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EPR-Relations, von Neumann's Standard Forms and a Proof Concerning a Conjecture of E. Scheibe

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Abstract: In the first part of this paper it is shown how EPR-situations are correlated with von Neumann's standard form of quantum mechanical states describing a system consisting of two dynamical independent subsystems. These standard forms are the mathematical tools for a proof of a conjecture of E. Scheibe: If the 4 selfadjoint operators in Bell's – inequality are pairwise EPR – related, then this inequality is valid in the strong form (with the same upper bound as in statistical mechanics). In the last section the question is discussed whether observations made on the two subsystems together with EPR-relations between them, determine the state of the composed system.

1. Introduction

If a physical system S consists of two subsystems S_1 resp. S_2, then there exist in general correlations between the results of pairs of measurements made on S_1 resp. S_2. This might be also the case if the two subsystems are dynamically independent of each other. Dynamical independency is assumed for the following remarks and also that S, S_1 resp. S_2 are quantum mechanical systems. The pure states of these systems are elements of Hilbertspaces $\mathbf{H}$, $\mathbf{H_1}$ resp. $\mathbf{H_2}$. $\mathbf{H}$ is constructed from $\mathbf{H_1}$ and $\mathbf{H_2}$ by taking the tensor product of $\mathbf{H_1}$ and $\mathbf{H_2}$ and by completing this tensor product in the standard procedure. Measurements on S may be classified into those which are performed only on S_1, S_2 resp. S_1 and S_2; they are represented by selfadjoint operators $A \otimes 1$, $1 \otimes B$ resp. $A \otimes B$. 1 is the identity-operator in $\mathbf{H_2}$ resp. in $\mathbf{H_1}$. We restrict our attention to pure states Φ of S with the aim to give a rather short discussion.

Correlations between S_1 and S_2 are here our main interest; they belong to pairs of observables $A \otimes B$. They appear in a simple form, as correlations between eigenvalues, if the corresponding selfadjoint operators have pure point-spectra; this property will

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be assumed as the basis for the following remarks. There are states $\mathbf{\Phi}$ and observables $A \otimes 1$ resp. $1 \otimes B$, so that for the measurement of $A \otimes B$ there exist simple strong correlations. If a_r is the measured eigenvalue of A, then the eigenvalue b_r is the result for the measurement of B and vice versa; and this one-to one correlation is valid for each a_r resp. b_r. In this case one calls A and B an EPR-pair in the state $\mathbf{\Phi}$ [1].

In Sect. 2 we repeat some facts about von Neumann's standard forms [2] concerning the pure states $\{\Phi_j\}$ of S. They are especially suitable for the representation of EPR-correlations. On the one side one has to describe the class $\{\mathbf{\Phi}_j\}$ of pure states of S in which a given pair of observables $A \otimes 1$, $1 \otimes B$ are an EPR-pair. On the other hand one knows that there are states $\mathbf{\Phi}$, where the standard form is not unique. Since this fact is important for a situation which E. Scheibe had considered for his conjecture, one has to characterise the manifold of standard forms if $\mathbf{\Phi}$ is given.

Section 3 contains the proof of the conjecture of E. Scheibe: Bell's inequality is valid with the strong bound (≤ 2) if the four observables A, B, A', B' are pairwise in EPR-correlation A with B and A' with B'; thereby A and A' and B and B' dont commute in general. $[A, A'] \neq 0$, $[B, B'] \neq 0$. In the special case, where only two eigenvalues of each operator are involved, E. Scheibe has given the proof elsewhere [4].

In Sect. 4 one finds some remarks about the question, whether a state Φ of S is determined if the statistical operators $w^{(1)}$ resp. $w^{(2)}$ of S_1 resp. S_2 are given and also the EPR-correlations for $\mathbf{\Phi}$ are known.

