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Solvability of the Localized Induction Equation for Vortex Motion

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Abstract: The initial and the initial-boundary value problems for the localized induction equation which describes the motion of a vortex filament are considered. We prove the existence of solutions of both problems globally in time in the sense of distribution by the method of regularization.

1. Introduction

The localized induction equation which describes the motion of a smooth thin vortex filament in three-dimensional perfect fluid is derived from some physical approximations of the Biot–Savart law ([1, 4]). It is formulated as

$$\mathbf{x}_t = \mathbf{x}_s \times \mathbf{x}_{ss} , \tag{1.1}$$

where $\mathbf{x} = \mathbf{x}(s, t)$ denotes the coordinate of a point on the filament in $\mathbb{R}^3$ as a vector-valued function of arclength $s \in \mathbb{R}$ and time t, and the subscripts mean the partial differentiation with respect to the corresponding variables.

Some exact solutions of (1.1) are known ([7]): the trivial type ($\mathbf{x}_s \times \mathbf{x}_{ss} = 0$), the circular and the helical ones ($|\mathbf{x}_s \times \mathbf{x}_{ss}| = \text{const.}$), the elastic one rotating about an axis without changing its own form, etc.

Besides, Hasimoto indicated in [5] that (1.1) can be transformed by means of the Frenet–Serret formulae into the nonlinear Schrödinger equation,

$$-i\Psi_t = \Psi_{ss} + (1/2)|\Psi|^2\Psi \tag{1.2}$$

for $\Psi = \kappa(s, t)\exp\{i\int_0^s \tau(s,t)\,ds - i\int_0^t a(t)/2\,dt\}$. Here κ and τ are the curvature and the torsion of the filament respectively, i.e.,

$$\kappa = |\mathbf{x}_{ss}|, \quad \tau = \mathbf{x}_s \cdot (\mathbf{x}_{ss} \times \mathbf{x}_{sss})/|\mathbf{x}_{ss}|^2 ,$$

and

$$a(t) = 2\int_0^s \tau_t ds + 2\tau^2 - \kappa^2 - 2\kappa_{ss}/\kappa ,$$

which is proved to be independent of s. Then he showed that (1.2) has the 1-soliton solution corresponding to a vortex filament with a local loop on it. This result was extended to the N-soliton solutions in [2].

The unique solvability of the initial value problem for (1.2) was established by several authors. For example, Ginibre and Velo [3] proved it in the space $C_b^0(\mathbb{R}; W_2^1(\mathbb{R}))$. Hayashi et al. [6] showed a regularizing property of (1.2) that the solution for a nonsmooth initial condition is smooth for $t \neq 0$.

However, we must point out that (1.1) and (1.2) are *not* always equivalent from a mathematical point of view. Indeed, in transforming (1.1) into (1.2) we assume, for almost every (s, t), that $|\mathbf{x}_{sss}|$ and $|\mathbf{x}_{ssss}|$ remain bounded and that κ cannot be zero. This is the reason why in this paper we study the initial and the initial-boundary value problems for (1.1), not for (1.2). Our result is that there exist solutions to these problems in a weak sense. Unlike the results for (1.2), it seems to be difficult to say something about their uniqueness and smoothness by our method (see Remark 4.2 below).

Since $\mathbf{x}$ and $\mathbf{x}_s$ are the position and the unit tangential vectors respectively, they are not square integrable with respect to $s \in \mathbb{R}$. Thus, taking account of several numerical experiments ([1, 4, 11]), we consider the following situations:

(I) the curvature $|\mathbf{x}_{ss}| \to 0$ as $s \to \pm\infty$.
(II) $\mathbf{x}(s,t)$ approaches an exact solution $\mathbf{y}(s,t)$ with $\mathbf{y}_{ss}(\pm\infty, t) \neq 0$, such as a helix or an elastica, as $s \to \pm\infty$.
(III) $\mathbf{x}_{ss}(\pm 1, t) = 0$ is satisfied when the domain of s is restricted to $J = (-1, 1)$.
(IV) the filament is closed: $\mathbf{x}(s-1, t) = \mathbf{x}(s+1, t)$ for $s \in \mathbb{R}$.

Then in the cases (I) and (II) the problems become the initial value ones for the functions $\mathbf{X} = \mathbf{x} - \mathbf{x}_0$ (with $\mathbf{x}(s, 0) = \mathbf{x}_0(s)$) and $\mathbf{Y} = \mathbf{x} - \mathbf{y}$ respectively. The reason to distinguish (II) from (I) is that $\mathbf{x}_{0ss}$ can be assumed to belong to $L^2(\mathbb{R})$ but $\mathbf{y}_{ss}(\cdot, t)$ cannot. We do not intend to discuss the stability problem at present. The numerical simulations in [4] for the motion of a vortex filament whose initial form is plane parabolic, hyperbolic or exponential correspond to (I). On the other hand, the condition (III) is equivalent to $\mathbf{x}(\pm 1, t) = \mathbf{x}_0(\pm 1)$. In other words it means that the filament is fixed at the points $s = \pm 1$ such as an initially sinusoidal curve in the numerical experiment [11]. Therefore the problem under this condition becomes the initial-boundary value one. The study of (IV) is mathematically similar to and simpler than that of (III), and we mention it only at the end of this paper. In any case, there is little essential difference among the final results for (I)–(IV).

Since the eigenvalues of the matrix A, defined by $A\mathbf{x}_{ss} = \mathbf{x}_s \times \mathbf{x}_{ss}$, are 0, $\pm i|\mathbf{x}_s|$, (1.1) is not of parabolic type. Thus in this paper we first discuss the parabolic regularizations

$$\mathbf{x}^\varepsilon_t = \mathbf{x}^\varepsilon_s \times \mathbf{x}^\varepsilon_{ss} + \varepsilon \mathbf{x}^\varepsilon_{ss} \quad \text{in cases (I) and (III)}, \tag{1.3}$$

and

$$\mathbf{x}^\varepsilon_t = \mathbf{x}^\varepsilon_s \times \mathbf{x}^\varepsilon_{ss} + \varepsilon\ (\mathbf{x}^\varepsilon - \mathbf{y})_{ss} \quad \text{in case (II)} \tag*{(1.3)'}$$

for $\varepsilon > 0$. In Sect. 2 we give a priori estimates for them and in Sect. 3 we show the existence and uniqueness of their solutions. Then we establish the solvability of (1.1) for all the types (I)–(IV) in the sense of distribution of Sect. 4.

