.

The 2-points Schwinger function $S(k,k) \equiv S(k)$ is given by

$$S(k) = \frac{h(k_0^2 + E(\vec{k})^2)}{-ik_0 + E(\vec{k})} + S^{\leqq 0}(k)(1 - h(k_0^2 + E(\vec{k})^2)) \,,$$

where $S^{\leqq 0}(k)$ in terms of Fourier transform

$$\hat{S}^{\leqq 0}(x-y) = \frac{\partial}{\partial\phi^+(x)\partial\phi^-(y)}\bigg|_{\phi=0} \log \frac{\mathcal{N}_0}{\mathcal{N}} \int P_{Z_0}(d\psi^{\leqq 0})e^{-V^0(\sqrt{Z_0}\psi)+(\psi^+,\phi^-)+(\phi^+,\psi^-)} \,. \tag{73}$$

In order to study $\hat{S}^{\leqq 0}(x-y)$ one shall study a tree expansion similar to that one studied in this and in the preceding section, generated integrating step by step the fields with decreasing frequency. This expansion is described in all details in

[B.G.P.S.]. The bound Eq. (48) can be easily converted into a bound for the functional derivative of Eq. (73) finding that Schwinger functions are analytic functions of the running couplings and

$$\hat{S}(x-y) = \sum_{h=-\infty}^{0} \frac{1}{Z_h}(g^h + \varepsilon \bar{g}^h)\,, \tag{74}$$

where ε is supposed small enough and $|\bar{g}^h(x-y)| \leqq \frac{\gamma^h}{(1+((x_0)_\pi^2+(\vec{x})_\pi^2)^N}$. Of course $S(k)$ is analytic in λ if $|\lambda - \varepsilon/2| \leqq \varepsilon/2$.

From Eq. (74) it follows that, if $\frac{Z_h}{Z_{h-1}} \to_{h\to-\infty} \gamma^{-2\eta}$ then $\hat{S}$ decays, for $T, L \to \infty, |x-y| \to \infty$ as $|x-y|^{-1-2\eta}$. Then, from Eq. (72) and performing the Fourier transform to the two-points Schwinger function we have Theorem 1.2. A simple corollary of this theorem is that the two point Schwinger function $S(x,y)$ behaves, for $|x-y| \to \infty$, as

$$S(x,y) = \int \phi(\vec{k},x)\phi(-\vec{k},y)e^{-ik_0(x_0-y_0)}S(k) \simeq (1-A_0(\lambda))\frac{S_0(x,y)}{|x-y|^{2\eta(\lambda)}}$$
$$+ A_1(\lambda)\frac{1}{|x-y|^{1+2\eta(\lambda)}}$$

with $A_0(\lambda)$ independent from x and y and with S_0 being the free pair Schwinger functions.

It remains to discuss the Borel summability. We remember that if for some $\varepsilon > 0$ in the domain $|\lambda - \varepsilon/2| \leqq \varepsilon/2$ a function $f(\lambda)$ of complex λ is such that:

1. $f(\lambda)$ is analytic,
2. $|f(\lambda) - \sum_{k=0}^{n} \frac{\lambda^k}{k!} f^k(0)| \leqq C^n |\lambda|^{n+1} n!$,

then $f(\lambda)$ is Borel summable in the given domain.

We known that $v_0(\lambda)$ and $S(x)$ verifies the first condition. Noting that $|\sum_{h=-\infty}^{0} h^n\gamma^h| \leqq n!C^n$ and remembering Eq. (53) an estimate on v_h of the form $|\frac{\partial^n \vec{v}_h}{\partial\lambda^n}| \leqq n!h^nC^n$ uniform in λ and h, which should be clear from the consideration above, seems to be enough to prove Borel summability of $v_0(\lambda)$. Without anomalous scaling this would be enough to prove Borel summability for the Schwinger function. But the presence of anomalous scaling requires also that

$$\left|\frac{\partial^n \eta(\lambda)}{\partial\lambda^n}\right| \leqq (n!)^2 C^n\,.$$

Some cancellations in Eq. (49) strongly support the validity of this estimate, but its rigorous proof requires some extra work.