2. Von Neumann's Standard Forms and EPR-Correlations

Each normalized state Φ in $\mathbf{H}$ can be written in the form

$$\Phi = \sum_{i,k} c_{ik} \psi_i \otimes \xi_k \,.$$

$\{\psi_i\}$ is a an orthonormal system in $\mathbf{H_1}$, $\{\xi_i\}$ is such a system in $\mathbf{H_2}$ and $\sum_{i,k} |c_{ik}|^2 = 1$. J. von Neumann has proven that one can find orthonormal-systems $\{\phi_i\}$ in $\mathbf{H_1}$, resp. $\{\eta_i\}$ in $\mathbf{H_2}$ (depending on $\mathbf{\Phi}$) that $\mathbf{\Phi}$ can be given in a standard form

$$\mathbf{\Phi} = \sum_r \sqrt{w_r} \phi_r \otimes \eta_r \quad \text{with} \quad w_r > 0\,, \quad \sum_r w_r = 1\,. \tag{1}$$

In the case of no degeneracy: $w_r \neq w_s$ for $r \neq s$ (we regard only the $w_r > 0$) one has only one standard form. The case of degeneracy leads to several standard forms. To characterize this set, we begin with the case of complete degeneration:

$$\hat{\mathbf{\Phi}} = \frac{1}{\sqrt{n}} \sum_{r=1}^{n} \phi_r \otimes \eta_r \,. \tag{2}$$

The substitutions

$$\phi_r = \sum_{s=1}^{n} u_{sr} \phi'_s \,, \qquad \eta_r = \sum_{s=1}^{n} \bar{u}_{sr} \eta'_s \tag{3}$$

with $\{\phi'_s\}$ orthonormal system in $\mathbf{H_1}$, $\{\eta'_s\}$ orthonormal system in $\mathbf{H_2}$ and

$$\begin{aligned} u_{sr} &= (\phi_r, \phi'_s)\,, & \bar{u}_{sr} &= (\eta_r, \eta'_s)\,, \\ \bar{u}_{rs} &= (\phi'_r, \phi_s)\,, & u_{rs} &= \eta'_r, \eta_s)\,, \end{aligned} \tag{4}$$

and (u_{rs}) as unitary $(n \times n)$-matrix, and $(\bar{u}_{rs})$ the complex-conjugate to (u_{rs}) lead to the new standard form

$$\widehat{\mathbf{\Phi}} = \frac{1}{\sqrt{n}} \sum_{r=1}^{n} \phi'_r \otimes \eta'_r\,, \quad \text{so far} \quad n \geq 2\,. \tag{5}$$

The (infinite) set of standard forms corresponds with the (infinite) set of unitary $(n \times n)$-matrices. For one can show that one gets each standard form by a pair of unitary substitutions, which are complex-conjugate to each other. The proof goes as follows: If $\widehat{\mathbf{\Phi}}$ is given by two different standard forms

$$\widehat{\mathbf{\Phi}} = \frac{1}{\sqrt{n}} \sum_{r} \phi'_r \otimes \eta'_r\,;$$

one sees immediately, that the $\{\phi'_r\}$ appearing in the second standard form have to span the same subspace of $\mathbf{H_1}$ as the ϕ_r, and the $\{\eta'_r\}$ the same subspace of $\mathbf{H_2}$ as the η_r. From this one sees that there exists a substitution expressed by a unitary $(n \times n)$-matrix (u_{rs}) leading from the $\{\phi_r\}$ to the $\{\phi'_s\}$, and a substitution expressed by another unitary $(n \times n)$-matrix (v_{rs}) leading from $\{\eta_r\}$ to $\{\eta'_s\}$. Now (v_{rs}) must be complex-conjugate to (u_{rs}) otherwise one would not get a new standard form.

If for $\mathbf{\Phi}$ there exist several different degenerations in the $\{w_r\}$ one has by a new ordering and with new indices a standard form

$$\mathbf{\Phi} = \frac{\sqrt{p_1}}{\sqrt{n_1}} \sum_{r=1}^{n_1} \phi_r \otimes \eta_r + \frac{\sqrt{p_2}}{\sqrt{n_2}} \sum_{r=n_1+1}^{n_1+n_2} \phi_r \otimes \eta_r + \ldots \quad \text{with } p_l > 0\,, \quad \sum_l p_l = 1\,. \tag{6}$$

From this expression one gets the other standard forms by a set of pairs of unitary matrices $\{(u_{rs})^{(1)}, (\bar{u}_{rs})^{(1)}; \{(u_{rs})^{(2)}, (\bar{u}_{rs})^{(2)}; \ldots\}$; each pair represents the substitutions within the corresponding sum.

