Long Time Behavior for the Equation of Finite-Depth Fluids

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Abstract: In this paper we study the Cauchy problem for the generalized equation of finite-depth fluids

$$\partial_t u - G(\partial_x^2 u) - \partial_x \left(\frac{u^p}{p} \right) = 0 \, ,$$

where $G(\cdot)$ is a singular integral, and p is an integer larger than 1. We obtain the long time behavior of the fundamental solution of linear problem, and prove that the solutions of the nonlinear problem with small initial data for $p > 5/2 + \sqrt{21}/2$ are decay in time and freely asymptotic to solutions of the linear problem. In addition we also study some properties of the singular integral $G(\cdot)$ in $L^q(R)$ with $q > 1$.

1. Introduction

In this paper we shall consider the Cauchy problem for the generalized equation of finite-depth fluids

$$\partial_t u - G(\partial_x^2 u) + \partial \left(\frac{u^p}{p} \right) = 0 \, , \tag{1}$$

where p is an integer larger than 1, $G(f) = \lim_{\varepsilon \to 0} \int_{|y| \geq \varepsilon > 0} f(x-y)K(y)dy$ with $K(y) = \frac{1}{2\delta} (\coth \frac{\pi y}{2\delta} - \operatorname{sign} y)$ is a singular integral, here δ is a positive real which characterizes the depth of the fluid layer. Equation (1) was first derived by Joseph [5, 10] to describe the propagation of internal waves in the stratified fluid of finite depth. It is known [1] that Eq. (1) reduces to the nonlinear Korteweg–de Vries (KdV) equation

$$\partial_t u - \partial_x^3 u + \partial \left(\frac{u^p}{p} \right) = 0 \, ,$$

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and the Benjamin–Ono (BO) equation

$$\partial_t u - H(\partial_x^2 u) + \partial\left(\frac{u^p}{p}\right) = 0$$

for $p = 2$ as the depth δ tends respectively to zero and infinity, where $H(\cdot)$ denotes the Hilbert transform. From the view point of mathematics, there is an amount of work devoted to studying large time behavior problem for the solutions of the nonlinear KdV equation and BO equation (see [3, 6, 7, 11, 12] and the references therein). In particular, if we denote by $U(\cdot)$ and $V(\cdot)$ the free evolution group which solve respectively the Cauchy problems of linear KdV equation $\partial_t u - \partial_x^3 u = 0$, and the linear BO equation $\partial_t u - H(\partial_x^2 u) = 0$, then one has the following available results:

$$\|U(t)f\|_p \leq Ct^{-\frac{1}{3}\left(1-\frac{2}{p}\right)}\|f\|_{p'} , \tag{2a}$$

and

$$\|V(t)f\|_p \leq Ct^{-\frac{1}{2}\left(1-\frac{2}{p}\right)}\|f\|_{p'} \tag{2b}$$

for all $t \geq 1$, and $p \geq 2$, $p' = p/(p-1)$. In the present paper we shall obtain a similar decay estimate for the fundamental solution of the linear problem

$$\partial_t u - G(\partial_x^2 u) = 0, \quad u(x,0) = f(x) , \tag{3}$$

i.e. we gave

$$\|u\|_p \leq C(t^{-\frac{1}{2}\left(1-\frac{2}{p}\right)} + (\delta t)^{-\frac{1}{3}\left(1-\frac{2}{p}\right)})\|f\|_{p'} \tag{4}$$

for all $t \geq 1$. Furthermore, we shall substitute the decay estimate (4) into the integral equation associated with the nonlinear equation (1) to obtain long time behavior for the nonlinear problem (1). However, as we shall see, because of the complication of the symbol $P(\xi)$, which characterizes the dispersive relation of Eq. (1), it is much more difficult to prove the decay estimate (4) for the linear problem (3), and requires more a elaborate calculation than to obtain the estimates (2) for the linear KdV equation and linear BO equation. In fact the dispersive relation $P(\xi)$ for Eq. (1) is $(2\pi\xi)^2\left(\coth(2\pi\delta\xi) - \frac{1}{2\pi\delta\xi}\right)$, while for nonlinear KdV equation and BO equation the dispersive relations are respectively the simple forms ξ^3 and $|\xi|\xi$.

This paper is organized as follows. In Sect. 2 we state some basic lemmas, and give certain properties for the singular integral $G(\cdot)$. In Sect. 3 we consider the fundamental solution of the linear problem (3). The decay estimate (4) is proved by applying the Van der Corput Lemma [14]. Finally in Sect. 4 we exploit the results of Sects. 2 and 3 to derive the time decay estimate and free asympotic property for the nonlinear problem (1).

To conclude this section, we give the main notation used in this paper. By C we denote various positive constants which may be different from line to line, and is independent of time t and the functions to be estimated. For all $p, 1 \leq p \leq \infty$, we denote by $\|\cdot\|_p$ the norm in $L^p(R)$.

2. Preliminary Results

We denote by $\hat{f}$ and $\check{f}$ the Fourier transform and its inversion for function $f(x)$, i.e.

$$\hat{f}(\xi) = \int_R f(x)e^{-2\pi i\xi x}dx, \; \check{f}(x) = \int_R f(\xi)e^{2\pi i\xi x}d\xi \;.$$

For the singular integral $G(\cdot)$, we can derive the following properties

Lemma 1. *For any function* $f(x)$, $g(x) \in C_0^\infty(R)$, *we have*

   a. $[G(f)]^\wedge = -i\left[\coth(2\pi\delta\xi) - \frac{1}{2\pi\delta\xi}\right]\hat{f}$,
   b. $G(\partial_x f) = \partial_x G(f)$,
   c. $\int_R f\,G(g)dx = -\int_R g G(f)dx$,
   d. $\int_R f_x(G(f_x))^2 dx + \frac{2}{\delta}\int_R f f_x G(f_x)dx = \frac{1}{3}\int_R (f_x)^3 dx$,
   e. $\|G(f)\|_2 \leqq \|f\|_2$.