6. Conclusions

In the preceding sections we find that the Fermi surface is anomalous both if the fermions are spinless and the conduction band is not filled or if the fermions are spinning, the interaction repulsive and the band neither filled nor half filled.

Changing the form in the interaction it is possible to have a normal Fermi surface, i.e. $\eta(\lambda) \equiv 0$. We can consider in fact a slightly different model in which

$\hat{v}_n(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4)$ in Eq. (6) is a function with support strictly contained in the region where the $\vec{k}_i$ have the same sign and are non-zero. Note that in this case the interaction in the hamiltonian Eq. (1) is not a pair potential, *i.e.* it has the form $W(x_1, x_2, x_3, x_4)$. Then it is easy to see, by symmetry reasons, that there are no graphs contributing to $g_{1,h}$ or $g_{2,h}$ for every h. In fact in the graphs contributing to $g_{1,h}$ or to $g_{2,h}$ there are necessarily vertices in which the incoming or outcoming fields have momenta with different signs. If $g_{2,h} = g_{1,h} = 0, |v_{i,0}| \leqq \varepsilon, i = 4, 5$ it follows that $|v_{i,h} - v_{i,0}| \leqq c\varepsilon^2$ and

$$g_{4,h} \rightarrow_{h \rightarrow -\infty} g_{4,\infty}, \quad \delta_h \rightarrow_{h \rightarrow -\infty} \delta_\infty, \quad \frac{Z_h}{Z_{h-1}} \rightarrow_{h \rightarrow -\infty} 1, \tag{75}$$

where $g_{4,\infty}, \delta_\infty$ are bounded functions in $v_{0,i}$ and $O(\varepsilon)$. It is not clear if an Hamiltonian with such a potential should be considered the model for some physical situation.

We discuss briefly the cases not covered by Theorems 1.1, 1.2. We start from the case in which $p_F = \pi/2a$. From Sect. (3) we know that in this case there is another relevant running coupling, $g_{3,h}$. In order to study the Beta function at the second order, it is convenient to introduce a new coupling $\tilde{g}_{2,h} = g_{1,h} - 2g_{2,h}$ so that

$$\begin{aligned}
g_{4,h-1} &= g_{4,h}, \quad g_{3,h-1} = g_{3,h} - \beta g_{3,h}\tilde{g}_{2,h}, \\
\tilde{g}_{2,h-1} &= \tilde{g}_{2,h}, \quad g_{1,h-1} = g_{1,h} - \beta g_{1,h}^2, \\
\alpha_{h-1} &= \alpha_h + \tilde{\beta}_1 g_{1,h}^2 + 1/4\tilde{\beta}_2(g_{1,h} - 2\tilde{g}_{2,h})^2 + \tilde{\beta}_3 g_{3,h}^2, \\
\zeta^{h-1} &= \zeta^h + \tilde{\beta}_1 g_{1,h}^2 + 1/4\tilde{\beta}_2(g_{1,h} - 2\tilde{g}_{2,h})^2 + \tilde{\beta}_3 g_{3,h}^2, \\
v^{h-1} &= \gamma v^h.
\end{aligned} \tag{76}$$

It is possible to choose $|v_{i,0}| \geqq \varepsilon$ so that, if $g_{1,0}, \tilde{g}_{2,0} \geqq 0$, then $\tilde{g}_{2,\infty} = \tilde{g}_{2,0} = O(\varepsilon)$ and $g_{3,h}, g_{1,h} \rightarrow_{h \rightarrow -\infty} 0$. The behaviour of the flow at the second order suggests that we try an anomalous scaling. One can repeat the consideration in Sect. (4) and write the analogues of Eq. (49) for the case $p_F \neq \pi/2a$. However the equations are too difficult and we are not able to prove that the flow is bounded at any order.