We introduce the notation as follows: Let Ω be a domain in $\mathbb{R}$, T be any positive number and $\Omega_T = \Omega \times (0,T)$. The norm in $L^2(\Omega)$ (resp. $L^2(\Omega_T)$) is denoted by $\|\cdot\|_\Omega$ (resp. $|\cdot|_{\Omega_T}$). If $\Omega = \mathbb{R}$, we write them simply as $\|\cdot\|$ and $|\cdot|$ respectively. By $W_2^\alpha(\Omega)$ ($\alpha > 0$) we mean the Sobolev–Slobodetskiĭ space ([9]) in which the norm of an element u is defined by

$$(\|u\|_\Omega^{(\alpha)})^2 = \sum_{\substack{0 \leqq k < \alpha \\ k \in \mathbf{Z}}} \|D^k u\|_\Omega^2 + (\langle\langle u \rangle\rangle_\Omega^{(\alpha)})^2 ,$$

where

$$(\langle\langle u \rangle\rangle_\Omega^{(\alpha)})^2 = \begin{cases} \|D^\alpha u\|_\Omega^2 \text{ if } \alpha = [\alpha] , \\ \int\limits_\Omega \int\limits_\Omega \dfrac{|D^{[\alpha]} u(s) - D^{[\alpha]} u(s')|^2}{|s-s'|^{1+2(\alpha-[\alpha])}} \, ds ds' \quad \text{if } \alpha \neq [\alpha] . \end{cases}$$

If $\Omega = \mathbb{R}$, we write the norm simply as $\|\cdot\|^{(\alpha)}$. We define the norm in the anisotropic space $W_2^{\alpha,\alpha/2}(\Omega_T) = L^2(0,T; W_2^\alpha(\Omega)) \cap L^2(\Omega; W_2^{\alpha/2}(0,T))$ of functions $u(s,t)$ ($s \in \Omega, t \in (0,T)$) by

$$(|u|_{\Omega_T}^{(\alpha)})^2 = \int\limits_0^T (\|u(\cdot,t)\|_\Omega^{(\alpha)})^2 dt + \int\limits_\Omega (\|u(s,\cdot)\|_{(0,T)}^{(\alpha/2)})^2 \, ds .$$

The subscript is also omitted if $\Omega = \mathbb{R}$. Let $C^n(\Omega)$ (resp. $C_b{}^n(\Omega)$) be the set of all n-times continuously (resp. bounded-continuously) differentiable vector-valued functions on Ω. Positive constants, denoted by C (independent of ε), C_ε (dependent on ε) in Sect. 2 and c in Sect. 3, change from line to line. Moreover, C_ε has the property: $C_\varepsilon \to 0$ as $\varepsilon \to 0$.

2. A Priori Estimates

We first consider the case (I).

Lemma 2.1. *Let $\mathbf{x}_0(s)$ be a function on $\mathbb{R}$ satisfying $\mathbf{x}_{0ss} \in W_2{}^2(\mathbb{R})$, ε be fixed as $0 < \varepsilon < 1$ and T be an arbitrary finite positive number. Assume that $\mathbf{x}^\varepsilon$ is a solution of* (1.3) *with the initial condition $\mathbf{x}^\varepsilon(s,0) = \mathbf{x}_0$ such that $\mathbf{x}^\varepsilon - \mathbf{x}_0$ belongs to $W_2^{2+\alpha,1+\alpha/2}(\mathbb{R}_T)$ ($0 < \alpha \leqq 1$). Then for $\mathbf{X}^\varepsilon = \mathbf{x}^\varepsilon - \mathbf{x}_0$ the estimate*

$$|\mathbf{X}^\varepsilon|^{(2)} + C_\varepsilon |\mathbf{X}^\varepsilon|^{(2+\alpha)} \leqq C \tag{2.1}$$

is valid, where the constants C and C_ε depend on $\mathbf{x}_0$ and T.

Proof. Substituting $\mathbf{x}^\varepsilon$ with $\mathbf{X}^\varepsilon + \mathbf{x}_0$ in (1.3), we have

$$\mathbf{X}^\varepsilon{}_t = (\mathbf{X}^\varepsilon + \mathbf{x}_0)_s \times \mathbf{X}^\varepsilon{}_{ss} + \varepsilon \mathbf{X}^\varepsilon{}_{ss} + (\mathbf{X}^\varepsilon + \mathbf{x}_0)_s \times \mathbf{x}_{0ss} + \varepsilon \mathbf{x}_{0ss} . \tag{2.2}$$

By the density theorem it is sufficient to assume that $\mathbf{X}^\varepsilon$ is smooth and has a compact support. Multiplying (2.2) by $\mathbf{X}^\varepsilon_{ss}$ and $\mathbf{X}^\varepsilon_{ssss}$ and integrating by parts, we have, respectively,

$$
\begin{aligned}
(1/2)(\|\mathbf{X}^{\varepsilon}{}_{s}\|^2)_t + \varepsilon\|\mathbf{X}^{\varepsilon}{}_{ss}\|^2 & \\
&= \int_{\mathbb{R}} (\mathbf{x}_{0ss} \times \mathbf{X}^{\varepsilon}{}_{s} + \mathbf{x}_{0ss} \times \mathbf{x}_{0s} - \varepsilon\ \mathbf{x}_{0ss}) \cdot \mathbf{X}^{\varepsilon}{}_{ss} ds \\
&\leqq \sup_s|\mathbf{x}_{0ss}| \cdot (\|\mathbf{X}^{\varepsilon}{}_{s}\|^2 + \|\mathbf{X}^{\varepsilon}{}_{ss}\|^2) \\
&\quad + (1+\varepsilon)\ (\|\ \mathbf{x}_{0ss}\|^2 + \|\mathbf{X}^{\varepsilon}{}_{ss}\|^2) \\
&\leqq C(\|\ \mathbf{X}^{\varepsilon}{}_{s}\|^2 + \|\mathbf{X}^{\varepsilon}{}_{ss}\|^2) + C
\end{aligned} \tag{2.3}
$$

and

$$
\begin{aligned}
(1/2)\ (\|\mathbf{X}^{\varepsilon}{}_{ss}\|^2)_t + \varepsilon\|\mathbf{X}^{\varepsilon}{}_{sss}\|^2 &\leqq \sup_s|\mathbf{x}_{0ss}| \cdot (\|\mathbf{x}_{0sss}\|^2 + \|\mathbf{X}^{\varepsilon}{}_{ss}\|^2) \\
&\quad + \sup_s|\mathbf{X}^{\varepsilon}{}_{s}| \cdot \|\mathbf{x}_{0ssss}\|\ \|\mathbf{X}^{\varepsilon}{}_{ss}\| \\
&\quad + (1+\varepsilon)\ (\|\mathbf{x}_{0ssss}\|^2 + \|\mathbf{X}^{\varepsilon}{}_{ss}\|^2) \\
&\leqq C(\|\mathbf{X}^{\varepsilon}{}_{s}\|^2 + \|\mathbf{X}^{\varepsilon}{}_{ss}\|^2) + C\,.
\end{aligned} \tag{2.4}
$$

