Commun. Math. Phys. 168, 441 – 453 (1995)

Communications in
Mathematical
Physics

Yang–Mills and Dirac Fields in a Bag, Existence and Uniqueness Theorems

Günter Schwarz[1], **Jędrzej Śniatycki**[2]
Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4

Received: 25 February 1994

Abstract: The Cauchy problem for the Yang–Mills–Dirac system with minimal coupling is studied under the MIT quark bag boundary conditions. An existence and uniqueness theorem for the free Dirac equation is proven under that boundary condition. The existence and uniqueness of the classical time evolution of the Yang–Mills–Dirac system in a bag is shown. To ensure sufficient differentiability of the fields we need additional boundary conditions. In the proof we use the Hodge decomposition of Yang–Mills fields and the theory of non-linear semigroups.

1. Introduction

The present paper is part if a series devoted to the study of the classical theory of Yang–Mills fields. Its aim is to establish the existence and uniqueness theorem for Yang–Mills–Dirac fields satisfying modified bag boundary conditions on a contractible bounded domain $M \subset \mathbb{R}^3$. Since the domain M is fixed, our result corresponds to a static bag with zero tension. In Minkowski space the classical Yang–Mills equations have been studied in refs. [1–3]. The existence and uniqueness result for the pure Yang–Mills theory under bag boundary conditions was obtained in [4]. Here, we extend that result to include minimal interaction between the Yang–Mills field and the Dirac field.

Since classical non-abelian Yang–Mills fields are not observed in nature, one may argue that the classical Yang–Mills theory is not relevant to physics. However, the understanding of many physical phenomena in gauge theory, like conservation laws for colour charges, are based on the classical notions for the Yang–Mills theory. It is the knowledge of the classical structure of the theory, together with an appropriate understanding of the process of quantization, which enables one to arrive at a proper description of possible quantum phenomena.

---

[1] Work supported by DFG Grant Schw 485/2-1. Present address: Department of Mathematics and Computer Science, University of Mannheim, 6800 Mannheim, Germany
[2] Work partially supported by NSERC Research Grant A 8091.

One of the most fundamental aspects of a classical field theory is a complete description of its phase space. This is relatively easy in linear theories, though the gauge invariance of electrodynamics leads to some difficulties which we have learned to handle. Yang–Mills theory is both nonlinear and gauge invariant. In order to describe its phase space one needs an existence and uniqueness theorem for the evolution part of the Yang–Mills equations as well as the precise knowledge of the structure of the constraints.

Our interest in studying the Yang–Mills–Dirac system in the space-time of the form $X = \mathbb{R} \times M$, where M is a bounded domain in $\mathbb{R}^3$, is motivated by the following arguments. First, the structure of the phase space of the classical Yang–Mills fields exhibits some remarkable differences between the theory in $\mathbb{R}^4$ and in a tube $X = \mathbb{R} \times M$. The corresponding results rely on the respective existence and uniqueness theorems for the classical dynamics, and will be presented in a subsequent paper [5]. Second, there are several approaches to understand nature of hadrons (and nuclei) in terms of a field theory of the gluon and quark fields in such a tube, among these the celebrated MIT bag model. Finally, one might argue that in a real experiment the field are always (spatially) constrained to a bounded domain $M \subset \mathbb{R}^3$.

The system we are dealing with here is the standard Yang–Mills–Dirac theory with minimal coupling:

$$\nabla_\mu^A F^{\mu\nu} = J^\nu \text{ and } (\gamma^\mu \nabla_\mu^A + im)\Psi = 0 ,$$

where the superscript ^{A} refers to the gauge field A used to define the operator of covariant differentiation. Rewriting these equations as dynamical system yields

$$\begin{aligned} \partial_t A &= E + \operatorname{grad}\Phi - [\Phi, A] , \\ \partial_t E &= -\operatorname{curl} B - [A\times, B] - [\Phi, E] + J , \\ \partial_t \Psi &= -\gamma^0(\gamma^j \nabla_j^A + \gamma^0 \Phi + im)\Psi , \end{aligned} \tag{1.1}$$

where the gauge field A_μ is split into the scalar potential $\Phi = A_0$ and the vector potential $A = (A_1, A_2, A_3)$, while $E_j = F_{0j}$ and $B_j = \frac{1}{2}\varepsilon_j^{kl} F_{kl}$ denote the "electric" and the "magnetic" component of the field strength tensor, respectively. We study this dynamical system under the following boundary conditions. For the Yang–Mills fields we choose

$$nA = 0,\ nE = 0 \text{ and } tB = 0 , \tag{1.2a}$$

where nA and nE denote the normal components and tB the tangential component. On the spinor fields we impose the conditions

$$(i\gamma^k nk\Psi) \mid_{\partial M} = \Psi \mid_{\partial M} \text{ and } (i\gamma^k nk(\mathscr{D}\Psi)) \mid_{\partial M} = (\mathscr{D}\Psi) \mid_{\partial M} , \tag{1.2b}$$

where $\mathscr{D} = -\gamma^0(\gamma^j \partial_j + im)$ is the free Dirac operator.

We show that these boundary conditions are preserved under the time evolution. Furthermore they are physically reasonable by maintaining the "physical content" of the bag, in the sense that there is no flux of matter or energy through ∂M. They, in fact, turn out to be a modification of the boundary conditions of the original MIT quark bag model [6]. The MIT bag boundary conditions for the Yang–Mills fields coincide with the last two conditions of (1.2a). The first condition of (1.2a) is a partial gauge fixing. The MIT condition on the Dirac field coincides with the first one in (1.2a). As we shall show this condition suffices to guarantee that the initial value problem for the free Dirac equation has a unique global solution in the Sobolev space H^1. To handle the nonlinearity, however, one has to demand higher order of differentiability of Ψ, which enforces the stronger boundary condition (1.2b). Under these conditions the initial value problem for the free Dirac equation is uniquely solvable in the Sobolev space H^2.