Going back to the case $p_F \neq \pi/2a$ if $g_{1,0} < 0$ the running coupling constants exit from the convergence circle of the beta function in finite many steps. The only hope in order to have a bounded flow that can be studied by perturbation theory is that the Beta function have a non-trivial fixed point:

$$v^* = \beta(v^*).$$

Some heuristic consideration [L.E.,S.] and a third order analysis lead to the conjecture that

$$g_{1,h}, \tilde{g}_{2,h}, g_{3,h} \rightarrow_{h \rightarrow -\infty} g_{1,\infty}^*, \tilde{g}_{1,\infty}^*, \tilde{g}_{2,\infty}^*, g_{3,\infty}^*, \quad \delta_h \rightarrow_{h \rightarrow -\infty} \delta_\infty^*,$$

$$\frac{Z_h}{Z_{h-1}} \rightarrow_{h \rightarrow -\infty} \gamma^{-2\eta^*}, \tag{77}$$

where $g_{1,h}^*, g_{2,h}^*, g_{3,h}^*, g_{4,h}^*, \eta^*$ are constants independent from the interaction λ and $O(1)$. A similar flow cannot be studied very likely by our perturbative expansion as the radius of convergence is very small. Perhaps one could try to use better estimates than those in Sect. (4) or use other techniques in order to enlarge the convergence domain and reach this fixed point (if it exists at all).

Finally we note that, if $p_F = \pi/a$, the propagator is given by (if $\vec{k'} = \vec{k} + \pi/a$):

$$g^h(k') = \gamma^{-h}\tilde{g}^h(\gamma^{-h}k_0, \gamma^{-h/2}\vec{k'}) = \gamma^{-h}\overline{g}_I(\gamma^{-h}k_0, \gamma^{-h/2}\vec{k'}) + \gamma^{-h/2}C_h(\gamma^{-h}k_0, \gamma^{-h/2}\vec{k'}))\,,$$

where

$$g_I(\gamma^{-h}k_0, \gamma^{-h/2}\vec{k'}) = \frac{f(\gamma^2([\gamma^{-h}k_0]^2 + (\beta[\gamma^{-h}\vec{k'}]^4)}{-i[\gamma^{-h}k_0] + \tilde{\beta}[\gamma^{-h}\vec{k'}]^2}\,,$$

and $C_h(t)$ weakly dependent on h, T, L. One can estimate by a power counting argument like in Sect. (3) the generic graph contributing to V^k: in this case one obtains that the size of the graph is bounded by $\gamma^{-k(m_4+\frac{m_2}{2})}$, where m_4 and m_2 are the number of the vertex with two or four external lines in the graph. This is the behaviour of *not renormalizable* field theories. It is unclear whether techniques of [F.G.,F.] for a non-renormalizable field theory can be of any use.

A. Appendix

We state first some easy consequence of the results in [Ko.], proved in [B.M.]

Lemma A.1. *We have*

$$\lim_{\tau\to n\pi/a^\pm} E(\tau + ih_I) = \lim_{\tau\to n\pi/a^\pm} E(-\tau + ih_I)\,,$$
$$\lim_{\tau\to n\pi/a^\pm} \phi(\tau + ih_I, \vec{x})\phi(\tau + ih_I, \vec{y}) = \lim_{\tau\to n\pi/a^\pm} \phi(-\tau + ih_I, \vec{x})\phi(-\tau + ih_I, \vec{y})\,, \quad (78)$$

where $h_I \leqq h_{n-1}$.

Lemma A.2. *If $\overline{k} = g + i\overline{h}$ where $\overline{h} \mp h_n$ then $\varepsilon(\vec{\overline{k}}) = \vec{\overline{k}}^2 + \varepsilon_1(\vec{\overline{k}})$, where $\lim_{g\to\infty} \frac{\varepsilon_1(\overline{k})}{k} = 0$, and $|\sqrt{\varepsilon(\overline{k})} - \overline{k}| \leqq \frac{c}{\sqrt{\varepsilon(\overline{k})}}$. Moreover there exist a constant K such that $\phi(\overline{k}, x) < Ke^{-\overline{h}a\vec{x}}$ and $|\phi(\overline{k}, \vec{x}) - e^{i\overline{k}\vec{x} + i\tilde{\varepsilon}(\overline{k})t}| \leqq \frac{Ke^{-\overline{h}\vec{x}}}{\sqrt{\varepsilon(\overline{k})}}$.*