The next concern is the relationship between the EPR-correlations and the standard forms of states. If 2 selfadjoint operators with point spectra $A \otimes 1$ and $1 \otimes B$ are given one can ask for all pure states $\mathbf{\Phi} \in \mathbf{H}$ which show an EPR-correlation for $A \otimes B$. Here it is assumed that the EPR-correlation is one to one: If A has the eigenvalues $\{a_r\}$, B the eigenvalues $\{b_r\}$ the set of correlated pairs $(a_1, b_1), (a_2, b_2), (a_3, b_3), \ldots$ can be written with the short notation $\{a_r, b_r\}$. If one wishes also to specify the probabilities for the different pairs, one can do this with the notation $[w_r\{a_r, b_r\}]$.

At first one can assume that A and B are not degenerate; then all states of the form

$$\{\phi\} = \left\{ \sum_r \sqrt{w_r}\, e^{i\alpha_r} \phi_r \otimes \eta_r \right\}, \tag{7}$$

with w_r arbitrary however restricted by $\sum_r w_r = 1$ have the correlation $\{a_r, b_r\}$. In the case, that one is interested in the stronger correlation $[w_r, \{a_r, b_r\}]$, then the $\{w_r\}$ are fixed, however the phases $\{\alpha_r\}$ in (7) remain arbitrary as for $\{a_r, b_r\}$.

In the case that degenerate operators in $C \times D$ shall be in an EPR-correlation, one has to realize that the degeneration of C resp. D must appear simultaneously, otherwise the EPR-correlation would not be one-to-one. For this case of degeneration we only remark that the set $\{\phi\}$ of states, which show $\{c_r, d_r\}$ is enlarged. The reason for that lies in the fact that one can choose an arbitrary basis of eigenvectors within each subspace of degeneration; each choice leads to standard forms like in (7) representing the correlation.

3. Proof of a Conjecture of E. Scheibe

In his paper "J. v. Neumann and J. S. Bell's Theorem. Ein Vergleich" [3] E. Scheibe has discussed different cases in which Bell's inequality (with the bound ≤ 2) is also valid in quantum-mechanics. One of these cases is characterised by a constellation where the 4 operators appearing in Bell's inequality are 2 EPR-pairs. This was formulated by E. Scheibe in general as a conjecture, however, proven by him [4] for the special case, that in the EPR-correlations only 2 eigenvalues are involved.

Using J. v. Neumann's standard forms one can give the following proof for E. Scheibe's general conjecture: A and B resp. A' and B' may be two EPR-pairs; written as operators in $\mathbf{H}$ one has

$$\begin{array}{llll} A \otimes 1 & \text{and} & 1 \otimes B & \text{with the correlation} \quad \{a_r, b_r\}\,, \\ A' \otimes 1 & \text{and} & 1 \otimes B' & \text{with the correlation} \quad \{a'_r, b'_r\}\,. \end{array} \tag{8}$$

It is assumed, that the norms of the selfadjoint operators A, B, A', B' are bounded by 1. This means for the eigenvalues

$$|a_r| \leq 1\,, \qquad |b_r| \leq 1\,, \qquad |a'_r| \leq 1\,, \qquad |b'_r| \leq 1\,. \tag{9}$$

Besides these conditions one assumes

$$||\mathbf{\Phi}|| = 1\,.$$

Since the commutativity: $[A, A']_- = 0$, $[B, B']_- = 0$ leads to a structure similar to one in classical statistical mechanics, one is here especially interested in cases where one has

$$(\mathbf{\Phi}, [A \otimes 1, A' \otimes 1]_- \mathbf{\Phi}) \neq 0 \quad \text{and} \quad (\mathbf{\Phi}, [1 \otimes B, 1 \otimes B']_- \mathbf{\Phi}) \neq 0\ . \tag{10}$$

The EPR-correlation of A and B leads corresponding to (1) to $\mathbf{\Phi} = \sum_r \sqrt{w_r}\,\phi_r \otimes \eta_r$ with $\{\phi_r\}$ resp. $\{\eta_r\}$ as eigenstates of A resp. B. And this feature remains the same also for degenerate A resp. B; only the class of states fulfilling $\{a_r, b_r\}$ is enlarged. To get a state $\mathbf{\Phi}$ which allows one to satisfy the conditions in (8) and (10) simultaneously, one has to assume that $\mathbf{\Phi}$ has a form as in (6).