Based on that existence, uniqueness and regularity result for the free Dirac equation, our results on the pure Yang–Mills system [4], and the theory of nonlinear semigroups [7] we will prove as the main result of this paper:

The Cauchy problem for the Yang–Mills–Dirac dynamics given by Eqs. (1.1) *under the boundary conditions* (1.2) *has, for any initial condition* $(A(0), E(0), \Psi(0))$, *a unique solution in an appropriate Sobolev space.*

In order to obtain this we introduce in Sect. 2 the phase space for the Yang–Mills–Dirac equations, written as a dynamical system, by fixing the boundary conditions and choosing appropriate Sobolev classes for the respective fields A, E and Ψ of the theory. In Sect. 3 we eliminate the scalar potential Φ from the dynamical system by choosing an adequate gauge fixing. Linearizing the system we give an existence and uniqueness result for the free Yang–Mills evolution equation. In Sect. 4 the existence and uniqueness of solutions for the free Dirac equation is proven. Together with the results of [4] on the pure Yang–Mills dynamics and some analytic properties of the nonlinearity of the coupled system we then establish the existence and uniqueness theorem for the nonlinear evolution in Sect. 5. Section 6 is devoted to the study of the conservation of the Gauß law constraint under the time evolution of that system. In an appendix we give a number of estimates used in the proofs.

The authors are indebted to R. Racke for his remarks on the choice of boundary conditions for the Dirac equation.

2. The Cauchy Problem for the Bag

To study the coupled Yang–Mills–Dirac equations we denote by M a fixed contractible bounded domain in $\mathbb{R}^3$ describing the bag. We consider here only static bags, which implies that the part X of the space-time accessible to the fields is the product $X = \mathbb{R} \times M$ of the time $\mathbb{R}$ and the space M– the usual $(3+1)$-splitting. We equip X with a Lorentzian metric $\eta^{\mu\nu} = \mathrm{diag}(+,-,-,-)$. For our choice of convention the Dirac matrices obey $(\gamma^0)^\dagger = \gamma^0$, $(\gamma^i)^\dagger = -\gamma^i$ $(i = 1,2,3)$, and the anti-commutation relations $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$.

By G we denote the structure group of the theory and by $\mathfrak{Y}$ its Lie algebra. The generators of $\mathfrak{Y}$ denoted by T_a, act as matrices on a vector space V and the structure constants are given by $[T_a, T_b] = f^c_{ab} T_c$. We assume $\mathfrak{Y}$ to be equipped with an ad-invariant metric given by the trace of $(T_a^\dagger \cdot T_b)$, which is used to raise and lower the Lie algebra indices. To formulate the corresponding gauge theory we consider a right principal G-fibre bundle over X with Yang–Mills connection $A_\mu = A^a_\mu T_a$ and the covariant derivative ∇^A_μ. With Dirac spinors $\Psi : X \to \mathbb{C}^4 \otimes V$ as the matter fields of the theory, the Yang–Mills–Dirac system is

$$\nabla^A_\mu F^{\mu\nu} = J^\nu \,,$$
$$(\gamma^\mu \nabla^A_\mu + im)\Psi = 0 \,, \tag{2.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ is the Yang–Mills field strength tensor and $J^\mu = \Psi^\dagger \gamma^0 \gamma^\mu T_a \Psi T^a$ is the current density of the matter field.

To study the existence and uniqueness of solutions of these equations it is convenient to reformulate them as a dynamical system. The $(3+1)$-splitting $X = \mathbb{R} \times M$ leads to the usual splitting of the Yang–Mills field A^a_μ into the scalar potential $\Phi^a = A^a_0$ and the vector potential $A^a = (A^a_1, A^a_2, A^a_3)$. Similarly, the field strength $F^a_{\mu\nu}$ splits into the "electric" field E^a and the "magnetic" field B^a with components

$$E^a_j = F^a_{0j} = \partial_0 A^a_j - \partial_j \Phi^a + [\Phi, A_j]^a ,$$
$$B^a_j = \frac{1}{2}\varepsilon_j^{kl} F^a_{kl} = (\mathrm{curl} A^a)_j + [A\times, A]^a_j . \tag{2.2}$$

Here the bracket terms are to be understood as $[\Phi, A_j]^a = f^a_{bc}\Phi^b A^c_j$, and $[A\times, A]^a_j = \frac{1}{2} f^a_{bc}\varepsilon_j^{kl} A^b_k A^c_l$. Furthermore $(\mathrm{curl} A^a)_j = \varepsilon_j^{kl}\partial_k A^a_l$. The fields A^a, E^a, and B^a are treated as time dependent vector fields on M. The current density J^μ determines a scalar density ρ_a and a 3-current J^j_a on M given by

$$\rho_a = \Psi^\dagger T_a \Psi \text{ and } J^j_a = \Psi^\dagger \gamma^0 \gamma^j T_a \Psi . \tag{2.3}$$

In terms of the fields (Φ, A, E, Ψ) and the quantities derived from them, the Yang–Mills–Dirac equations (2.1) determine a set of evolution equations

$$\partial_t A^a_j = E^a_j + \partial_j \Phi^a - [\Phi, A_j]^a , \tag{2.4a}$$
$$\partial_t E^a_j = -(\mathrm{curl} B^a)_j - [A\times, B]^a_j - [\Phi, E_j]^a + J^a_j , \tag{2.4b}$$
$$\partial_t \Psi = -\gamma^0(\gamma^j \partial_j + im + \gamma^0 \Phi^a T_a + \gamma^j A^a_j T_a)\Psi ; \tag{2.4c}$$

Eq. (2.1) also contains the (non-dynamical) Gauß law constraint

$$\partial_j (E^a)^j + [A\cdot, E]^a = \rho^a , \tag{2.4d}$$

where $[A\cdot, E]^a = f^a_{bc} A^b_j E^{cj}$. In the sequel we will skip the Lie algebra index on all the fields defined above by identifying $A = A^a T_a$, etc.

To get a complete formulation of the Cauchy problem of a dynamical system one has to specify the boundary behaviour of the fields involved. Here we study the existence and uniqueness of solutions of the Yang–Mills–Dirac system on the M under the boundary conditions given by (1.2). To clear the notion we denote by $\vec{n}$ the outward pointing unit normal vector field on ∂M. Writing n_j for its components, for every vector field W, we call $nW = \vec{n}(W^j|_{\partial M} n_j)$ the normal and $tW = W|_{\partial M} - nW$ the tangential component of W. To get a short notion for the boundary condition on the spinors we furthermore define the boundary operator $\mathscr{B} = i\gamma^j n_j$ acting on the Dirac fields restricted to ∂M.