Proof. See Zhou Yulin, et al. [17].

The above lemma shows that the singular integral $G(\cdot)$ is a linear bounded transform from Hilbert space $H^s(R)$ into itself for $s \geqq 0$. It is to be noted that the inequality (e) in fact can be extended to the case of a function taking value in Sobolev space $L^p(R)$, *for* $1 < p < \infty$.

Lemma 2. *For any function* $f \in L^p(R)$, $1 < p < \infty$, *we have*

$$\|G(f)\|_p \leqq C(p)\|f\|_p \;,$$

where the positive constant C is independent of f.

In order to prove Lemma 2, we need the following results.

Lemma 3. *Suppose the kernel $K(x)$ satisfies the following conditions:*

$$|K(x)| \leqq B|x|^{-1}, \quad \text{for } |x| > 0 \;, \tag{5}$$

$$\int_{R_1 < |x| < R_2} K(x)dx = 0, \quad \text{for } 0 < R_1 < R_2 < \infty \;, \tag{6}$$

$$\int_{|x| \geqq 2|y|} |K(x - y) - K(x)|dx \leqq B, \quad \text{for } |y| > 0 \;, \tag{7}$$

where B is a positive constant. Then for any $f \in L^p(R)$, $1 < p < \infty$, we have

$$\|T(f)\|_p \leqq C(p)\|f\|_p \;,$$

where $T(f)(x) = \lim_{\varepsilon \to 0} \int_{|y| \geqq \varepsilon > 0} f(x - y)K(y)dy.$

Proof. See E.M. Stein [13]

Lemma 4. *For* $|y| \neq 0$, *we have*

$$y^2\coth y = y + \frac{1}{3}y^3 - 2y^3\sum_{k=1}^\infty \frac{y^2}{k^2\pi^2(k^2\pi^2 + y^2)} \;, \tag{8}$$

$$y^2 \coth y = y|y| \left(1 + 2\sum_{k=1}^{\infty} e^{-2k|y|} \right) . \tag{9}$$

Proof. The first expansion (8) was given in [1]. Here we need only to show the second expansion of $y^2 \coth y$. In fact, for $y > 0$,

$$y^2 \coth y = y^2 \frac{e^y + e^{-y}}{e^y - e^{-y}} = y^2 \left(1 + \frac{2e^{-2y}}{1 - e^{-2y}} \right)$$

$$= y^2 \left(1 + 2e^{-2y} \sum_{k=0}^{\infty} e^{-2ky} \right) = y^2 \left(1 + 2\sum_{k=1}^{\infty} e^{-2ky} \right) .$$

And for $y < 0$ we can obtain

$$y^2 \coth y = -y^2 \left(1 + 2\sum_{k=1}^{\infty} e^{2ky} \right) .$$

Therefore, we obtain (9).

Now we are in a position to prove Lemma 2. On account of Lemma 3, we need only to check that the kernel $K(x) = \frac{1}{2\delta}(\coth \frac{\pi y}{2\delta} - \mathrm{sign} y)$ satisfies the conditions (5)–(7).

Proof of Lemma 2. First, for $|x| \neq 0$, By using lemma 4, we see that

$$K(x) = \delta^{-1} \mathrm{sign} x \sum_{k=1}^{\infty} e^{-\delta^{-1}\pi k|x|} .$$

This gives us

$$|xK(x)| = \delta^{-1}|x| \sum_{k=1}^{\infty} e^{-\delta^{-1}\pi k|x|} = \frac{\delta^{-1}|x|}{e^{\delta^{-1}\pi|x|} - 1} \leqq \frac{1}{\pi} .$$

Next, for $0 < R_1 < R_2 < \infty$, one can easily check (6) by using the expansion (9).

Finally, we prove condition (7). For $|y| \neq 0$, we have

$$\int_{|x| \geqq 2|y|} |K(x - y) - K(x)| dx = \delta^{-1} \int_{|x| \geqq 2|y|} |\mathrm{sign}(x - y) \sum_{k=1}^{\infty} e^{-k\delta^{-1}\pi|x-y|}$$

$$- \mathrm{sign} x \sum_{k=1}^{\infty} e^{-k\delta^{-1}\pi|x|}| dx$$

$$= \delta^{-1} \sum_{k=1}^{\infty} [\int_{2|y|}^{\infty} |e^{-k\delta^{-1}\pi(x-y)} - e^{-k\delta^{-1}\pi x}| dx$$

$$+ \int_{2|y|}^{\infty} |e^{-k\delta^{-1}\pi x} - e^{-k\delta^{-1}\pi(x+y)}| dx]$$

$$= I(y) .$$

We now split the proof into two cases. First, for $y > 0$, we have

$$I(y) = \delta^{-1} \sum_{k=1}^{\infty} \int_{2y}^{\infty} e^{-k\delta^{-1}\pi x} dx \left(e^{k\delta^{-1}\pi y} - e^{-k\delta^{-1}\pi y} \right)$$

$$= \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \left(e^{-k\delta^{-1}\pi y} - e^{-3k\delta^{-1}\pi y} \right)$$

$$= \frac{1}{\pi} \ln \frac{(e^{3\delta^{-1}\pi y} - 1)e^{\delta^{-1}\pi y}}{e^{3\delta^{-1}\pi y}(e^{\delta^{-1}\pi y} - 1)} \leq \frac{3}{\pi} . \tag{10}$$

By a similar manner, we can also see that the inequality (10) valid for $y < 0$. The proof of Lemma 2 is then complete.