It is preferable to begin with the case of total degeneracy of the $\{w_r\}$:

$$\widehat{\mathbf{\Phi}} = \frac{1}{\sqrt{n}} \sum_{r=1}^{n} \phi_r \otimes \eta_r\,;$$

thereby we assume

$$A\phi_r = a_r \phi_r\,, \qquad B\eta_r = b_r \eta_r\,, \qquad r = 1, 2, \ldots, n\,. \tag{11}$$

With the eigenstates ϕ'_r resp. η'_r from

$$A'\phi'_r = a'_r \phi'_r \quad \text{and} \quad B'\eta'_r = b'_r \eta'_r\,, \qquad r = 1, 2, \ldots, n\,, \tag{12}$$

one constructs the standard form of

$$\widehat{\mathbf{\Phi}} = \frac{1}{\sqrt{n}} \sum_{r=1}^{n} \phi'_r \otimes \eta'_r\,.$$

The two standard forms of $\widehat{\mathbf{\Phi}}$ are related to each other through unitary substitutions as in (3) and (4). In the expression Δ^q of Bell's inequality the two standard forms are used in a suitable manner:

$$\begin{aligned}\Delta^q &= |(\widehat{\mathbf{\Phi}}, A \otimes B\widehat{\mathbf{\Phi}}) - (\widehat{\mathbf{\Phi}}, A \otimes B'\widehat{\mathbf{\Phi}})| + |(\widehat{\mathbf{\Phi}}, A' \otimes B'\widehat{\mathbf{\Phi}}) + (\widehat{\mathbf{\Phi}}, A' \otimes B\widehat{\mathbf{\Phi}})| \\ &= \frac{1}{n}\left|\left(\sum_r \phi_r \otimes \eta_r, A \otimes B \sum_s \phi_s \otimes \eta_s\right)\right. \\ &\quad \left. - \left(\sum_r \phi_r \otimes \eta_r, A \otimes B' \sum_s \phi'_s \otimes \eta'_s\right)\right| \\ &\quad + \frac{1}{n}\left|\left(\sum_r \phi'_r \otimes \eta'_r, A' \otimes B' \sum_s \phi'_s \otimes \eta'_s\right)\right. \\ &\quad \left. + \left(\sum_r \phi'_r \otimes \eta'_r, A' \otimes B \sum_s \phi_s \otimes \eta_s\right)\right|, \\ \Delta^q &= \frac{1}{n}\left|\sum_r \left\{a_r b_r - a_r \sum_s b'_s (\phi_r, \phi'_s)(\eta_r, \eta'_s)\right\}\right| \\ &\quad + \frac{1}{n}\left|\sum_r \left\{a'_r b'_r + a'_r \sum_s b_s (\phi'_r, \phi_s)(\eta'_r, \eta_s)\right\}\right|.\end{aligned} \tag{13}$$

Regarding (4) one gets from (13)

$$\Delta^q = \frac{1}{n}\left|\sum_r \left\{a_r b_r - a_r \sum_s b'_s |u_{sr}|^2\right\}\right| + \frac{1}{n}\left|\sum_r \left\{a'_r b'_r + a'_r \sum_s b_s |u_{rs}|^2\right\}\right|.$$

With the notation:

$$|u_{rs}|^2 = p_{rs},$$

the following equations are evident:

$$p_{rs} \geq 0, \qquad \sum_r p_{rs} = 1, \qquad \sum_s p_{rs} = 1.$$

Introducing $b_r = \sum_s p_{sr} b_r$ and $b'_r = \sum_s p_{sr} b'_r$ one gets

$$\Delta^q = \frac{1}{n}\left|\sum_r \left\{a_r \sum_s p_{sr}(b_r - b'_s)\right\}\right| + \frac{1}{n}\left|\sum_r \left\{a'_r \sum_s p_{rs}(b'_r + b_s)\right\}\right|.$$

and from that the inequality

$$\Delta^q \leq \frac{1}{n}\sum_r \left(|a_r| \left|\sum_s p_{sr}(b_r - b'_s)\right|\right) + \frac{1}{n}\sum_r \left(|a'_r| \left|\sum_s p_{rs}(b'_r + b_s)\right|\right).$$

With (9) and $p_{rs} \geq 0$ follows

$$\Delta^q \leq \frac{1}{n}\sum_{r,s} p_{sr}|b_r - b'_s| + \frac{1}{n}\sum_{r,s} p_{rs}|b'_r + b_s| = \frac{1}{n}\sum_{r,s} p_{rs}(|b_s - b'_r| + |b_s + b'_r|).$$

Again from (9)

$$|b_s - b'_r| + |b_s + b'_r| \leq 2\,,$$

and with

$$\sum_{r,s} p_{rs} = n$$

follows the final result

$$\Delta^q \leq 2\,.$$

The generalisation of the proof for the general case (6) is evident. The correctness of Scheibe's conjecture has therefore been confirmed: The correlations between 2 pairs of EPR-observables dont allow to discriminate between classical and quantum mechanical statistical behaviour.