In order to prove Eq. (15), we start proving the following lemma:

Lemma A.3.

$$g_{u.v.}(x, y) = S(x, y) + R_1(x, y) + R_2(x, y)\,,$$

$$S(x, y) = H(\vec{x} - \vec{y})H(x_0 - y_0)\theta(x_0 - y_0) \int d\vec{k}\phi(\vec{k}, -\vec{x})\phi(\vec{k}, \vec{y})e^{-E(\vec{k})(x_0 - y_0)}\,, \quad (79)$$

$$|R_1(x, y)| \leqq \frac{C_N}{1 + |x - y|^N}\,, \quad |R_2(x, y)| \leqq Ce^{-a|x-y|}\,,$$

where $E(\vec{k}) = \varepsilon(\vec{k}) - \mu$, N is an integer, $H(t)$ is a C^∞ compact support function such that $H(t) = 1$ for $|t| \leqq 1$, $H(t) = 0$ for $|t| \geqq \gamma$, $\gamma > 1$ and θ is the step function.

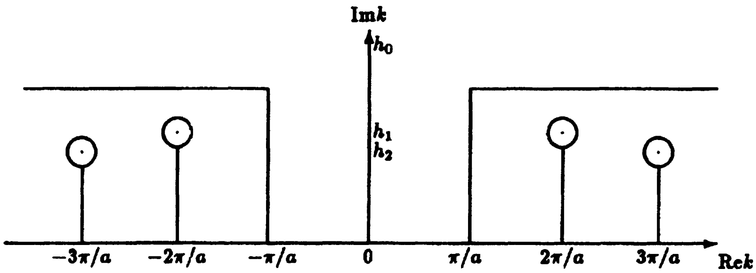

Fig. 2.

Proof. Let $g_{u.v.}(x,y) = g_1(x,y) + g_2(x,y)$, where in g_1 the integral is restricted on $k \leqq \pi/a$ and in g_2 on $k \geqq \pi/a$. It is easy to see that g_1 belong to R_1.

For $|\vec{k}| \geqq \pi/a$ we have $E(\vec{k}) \geqq 0$ and $h(c^2(k_0^2 + E(\vec{k})^2) = 1$, so we can write

$$g_2(x,y) = \theta(x_0 - y_0) \int_{|\vec{k}| \geqq \frac{\pi}{a}} d\vec{k} e^{-E(\vec{k})(x_0 - y_0)} \phi(\vec{k}, -\vec{x}) \phi(\vec{k}, \vec{y}) \, .$$

Let

$$g_2(x,y) = H(\vec{x} - \vec{y}) H(x_0 - y_0) g_2(x,y) + (1 - H(\vec{x} - \vec{y}) H(x_0 - y_0)) g_2(x,y) \, . \tag{80}$$

We will show that the second term in Eq. (80) belongs to $R_2(x,y)$.

It is always possible to choose $\bar{h} \neq h_n, \bar{h} < h_0$ such that $ReE(g + i\bar{h}) > 0$ for $|g| \geqq \pi/a$. Let us remember that $\lim_{n \to \infty} h_n = 0$. We can shift the integral of $g_2(x,y)$ to a line with a complex part $\bar{h}$ considering the following integral in the complex plane:

$$\int_{\gamma_1} \phi(k, -\vec{x}) \phi(k, \vec{y}) e^{-E(k)(x_0 - y_0)} \, ,$$

where γ_1 is a connected path such that the function is analytic in its interior and $-[\tilde{n} + 1/2]\pi/a \leqq \mathrm{Re}k \leqq [\tilde{n} + 1/2]\pi/a, \tilde{n} \to \infty$ (see the picture)

We can eliminate the integral along the part of the path perpendicular to the real axis using periodicity Eq. (78). The integrals on the circle around the non-analyticity points $\vec{k}_n$ give a vanishing contribution when the radius is sent to zero by Eq. (3), (2).