Lemma 5. *Let* $\phi(y) = xy + y^2 \coth y$, *then for* $|y| \neq 0$, *we have* $\phi'(y) = \phi'(-y)$, $|\phi''(y)| = \phi''(|y|)$. *Moreover, for* $|y| > 2$, *we have* $|\phi''(y)| > 2$.

Proof. From Lemma 4, we can easily deduce that

$$\phi'(y) = x + 2|y| + 4\sum_{k=1}^{\infty} (|y| - ky^2)e^{-2k|y|} , \tag{11}$$

$$\phi''(y) = \text{sign} y \left(2 + 8\sum_{k=1}^{\infty} \left(y^2 k^2 - 2|y|k + \frac{1}{2} \right) e^{-2k|y|} \right) , \tag{12}$$

and

$$\phi''(y) = 2y - 2y \sum_{k=1}^{\infty} \frac{20k^4\pi^4 y^2 + 18k^2\pi^2 y^4 + 6y^6}{k^2\pi^2(k^2\pi^2 + y^2)^3} \tag{13}$$

Furthermore if we set $h(y) = \dfrac{20a^2 y + 18ay^2 + 6y^3}{(a+y)^3}$, for $a > 0, y > 0$, then we have

$$h'(y) = \frac{4a^2(5a - y)}{(a+y)^4} . \tag{14}$$

Therefore, the results of the lemma can be derived simply by using (11)–(14).

Lemma 6. *For any function* $f \in C_0^{\infty}(R)$, *we have*

$$\int_R f\, G(\partial_x f)\, dx \geq A\|f\|_{H^{\frac{1}{2}}(R)} - B\|f\|_2 ,$$

where the positive constants A *and* B *are independent of* f.

Proof. On account of Lemma 1, and by using the Parseval identity

$$\int_R f\, G(\partial_x f)\, dx = \int_R 2\pi\xi \left(\coth(2\pi\delta\xi) - \frac{1}{2\pi\delta\xi} \right) |\hat{f}|^2 d\xi$$

$$= 2\pi \left(\int_{|\xi|>k_0} + \int_{|\xi|\leq k_0} \right) , \tag{15}$$

where k_0 is a positive number to be chosen later.

Since $\lim_{|\xi|\to\infty} \left(\coth(2\pi\delta\xi) - \frac{1}{2\pi\delta\xi} \right) = \text{sign}(\xi)$, thus there exists a sufficient large constant $k_0 > 0$, such that, for all $|\xi| > k_0$, the first integral in (15) satisfies

$$2\pi \int\limits_{|\xi|>k_0} \xi \left(\coth(2\pi\delta\xi) - \frac{1}{2\pi\delta\xi}\right) |\hat{f}|^2 d\xi \geq \pi \int\limits_{|\xi|>k_0} |\xi||\hat{f}|^2 d\xi . \tag{16}$$

Moreover, in order to handle the second integral in (15), we apply Lemma 4,

$$2\pi \int\limits_{|\xi|\leq k_0} \xi \left(\coth(2\pi\delta\xi) - \frac{1}{2\pi\delta\xi}\right) |\hat{f}|^2 d\xi$$

$$= \int\limits_{|\xi|\leq k_0} \xi^2 \left(\frac{4\pi^2\delta}{3} - 8\delta\sum_{n=1}^{\infty} \frac{4\delta^2\xi^2}{n^2(n^2 + 4\delta^2\xi^2)}\right) |\hat{f}|^2 d\xi$$

$$\geq \int\limits_{|\xi|\leq k_0} \xi^2 \left(\frac{4\pi^2\delta}{3} - 8\delta\sum_{n=1}^{\infty} \frac{4\delta^2 k_0^2}{n^2(n^2 + 4\delta^2 k_0^2)}\right) |\hat{f}|^2 d\xi$$

$$= \frac{4\pi^2\delta}{3}\varepsilon_0 \int\limits_{|\xi|\leq k_0} \xi^2|\hat{f}|^2 d\xi ,$$

where we have used the identity: $\sum_{k=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, and the ε_0 is a positive constant depending only on δ and k_0.

Therefore, from (16), (17) it follows that

$$\int\limits_{R}\int f\, G(\partial_x f)\, dx \geq \pi \int\limits_{|\xi|>k_0} |\xi||\hat{f}|^2 d\xi + \frac{4\pi^2\delta\varepsilon_0}{3} \int\limits_{|\xi|\leq k_0} \xi^2|\hat{f}|^2 d\xi$$

$$= \pi \int\limits_{|\xi|>k_0} |\xi||\hat{f}|^2 d\xi + \frac{4\pi^2\delta\varepsilon_0}{3} \int\limits_{|\xi|\leq k_0} (1 + \xi^2)|\hat{f}|^2 d\xi - \frac{4\pi^2\delta\varepsilon_0}{3} \int\limits_{|\xi|\leq k_0} |\hat{f}|^2 d\xi$$

$$\geq A \int\limits_{R} (1 + |\xi|)|\hat{f}|^2 d\xi - B \int\limits_{R} |\hat{f}|^2 d\xi$$

$$= A\|f\|_{H^{1/2}(R)} - B\|f\|_2 ,$$

where $A = \min\{\pi, 8\pi^2\delta\varepsilon_0/3\}$, $B = A + 4\pi^2\delta\varepsilon_0/3$.

3. Linear Estimates

In this section we shall obtain the decay estimate concerning the fundamental solution of the linear problem

$$\partial_t u - G(\partial_x^2 u) = f , \tag{18}$$

$$u(x, 0) = u_0(x) . \tag{19}$$

Define the unitary group $W(t)u_0 = g_t * u_0$, where

$$g_t(x) = \int\limits_{R} e^{i(2\pi\xi)^2 \left(\coth(2\pi\delta\xi) - \frac{1}{2\pi\delta\xi}\right)t + 2\pi i\xi x} d\xi , \tag{20}$$

then the solution of (18), (19) can be written as

$$u(x, t) = W(t)u_0(x) + \int\limits_{0}^{t} W(t - s)f(x, s)ds . \tag{21}$$

Our purpose in this section is to estimate the oscillatory integral $g_t(x)$ for all $t \geqq 0$.