To formulate proper existence and uniqueness results for that Cauchy problem we finally have to impose appropriate differentiability conditions for the fields involved. These are given in terms of the Sobolev spaces $H^k(M)$, consisting of the Lie algebra valued vector fields and V-valued spinor fields on M, respectively, which are square integrable together with their derivatives up to order k. The scalar products on these spaces $H^k(M)$ is given in terms of the usual scalar product on $M \subset \mathbb{R}^3$, the ad-invariant metric on $\mathfrak{Y}$ and a $\mathfrak{Y}$-invariant scalar product on V.

In this setting the phase space of our system is given by $\mathbf{P} = \mathbf{P}_{YM} \times \mathbf{P}_D$, where

$$\mathbf{P}_{YM} = \{(A, E) \in H^2(M) \times H^1(M) \mid nA = 0,\ t(\mathrm{curl} A) = 0, nE = 0\} \text{ and}$$
$$\mathbf{P}_D = \{\Psi \in H^2(M) \mid \mathscr{B}\Psi|_{\partial M} = \Psi|_{\partial M} \text{ and } \mathscr{B}\mathscr{D}\Psi|_{\partial M} = \mathscr{D}\Psi|_{\partial M}\} . \tag{2.5}$$

In view of the boundary conditions (1.2a) we note that $nA = 0$ implies $tB = t\mathrm{curl}A$. As a state of this classical system we denote a triple $(A, E, \Psi) \in \mathbf{P}$. It is crucial to note $\mathbf{P}$ is by its definition a Hilbert space.

3. Gauge Fixing, Linearization and Hodge-Decomposition

In the Yang–Mills–Dirac equation written as a dynamic system (2.4), the scalar potential Φ does not appear as an independent dynamical degree of freedom. The gauge group acts transitively on the space of scalar potentials [1]. Hence, using appropriate gauge transformations, we can fix the field Φ at all times. The most common gauge fixing for studies of Yang–Mills field as a dynamical system is the temporal gauge $A_0 \equiv 0$. For our approach, however, it is much more convenient to use the gauge condition giving the scalar potential Φ to be the solution of the Neumann problem

$$\Delta\Phi = -\mathrm{div}E \quad \text{and} \quad n(\mathrm{grad}\Phi) = 0 \quad \text{with} \quad \int_M \Phi d^3x = 0 \,. \tag{3.1}$$

From the theory of partial differential equations [8] the unique solvability of this problem is guaranteed by the boundary condition $nE = 0$.

Linearizing the evolution equations in the fields A, E, and Ψ, and observing that Φ depends linearly on E by construction, we obtain

$$\partial_t A = E + \mathrm{grad}\Phi \tag{3.2a}$$
$$\partial_t E = -\mathrm{curl}\,\mathrm{curl}A \,, \tag{3.2b}$$
$$\partial_t \Psi = \mathscr{D}\Psi = -\gamma^0(\gamma^j\partial_j + im)\Psi \,. \tag{3.2c}$$

In order to solve (3.2) under the boundary conditions imposed here, we will use the Helmholz–Hodge decomposition theorem [9] for vector fields. Its statement is that each vector field W on the bounded domain M can be uniquely decomposed into $W = W^L + W^T$ – called in resemblance to electrodynamics the longitudinal and transversal component of the field – such that

$$\mathrm{curl}W^L = 0 \quad \text{and} \quad W^T = \mathrm{curl}U \quad \text{for some vector field } U \text{ with } tU = 0 \,.$$

In this way we split the gauge fields into $A = A^L + A^T$ and $E = E^L + E^T$. Observing that $nW = nW^L$ for any vector field W, our specific choice of the gauge fixing (3.1), and the uniqueness of the solution of the Neumann problem yield

$$\mathrm{grad}\Phi = -E^L \,.$$

Furthermore $(\mathrm{curl}\,\mathrm{curl}A)^T = \mathrm{curl}\,\mathrm{curl}A^T$ because $t\mathrm{curl}A = 0$. Since the operator $(-\mathrm{curl}\,\mathrm{curl})$ acts as the Laplacian Δ on A^T, Eqs. (3.2a, b) split into two linear systems, one for the longitudinal and the other for the transversal components of the Yang–Mills fields:

$$\begin{aligned} &\partial_t A^L = 0 \quad \text{and} \quad \partial_t E^L = 0 \,, \\ &\partial_t A^T = E^T \quad \text{and} \quad \partial_T E^T = \Delta A^T \,. \end{aligned} \tag{3.3}$$

The pure Yang–Mills system under the boundary conditions (1.2a) has been studied by the authors in [4]. Let $\mathscr{S}$ be the evolution operator for the linearized system, determined by Eq. (3.3) with $\mathcal{S}(A^L,A^T;E^L,E^T) = \partial_t(A^L,A^T;E^L,E^T)$. The result of [4] can be written as:

Proposition 1. *The linearized Yang–Mills operator $\mathcal{S}$, defined by Eq.* (3.3) *with domain*

$$\mathbf{D}_{YM} = \{(A,E) \in H^2(M) \times H^1(M) \mid nA = 0,\ t(\mathrm{curl}A) = 0,\ nE = 0\}$$

generates a one-parameter group $\exp(t\mathcal{S})$ *of unitary transformations in the Hilbert space*

$$\mathbf{H}_{YM}\{(A,E) \in H^1(M) \times L^2(M) | A^L \in H^2(M),\ E^L \in H^1(M),\ nA^L = nE^L = 0\}\ .$$

This induces a one-parameter group $\exp(t\mathcal{S})$ *of continuous linear transformations on the domain* $\mathbf{D}_{YM}$ *of* $\mathcal{S}$.