Here we used the fact that $\mathbf{x}_{0s} \in C^2(\mathbb{R})$ is the unit tangential vector of the initial vortex filament: $|\ \mathbf{x}_{0s}\ |= 1$, the multiplicative inequality and Young's. From adding (2.3) to (2.4) and applying the Gronwall inequality we derive the estimate

$$
\|\mathbf{X}^{\varepsilon}{}_{s}\| + \|\mathbf{X}^{\varepsilon}{}_{ss}\| \leqq C\,. \tag{2.5}
$$

It follows from (2.4) and (2.5) that

$$
|\mathbf{X}^{\varepsilon}{}_{sss}| \leqq C\varepsilon^{-1/2}\,. \tag{2.6}
$$

In the same way as above it is not difficult to obtain from (2.5),

$$
\begin{aligned}
(1/2)\ (\|\mathbf{X}^{\varepsilon}\|^2)_t + \varepsilon\ \|\mathbf{X}^{\varepsilon}{}_{s}\|^2 &= \int_{\mathbb{R}} \{(\mathbf{X}^{\varepsilon} + \mathbf{x}_0)_s \times (\mathbf{X}^{\varepsilon} + \mathbf{x}_0)_{ss} + \varepsilon \mathbf{x}_{0ss}\} \cdot \mathbf{X}^{\varepsilon} ds \\
&\leqq C\|\mathbf{X}^{\varepsilon}\|^2 + C\,,
\end{aligned}
$$

hence

$$
\|\mathbf{X}^{\varepsilon}\| \leqq C\,. \tag{2.7}
$$

It is obvious that the estimate

$$
\begin{aligned}
\|\mathbf{X}^{\varepsilon}{}_{t}\| &\leqq \|(\mathbf{X}^{\varepsilon} + \mathbf{x}_0)_s \times \mathbf{X}^{\varepsilon}{}_{ss}\| + \|(\mathbf{X}^{\varepsilon} + \mathbf{x}_0)_s \times \mathbf{x}_{0ss}\| \\
&\quad + \varepsilon\|\mathbf{X}^{\varepsilon}{}_{ss}\| + \varepsilon\|\mathbf{x}_{0ss}\| \leqq C
\end{aligned} \tag{2.8}
$$

follows from (2.2).

Therefore (2.5), (2.7) and (2.8) yield

$$
|\mathbf{X}^{\varepsilon}|^{(2)} \leqq C\,. \tag{2.9}
$$

Finally, differentiating (2.2) with respect to s leads to

$$
\begin{aligned}
|\mathbf{X}^{\varepsilon}{}_{st}| &\leqq |(\mathbf{X}^{\varepsilon} + \mathbf{x}_0)_s \times \mathbf{X}^{\varepsilon}{}_{sss}| + |(\mathbf{X}^{\varepsilon} + \mathbf{x}_0)_s \times \mathbf{x}_{0sss}| \\
&\quad + \varepsilon|\mathbf{X}^{\varepsilon}{}_{sss}| + \varepsilon\ |\mathbf{x}_{0sss}| \\
&\leqq (\delta^{1/2}|\mathbf{X}^{\varepsilon}{}_{s}|^{(2)} + C\delta^{-3/2}|\mathbf{X}^{\varepsilon}{}_{s}| + 1)\ C\varepsilon^{-1/2} + C \\
&\leqq C\varepsilon^{-1/2}\delta^{1/2}|\mathbf{X}^{\varepsilon}{}_{st}| + C_{\varepsilon}{}^{-1}
\end{aligned}
$$

with δ being an arbitrary positive constant. If we take δ so small that $C\varepsilon^{-1/2}\,\delta^{1/2} < 1$ holds, then we obtain

$$|\mathbf{X}^{\varepsilon}_{st}| \leqq C_{\varepsilon}^{-1}. \tag{2.10}$$

Hence from (2.6), (2.9) and (2.10), the inequality

$$|\mathbf{X}^{\varepsilon}_{s}|^{(2)} \leqq C_{\varepsilon}^{-1} \tag{2.11}$$

follows. It is clear to get

$$|\mathbf{X}^{\varepsilon}_{t}|^{(1)} \leqq C_{\varepsilon}^{-1} \tag{2.12}$$

by using (2.2) and (2.11).

The relations (2.9), (2.11) and (2.12) yield (2.1) with $\alpha = 1$, which immediately gives (2.1) with $\alpha \in (0, 1)$. □

Similarly, we get a priori estimates for the cases (II) and (III).

Lemma 2.2. *Let ε and T be as in Lemma* 2.1, $\mathbf{y}$ *be an exact solution for* (1.1) *with the property* $\mathbf{y}_{ss} \in C^0(0, T; C_b^{\,2}(\mathbb{R}))$ *and* $\mathbf{x}_0(s)$ *satisfy* $\mathbf{x}_0 - \mathbf{y}(\cdot, 0) \in W_2^{\,2}(\mathbb{R})$. *Suppose that* $\mathbf{x}^{\varepsilon}$ *is a solution of* (1.3)′ *with the initial condition* $\mathbf{x}^{\varepsilon}(s, 0) = \mathbf{x}_0$ *such that* $\mathbf{x}^{\varepsilon} - \mathbf{y} \in W_2^{2+\alpha,1+\alpha/2}(\mathbb{R}_T)$ $(0 < \alpha \leqq 1)$. *Then the estimate*

$$|\mathbf{Y}^{\varepsilon}|^{(2)} + C_{\varepsilon}|\mathbf{Y}^{\varepsilon}|^{(2+\alpha)} \leqq C \tag{2.13}$$

is valid for $\mathbf{Y}^{\varepsilon} = \mathbf{x}^{\varepsilon} - \mathbf{y}$, *where C and C_{ε} depend on* $\mathbf{y}$, $\mathbf{x}_0$ *and T.*