Lemma 7. *Let $\delta > 0$ be a constant, then*

$$\|g_t(x)\|_\infty \leqq C(t^{-1/2} + (\delta t)^{-1/3}) \,,$$

for all $t \geqq 1$, where the constant C is independent of t and δ.

Let $\eta = 2\pi\delta\xi$, $\tau = \delta^{-2}t$, and $y = \frac{x}{\tau\delta} - 1$, then $g_t(x)$ can be rewritten as

$$S_\tau(y) = 2\pi\delta g_t(x) = \int_R e^{i\tau(\eta^2\coth\eta + y\eta)}\,d\eta \,. \tag{22}$$

In order to bound $S_\tau(y)$ for all $y \in R$, we need the following classical Van der Corput lemma [14]

Lemma 8. *Suppose that $\psi \in C_0^1(R)$ and $\phi \in C^2(R)$ such that $|\phi''(\xi)| \geqq 1$ on the support of ψ. Then*

$$\left| \int_R e^{i\lambda\phi(\xi)}\psi(\xi)\,d\xi \right| \leqq C\lambda^{-1/2}\{\|\psi\|_\infty + \|\psi'\|_1\} \,,$$

where the constant C is independent of λ, ϕ and ψ. The prime denotes the derivative.

Proof of Lemma 7. We write $S_t(x)$ as

$$S_t(x) = \int_{|\eta| \leqq 3} e^{it\phi(\eta)}\,d\eta + \int_{|\eta| > 3} e^{it\phi(\eta)}\,d\eta = S_t^1(x) + S_t^2(x) \,, \tag{23}$$

where $\phi(\eta) = x\eta + \eta^2\coth\eta$.

We shall split the proof into two parts to bound respectively $S_1(x)$ and $S_2(x)$ for all $x \in R$.

Part 1. Taking the variable-transform $\xi = t^{1/3}\eta$, $y = t^{2/3}(1 + x)$, and applying Lemma 4, we have

$$S_t^1(x) = t^{-1/3} \int_{|\xi| \leqq 2t^{1/3}} e^{i\bar\phi(\xi)}\,d\xi \,, \tag{24}$$

where $\bar\phi(\xi) = y\xi + \frac{1}{3}\xi^3 - 2\xi^3 \sum_{k=1}^\infty \frac{(t^{-1/3}\xi)^2}{\pi^2 k^2(\pi^2 k^2 + (t^{-1/3}\xi)^2)}$.

In what follows, we shall show that the identity

$$\int_{|\xi| \leqq 3t^{1/3}} e^{i\bar\phi(\xi)}\,d\xi \tag{25}$$

is bounded uniformly for all $t \geqq 0$, $y \in R$.

In fact, one can easily see that the boundedness of (25) is equivalent with the boundedness of the following integral for all $t \geqq 1$ and $y \in R$:

$$\int_R e^{i\bar\phi(\xi)}\bar\varphi(\xi)\,d\xi \,, \tag{26}$$

where the cut-off function $\bar\varphi(\xi) \in C^\infty(R)$, satisfies $0 \leqq \bar\varphi \leqq 1$, and $\bar\varphi \equiv 1$ for $2 \leqq |\xi| \leqq 3t^{1/3}$, $\bar\varphi \equiv 0$ for $|\xi| < 1$ and $|\xi| > 3t^{1/3} + 1$, and such that $|\bar\varphi(\xi)| \leqq C$.

On account of Lemma 8, the boundedness of (26) can be immediately obtained by checking the positive upper estimate of $|\bar{\phi}''(\xi)|$ in the support of $\bar{\varphi}(\xi)$, i.e. $|\bar{\phi}''(\xi)| \geqq \lambda_0$, for all $\xi \in$ supp $\{\bar{\varphi}(\xi)\}$, and λ_0 is a positive constant. In fact, by using Lemma 5,

$$
\begin{aligned}
|\bar{\phi}''(\xi)| &= t^{1/3}|\phi''(\eta)| \\
&\geqq 2|\xi| \left(1 - \sum_{k=1}^{\infty} \frac{20k^4\pi^4(t^{-1/3}\xi)^2 + 18k^2\pi^2(t^{-1/3}\xi)^4 + 6(t^{-1/3}\xi)^6}{k^2\pi^2(k^2\pi^2 + (t^{-1/3}\xi)^2)^3} \right) \\
&\geqq 2 \left(1 - \sum_{k=1}^{\infty} \frac{20k^4\pi^4 \cdot 16 + 18k^2\pi^2(16)^2 + 6(16)^2}{k^2\pi^2(k^2\pi^2 + 16)^3} \right) \\
&= \lambda_0 > 2 \left(1 - \sum_{k=1}^{\infty} \frac{6}{k^2\pi^2} \right) = 0 \, ,
\end{aligned}
$$

for all $\xi \in$ supp $\{\bar{\varphi}(\xi)\}$, and $t \geqq 1$. Therefore, we obtain

$$|S_t^1(x)| \leqq Ct^{-1/3} \, , \tag{27}$$

for $t > 0$, $x \in R$.