4. The Dirac Equation under Bag Boundary Conditions

In order to prove an existence and uniqueness theorem for the coupled Yang–Mills–Dirac system we next study the free Dirac dynamics under the boundary condition given by Eq. (1.2b). Therefore we introduce the Hilbert spaces of spinors as

$$\mathbf{H}_D = \{\Psi \in L^2(M)\} \text{ with scalar product } \langle\langle \Xi,\, \Psi\rangle\rangle_{L^2} = \int_M \Xi^\dagger \cdot \Psi d^3x\ , \tag{4.1}$$

where $\cdot$ denotes the $\mathfrak{Y}$-invariant product on V, and † denotes the Hermitian adjoint. Considering the boundary conditions (1.2b) imposed on Ψ we observe that

$$\mathcal{B} := i\gamma^j n_j : L^2(\partial M) \longrightarrow L^2(\partial M)$$

defines a self-adjoint operator, and $\mathcal{B}^2 = 1$. The first of the conditions (1.2b) corresponds to choosing the eigenspace

$$\mathbf{B}^+ = \{\Psi \mid \mathcal{B}\Psi\ |_{\partial M} = \Psi\ |_{\partial M}\}$$

for the Dirac fields. So we let

$$\mathbf{D}_D = H^1(M) \cap \mathbf{B}^+ \tag{4.2}$$

be the domain of the Dirac operator $\mathcal{D} = -\gamma^0(\gamma^j\partial_j + im)$.
From the basic properties of the γ-matrices, and by integration by parts we get

$$\begin{aligned} \langle\langle \Xi,\, \mathcal{D}\Psi\rangle\rangle_{L^2} &= -\int_M \left((\gamma^0\gamma^j\Xi)^\dagger \cdot \partial_j\Psi - (im\gamma^0\Xi)^\dagger \cdot \Psi\right) d^3x \\ &= -\langle\langle \mathcal{D}\Xi,\, \Psi\rangle\rangle_{L^2} - \int_{\partial M} (\gamma^0\gamma^j n_j\Xi)^\dagger \cdot \Psi d^2x \end{aligned} \qquad \forall\Psi,\, \Xi \in H^1(M)\ . \tag{4.3}$$

Considering the boundary integral we find by taking the Hermitian conjugate

$$(\gamma^0\gamma^j n_j\Xi)^\dagger \cdot \Psi = +\left((\gamma^0\gamma^j n_j\Psi)^\dagger \cdot \Xi\right)^* \quad \forall\Psi,\, \Xi \in H^1(M)\ . \tag{4.4}$$

On the other hand, if both spinor fields Ψ and Ξ obey the boundary condition $\mathcal{B}\Psi\ |_{\partial M} = \Psi\ |_{\partial M}$, we have

$$(\gamma^0\gamma^j n_j \Xi)^\dagger \cdot \Psi = -(i\gamma^0 \Xi)^\dagger \cdot \Psi = -\left((\gamma^0\gamma^j n_j \Psi)^\dagger \cdot \Xi\right)^* . \tag{4.5}$$

Thus the boundary integral in (4.3) vanishes, which implies the skew-symmetry of the Dirac operator, i.e.

$$\langle\!\langle \Xi, \mathscr{D}\Psi \rangle\!\rangle_{L^2} = -\langle\!\langle \mathscr{D}\Xi, \Psi \rangle\!\rangle_{L^2} \quad \forall \Psi, \Xi \in \mathbf{D}_D .$$

The boundary operator $\mathscr{B}$ in self-adjoint on ∂M, and the decomposition $L^2(\partial M) = \mathbf{B}^+ \oplus \mathbf{B}^-$ is orthogonal. Furthermore γ^0 maps $\mathscr{B}^+$ onto $\mathscr{B}^-$, and $\mathscr{B}^-$ onto $\mathscr{B}^+$. Therefore, the adjoint $\mathscr{D}^*$ of the Dirac operator is given by

$$\mathscr{D}^* = -\mathscr{D} \text{ with domain } \mathbf{D}_{D^*} = \{\Xi \in \mathbf{H}_D \mid \mathscr{D}\Xi \in L^2(M) \text{ and } \mathscr{B}\Xi \mid_{\partial M} = \Xi \mid_{\partial M}\} .$$

On the other hand $\Xi \in \mathbf{D}_{D^*}$ implies $\Xi \in H^1(M)$, which follows from the estimate

$$\|\Xi\|_{H^1} \leqq C_1 \left(\|\mathscr{D}\Xi\|_{L^2} + \|\Xi\|_{L^2}\right) , \tag{4.6}$$

proven in the appendix. This shows that the Dirac operator with domain $\mathbf{D}_D$ is skew-adjoint. Also in the appendix we prove that

$$\|\Psi\|_{H^2} \leqq C_2 \left(\|\mathscr{D}\mathscr{D}\Psi\|_{L^2} + \|\Psi\|_{L^2}\right) \quad \forall \Psi \in \mathbf{P}_D , \tag{4.7}$$

where the space $\mathbf{P}_D$ is given by Eq. (2.5). Since the one-parameter semigroup $\exp(t\mathscr{D})$ preserves the domain of its generator, and commutes with $\mathscr{D}$, cf. [10], and since $\mathscr{D}\Psi \in \mathbf{D}_D$ for $\Psi \in \mathbf{P}_D$ we get

$$\|\Psi\|_{H^2} \geqq C_3 \|\mathscr{D}\mathscr{D}\Psi\|_{L^2} = C_3 \|\exp(t\mathscr{D})\mathscr{D}\mathscr{D}\Psi\|_{L^2} = C_3 \|\mathscr{D}\mathscr{D}\exp(t\mathscr{D})\Psi\|_{L^2} ,$$

and consequently

$$\|\Psi\|_{H^2} \geqq C_4 \left(\|\mathscr{D}\mathscr{D}\exp(t\mathscr{D})\Psi\|_{L^2} + \|\exp(t\mathscr{D})\Psi\|_{L^2}\right) \geqq C_5 \|\exp(t\mathscr{D})\Psi\|_{H^2} .$$

Thus $\exp(t\mathscr{D})$ acts as a family of bounded operators on $\mathbf{P}_D$. The group properties of $\exp(t\mathscr{D})$ on $\mathbf{D}_D$ and on $\mathbf{P}_D$, respectively, follow from the group property on $\mathbf{H}_D$. Finally, using the same arguments as above one shows that

$$\lim_{t \to 0} \|\exp(t\mathscr{D})\Psi - \Psi\|_{H^2} = 0 \quad \forall \Psi \in \mathbf{P}_D ,$$

which implies that the map $t \mapsto \exp(t\mathscr{D})$ determines a continuous family of bounded operators in $\mathbf{P}_D$. So we have proven:

Proposition 2. *On the Hilbert space* $\mathbf{H}_D$ *the operator* $\mathscr{D} = -\gamma^0(\gamma^j\partial_j + im)$ *with domain* $\mathbf{D}_D$ *given by* (4.2) *generates a one-parameter group of unitary transformations*

$$\Psi(0) \mapsto \Psi(t) = \exp(t\mathscr{D})\Psi(0) ,$$

which preserves the domain $\mathbf{D}_D$*. This induces a continuous group of bounded transformations in the space* $\mathbf{P}_D$*.*

As it stands this theorem gives a (global) existence and uniqueness result for the Dirac equation in $H^1(M)$ under the boundary condition $\mathscr{B}\Psi \mid_{\partial M} = \Psi \mid_{\partial M}$. In order to guarantee a dynamics in $H^2(M)$, however, we have to impose the stronger boundary condition. this is inevitable in view of the subsequent estimates for the nonlinear coupling of the spinors Ψ to the Yang–Mills fields (A, E).

Theorem 1. *The linearized Yang–Mills–Dirac system* (3.2c) *and* (3.3) *defines an operators* $\mathscr{T}$ *given by*

$$\mathscr{T}(A^L, A^T; E^L, E^T; \Psi) = (0, E^T; 0, \Delta A^T; \mathscr{D}\Psi)\,,$$

which is skew adjoint on the Hilbert space $\mathbf{H}_{YM} \times \mathbf{H}_D$ *and has as its domain the space* $\mathbf{D} = \mathbf{D}_{YM} \times \mathbf{D}_D$. *The operator* $\mathscr{T}$ *generates a one-parameter group* $\exp(t\mathscr{T})$ *of unitary transformations in* $\mathbf{H}_{YM} \times \mathbf{H}_D$. *This induces a one-parameter group* $\mathscr{U}(t)$

$$(A(0),\, E(0),\, \Psi(0)) \longmapsto \mathscr{U}(t)(A(0),\, E(0),\, \Psi(0)) = (A(t),\, E(t),\, \Psi(t))$$

of continuous linear transformations of the phase space $\mathbf{P}$ *defined by Eq.* (2.5).

5. The Non-linear Yang–Mills–Dirac Dynamics

Considering the nonlinear dynamics we rewrite the evolution equation for curves $(A, E, \Psi)_t = (A(t), E(t), \Psi(t))$ of states as

$$\partial_t(A, E, \Psi)_t = \mathscr{T}((A, E, \Psi)_t) + F((A, E, \Psi)_t)\,, \tag{5.1}$$

with a mapping $F = F_{YM} + F_C : \mathbf{P} \to \mathbf{P}$ given as follows: Let F_{YM} describe the nonlinearity of the pure Yang–Mills theory and the F_C the coupling between Yang–Mills field and matter fields. Then we find from (2.4),

$$\begin{aligned} F_{YM}(A, E, \Psi) &= (-[\Phi, A]; -[A\times, B] - \mathrm{curl}[A\times, A] - [\Phi, E];\, 0)\,, \\ F_C(A, E, \Psi) &= (0; J; -\Phi\Psi - \gamma^0\gamma^j A_j \Psi)\,, \end{aligned} \tag{5.2}$$

where the fields B and Φ are uniquely determined from A and E by (2.2) and (3.1). First we observe that the boundary conditions (1.2) are preserved by these nonlinear maps: for the Yang–Mills term F_{YM} we refer to [4]. Concerning the coupling term it follows from the argument above (Eq. (4.4) and (4.5)) that

$$nJ = n_j(\Psi^\dagger \gamma^0 \gamma^j \Psi) = 0 \quad \forall \Psi \in \mathbf{D}_D\,.$$

On the other hand we observe that $(\mathscr{B}\Phi\Psi)\,|_{\partial M} = (\Phi\mathscr{B}\Psi)\,|_{\partial M}$, and

$$\mathscr{B}(\gamma^0\gamma^j A_j \Psi) = -\gamma^0 n A \Psi + i\gamma^0\gamma^j A_j \gamma^k n_k \Psi = \gamma^0\gamma^j A_j(\mathscr{B}\Psi)\,.$$

This obviously also holds for $\Xi = \mathscr{D}\Psi$ with $\Psi \in \mathbf{P}_D$. The statement that F maps the $\mathbf{P}$ onto itself furthermore requires to prove that its components have values in the right Sobolev spaces. To characterize the analytic properties of F, which also includes that statement, we formulate:

Proposition 3. *The nonlinear mapping* $F = F_{YM} + F_C$ *defined on the phase space* $\mathbf{P}$, *with components given by* (5.2) *has the following properties*:

1) *The range of this map is a subset of* $\mathbf{P}$, *i.e.* $F : \mathbf{P} \to \mathbf{P}$.
2) F *is continuous with respect to the norm on* $\mathbf{P}$ *given by*

$$\|(A, E, \Psi)\|^2_{\mathbf{P}} = \|A\|^2_{H^2} + \|E\|^2_{H^1} + \|\Psi\|^2_{H^2}\,.$$

3) F *is a smooth map with respect to that norm.*

A proof of the properties of the operator F_{YM}, i.e. for the pure Yang–Mills case, was given in [4]. The required estimates for the coupling term F_C can be found in the appendix.