Since in the limit $\tilde{n} \to \infty$ the integral on the path from $\tilde{n} + 1/2$ and $(\tilde{n} + 1/2) + i\bar{h}$ is vanishing we have

$$\int_{|\vec{k}| > \pi/a} d\vec{k} \phi(\vec{k}, \vec{x}) \phi(-\vec{k}, \vec{y}) e^{-E(\vec{k})(x_0 - y_0)} = \int_{|\vec{k}| > \pi/a} d\vec{k} \phi(\vec{k} + i\bar{h}, \vec{x}) \phi(-\vec{k} - i\bar{h}, \vec{y}) \; e^{-E(\vec{k} + i\bar{h})(x_0 - y_0)}$$

so that, using the properties of Block wave listed in Lemma A.2, we have that g_2 belong to R_2. Finally summing and subtracting:

$$H(\vec{x} - \vec{y}) H(x_0 - y_0) \theta(x_0 - y_0) \int_{|\vec{k}| \leqq \pi/a} d\vec{k} \phi(\vec{k}, \vec{x}) \phi(\vec{k}, -\vec{y}) e^{-E(\vec{k})(x_0 - y_0)} \, ,$$

and noting that it belongs to $R_1(x,y)$, we have the lemma.
Q.E.D.

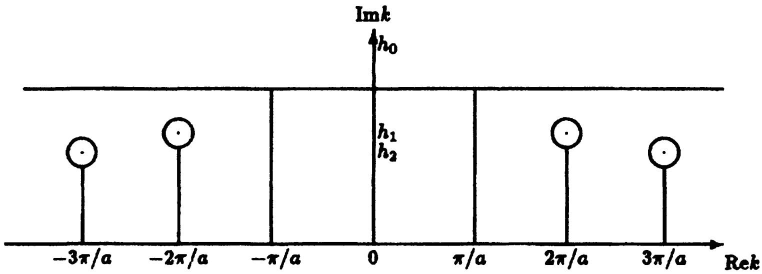

Fig. 3.

We regularize $S(x,y)$ replacing in Eq. (79) $\theta(x_0-y_0)$ by $\theta_N(x_0-y_0)$, where $\theta_N(t)$ is a smooth function with support in (γ^{-N},γ), where N is a positive integer. $\theta_N(t)$ can be written as: $\theta_N(t)=\sum_i^N f(\gamma^i t)$ with $f(t)=(H(t/\gamma)-H(t))\theta(t)$, so that $\theta(t)h(t)=\lim_{N\to\infty}\theta_N(t)$, $\lim_{N\to\infty}S_N(x,y)=S(x,y)$. We write $S_N(x,y)=\sum_{h=1}^N C_h(x,y)$, where

$$C_h(x,y)=H(\vec{x}-\vec{y})f(\gamma^h(x_0-y_0))\int d\vec{k}\phi(\vec{k},\vec{x})\phi(\vec{k},-\vec{y})e^{-E(\vec{k})(x_0-y_0)}\,. \tag{81}$$

Lemma A.4. *$C_h(x,y)$ can be written in the following way:*

$$C_h(x,y)=\gamma^{h/2}C_{1,h}(\gamma^h(x_0-y_0),\gamma^{h/2}(\vec{x}-\vec{y}))+C_{2,h}(\gamma^h x_0,\gamma^{h/2}\vec{x};\gamma^h y_0,\gamma^h\vec{y})\,, \tag{82}$$

$$\begin{aligned}\gamma^{h/2}C_{1,h}(\gamma^h(x_0-y_0),\gamma^{h/2}(\vec{x}-\vec{y}))=&H(\vec{x}-\vec{y})f(\gamma^h(x_0-y_0))\gamma^{h/2}\\&\times\int dke^{-i\vec{k}\gamma^{h/2}(\vec{x}-\vec{y})}e^{-(\vec{k}^2\gamma^h-\mu)(x_0-y_0)}\,,\end{aligned}$$

and $C_{1,h},C_{2,h}$ are smooth functions such that $|C_{1,h}(x-y)|,|C_{2,h}(x_0,x;y_0,y)|\leqq Ce^{-a|x-y|}$.