Part 2. In order to bound the integral

$$S_t^2(x) = \int_{|\eta|>3} e^{it\phi(\eta)} \, d\eta, \quad (\phi(\eta) = x\eta + \eta^2\coth\eta)$$

for $t > 0$, $x \in R$. We set a cut-off function $\varphi_0(\eta) \in C^{\infty}(R)$, which satisfies $0 \leqq \varphi_0(\eta) \leqq 1$, $\varphi_0(\eta) \equiv 1$ for $|\eta| > 3$, $\varphi_0(\eta) \equiv 0$ for all $|\eta| < 3 - \varepsilon$, and such that

$$|\partial_\eta^k \varphi_0(\eta)| \leqq C(k)\varepsilon^{-k} \, ,$$

for $k = 1$, 2, where $\varepsilon \in (0,1)$ is a number to be chosen later. Therefore

$$
\begin{aligned}
|S_t^2(x)| &= |\int_R e^{it\phi(\eta)} \varphi_0(\eta)d\eta - \int_{|\eta| \leqq 3} e^{it\phi(\eta)} \varphi_0(\eta) \, d\eta| \\
&\leqq |\int_R e^{it\phi(\eta)} \varphi_0(\eta) \, d\eta| + 2\varepsilon \, .
\end{aligned}
\tag{28}
$$

In what follows we shall split the proof into two cases to bound the integral in (28).

Case 1. Assume that $x > -(3 - 8\sum_{k=1}^{\infty}(2k-1)e^{-4k})$. Then from Lemma 5 we can verify that $|\phi''(\eta)| = \phi''(|\eta|) > 2$, and $\phi'(\eta) = \phi'(|\eta|) > \phi'(2) > 1$ for all $|\eta| > 2$. Therefore

$$
\begin{aligned}
\left| \int_R e^{it\phi(\eta)} \varphi_0(\eta)d\eta \right| &\leqq \frac{1}{t} \left| \int_R e^{it\phi(\eta)} \frac{\varphi_0'(\eta)}{\phi'(\eta)} \, d\eta \right| + \frac{1}{t} \int_R \frac{|\varphi_0(\eta)\phi''(\eta)|}{(\phi'(\eta))^2} \, d\eta \\
&\leqq Ct^{-3/2} \left(\left\| \frac{\varphi_0'(\eta)}{\phi'(\eta)} \right\|_{\infty} + \left\| \left(\frac{\varphi_0'(\eta)}{\phi'(\eta)} \right)' \right\|_1 \right) + t^{-1} \left\| \frac{\varphi_0(\eta)\phi''(\eta)}{(\phi'(\eta))^2} \right\|_1 \\
&\leqq Ct^{-3/2}(\|\varphi_0'(\eta)\|_{\infty} + \|\varphi_0''(\eta)\|_{\infty}\text{mes}\{3 - \varepsilon < |\eta| < 3\} \\
&\quad + \|\varphi_0'(\eta)\|_{\infty} \int_{3-\varepsilon<|\eta|<3} \frac{|\phi''(\eta)|}{(\phi'(\eta))^2}d\eta) + t^{-1} \int_{|\eta|>3-\varepsilon} \frac{|\phi''(\eta)|}{(\phi'(\eta))^2}d\eta \\
&\leqq C(t^{-1} + \varepsilon^{-1}t^{-3/2}) \, ,
\end{aligned}
\tag{29}
$$

where we have used integration by part, and Lemma 8.

Case 2. Assume that $x \leqq -(3 - 8\sum_{k=1}^{\infty}(2k-1)e^{-4k})$. We note that $-(3 - 8\sum_{k=1}^{\infty}(2k-1)e^{-4k}) = -\left(3 - \frac{8(e^4+1)}{(e^4-1)^2}\right) = -x_0 < -2$. Therefore, in order to bound the integral $S_I^2(x)$ in this case, we consider two domains

$$\Omega_1 = \{\eta; |\phi'(\eta)| \leqq |x|/2\}, \tag{30}$$

$$\Omega_2 = \{\eta; |\phi'(\eta)| \leqq |x|/3\}, \tag{31}$$

where $\phi'(\eta) = (x\eta + \eta^2\coth\eta)' = x + 2|\eta| + 4\sum_{k=1}^{\infty}(|\eta| - k\eta^2)e^{-2k|\eta|}$.

The proof in this case is based on the version of the following three lemmas concerning the two domains Ω_1 and Ω_2.

Lemma 9. *Let $x \leqq -x_0$, then there exists a constant $\delta_0 > 0$ such that $\{\eta; |\eta| < \delta_0\} \subset (\bar{\Omega}_1)^c$.*

Proof. Since

$$\phi'(\eta) = (\eta^2\coth\eta + x\eta)' = x + 2\eta + \frac{4\eta}{e^{2\eta}-1} - \frac{4\eta^2 e^{2\eta}}{(e^{2\eta}-1)^2},$$

and

$$\lim_{|\eta|\to 0}\left(2\eta + \frac{4\eta}{e^{2\eta}-1} - \frac{4\eta^2 e^{2\eta}}{(e^{2\eta}-1)^2}\right) = 1,$$

thus there exists a constant $\delta_0 > 0$ such that, for all $|\eta| < \delta_0$, we have

$$|\phi'(\eta)| > |x| - \left|2\eta + \frac{4\eta}{e^{2\eta}-1} - \frac{4\eta^2 e^{2\eta}}{(e^{2\eta}-1)^2}\right| > |x| - x_0/2 \geqq |x|/2.$$

This implies the result of the lemma.

Lemma 10. *There exists a constant C_0, such that*

$$\text{mes}\{\Omega_1 \cap \{|\eta| > 2\}\} \leqq |x| + C_0$$

for $x \leqq -x_0$.