In terms of the continuous one-parameter group $\mathscr{U}(t)$ of linear transformations on $\mathbf{P}$, determined by Theorem 1, we rewrite the evolution equation (5.1) together with the initial condition $(A,E,\Psi)_0$ in the integral form

$$(A,E,\Psi)_t = \mathscr{U}(t)(A,E,\Psi)_0 + \int_0^t \mathscr{U}(t-s)F((A,E,\Psi)_s)ds\,. \tag{5.3}$$

On the basis of this we can apply the theory of nonlinear semigroups. The corresponding statement of [7] is that each initial state $x_0 \in \mathbf{P}$ determines a unique curve of solutions of the integral equation (5.3). Furthermore, since the nonlinearity $F : \mathbf{P} \to \mathbf{P}$ of the system is a differentiable map, the solution curve is also t-differentiable and solves the corresponding differential equation[3]

Theorem 4. *For every initial condition $(A,E,\Psi)_0 \in \mathbf{P}$ of gauge and matter fields on M there exists a unique continuous curve $(A,E,\Psi)_t$ in $\mathbf{P}$, satisfying the integral equation* (5.3). *This time evolution is well defined for all $t \in [0,T)$, where the maximal time of existence $0 < T \leqq \infty$ is determined by the initial condition. Furthermore the curve $(A(t),E(t),\Psi(t))$ is continuously differentiable and solves the Cauchy problem for the Yang–Mills–Dirac evolution, given by Eqs.* (2.4a–c), *for all $t \in [0,T)$ with initial condition indicated.*

One should remark that the time evolution for the Yang–Mills–Dirac system not only yields a curve of solutions for any initial condition, but also determines a diffeomorphism on the phase space $\mathbf{P}$. To see this we differentiate the map $(A,E,\Psi)_0 \mapsto (A,E,\Psi)_t$, given by Eq. (5.3) in the direction of an arbitrary $(a,e,\psi) \in \mathbf{P}$. This yields

$$(a,e,\psi) \mapsto \mathscr{U}(t)(a,e,\psi) + \int_0^t \mathscr{U}(t-s)D(F((A,E,\Psi)_s))(a,e,\psi)ds\,,$$

which is a smooth map by Proposition 3. Since the dynamics is reversible, this shows that the time evolution determines a diffeomorphism.

6. The Constraint Equation

In order to complete our results on the existence and uniqueness of solutions of the Yang–Mills–Dirac system (2.1) we finally have to study the Gauß law constraint, Eq. (2.4d),

$$\nabla_j^A E^j - \rho = 0 \tag{6.1}$$

under the dynamics determined above. Therefore we compute the covariant time derivative ∇_0^A of that expression, and find direct from the Yang–Mills equations (2.4a, b):

$$\begin{aligned}\nabla_0^A(\nabla_j^A E^j - \rho) &= \nabla_j^A(\partial_t E^j) + [\partial_t A\cdot, E] + [\Phi, \nabla_j^A E^j] - \nabla_0^A \rho \\ &= \nabla_j^A J^j - \nabla_0^A \rho\,.\end{aligned}$$

[3] In fact it suffices, as proven in [11], to show that the nonlinearity F is Lipschitz.

This expression vanishes due to the continuity equation for the current density associated to the Dirac fields. Since the Hilbert space structure on $\mathbf{H}_{YM} \times \mathbf{H}_D$ relies on an ad-invariant scalar product in $\mathfrak{Y}$, we get

$$\frac{d}{dt}\|\nabla_j^A E^j - \rho\|_{L^2}^2 = 2\langle\langle \nabla_0^A(\nabla_j^A E^j - \rho),\ \nabla_j^A E^j - \rho\rangle\rangle_{L^2} = 0\,.$$

Thus $\|\nabla_j^A E^j - \rho\|_{L^2}$ is a constant of motion, and in particular, the Gauß law constraint (6.1) is preserved under the evolution of the system.

Appendix

In order have a complete proof of Proposition 2 we are left with proving the estimates (4.6) and (4.7). In Sect. 4 we introduced the Hilbert space $\mathbf{H}_D$ of square integrable spinor fields with scalar product given by (4.1); and its subspace $\mathbf{D}_D$ of fields obeying the $\mathscr{B}\Psi\mid_{\partial M} = \Psi\mid_{\partial M}$, and $\mathbf{P}_D$ of H^2-fields obeying the full boundary condition (1.2b).

Lemma A.1. *For the operator* $\mathscr{D} = -\gamma^0(\gamma^j\partial_j + im)$ *the following estimates hold*:

$$\|\Psi\|_{H^1} \leqq C_1(\|\mathscr{D}\Psi\|_{L^2} + \|\Psi\|_{L^2}) \qquad \forall \Psi \in \mathbf{D}_D\,, \tag{A.1}$$

$$\|\Psi\|_{H^2} \leqq C_2(\|\mathscr{D}(\mathscr{D}\Psi)\|_{L^2} + \|\Psi\|_{L^2}) \quad \forall \Psi \in \mathbf{P}_D\,. \tag{A.2}$$

Proof.

(i) For any $\Psi \in \mathbf{D}_D$ we compute with (4.1) that

$$\|\mathscr{D}\Psi\|_{L^2}^2 = -\langle\langle \gamma^k\gamma^j\partial_j\Psi,\ \partial_k\Psi\rangle\rangle_{L^2} + A(\Psi) + m^2\|\Psi\|_{L^2}\,,$$
$$\text{where } A(\Psi) = im(\langle\langle \gamma^j\partial_j\Psi,\ \Psi\rangle\rangle_{L^2} - \langle\langle \Psi,\ \gamma^j\partial_j\Psi\rangle\rangle_{L^2})\,.$$

Furthermore, $\gamma^k\gamma^j = -\delta^{kj} + \frac{1}{2}[\gamma^k,\ \gamma^j]$, and hence

$$\|\mathscr{D}\Psi\|_{L^2}^2 = \sum_{j=1}^{3}\|\partial_j\Psi\|_{L^2}^2 - B(\Psi) + A(\Psi) + m^2\|\Psi\|_{L^2}\,,$$
$$\text{where } 2B(\Psi) = \langle\langle [\gamma^k, \gamma^j]\partial_j\Psi,\ \partial_k\Psi\rangle\rangle_{L^2}\,.$$