Proof. If $\vec{x}=u+n_xa,\vec{y}=v+n_ya$, we write

$$\phi(\vec{k}+i\bar{h},\vec{x})=e^{i(\vec{k}+i\bar{h})\vec{x}+i\tilde{\varepsilon}(\vec{k}+i\bar{h})u}+\phi_1(\vec{k}+i\bar{h},\vec{x})\,,$$

$$\bar{\varepsilon}(\vec{k}+i\bar{h})=(\vec{k}+i\bar{h})^2=(\vec{k}+i\bar{h})^2+\varepsilon_1(\vec{k}),\quad\sqrt{\bar{\varepsilon}(\vec{k}+i\bar{h})}=(\vec{k}+i\bar{h})+\tilde{\varepsilon}(\vec{k}+i\bar{h})\,.$$

We can shift the integral of $C_{2,h}$ to a line with imaginary part $\gamma^{h/2}\bar{h},\bar{h}>max_nh_n$ using a connected path γ_2 (see the picture)

We have

$$C_{2,h}(\gamma^h x_0,\gamma^{h/2}\vec{x},\gamma^h y_0,\gamma^h\vec{y})=H(\vec{x}-\vec{y})f(\gamma^h(x_0-y_0))$$

$$\int_0^1 d\sigma\int dk\frac{\partial}{\partial\gamma^{-h/2}\sigma}[e^{i(\vec{k}+i\bar{h})\vec{y}\gamma^{h/2}+i[\sigma\gamma^{-h/2}]\tilde{\varepsilon}(\gamma^{h/2}(k+i\bar{h})\gamma^{h/2}\vec{y}}+\sigma\gamma^{-h/2}(\phi_1(\gamma^{h/2}(\vec{k}+i\bar{h}),\vec{y})\gamma^{h/2})]$$

$$[e^{-i(\vec{k}+i\bar{h})\vec{x}\gamma^{h/2}-i[\sigma\gamma^{-h/2}]\tilde{\varepsilon}(\gamma^{h/2}(k+i\bar{h}))\gamma^{h/2}\vec{x}}+\sigma\gamma^{-h/2}(\phi_1(\gamma^{h/2}(\vec{k}+i\bar{h}),\vec{x})\gamma^{h/2})]$$

$$e^{-[(\vec{k}+i\bar{h})^2+(\sigma\gamma^{-h/2})\gamma^{-h/2}\varepsilon_1(\gamma^{h/2}(\vec{k}+i\bar{h}))]\gamma^h(x_0-y_0)}\,. \tag{83}$$

The lemma is proved noting that, from Lemma A.2:

$$\lim_{\vec{k}\to\infty} |\vec{k}\phi_1(\vec{k}+i\bar{h},\vec{x})| = e^{-\bar{h}\vec{x}}C \quad \lim_{\vec{k}\to\infty} \frac{\varepsilon_1(\vec{k}+i\bar{h})}{\vec{k}} = 0 \quad \lim_{\vec{k}\to\infty} \tilde{\varepsilon}(\vec{k}+i\bar{h})\vec{k} = const$$

Q.E.D.

Summing over C_h and making the limit $N \to \infty$ we have Eq. (16).

B. Appendix 2

Consider $V^k(\tau, P_{v_0}, x_{v_0})$ on each vertex of τ with the action of $\mathscr{R}$ given by Eq. (32) and analogues; if the delta-functions are not integrated away there are no zeros or interpolated points. In order to obtain Eq. (43) we write the renormalization using Eq. (33) by integrating the delta-functions. We start from the first (climbing the tree from the root) non-trivial vertex v in which the action $\mathscr{R} \neq 1$. The renormalization produces a zero $(x-y)^x$, if $z = 1,2$ and x, y are points in the cluster v, that we can write as

$$(x-y)^x = \left(\sum_{\imath,j,i',j'} (x_{i,j} - x_{i',j'}) \right)^z , \tag{84}$$

Where $x_{i,j} - x_{i',j'}$ are defined in the following way:

1. $x_{\imath,j}, x_{i',j'}$ is the argument of a propagator belonging to $\tilde{T}_v$,
2. otherwise $x_{i,j}, x_{i',j'}$ are the coordinates of some field with label in the set P_{v_i} if $v_\imath$ is the frequency label of the generic subtrees coming from v.