Proof. We can rewrite (1.3)′ in the form

$$\mathbf{Y}^{\varepsilon}_{t} = (\mathbf{Y}^{\varepsilon} + \mathbf{y})_s \times \mathbf{Y}^{\varepsilon}_{ss} + \varepsilon \mathbf{Y}^{\varepsilon}_{ss} + \mathbf{Y}^{\varepsilon}_{s} \times \mathbf{y}_{ss}\,. \tag{2.14}$$

In just the same way as in the proof of Lemma 2.1 we derive the inequality (2.13) from the fact $|\,\mathbf{y}_s\,| = 1$ yielded by

$$(1/2)(|\mathbf{y}_s|^2)_t = \mathbf{y}_s \cdot \mathbf{y}_{st} = \mathbf{y}_s \cdot (\mathbf{y}_s \times \mathbf{y}_{sss}) = 0$$

and $|\,\mathbf{y}_s(\cdot, 0)\,| = 1$. □

Estimating $\mathbf{x}^{\varepsilon}$ in (1.3) directly without using (2.2), then we have

Lemma 2.3. *Let ε and T be as in Lemma* 2.1 *and* $\mathbf{x}_0(s)$ *be a function on J satisfying* $\mathbf{x}_{0ss} \in L^2(J)$. *Suppose that* $\mathbf{x}^{\varepsilon} \in W_2^{2+\alpha,1+\alpha/2}(J_T)$ $(0 < \alpha \leqq 1)$ *is a solution of* (1.3) *with the initial-boundary conditions* $\mathbf{x}^{\varepsilon}(s, 0) = \mathbf{x}_0$, $\mathbf{x}^{\varepsilon}_{ss}(\pm 1, t) = 0$. *Then the estimate*

$$|\mathbf{x}^{\varepsilon}|^{(2)}_{J_T} + C_{\varepsilon}|\mathbf{x}^{\varepsilon}|^{(2+\alpha)}_{J_T} \leqq C \tag{2.15}$$

is valid with some constants C, C_{ε} depending on $\mathbf{x}_0$ *and T.*

3. Existence and Uniqueness of Solution for (1.3)

In order to establish the existence and uniqueness of solution for (1.3) with $\varepsilon > 0$ fixed it is convenient to introduce the weighted anisotropic Sobolev–Slobodetskiĭ space, the norm $|\!|\!|\cdot|\!|\!|_{\gamma}^{(\alpha)}$ and the lemma relevant to them ([10]).

Definition 3.1. $H_\gamma^{\alpha,\alpha/2}(\Omega_T)$ $(\gamma \geqq 0,\ \alpha > 0)$ *denotes the space of three-dimensional vector functions on* Ω_T *with the finite norm defined by*

$$(|u|_{\gamma,\Omega_T}^{(\alpha)})^2 = \begin{cases} \int_0^T e^{-2\gamma t}\{(\langle\langle u\rangle\rangle_\Omega^{(\alpha)})^2 + \gamma^\alpha \|u\|_\Omega^2 + \int_0^\infty \|D_t^{[\alpha/2]}u_0(\cdot, t-\tau) \\ -D_t^{[\alpha/2]}u_0(\cdot,t)\|_\Omega^2 \tau^{-1-\alpha+2[\alpha/2]}d\tau\}dt \quad \text{if } \alpha/2 \neq [\alpha/2]\,, \\ \int_0^T e^{-2\gamma t}\{(\langle\langle u\rangle\rangle_\Omega^{(\alpha)})^2 + \gamma^\alpha\|u\|_\Omega^2 + \|D_t^{\alpha/2}u\|_\Omega^2\}dt \quad \text{if } \alpha/2 = [\alpha/2]\,, \end{cases}$$

where $u_0 = u$ (resp. $u_0 = 0$) *when* $t > 0$ (resp. $t < 0$), *and in the second case we assume that* $D_t^j u(s,0) = 0$ $(j = 0, \ldots, \alpha/2 - 1)$ *are satisfied.*

Definition 3.2. *For* $u \in H_\gamma^{\alpha,\alpha/2}(\mathbb{R}_\infty)$ *we define its Fourier–Laplace transformation by*

$$\hat{u}(\xi,\ \sigma) = \int_0^\infty e^{-\sigma t}\left(\int_{\mathbb{R}} u(s,t)e^{-is\xi}\, ds\right) dt\,,$$

and the norm $|||\cdot|||_\gamma^{(\alpha)}$ *by*

$$(|||u|||_\gamma^{(\alpha)})^2 = \int_{\mathbb{R}}\int_{\mathbb{R}} |\hat{u}(\xi,\ \sigma)|^2(|\sigma| + \xi^2)^\alpha d\xi\ d\zeta\,,$$

where $\sigma = \gamma + i\zeta$. *For* $u \in H_\gamma^{\alpha,\alpha/2}(D)$, *where* $D = \mathbb{R}^+ \times \mathbb{R}^+$, *we define its Laplace transformation by*

$$\tilde{u}(s,\sigma) = \int_0^\infty e^{-\sigma t}u(s,t)dt\,,$$

and the norm $|||\cdot|||_{\gamma,D}^{(\alpha)}$ *by*

$$(|||u|||_{\gamma,D}^{(\alpha)})^2 = \sum_{k<\alpha}\int_{\mathbb{R}} \|D_s^k\ \tilde{u}(\cdot,\sigma)\|_{\mathbb{R}^+}^2 |\sigma|^{\alpha-k}d\zeta + \int_{\mathbb{R}}(\langle\langle\tilde{u}(\cdot,\sigma)\rangle\rangle_{\mathbb{R}^+}^{(\alpha)})^2 d\zeta\,,$$

where $\sigma = \gamma + i\zeta$.

Lemma 3.1. ([10, Lemmas 2.1, 2.3]) *For every* $\gamma \geqq 0$ *the norm* $|||\cdot|||_\gamma^{(\alpha)}$ (resp. $|||\cdot|||_{\gamma,D}^{(\alpha)}$) *is equivalent to* $|\cdot|_{\gamma,\mathbb{R}_\infty}^{(\alpha)}$ (resp. $|\cdot|_{\gamma,D}^{(\alpha)}$).

From now on, in this section we denote the constants which may depend on ε by c.