Since $C^\infty(M)$ is dense in $H^1(M)$ it follows by integration by parts that

$$2B(\Psi) = \int_{\partial M} (n_k[\gamma^k,\ \gamma^j]\partial_j\Psi)^\dagger \cdot \Psi d^2x$$

for all $\Psi \in H^1(M)$, where the surface integral makes sense since the restriction of Ψ to ∂M is in $H^{1/2}(\partial M)$. To handle this expression we take $\Psi_0 \in C^\infty(M)$, and introduce an orthonormal non-holonomic frame $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ along the boundary ∂M, such that $\vec{e}_3 = \vec{n}$. In terms of this we have $[\gamma^3, \gamma^s] = 2\gamma^3\gamma^s$. Furthermore $\gamma^j\partial_j = \gamma^s\nabla_s + \gamma^3\nabla_3$, where ∇_s denotes the covariant derivative induced by the Levi-Civita connection of the metric g_{rs} on ∂M. Then we find from the boundary condition (1.2b) that

$$B(\Psi_0) = \int_{\partial M} (n_3\gamma^3\gamma^s\nabla_s\Psi_0)^\dagger \cdot \Psi_0(\det(g_{rs}))^{1/2}d^2y$$
$$= i\int_{\partial M} (\gamma^s\nabla_s\Psi_0)^\dagger \cdot \Psi_0(\det(g_{rs}))^{1/2}d^2y\,.$$

Since $B(\Psi_0)$ is real, the integral has to be purely imaginary. Taking into account that γ^3 is covariantly constant, we obtain by integration by parts

$$\int_{\partial M} (\gamma^s \nabla_s \Psi_0)^\dagger \cdot \Psi_0 (\det(g_{rs}))^{1/2} d^2 y = \int_{\partial M} \Psi_0^\dagger \cdot \gamma^s \nabla_s \Psi_0 (\det(g_{rs}))^{1/2} d^2 y$$
$$= \left(\int_{\partial M} (\gamma^s \nabla_s \Psi_0)^\dagger \cdot \Psi_0 (\det(g_{rs}))^{1/2} d^2 y \right)^* .$$

This implies that the surface integral vanishes for all smooth spinor fields Ψ_0 on ∂M, satisfying the boundary condition (1.2b). Since $C^\infty(M)$ is dense in $H^{1/2}(\partial M)$, it follows that the expression $B(\Psi)$ has to vanish for all $\Psi \in \mathbf{D}_D$. This shows that

$$\|\Psi\|_{H^1}^2 \leqq \|\mathscr{D}\Psi\|_{L^2}^2 + | A(\Psi) | + C\|\Psi\|_{L^2}^2 .$$

The term $A(\Psi)$ can be estimated by means of the Cauchy–Schwarz inequality as

$$|A(\Psi)| \leqq 2m\|\Psi\|_{L^2}\|\Psi\|_{H^1} .$$

Hence there is a constant $C_1 > 0$ such that

$$\|\Psi\|_{H^1} \leqq C_1(\|\mathscr{D}\Psi\|_{L^2} + \|\Psi\|_{L^2}) .$$

(ii) In order to prove (A.2) we show that the operator $\mathscr{D}\mathscr{D}$, accompanied by the boundary conditions (1.2b) is elliptic. The square of the Dirac operator corresponds to the Laplacian, i.e. $\mathscr{D}\mathscr{D}\Psi = (\Delta - m^2)\Psi$. To prove ellipticity we have to show that the boundary conditions (1.2b) satisfies the Lopatinskiĭ-Šapiro condition [12][4] for the Laplace operator. Using for $p \in \partial M$ the coordinates $(y_s; y_3)$ with y_3 parallel to $\vec{n}$, we Fourier transform the equation $\Delta\Psi = 0$ with respect to (y_1, y_2). This yields

$$(- \mid \xi \mid^2 + \partial_{ys}^2)\tilde{\Psi}(\xi_1, \xi_2, y_3) = 0 ,$$

where $\mid \xi \mid^2 = \xi_1^2 + \xi_2^2$. As the set of the solution of this (ordinary) differential equation, which decay at infinity, we get

$$\mathscr{U}^+ = \{\hat{\Psi} \exp(- \mid \xi \mid y_3) \mid \hat{\Psi} \in V\} .$$

Under the Fourier transformation the boundary conditions turn into

$$\gamma^3 \tilde{\Psi}(\xi_1, \xi_2, 0) = -i\tilde{\Psi}(\xi_1, \xi_2, 0) , \tag{A.3}$$
$$\gamma^3 \widetilde{\mathscr{D}\Psi}(\xi_1, \xi_2, 0) = -i\widetilde{\mathscr{D}\Psi}(\xi_1, \xi_2, 0) , \tag{A.4}$$

where the Fourier transformation of $\mathscr{D}\Psi$ is

$$\widetilde{\mathscr{D}\Psi} = -i(\gamma^0\gamma^s\xi_s + m)\tilde{\Psi} - \gamma^0\gamma^3\partial_{y_3}\tilde{\Psi} .$$

Therefore

$$\gamma^3\widetilde{\mathscr{D}\Psi} = -i(\gamma^0\gamma^s\xi_s + m)\gamma^3\tilde{\Psi} + i\gamma^0\gamma^3\partial_{y3}\tilde{\Psi} - i\gamma^0\gamma^3\partial_{y3}\tilde{\Psi} - \gamma^0\partial_{y3}\tilde{\Psi} ,$$

and by using (A.3), the condition (A.4) turns into

$$i\gamma^0\gamma^3\partial_{y3}\tilde{\Psi}(\xi_1, \xi_2, 0) = -\gamma^0\partial_{y3}\tilde{\Psi}(\xi_1, \xi_2, 0) . \tag{A.5}$$

[4] See also [8] for a more explicit version of that condition.