If $|P_{v_\imath}|$ is equal to 2 or 4 the renormalization, acting on $v_\imath$, can produce some other factor $(x_{i,j} - x_{\imath',j'})^z$. However this does not happen. In fact if we call the effective potential on which $\mathscr{R}$ act in a non-trivial way as $V^{h_{v_i}}(\tau_{v_\imath}, P_{v_\imath}; x_1 - x_2, x_2 - x_3, x_3 - x_4)$ or $V^{h_{\iota_\imath}}(\tau_{v_\imath}, P_{v_\imath}; x_1 - x_2)$ we have

$$\begin{aligned}
&\int_\Lambda \left(\prod_{i=1} dx_\imath \right) (x_\imath - x_j)^\alpha \mathscr{R} V^{h_{v_i}}(\tau_{v_\imath}, P_{v_\imath}; x_1 - x_4, x_2 - x_4, x_3 - x_4) \\
&\qquad \times \hat{\psi}^+_{x_1,\vec{\omega}_1,\sigma} \hat{\psi}^+_{x_2,\vec{\omega}_2,\sigma'} \hat{\psi}^-_{x_3,\vec{\omega}_3,\sigma'} \hat{\psi}^-_{x_4,\vec{\omega}_4,\sigma} = \\
&\int_\Lambda \left(\prod_{\imath=1} dx_i \right) (x_\imath - x_\jmath)^\alpha V^{h_{v_i}}(\tau_{v_i}, P_{v_\imath}; x_1 - x_4, x_2 - x_4, x_3 - x_4) \\
&\qquad \times \hat{\psi}^+_{x_1,\vec{\omega}_1,\sigma} \hat{\psi}^+_{x_2,\vec{\omega}_2,\sigma'} \hat{\psi}^-_{x_3,\vec{\omega}_3,\sigma'} \hat{\psi}^-_{x_4,\vec{\omega}_4,\sigma} ,
\end{aligned} \tag{85}$$

$$\begin{aligned}
&\int_\Lambda dx_1 dx_2 (x_1 - x_2)^\beta \hat{\psi}^+_{x_1,\vec{\omega},\sigma} \hat{\psi}^-_{x_2,\vec{\omega},\sigma} \mathscr{R} V^{h_{v_i}}(\tau_{v_\imath}, P_{v_\imath}; x_1 - x_2) = \\
&\int_\Lambda dx_1 dx_2 (x_1 - x_2)^\beta \hat{\psi}^+_{x_1,\vec{\omega}} \hat{\psi}^-_{x_2,\vec{\omega}} V^{h_{v_i}}(\tau_{v_i}, P_{v_i}; x - y) ,
\end{aligned} \tag{86}$$

$$\begin{aligned}
&\int_\Lambda dx_1 dx_2 (\vec{x}_1 - \vec{x}_2) \hat{\psi}^+_{x_1,\vec{\omega},\sigma} \hat{\psi}^-_{x_2,\vec{\omega},\sigma} \mathscr{R} V^{h_{v_i}}(\tau_{v_\imath}, P_{v_\imath}; x_1 - x_2) = \\
&\quad \int_\Lambda dx_1 dx_2 (\vec{x}_1 - \vec{x}_2) \hat{\psi}^+_{x_1,\vec{\omega},\sigma} \hat{\psi}^-_{x_2,\vec{\omega},\sigma}
\end{aligned}$$

$$\times V^{h_{v_l}}(\tau_{v_l},P_{v_l};x_1-x_2)-\int_\Lambda dt V^{h_{v_l}}(\tau_{v_l},P_{v_l};t)\frac{L}{\pi}\sin\left(\vec{t}\frac{\pi}{L}\right)\cos\frac{\pi}{\beta}t_0$$
$$\times\int_\Lambda dx_1dx_2\hat\psi^+_{x_1,\vec\omega,\sigma}\hat\psi^-_{x_2,\vec\omega,\sigma}\delta(x_1-x_2)-$$
$$\int_\Lambda dx_1dx_2\hat\psi^+_{x_1,\vec\omega,\sigma}\vec\omega\Lambda\psi_{x_1,\vec\omega,\sigma}(\vec x_1-\vec x_2)V^{h_{v_l}}(\tau_{v_l},P_{v_l};x_1-x_2)$$
$$\times\frac{L}{\pi}\sin(\vec\omega(\vec x_1-\vec x_2))\frac{\pi}{L}\cos\frac{i\pi}{\beta}(x_{1,0}-x_{2,0}), \tag{87}$$