Proof. From (30), we have

$$\Omega_1 \cap \{|\eta| > 2\} = \left\{\eta; \frac{|x|}{4} - 2\sum_{k=1}^{\infty}(k\eta^2 - |\eta|)e^{-2k|\eta|}\right.$$

$$\leqq |\eta| \leqq \frac{3|x|}{4} + 2\sum_{k=1}^{\infty}(k\eta^2 - |\eta|)e^{-2k|\eta|}$$

$$\left. \text{and} \quad |\eta| > 2 \right\}.$$

This gives us

$$\text{mes}\{\Omega_1 \cap \{|\eta| > 2\}\} \leqq 2 \sup_{|\eta|>2} \left\{ \frac{|x|}{2} + 4\sum_{k=1}^{\infty}(k\eta^2 - |\eta|)e^{-2k|\eta|} \right\}$$

$$= 2\left\{ \frac{|x|}{2} + \sum_{k=1}^{\infty}(k\eta^2 - |\eta|)e^{-2k|\eta|} \right\}\Bigg|_{|\eta|=2} = |x| + C_0 .$$

Lemma 11. *Let* $r_0 = \text{dist}\{\Omega_2, (\bar{\Omega}_1)^c\}$, *then there exists a positive constant* $C_1 = C_1(\delta_0)$ *such that* $r_0 > C_1|x|$ *for* $x \leqq -x_0$.

Proof. For any $\xi_1 \in (\bar{\Omega}_1 \cup \{|\eta| < \delta_0\})^c$, and $\xi_2 \in \Omega_2$, we have $|\phi'(\xi_1)| > |x|/2$, $|\phi'(\xi_2)| \leqq |x|/3$, which gives us $|\phi'(\xi_1) - \phi'(\xi_2)| > |x|/6$, and

$$r = |\xi_1 - \xi_2| > \frac{|x|}{6}|\phi''(\xi_0)|^{-1} ,$$

where the point ξ_0 is on the line connecting ξ_1 with ξ_2. Therefore

$$r_0 = \inf_{\xi_1 \in (\bar{\Omega}_1)^c; \xi_2 \in \Omega_2} |\xi_1 - \xi_2| \geqq \frac{|x|}{6} \inf_{|\xi|>\delta_0}|\phi''(\xi)|^{-1} = \frac{|x|}{6}\frac{1}{\sup_{|\xi|>\delta_0}|\phi''(\xi)|} ,$$

where

$$\sup_{|\xi|>\delta_0} |\phi''(\xi)| \leqq \sup_{|\xi|>\delta_0} (2 + 8\sum_{k=1}^{\infty}(\xi^2 k^2 + 2|\xi|k + 1/2)e^{-2k|\xi|}) = C(\delta_0) > 0 ,$$

and which finishes the proof of the lemma.

Now we are in a position to consider Case 2. On account of the above three lemmas, we construct a unitary decomposition of R, namely we expect two functions such that $\varphi_1(\eta)$, $\varphi_2(\eta) \in C^{\infty}(R)$, $0 \leqq \varphi_1 \leqq 1$, $0 \leqq \varphi_2 \leqq 1$, and $\varphi_1(\eta) + \varphi_2(\eta) \equiv 1$ for all $\eta \in R$. In addition, supp $\{\varphi_1(\eta)\}$ is contained in Ω_1, $\varphi_2(\eta) \equiv 0$ for all $\eta \in \Omega_2$, and

$$|\partial_\eta^k \varphi_i(\eta)| \leqq C(k)r_0^{-k} \leqq C|x|^{-k} ,$$

for $k = 1, 2$, and all $\eta \in R$.

The integral in (28) can be then written as

$$\begin{aligned}|\int_R e^{it\phi(\eta)}\,\varphi_0(\eta)\,d\eta| &\leqq |\int_R e^{it\phi(\eta)}\,\varphi_0(\eta)\varphi_1(\eta)\,d\eta| \\ &\quad + |\int_R e^{it\phi(\eta)}\,\varphi_0(\eta)\varphi_2(\eta)\,d\eta| . \end{aligned} \qquad (32)$$

If $\varphi_2(\eta) \neq 0$, i.e. $\eta \notin \Omega_2$, then we have $|\phi'(\eta)| > \frac{|x|}{3}$. Note that $|x| + 4\sum_{k=1}^{\infty}(-k\eta^2 + |\eta|)e^{-2k|\eta|} > 0$ for all $|\eta| > 2$, we can thus obtain

$$|\phi'(\eta)| \geqq C'(|x| + 2|\eta| + 4\sum_{k=1}^{\infty}(-k\eta^2 + |\eta|)e^{-2k|\eta|}) \geqq 4C' ,$$

for all $|\eta| > 0$ and $\eta \notin \Omega_2$, where C' is a positive constant.

Therefore, by applying integration by part and Lemma 8, we have

$$\Big| \int_R e^{it\phi(\eta)} \varphi_0(\eta)\varphi_2(\eta)d\eta \Big| = t^{-1} \Big| \int_R e^{it\phi(\eta)} \left(\frac{\varphi_0(\eta)\varphi_2(\eta)}{\phi'(\eta)} \right)' d\eta \Big|$$

$$\leqq Ct^{-1} \Bigg\{ t^{-1/2} \left(\left\| \frac{\varphi_0(\eta)\varphi_2(\eta)}{\phi'(\eta)} \right\|_\infty + \left\| \left(\frac{\varphi_0'(\eta)\varphi_2(\eta)}{\phi'(\eta)} \right)' \right\|_1 \right)$$

$$+ t^{-1/2} \left(\left\| \frac{\varphi_0(\eta)\varphi_2'(\eta)}{\phi'(\eta)} \right\|_\infty + \left\| \left(\frac{\varphi_0(\eta)\varphi_2'(\eta)}{\phi'(\eta)} \right)' \right\|_1 \right)$$

$$+ \left\| \frac{\phi''(\eta)\varphi_0(\eta)\varphi_2(\eta)}{(\phi'(\eta))^2} \right\|_1 \Bigg\}$$

$$\leqq Ct^{-3/2} \{ \|\varphi_0'(\eta)\|_\infty$$

$$+ \|\varphi_2'(\eta)\|_\infty + \|\varphi_0''(\eta)\|_\infty \mathrm{mes}\{3-\varepsilon < |\eta| < 3\}$$