A. The Initial Value Problem

In this subsection we consider the initial value problem for (1.3) related to the case (I)

$$\mathbf{X}^\varepsilon{}_t = (\mathbf{X}^\varepsilon + \mathbf{x}_0)_s \times \mathbf{X}^\varepsilon{}_{ss} + \varepsilon\mathbf{X}^\varepsilon{}_{ss} + (\mathbf{X}^\varepsilon + \mathbf{x}_0)_s \times \mathbf{x}_{0ss} + \varepsilon\mathbf{x}_{0ss} \quad (s \in \mathbb{R}, t > 0)\,,$$
$$\mathbf{X}^\varepsilon(s,0) = 0 \quad (s \in \mathbb{R})\,. \tag{3.1}$$

As preliminaries we consider the linearized problems; first

$$u_t = a \times u_{ss} + \varepsilon\ u_{ss} + f(s,t) \quad (s \in \mathbb{R},\ t > 0)\,,$$

$$u(s,\ 0) = 0 \quad (s \in \mathbb{R})\,, \tag{3.2}$$

with a constant vector a in $\mathbb{R}^3$, and second

$$u_t = a(s,t) \times u_{ss} + \varepsilon u_{ss} + f(s,t)\,,$$
$$u(s,0) = 0\,, \tag{3.3}$$

with a vector-valued function $a(s,t)$ in $\mathbb{R}^3$. Then we go to (3.1).

Proposition 3.1. *If* $\alpha > 0$, $\varepsilon > 0$, $\gamma > 0$ *and* $f \in H_\gamma^{\alpha,\alpha/2}(\mathbb{R}_\infty)$, *then there exists a unique solution* u *of* (3.2) *such that* $u \in H_\gamma^{2+\alpha,1+\alpha/2}(\mathbb{R}_\infty)$. *Moreover the following inequality is valid:*

$$|u|_{\gamma,\mathbb{R}_\infty}^{(2+\alpha)} \leqq c\,|f|_{\gamma,\mathbb{R}_\infty}^{(\alpha)}\,. \tag{3.4}$$

Proof. Let us write the right-hand side of the first equation of (3.2) as $A_0 u_{ss} + f$. By the Fourier–Laplace transformation we have formally

$$\hat{u} = (\sigma + A_0\xi^2)^{-1}\hat{f}\,. \tag{3.5}$$

Obviously (3.2) is satisfied with the function u obtained by the inverse transformation of $\hat{u}$.

On the other hand, since the eigenvalues of the matrix $\sigma + A_0\xi^2$ are $\sigma + \varepsilon\xi^2$, $\sigma + (\varepsilon \pm i|a|)\xi^2$, the norm of $(\sigma + A_0\xi^2)^{-1}$ is equal to

$$\mathrm{Max}(|\sigma + \varepsilon\xi^2\,|^{-1}, |\,\sigma + (\varepsilon \pm i|a|)\xi^2\,|^{-1})\,.$$

By virtue of the inequality

$$\begin{aligned}(|\sigma| + \xi^2)^2 &\leqq \{|\sigma + ib\xi^2| + (1 + |b|)\xi^2\}^2\\ &\leqq 2(1 + |b|)^2\mathrm{Max}(1,\ \varepsilon^{-2})|(\sigma + ib\xi^2) + \varepsilon\xi^2|^2\,,\end{aligned}$$

where $b = 0,\ \pm|a|$, (3.5) can be evaluated as follows:

$$\begin{aligned}(\|\!|\, u \,\|\!|_\gamma^{(2+\alpha)})^2 &\leqq \sum_b \int_{\mathbb{R}} d\xi \int_{\mathbb{R}} |\hat{f}(\xi,\sigma)|^2|\sigma + (\varepsilon + ib)\xi^2|^{-2}(|\sigma| + \xi^2)^{2+\alpha}\ d\zeta\\ &\leqq c(\|\!|\, f \,\|\!|_\gamma^{(\alpha)})^2\,.\end{aligned}$$

This leads to (3.4) according to Lemma 3.1. □

Proposition 3.2. *Assume that* $0 < \alpha < 1$, $\varepsilon > 0$, $f \in W_2^{\alpha,\alpha/2}(\mathbb{R}_T)$ *and* $a(s,\ t)$ *is Hölder continuous. Then for any finite positive* T *there exists a unique solution* u *of* (3.3) *such that* $u \in W_2^{2+\alpha,1+\alpha/2}(\mathbb{R}_T)$. *Moreover the inequality*

$$|u|_*^{(2+\alpha)} \leqq c(T)|f|_*^{(\alpha)} \tag{3.6}$$

is valid, where $c(T)$ *denotes a function monotonically increasing with respect to* T *and*

$$(|f|_*^{(\alpha)})^2 = (|f|^{(\alpha)})^2 + T^{-\alpha}|f|^2\,,$$

$$(|u|_*^{(2+\alpha)})^2 = (|u_t|_*^{(\alpha)})^2 + (|u_{ss}|_*^{(\alpha)})^2 + |u_s|^2 + |u|^2\,.$$

Proof. Using Proposition 3.1 and the regularizer method ([10, §4]), we can prove that, for $f \in H_\gamma^{\alpha,\alpha/2}(\mathbb{R}_\infty)$ with a large $\gamma \gg 1$, there exists a unique solution of (3.3) which belongs to $H_\gamma^{2+\alpha,1+\alpha/2}(\mathbb{R}_\infty)$ and satisfies the same inequality with (3.4).

Since every element of $H_\lambda^{\alpha,\alpha/2}(\mathbb{R}_T)$ $(\lambda \geqq 0)$ can be extended with preservation of the class into $\mathbb{R}_\infty$ ([10]), the estimate (3.6) is derived as

$$
\begin{aligned}
|u|_*^{(2+\alpha)} &\leqq c\left[|u| + |u_t|^{(\alpha)} + |u_{ss}|^{(\alpha)} + \left\{\int_0^\infty (\|u_t\|^2 + \|u_{ss}\|^2)\, t^{-\alpha} dt\right\}^{1/2}\right] \\
&\leqq c(|u| + |u_t|_{0,\mathbb{R}_\infty}^{(\alpha)} + |u_{ss}|_{0,\mathbb{R}_\infty}^{(\alpha)}) \leqq c(|u| + |u_t|_0^{(\alpha)} + |u_{ss}|_0^{(\alpha)}) \\
&\leqq ce^{\gamma T}|u|_\gamma^{(2+\alpha)} \leqq c(T)|f|_\gamma^{(\alpha)} \leqq c(T)(|f|_0^{(\alpha)} + \gamma^{\alpha/2}|f|) \\
&\leqq c(T)|f|_*^{(\alpha)},
\end{aligned}
$$

where $|\cdot|_\gamma^{(\alpha)} \equiv |\cdot|_{\lambda,\mathbb{R}_T}^{(\alpha)}$ $(\lambda = 0, \gamma)$, and the second and the last inequalities are obtained from Lemma 6.3 and its Corollary in [10], respectively. Now the proof is completed. □