4. Is the State $\mathbf{\Phi}$ of the System S determined by the Statistical Operators of the Subsystems S_1 and S_2 and the EPR-correlations between S_1 and S_2?

If one uses a standard form for the pure state $\mathbf{\Phi}$ of S

$$\mathbf{\Phi} = \sum_r \sqrt{w_r}\, \phi_r \otimes \eta_r\,,$$

one gets the statistical operator $w^{(1)}$ resp. $w^{(2)}$ of the subsystems S_1 resp. S_2 in the simple form

$$w^{(1)} = \sum_r w_r P_{\phi_r}$$

resp.

$$w^{(2)} = \sum_r w_r P_{\eta_r}$$

with

$$\{P_{\phi_r}\}\,, \qquad \{P_{\eta_r}\}$$

as projection operators. Both operators $w^{(1)}$ resp. $w^{(2)}$ operate in $\mathbf{H_1}$ resp. $\mathbf{H_2}$ in the same way and they describe mixtures in the case that $\mathbf{\Phi}$ does not factorize. (This discovery, that a pure state in $\mathbf{H}$ leads in general to mixture for the states of the subsystems in $\mathbf{H_1}$ resp. $\mathbf{H_2}$ is by a remark of J. v. Neumann due to the Russian physicist Landau.) J. v. Neumann on the other hand has shown that the statistical operator W of S is not uniquely determined by the statistical operators $w^{(1)}$ and $w^{(2)}$ of subsystems, if $w^{(1)}$ and $w^{(2)}$ describe mixtures.

One can put the following question: Is a pure state $\mathbf{\Phi}$ of S, which is not factorising (as the outgoing configuration after a scattering process) determined by the knowledge of $w^{(1)}$ and $w^{(2)}$ and the EPR-correlation between S_1 and S_2? It is easy to see that $w^{(1)}$ and $w^{(2)}$ together with the correlations expressed by the expectation-values of all possible pairs of observables $C_j \otimes D_k$ determine $\mathbf{\Phi}$ uniquely (up to the overall free phase). Here we restrict the attention to the most evident correlations: the EPR-correlations.

If only this restricted class of correlations is known together with the statistical operators $w^{(1)}$ and $w^{(2)}$ as in our question, the answer may be different for different situations:

(a) The statistical operators

$$w^{(1)} = \sum_r w_r P_{\phi_r}\,, \qquad w^{(2)} = \sum_r w_r P_{\eta_r}$$

are not degenerate in the sense, that

$$\text{if} \quad w_r, w_s > 0 \quad \text{and} \quad r \neq s\,, \quad \text{then} \quad w_r \neq w_s\,.$$

The class of states $\{\Phi\} = \left\{ \sum_r \sqrt{w_r}\, e^{i\alpha_r} \phi_r \otimes \eta_r \right\}$ induce in $\mathbf{H_1}$ and $\mathbf{H_2}$ the same given statistical operators $w^{(1)}$ and $w^{(2)}$. Because of non-degeneracy of $w^{(1)}$ and $w^{(2)}$ there exists only one independent EPR-correlation $\{a_r, b_r\}$ for suitable selfadjoint operators, which fulfill

$$A\phi_r = a_r \phi_r\,, \qquad B\eta_r = b_r \eta_r\,, \qquad r = 1, 2, \ldots\,.$$

One can assume: $a_r \neq a_s$ and $b_r \neq b_s$ for $r \neq s$ and $a_r, a_s > 0$ and $b_r, b_s > 0$.

All other EPR-correlations are dependent on $\{a_r, b_r\}$ by introducing suitable functions of A and B: $f(A)$ and $g(B)$. So one is not able to determine the arbitrary phases $\{\alpha_r\}$: $\mathbf{\Phi}$ is not uniquely determined in case (a) by the statistical operators $w^{(1)}$ and $w^{(2)}$ and the EPR-correlations.