Hence for $\tilde{\Psi} = \hat{\Psi} \exp(- \mid \xi \mid y_3) \in \mathcal{U}^+$ the conditions (A.3) and (A.5) yield

$$\gamma^3 \hat{\Psi} = -i\hat{\Psi} \quad \text{respectively} \quad \gamma^3 \hat{\Psi} = -i\hat{\Psi} .$$

Thus the (constant) spinor $\hat{\Psi} \in V$ has to vanish, which proves ellipticity of the operator $\mathscr{D}\mathscr{D}$ under the boundary imposed. Therefore the inequality (ii) follows from the usual a-priori estimate for elliptic boundary value problems. □

In order to show that the nonlinear coupling term F_C has the analytic properties demanded in Proposition 3 we are left with proving:

Proposition A.2. *The map* $F_C(A, E, \Psi) = (0, J, -\Phi\Psi - \gamma^0\gamma^j A_j \Psi)$ *maps* $\mathbf{P}$ *to* $\mathbf{P}$, *and is of class* C^∞ *on this space with respect to the norm given by*

$$\|(A, E, \Psi)\|^2_{\mathbf{P}} = \|A\|^2_{H^2} + \|E\|^2_{H^1} + \|\Psi\|^2_{H^2} .$$

Proof. Since, for $\dim M = 3$ the space $H^2(M)$ is a Banach algebra, cf. [13], $\Psi \in H^2(M)$ implies that the components of the current $J^j_a = \Psi^\dagger \gamma^0 \gamma^j T_a \Psi$ are of Sobolev class H^2. Hence $J \in H^2(M) \subset H^1(M)$. Since also the potential Φ and the field A are $H^2(M)$, the same argument holds for the terms $\Phi\Psi$ and $\gamma^0\gamma^j A_j \Psi$. Furthermore F_C preserves the boundary conditions (1.2) and hence maps $\mathbf{P}$ onto itself.

From the Banach algebra property of $H^2(M)$ we also immediately prove the continuity of F_C. For Ψ, $\hat{\Psi} \in H^2(M)$ we have

$$\|J^j_a - \hat{J}^j_a\|^2_{H^1} = \|\Psi^\dagger \gamma^0 \gamma^j T_a \Psi - \hat{\Psi}^\dagger \gamma^0 \gamma^j T_a \hat{\Psi}\|^2_{H^1} \leqq K_1 \|\Psi\|^2_{H^2} \|\Psi - \hat{\Psi}\|^2_{H^2} .$$

On the other hand we get with E, $\hat{E} \in H^1(M)$ and A, $\hat{A} \in H^2(M)$,

$$\|\Phi\Psi - \hat{\Phi}\hat{\Psi}\|^2_{H^2} \leqq K_2(\|\Phi - \hat{\Phi}\|^2_{H^2} \|\Psi\|^2_{H^2} + \|\Psi - \hat{\Psi}\|^2_{H^2} \|\Phi\|^2_{H^2}) ,$$

$$\|\gamma^0\gamma^j A_j \Psi - \gamma^0\gamma^j \hat{A}_j \hat{\Psi}\|^2_{H^2} \leqq K_3(\|A - \hat{A}\|^2_{H^2} \|\Psi\|^2_{H^2} + \|\Psi - \hat{\Psi}\|^2_{H^2} \|A\|^2_{H^2}) ,$$

Putting these terms together and observing that $\|\Phi\|_{H^2} \leqq K_4 \|E\|_{H^1}$ by its construction, we end up with the required estimate

$$\begin{aligned} &\|F_C(A, E, \Psi) - F_C(\hat{A}, \hat{E}, \hat{\Psi})\|^2_{\mathbf{P}} \\ &\leqq K_5 \|(A, E, \Psi) - (\hat{A}, \hat{E}, \hat{\Psi})\|^2_{\mathbf{P}} (1 + \|E\|_{H^1} + \|A\|_{H^2} + \|\Psi\|_{H^2})^2 , \end{aligned} \tag{A.7}$$

which proves the continuity of $F_C : \mathbf{P} \to \mathbf{P}$.

To show the differentiability of F_C, we write (a, e, ψ) for an arbitrary infinitesimal variation, and evaluate

$$DF_C(A, E, \Psi)(a, e, \psi) = \left(0, \psi^\dagger \gamma^0 \gamma^j \Psi + \Psi^\dagger \gamma^0 \gamma^j \psi, -(\varphi\Psi + \Phi\psi) - \gamma^0\gamma^j(a_j \Psi + A_j \psi)\right) ,$$

$$\text{where } \Delta\varphi = -\operatorname{div} e \text{ and } n \operatorname{grad} \varphi = 0 .$$

Since a, e, φ and ψ are of the same Sobolev classes as A, E, Φ and Ψ, respectively, all the estimates used to prove continuity of F_C also can be applied here. Hence one shows, literally as above, the $DF_C(A, E, \Psi)(a, e, \psi)$ is continuous. Analogous arguments also hold for the higher derivatives of J. Actually, derivatives of F_C of order $\geqq 3$ vanish identically. □

References

1. Segal I. The Cauchy problem for the Yang–Mills equation. J. Func. Anal. **33**, 175–194 (1979)
2. Ginibre J., Velo G.: The Cauchy problem for coupled Yang–Mills and scalar fields in the temporal gauge. Commun. Math. Phys. **82**, 1–28 (1981)
3. Eardley D. M., Moncrief V.: The global existence of Yang–Mills–Higgs fields in 4-dimensional Minkowski space. Commun. Math. Phys. **83**, 171–191 and 193–212 (1982)
4. Śniatycki J., Schwarz G.: The existence and uniqueness of solutions of the Yang–Mills equations with bag boundary conditions. Commun. Math. Phys. **159**, 593–604 (1994).
5. Śniatycki J., Schwarz G., Bates L.: Yang–Mills in Dirac Fields in a bag, constraints and reduction. In preparation
6. Johnson K.: The M.I.T. bag model. Acta Physica Polonica **B6**, 865–892 (1975)
7. Segal, I.: Non-linear semi-groups. Ann. of Math. **78**, 339–364 (1963)
8. Wloka, J.: Partial Differential Equations. Cambridge: Cambridge University Press, 1987
9. Abraham, R., Marsden, J. E., Ratiu, T.: Manifolds, Tensor Analysis, and Applications. Berlin, Heidelberg, New York: Springer 1988
10. Yosida K.: Functional Analysis. 6th Edition, Berlin, Heidelberg, New York : Springer 1980
11. von Wahl W.: Analytische Abbildungen und semilineare Differentialgleichungen in Banachräumen. Nachr. Akad. Wiss. Gött., **II**. Math. Phys. K1., 1–48 (1979)
12. Hörmander L.: The Analysis of Linear Partial Differential Operators III. Berlin, Heidelberg, New York: Springer 1985
13. Adams R. A.: Sobolev Spaces. New York: Academic Press, 1975

Communicated by S.-T. Yau