Proposition 3.3. *Let* $1/2 < \alpha < 1$, $\varepsilon > 0$, $\mathbf{x}_{0ss} \in W_2^\alpha(\mathbb{R})$ *and* T *be as in Proposition* 3.2. *Then for some* $T_0 \in (0, T]$ *there exists a unique solution* $\mathbf{X}^\varepsilon$ *of* (3.1) *which belongs to* $W_2^{2+\alpha,1+\alpha/2}(\mathbb{R}_{T_0})$. *Moreover the following inequality is valid:*

$$
|\mathbf{X}^\varepsilon|_{\mathbb{R}_{T_0}}^{(2+\alpha)} \leqq c\|\mathbf{x}_{0ss}\|^{(\alpha)}. \tag{3.7}
$$

Proof. Let $u^{(0)} = 0$ and $u^{(n)}$ $(n = 1, 2, \ldots)$ be a solution in Proposition 3.2 with $a = (u^{(n-1)} + \mathbf{x}_0)_s$, $f = (u^{(n-1)} + \mathbf{x}_0)_s \times \mathbf{x}_{0ss} + \varepsilon\ \mathbf{x}_{0ss}$, where $u^{(n-1)} \in W_2^{2+\alpha,1+\alpha/2}(\mathbb{R}_T)$. Then $u^{(n)}$ is well-defined for each n. Indeed, the imbedding theorem yields that a is Hölder continuous on $\mathbb{R}_T$ and $f \in W_2^{\alpha,\alpha/2}(\mathbb{R}_T)$. Thus from (3.6) we confirm

$$
\begin{aligned}
|u^{(n)}|_*^{(2+\alpha)} &\leqq c(T)|(u^{(n-1)} + \mathbf{x}_0)_s \times \mathbf{x}_{0ss} + \varepsilon\mathbf{x}_{0ss}|_*^{(\alpha)} \\
&\leqq c(T)(\sup_{s,t}|u^{(n-1)}{}_s| + 1)\|\mathbf{x}_{0ss}\|^{(\alpha)} \\
&\leqq c(T)(\delta|u^{(n-1)}|^{(2+\alpha)} + c_\delta|u^{(n-1)}| + \|\mathbf{x}_{0ss}\|^{(\alpha)}) \\
&\leqq c(T)(\delta|u^{(n-1)}|^{(2+\alpha)} + c_\delta T|u^{(n-1)}{}_t| + \|\mathbf{x}_{0ss}\|^{(\alpha)}) \\
&\leqq c(T)(\delta + c_\delta T)|u^{(n-1)}|_*^{(2+\alpha)} + c(T)\|\mathbf{x}_{0ss}\|^{(\alpha)},
\end{aligned}
$$

where δ is any positive constant and c_δ denotes a function with negative power of δ. Since we can choose sufficiently small $\delta = \delta_1$, $T = T_1$ such that $c(T_1)(\delta_1 + c_{\delta_1} T_1) < 1$, we have

$$
|u^{(n)}|_{*;\mathbb{R}_{T_1}}^{(2+\alpha)} \leqq c\|\mathbf{x}_{0ss}\|^{(\alpha)}. \tag{3.8}
$$

The norm in the left-hand side of this inequality is the same as that of (3.6) but the time T is replaced by T_1.

Next, subtracting the equation for $u^{(n-1)}$ from that for $u^{(n)}$ and setting $v^{(n)} = u^{(n)} - u^{(n-1)}$, we have

$$\begin{aligned} v^{(n)}{}_t =& u^{(n-1)}{}_s \times v^{(n)}{}_{ss} + \varepsilon v^{(n)}{}_{ss} + \mathbf{x}_{0s} \times v^{(n)}{}_{ss} \\ &+ v^{(n-1)}{}_s \times \mathbf{x}_{0ss} + v^{(n-1)}{}_s \times u^{(n-1)}{}_{ss}, \end{aligned}$$

$$v^{(n)}(s,0) = 0 .$$

Again by Proposition 3.2 and by (3.8) the function $v^{(n)}$ is estimated for any $t \in (0, T_1]$ as

$$\begin{aligned} |v^{(n)}|^{(2+\alpha)}_{*,\mathbb{R}_t} &\leqq c(t)|v^{(n-1)}{}_s \times \mathbf{x}_{0ss} + v^{(n-1)}{}_s \times u^{(n-1)}{}_{ss}|^{(\alpha)}_{*,\mathbb{R}_t} \\ &\leqq c(t)(\delta + c_\delta t)(1 + \| \mathbf{x}_{0ss} \|^{(\alpha)})|v^{(n-1)}|^{(2+\alpha)}_{*,\mathbb{R}_t} . \end{aligned}$$

If we choose $\delta = \delta_0$, $t = T_0 \leqq T_1$ so small that the coefficient of $|v^{(n-1)}|^{(2+\alpha)}_{*,\mathbb{R}_t}$ in the right-hand side is less than 1, then it follows that $\{u^{(n)}\}$ converges with respect to the norm $|\cdot|^{(2+\alpha)}_{*,\mathbb{R}_{T_0}}$ and (3.7) is satisfied with $\mathbf{X}^\varepsilon = \lim_{n\to\infty} u^{(n)}$ which is a solution of (3.1).

On the other hand, the uniqueness of the solution $\mathbf{X}^\varepsilon$ can be easily proved. □

Now, combining this proposition with Lemma 2.1, we establish

Theorem 3.1. *If* $0 < \varepsilon < 1$ *and* $\mathbf{x}_{0ss} \in W_2^2(\mathbb{R})$, *then for any* $T \in (0, \infty)$ *there exists a unique solution* $\mathbf{X}^\varepsilon$ *of* (3.1) *such that* $\mathbf{X}^\varepsilon \in W_2^{2+\alpha, 1+\alpha/2}(\mathbb{R}_T)$ $(1/2 < \alpha < 1)$.

Proof. We can easily confirm that if $\mathbf{X}^\varepsilon(\cdot, 0) = \mathbf{X}_0 \in W_2{}^{1+\alpha}(\mathbb{R})$ is nonzero in (3.1), then Proposition 3.3 is also valid with $c \| \mathbf{X}_0 \|^{(1+\alpha)}$ added to the right-hand side of (3.7). Hence from (2.1) we complete the proof. □

Besides, in the same way we obtain the following theorem related to (II).