$$+ \|\varphi_0'(\eta)\|_\infty \|\varphi_2'(\eta)\|_\infty \mathrm{mes}\{3-\varepsilon < |\eta| < 3\}$$

$$+ \|\varphi_2'(\eta)\|_\infty \mathrm{mes}\{\Omega_1 \cap \{|\eta| > 2\}\}$$

$$+ \|\varphi_0'(\eta)\|_\infty \left\| \frac{\phi''(\eta)}{(\phi'(\eta))^2} \right\|_{L^1(|\eta|>2)}$$

$$+ \|\varphi_2'(\eta)\|_\infty \left\| \frac{\phi''(\eta)}{(\phi'(\eta))^2} \right\|_{L^1(|\eta|>2)} \Bigg\}$$

$$+ Ct^{-1/2} \left\| \frac{\phi''(\eta)}{(\phi'(\eta))^2} \right\|_{L^1(|\eta|>2)}$$

$$\leqq Ct^{-3/2}(\varepsilon^{-1}+C) + Ct^{-1} . \tag{33}$$

Moreover, if $\eta \in \Omega_1$, then

$$\Big| \int_R e^{it\phi(\eta)} \varphi_0(\eta)\varphi_1(\eta)d\eta \Big| \leqq Ct^{-1/2}(\|\varphi_0(\eta)\varphi_1(\eta)\|_\infty + \|(\varphi_0(\eta)\varphi_1(\eta))'\|_1)$$

$$\leqq Ct^{-1/2}(1 + \|\varphi_0'(\eta)\|_\infty \mathrm{mes}\{3-\varepsilon < |\eta| < 3\}$$

$$+ \|\varphi_0'(\eta)\|_\infty \mathrm{mes}\{\Omega_1 \cap \{|\eta| > 2\}\})$$

$$\leqq Ct^{-1/2} . \tag{34}$$

Finally, combining (33), (34) with (28), and taking $\varepsilon = t^{-1/2}$, we thus have

$$|S_t^2(x)| \leqq Ct^{-1/2} , \tag{35}$$

for all $x \in R$ and $t > 1$.

Therefore, on account of the transform (22), we then obtain the desired decay estimate of Lemma 7 by combining (27), (35) and (23).

Theorem 1. *Suppose that $g_t(x)$ is the unitary group generated by the linear problem of Eq. (1). Let $u(x,t) = g_t(x) * u_0(x)$, then for $p \in (2,\infty)$,*

$$\|u(x,t)\|_p \leqq C \left(t^{\frac{1}{2}\left(1-\frac{2}{q}\right)} + (\delta t)^{-\frac{1}{3}\left(1-\frac{2}{p}\right)} \right) \|u_0(x)\|_q ,$$

for all $t \geqq 1$, with $p^{-1} + q^{-1} = 1$, where the constant C is independent of t and δ.

Proof. By using Lemma 7, and the $L^2(R)$-conservation law of the linear problem (3), the result of the theorem then follows from the Riesz–Thorin interpolation.

Remark. Since the constant C in Theorem 1 is independent of $\delta > 0$, we thus see that the decay estimate (2b) for the linear BO equation is a consequence of Theorem 1.

4. Decay Estimates for Nonlinear Problem

In this section we consider the following nonlinear problem:

$$\partial_t u - G(\partial_x^2 u) = \partial_x(u^p/p) , \tag{36}$$

$$u(x,0) = u_0(x) , \tag{37}$$

where $p \geqq 1$ is an integer. It is known [1, 17] that for any function $u_0(x) \in H^s(R)$ ($s \geqq 2$), there exists a positive constant T such that the nonlinear equation (1) with the initial data $u_0(x)$ admits a unique solution in $L^\infty(0, T; H^s(R))$.

Lemma 12. *For any initial data $u_0(x) \in H^k(R)$, then the solution $u = u(x,t)$ of problem (36), (37) such that*

$$\|u\|_{H^3} \leqq \|u_0\|_{H^3} \exp(C \int_0^t \|u(\tau)\|_{W^{2,q}(R)}^{p-1} d\tau) , \tag{38}$$

for $p \geqq 2$.

Proof. Since, for $k > 2$,

$$\frac{d}{dt} \int_R |\partial_x^k u|^2 \, dx = 2 \int_R \partial_x^k u [\partial_x^k(u^{p-1}\partial_x u) - u^{p-1}\partial_x^{k+1}u] dx$$

$$+ 2 \int_R \partial_x^k u \partial_x^{k+1} u \, u^{p-1} \, dx$$

$$\leqq C\|\partial_x^k u\|_2 (\|\partial_x(u^{p-1})\|_\infty \|\partial_x^k u\|_2 + \|\partial_x u\|_\infty \|\partial_x^k(u^{p-1})\|_2)$$

$$+ (p-1)\|\partial_x u\|_\infty \|u\|_\infty^{p-2} \|\partial_x^k u\|_2^2$$

$$\leqq C\|u\|_\infty^{p-2} \|\partial_x u\|_\infty \|\partial_x^k u\|_2^2$$

$$\leqq C\|u\|_{W^{2,q}(R)}^{p-1} \|\partial_x^k u\|_2^2 , \tag{39}$$

where we have used the calculus of inequalities [4].

By applying the Gronwall lemma, (39) is then implying the result of the lemma.

Theorem 2. *Let $\delta \in (0, \infty), q = 2p$, and $p > 5/2 + \sqrt{21}/2$. Assume that the initial data $u_0(x)$ is sufficiently small in $H^3(R) \cap W^{2,2p/(2p-1)}(R)$. Then the solution u of nonlinear problem (36), (37) such that*

$$\|u(t)\|_{W^{2,q}(R)} \leq C(1+t)^{-\frac{1}{3}\left(1-\frac{2}{q}\right)} \, ,$$

for all $t \geq 0$, where the constant C is independent of u and t.