(b) Here it is assumed that the statistical operators are completely degenerate:

$$w^{(1)} = \frac{1}{n} \sum_{r=1}^{n} P_{\phi_r}\,, \qquad w^{(2)} = \frac{1}{n} \sum_{r=1}^{n} P_{\eta_r}\,;$$

$\{\phi_r\}$ and $\{\eta_r\}$ may be EPR-correlated in the sense that $\{\phi_r\}$ resp. $\{\eta_r\}$ are eigenstates for a pair of non-degenerate selfadjoint operators $A \otimes B$ with eigenvalues $\{a_r\}$ resp. $\{b_r\}$.

Again all $\mathbf{\Phi}$ with

$$\{\mathbf{\Phi}\} = \left\{ \frac{1}{\sqrt{n}} \sum_r e^{i\alpha_r} \phi_r \otimes \eta_r \right\} \tag{14}$$

with (α_r) as arbitrary phases have the given statistical operators $w^{(1)}$ resp. $w^{(2)}$ and the EPR-correlation $\left[\frac{1}{n}, \{a_r, b_r\}\right]$. However now the standard form is not unique; so another standard form is given by

$$\mathbf{\Phi} = \frac{1}{\sqrt{n}} \sum_{r=1}^{n} \phi'_r \otimes \eta'_r = \frac{1}{\sqrt{n}} \sum_{r=1}^{n} \phi_r \otimes \eta_r\,,$$

where the $\{\phi'_r\}$ resp. the $\{\eta'_r\}$ are related to the $\{\phi_r\}$ resp. $\{\eta_r\}$ by a complex-conjugated pair of unitary substitutions

$$\phi_r = \sum_s u_{sr} \phi'_s\,, \qquad \eta_r = \sum_s \bar{u}_{sr} \eta'_s$$

as in (3).

Now one can use $\{\phi'_r\}$ and $\{\eta'_r\}$ also as eigenstates for a pair of EPR-correlated selfadjoint operators.

$$\{\Phi\}' = \left\{ \frac{1}{\sqrt{n}} \sum_r e^{i\beta_r} \phi'_r \otimes \eta'_r \right\} \tag{15}$$

is the set of states with this second EPR-correlation (and the given statistical operators $w^{(1)}$ and $w^{(2)}$).

The first and second EPR-correlations are simultaneously fulfilled by the intersection of the sets of states

$$\{\mathbf{\Phi}\} \cap \{\mathbf{\Phi}\}'$$

given in (14) resp. (15).

So one has to look for the set of states which fulfill

$$\frac{1}{\sqrt{n}} \sum_r e^{i\alpha_r} \phi_r \otimes \eta_r = \frac{1}{\sqrt{n}} \sum_s e^{i\beta_s} \phi'_s \otimes \eta'_s \tag{16}$$

if one chooses the phases appropriately. As indicated we use for the substitutions leading from $\{\phi_r\}$ to $\{\phi'_r\}$ resp. from $\{\eta_r\}$ to $\{\eta'_r\}$ those from (3) and (4) assume however in addition that $u_{rs} \neq 0$ for all r and s. (This means that we introduce certain conditions for the relationship of the two ERP-correlations to each other.) Constructing the Hermitian products with $\phi_t \otimes \eta_t$ one gets from (16) the system of equations

$$\left\{ e^{i\alpha_t} = \sum_s e^{i\beta_s} p_{st}, t = 1, 2, \ldots, n \right\}, \tag{17}$$

this is equivalent with the system

$$\left\{ 1 = \sum_s e^{i(\beta_s - \alpha_t)} p_{st}, t = 1, 2, \ldots, n \right\}. \tag{18}$$

Since $\sum_s p_{st} = 1$ for $t = 1, 2, \ldots, n$ and since we made the assumption $p_{st} > 0$ for each s and t one can fulfill (17) and therefore (16) then and only then, when all phases are equal $\alpha_s = \beta_t = \alpha$ for all s and t. This means that in the case (b) $\mathbf{\Phi}$ is (up to a phase) determined by the statistical operators $w^{(1)}$ and $w^{(2)}$ and the EPR-correlations. The proof shows that already 2 suitable chosen EPR-correlations (together with the knowledge of $w^{(1)}$ and $w^{(2)}$) are sufficient for this purpose.

It would not be difficult to discuss the cases where the statistical operators are only partially degenerate, which would be the case for states as in (6) described. However since no new insight is gained by that, we put an end to the discussion.

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