Theorem 3.2. *If* $0 < \varepsilon < 1$, $\mathbf{y}_{ss} \in C^0(0, T; C_b{}^2(\mathbb{R}))$ *for any* $T \in (0, \infty)$, *and* $\mathbf{x}_0 - \mathbf{y}(\cdot, 0) \in W_2{}^2$ $(\mathbb{R})$, *then there exists a unique solution* $\mathbf{Y}^\varepsilon$ *of the initial value problem for* (2.14) *such that* $\mathbf{Y}^\varepsilon \in W_2^{2+\alpha, 1+\alpha/2}(\mathbb{R}_T)$ $(1/2 < \alpha < 1)$.

B. The Initial-Boundary Value Problem

Similarly to Part A, we first consider the system

$$\begin{aligned} &u_t = a \times u_{ss} + \varepsilon u_{ss} + f\,(s > 0,\ t > 0) , \\ &u(s,0) = 0, \quad u(0,t) = g(t), \quad g(0) = 0 , \end{aligned} \tag{3.9}$$

where a is a constant vector.

Let $H_\gamma^\alpha(\mathbb{R}^+)$ be the space whose norm is defined for an element g by

$$(\| g \|_\gamma{}^{(\alpha)})^2 = \begin{cases} \int_0^\infty e^{-2\gamma t}(\gamma^{2\alpha}|g|^2 + \int_0^\infty |D^{[\alpha]}g_0(t-\tau) \\ \qquad -D^{[\alpha]}g_0(t)|^2 \tau^{-1-2\alpha+2\,[\alpha]}\, d\tau)dt \quad \text{if } [\alpha] \neq \alpha , \\ \int_0^\infty e^{-2\gamma t}(\gamma^{2\alpha}|g|^2 + |D^\alpha g|^2)dt \quad \text{if } [\alpha] = \alpha , \end{cases}$$

where $g_0 = g$ (resp. 0) when $t > 0$ (resp. $t < 0$).

Proposition 3.4. *If* $0 < \alpha < 1, \varepsilon > 0, \gamma > 0, f \in H_\gamma^{\alpha,\alpha/2}(D)$ *and* $g \in H_\gamma^{3/4+\alpha/2}(\mathbb{R}^+)$, *then there exists a unique solution* u *of* (3.9) *such that* $u \in H_\gamma^{2+\alpha,1+\alpha/2}(D)$. *Moreover the following inequality is valid*:

$$|u|_{\gamma,D}^{(2+\alpha)} \leqq c(|f|_{\gamma,D}^{(\alpha)} + \| g \|_\gamma^{(3/4+\alpha/2)}) . \tag{3.10}$$

Proof. Since we can extend f as $f \in H_\gamma^{\alpha,\alpha/2}(\mathbb{R}_\infty)$, we get, for u' defined by the inverse transformation of the right-hand side of (3.5),

$$|u'|_{\gamma,D}^{(2+\alpha)} \leqq |u'|_{\gamma,\mathbb{R}_\infty}^{(2+\alpha)} \leqq c|f|_{\gamma,\mathbb{R}_\infty}^{(\alpha)} \leqq c|f|_{\gamma,D}^{(\alpha)} . \tag{3.11}$$

Then we have only to consider the problem for $u'' = u - u'$,

$$u''_t = a \times u''_{ss} + \varepsilon u''_{ss} \equiv A_0 u''_{ss} ,$$

$$u''(s, 0) = 0, \quad u''(0, t) = g(t) .$$

By the Laplace transformation like that in Definition 3.2 it becomes

$$\tilde{u}''_{ss}(s,\sigma) = \sigma A_0^{-1}\tilde{u}'', \quad \tilde{u}''(0, \sigma) = \tilde{g}(\sigma), \quad \tilde{u}'' \to 0 \ (s \to \infty) .$$

Noticing that the eigenvalues of σA_0^{-1} are $\sigma\varepsilon^{-1}$ and $\sigma(\varepsilon \pm i|a|)^{-1}$ yields

$$(P\tilde{u}'')_k = G_k(\sigma)\exp(-\lambda_k s) , \tag{3.12}$$

where

$$\lambda_1 = (\sigma\varepsilon^{-1})^{1/2}, \ \lambda_2 = \{\sigma(\varepsilon + i|a|)^{-1}\}^{1/2}, \ \lambda_3 = \{\sigma(\varepsilon - i|a|)^{-1}\}^{1/2} ,$$

both Arg $\sigma^{1/2}$ and Arg$(\varepsilon \pm i\,|\,a\,|)^{1/2}$ belong to $(-\pi/4, \pi/4)$, P is an orthogonal matrix, and $G_k = (P\tilde{g})_k$ (resp. $(P\tilde{u}'')_k$) denotes the k^{th} component of $P\tilde{g}$ (resp. $P\tilde{u}''$).

In order to estimate $\| u'' \|_{\gamma,D}^{(2+\alpha)}$ we need

Lemma 3.2. ([10, Lemma 3.1]) *Let* $e_k = \exp(-\lambda_k s)$, $\sigma = \gamma + i\zeta$, $\gamma > 0$, $j \geqq 0$, $\beta \in (0, 1)$. *Then the following inequalities are satisfied:*

$$\int_0^\infty |D_s^j e_k|^2 ds \leqq c|\lambda_k|^{2j-1} ,$$

$$\iint_D |D_s^j e_k(s+z) - D_s^j e_k(s)|^2 z^{-1-2\beta} ds\, dz \leqq c|\lambda_k|^{2j+2\beta-1} .$$

From this lemma and (3.12) it follows that

$$\begin{aligned}(\| u'' \|_{\gamma,D}^{(2+\alpha)})^2 \leqq \sum_{k=1}^3 \Bigg\{ & \sum_{j=0}^2 \int_{\mathbb{R}} |\sigma|^{2+\alpha-j} |G_k(\sigma)|^2 d\zeta \int_0^\infty |D_s^j e_k|^2 ds \\ & + \int_{\mathbb{R}} |G_k|^2 d\zeta \iint_D |D_s^2 e_k(s+z) \\ & - D_s^2 e_k(s)|^2 z^{-1-2\alpha} ds\, dz \Bigg\} \\ \leqq c & \int_{\mathbb{R}} |\sigma|^{3/2+\alpha} |\tilde{g}(\sigma)|^2 d\zeta .\end{aligned} \tag{3.13}$$