Proof. Since the nonlinear problem (36), (37) can be written into the following formula:

$$u(t) = g_t * u_0 + \int_0^t g_{t-s} * \partial_x(u^p(s)/p)ds \, .$$

By using Theorem 1 and Hölder inequality, we obtain

$$\|u(t)\|_{W^{2,q}(R)} \leq \|g_t * u_0\|_{W^{2,q}(R)}$$

$$+ \int_0^t \|g_{t-s} * \partial_x(u^p/p)\|_{W^{2,q}(R)}ds$$

$$\leq C(1+t)^{-\frac{1}{3}\left(1-\frac{2}{q}\right)} \|u_0\|_{W^{2,q'}(R)} + C\int_0^t (t-s)^{-\frac{1}{3}\left(1-\frac{2}{q}\right)} \|\partial_x(u^p/p)\|_{W^{2,q'}(R)}ds$$

$$\leq C(1+t)^{-\frac{1}{3}\left(1-\frac{2}{q}\right)} \|u_0\|_{W^{2,q'}(R)} + C\int_0^t (t-s)^{-\frac{1}{3}\left(1-\frac{2}{q}\right)} \|u\|_{W^{2,q}(R)}^{p-1} \|u\|_{H^3(R)}ds$$

$$\leq C(1+t)^{-\frac{1}{3}\left(1-\frac{2}{q}\right)} \|u_0\|_{W^{2,q'}(R)} + C\|u_0\|_{H^3} \int_0^t (t-s)^{-\frac{1}{3}\left(1-\frac{2}{q}\right)} \|u\|_{W^{2,q}(R)}^{p-1}$$

$$\cdot \exp\left(C\int_0^s \|u(\tau)\|_{W^{2,q}(R)}^{p-1} d\tau \right) ds \, .$$

Let $M(t) = \sup_{0 \leq s \leq t}(1+s)^{\frac{1}{3}\left(1-\frac{2}{q}\right)} \|u(s)\|_{W^{2,q}(R)}$,   and $\bar{\delta} = \|u_0\|_{H^3(R)} + \|u_0\|_{W^{2,2p/(2p-1)}(R)}$. The above inequality then gives us

$$M(t) \leq C\bar{\delta} + C\bar{\delta}f(t)M^{p-1}(t) \exp(Ch(t)M^{p-1}(t)) \, , \tag{40}$$

where $f(t) = (1+t)^{\frac{1}{3}\left(1-\frac{1}{p}\right)} \int_0^t (t-s)^{-\frac{1}{3}\left(1-\frac{1}{p}\right)}(1+s)^{-\frac{p-1}{3}\left(1-\frac{1}{p}\right)} ds$, $h(t) = \int_0^t (1+s)^{-\frac{p-1}{3}\left(1-\frac{1}{p}\right)} ds$.

Note that $p > 5/2 + \sqrt{21}/2$, one can easily check that there exists a constant C such that $f(t) \leq C$, and $h(t) \leq C$, for all $t > 0$. Therefore, (40) gives us

$$M(t) \leq C\bar{\delta} + C\bar{\delta}M^{p-1}(t) \exp(CM^{p-1}(t)) \, . \tag{41}$$

Let $K(m) = c\bar{\delta}(1 + m^{p-1}\exp(Cm^{p-1})) - m$. Since $K(0) = C\bar{\delta} > 0$, and $K''(m) > 0$, for all $m > 0$, we take $\bar{\delta} > 0$ sufficiently small so that $K(m) = 0$ admits a positive zero m_1. Then set $C\bar{\delta} < m_1$, as $K(M(t)) \geq 0$ for all $t > 0$, and $M(0) = C\bar{\delta} < m_1$, so on account of the continuity of $K(M(t))$, we finally obtain that $M(t) \leq m_1$ for all $t \geq 0$, i.e. $\|u(t)\|_{W^{2,q}(R)} \leq m_1(1+t)^{-\frac{1}{3}\left(1-\frac{2}{q}\right)}$, for all $t \geq 0$, and $q = 2p > 5 + \sqrt{21}$.

Theorem 3. *Under the conditions of Theorem 2, the solution of the nonlinear problem (36), (37) is freely asymptotic to the solution of linear problem (3).*

Proof. Denoted by $u(t) \in L^\infty(R^+; H^3(R) \cap W^{2,2p/(2p-1)}(R))$ the solution of the non-linear problem (36), (37), we shall prove that there exists a function $u_+(t) \in L^\infty(R^+; H^2(R))$, such that

$$\partial_t u_+(t) - G(\partial_x^2 u_+(t)) = 0 , \tag{42}$$

and

$$\|u(t) - u_+(t)\|_{H^2(R)} \to 0 ,$$

as $t \to +\infty$. In fact, from Lemma 12 and Theorem 1, we have

$$\|u(t)\|_{H^3} \leqq \|u_0\|_{H^3} \exp(C \int_0^t \|u(\tau)\|_{W^{2,q}(R)}^{p-1} d\tau) \leqq C ,$$

Therefore, we define as in [7, 12]

$$u_+(t) = u(t) - \int_t^{+\infty} g_{t-s} * \partial_x(u^p/p)(s) ds .$$

One can easily find that above function satisfies the linear equation (42), and such that

$$\|u(t) - u_+(t)\|_{H^2} \leqq C \int_t^{+\infty} \|u(s)\|_{W^{2,q}}^{p-1} \|u(s)\|_{H^3} ds$$

$$\leqq C \int_t^{+\infty} (1+s)^{-\frac{p-1}{3}(1-\frac{2}{q})} ds = C \int_t^{+\infty} (1+s)^{-\frac{(p-1)^2}{3p}} ds \to 0 ,$$

as $t \to +\infty$, for $p > 5/2 + \sqrt{21}/2$. The proof of the theorem is then complete.

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