$$\int_\Lambda dx_1dx_2(x_{1,0}-x_{2,0})\hat\psi^+_{x_1,\vec\omega}\hat\psi^-_{x_2,\vec\omega}\mathscr{R}V^{h_{v_l}}(\tau_{v_l},P_{v_l};x_1-x_2)=$$
$$\int_\Lambda dx_1dx_2(x_{1,0}-x_{2,0})\hat\psi^+_{x_1,\vec\omega,\sigma}\hat\psi^-_{x_2,\vec\omega,\sigma}V^{h_{v_l}}(\tau_{v_l},P_{v_l};x_1-x_2)$$
$$-\int_\Lambda dt V^{h_{v_l}}(\tau_{v_l},P_{v_l};t)\frac{\beta}{\pi}\sin\left(t_0\frac{\pi}{\beta}\right)\cos\left(\frac{\pi}{L}\vec t\right)$$
$$\times\int_\Lambda dx_1dx_2\hat\psi^+_{x_1,\vec\omega,\sigma}\hat\psi^-_{x_2,\vec\omega,\sigma}\delta(x_1-x_2), \tag{88}$$

where $\beta>1$ and $\Lambda=\overline{\partial}-\partial$. The renormalization of the subtrees does not produce any factor (x_i-x_j) or $(x-y)$ in Eq. (85), (86) or only a factor $(x-y)$ in Eq. (87), (88). We can then repeat for the subtree v_l the considerations made for the cluster v, writing (x_l-y_l) or $(x-y)$ like in Eq. (84). Of course if in some subtree with frequency v_l, for some choice of P_{v_l} the renormalization acts in a non-trivial way, integrating the corresponding deltas and using Eq. (35), the arguments of the propagators which connect $v_1\dots,v_s$ form a tree T_v which joins simple or interpolated points.

Iterating this argument for all vertices v we have Eq. (43).

C. Appendix 3

We will give here a sketch of the proof. More detail can be found in [B.M.]. We want to prove that

$$\prod_{i=1}^n dx_i=\prod_{i=1}^{n-1}dy_l dx_1, \tag{89}$$

where x_1 is the root of the tree.

If we consider a minimal cluster, i.e. a cluster containing only points, there is no interpolated point inside it, so we have $\prod_i dx_i^{(j)}=dx_1^{(j)}\prod dy_l^{(j)}$, where j is the cluster index and $x_i^{(j)},y_l^{(j)}$ are points and lines internal to cluster j. We can now write

$$\prod_{i=1}^n dx_i=\prod_{j,l}dy_{l,j}\prod_j dx_{1,j}. \tag{90}$$

Let $\bar l$ be the line connecting the cluster 1 to cluster 2. We have $y_{\bar l}=x_1'-x_2'$ with $x_j'=\sum_i\lambda_{i,j}x_{i,j}$ for suitable interpolating parameters $\lambda_i^j(\sum_i\lambda_{i,j}=1$ see Sect. 4). So we have

$$\begin{aligned}x_{1,2}&=x_{1,2}+y_{\bar l}-x_2'+x_1'\\&=\sum_i\lambda_{i,2}(x_{1,2}-x_{i,2})+y_{\bar l}+x_1'.\end{aligned}$$

Now $x_{1,2} - x_{i,2}$ can be written in terms of $y_{l,2}$ and x_1' in terms of $y_{l,1}$ and $x_{1,1}$ so that we can fix $x_{1,1}$ and substitute $dx_{1,2}$ by $dy_{\bar{l}}$ in Eq. (90). By using this fact we can make the change of variables for the line connecting the minimal clusters. Clearly we can go on recursively on the level of the clusters and reach all the graphs.

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