On the other hand, from the relation between $g(t)$ and $\tilde{g}(\sigma)$,

$$2\pi e^{-\gamma t} g = \int_{\mathbb{R}} e^{i\zeta t} \tilde{g}(\gamma + i\zeta) d\zeta ,$$

and the Plancherel theorem we have

$$(\| g \|_{\gamma}^{(3/4+\alpha/2)})^2 = (2\pi)^{-2} \int_{\mathbb{R}} (| \sigma |^{3/2+\alpha} + \gamma^{3/2+\alpha}) | \tilde{g} |^2 d\zeta .$$

Hence this equality together with (3.11), (3.13) and Lemma 3.1 yields the assertion of the proposition. □

On the basis of this proposition we obtain, similarly to Part A,

Proposition 3.5. *Assume* $1/2 < \alpha < 1$, $\varepsilon > 0$, $\mathbf{x}_0 \in W_2^{\,1+\alpha}(J)$, *$T$ is as in Proposition* 3.2 *and* $g_\pm \in W_2^{\,3/4+\alpha/2}(0, T)$. *Then for a certain* $T_0 \in (0, T]$ *there exists, in* $W_2^{2+\alpha,1+\alpha/2}(J_{T_0})$, *a unique solution of* (1.3) *with the conditions*

$$\mathbf{x}^\varepsilon(s,0) = \mathbf{x}_0(s), \quad \mathbf{x}^\varepsilon(\pm 1, t) = g_\pm(t), \quad \mathbf{x}_0(\pm 1) = g_\pm(0) .$$

Moreover the following inequality is valid:

$$|\mathbf{x}^\varepsilon|_{J_{T_0}}^{(2+\alpha)} \leqq c(\| \mathbf{x}_0 \|_J^{(1+\alpha)} + \| g_\pm \|_{(0,T)}^{(3/4+\alpha/2)}) .$$

Since the condition $\mathbf{x}^\varepsilon{}_{ss}(\pm 1, t) = 0$ of (III) is equivalent to $\mathbf{x}^\varepsilon(\pm 1, t) = \mathbf{x}_0(\pm 1)$ by (1.3), we obtain the theorem related to (III) from Lemma 2.3 and Proposition 3.5:

Theorem 3.3. *If* $0 < \varepsilon < 1$ *and* $\mathbf{x}_{0ss} \in L^2(J)$, *then for any* $T \in (0, \infty)$ *there exists, in* $W_2^{2+\alpha,1+\alpha/2}(J_T)$ $(1/2 < \alpha < 1)$, *a unique solution of* (1.3) *with the conditions* $\mathbf{x}^\varepsilon(s,0) = \mathbf{x}_0(s)$, $\mathbf{x}^\varepsilon(\pm 1, t) = \mathbf{x}_0(\pm 1)$.

4. Final Results

On the basis of the results in Sects. 2 and 3 we obtain

Theorem 4.1. *If* $\mathbf{x}_{0ss}$ *belongs to* $W_2^2(\mathbb{R})$, *then for any* $T \in (0, \infty)$ *there exists, in the sense of distribution, a solution* $\mathbf{X}(s,t) \in W_2^{2,1}(\mathbb{R}_T)$ *of* (3.1) *with* $\varepsilon = 0$.

Proof. From Lemma 2.1, Theorem 3.1 and Rellich's theorem ([8]), it follows that there exist a subsequence $\mathbf{X}^{\varepsilon_j}$ and its limit function $\mathbf{X}$ such that $\mathbf{X}^{\varepsilon_j}{}_{ss} \to \mathbf{X}_{ss}$, $\mathbf{X}^{\varepsilon_j}{}_t \to \mathbf{X}_t$ weakly in $L^2(\mathbb{R}_T)$ and $\mathbf{X}^{\varepsilon_j}{}_s \to \mathbf{X}_s$ strongly in $L^2(K_T)$ when $\varepsilon_j \to 0$, where $K_T = K \times (0, T)$ and K is an arbitrary compact subset in $\mathbb{R}$.

Then the function $\mathbf{X}$ satisfies (3.1) with $\varepsilon = 0$ in the sense of distribution. In fact,

$$\int_{\mathbb{R}} \int_0^T (\mathbf{X}_s \times \mathbf{X}_{ss}) \cdot \Phi \, ds \, dt$$

is well-defined for every three-dimensional vector-valued function Φ which belongs to $C^\infty(\mathbb{R}_T)$ and whose support is compact, and

$$\left|\int_{\mathbb{R}}\int_0^T (\mathbf{X}^{\varepsilon_j}{}_s \times \mathbf{X}^{\varepsilon_j}{}_{ss} - \mathbf{X}_s \times \mathbf{X}_{ss}) \cdot \Phi \, ds\,dt\right|$$

$$= \left|\iint_{\mathrm{supp}[\Phi]} \{(\mathbf{X}^{\varepsilon_j}{}_s - \mathbf{X}_s) \times \mathbf{X}^{\varepsilon_j}{}_{ss} + \mathbf{X}_s \times (\mathbf{X}^{\varepsilon_j}{}_{ss} - \mathbf{X}_{ss})\} \cdot \Phi \, ds\,dt\right|$$

$$\leqq \sup_{s,t}|\Phi||\mathbf{X}^{\varepsilon_j}{}_{ss}||\mathbf{X}^{\varepsilon_j}{}_s - \mathbf{X}_s|_{\mathrm{supp}[\Phi]}$$

$$+ \left|\int_{\mathbb{R}}\int_0^T (\Phi \times \mathbf{X}_s) \cdot (\mathbf{X}^{\varepsilon_j}{}_{ss} - \mathbf{X}_{ss}) ds\,dt\right|$$

$$\to 0 \quad \text{if } \varepsilon_j \to 0. \quad \square$$

Similarly to Theorem 4.1 we establish the following theorems.

Theorem 4.2. *If* $\mathbf{y}_{ss} \in C^0(0, T; C_b^2(\mathbb{R}))$ *for any* $T \in (0, \infty)$, *and* $\mathbf{x}_0 - \mathbf{y}(\cdot, 0) \in W_2^2(\mathbb{R})$, *then there exists, in the sense of distribution, a solution* $\mathbf{Y} \in W_2^{2,1}(\mathbb{R}_T)$ *of the initial value problem for* (2.14) *with* $\varepsilon = 0$.

Theorem 4.3. *If* $\mathbf{x}_{0ss} \in L^2(J)$, *then for any* $T \in (0, \infty)$ *there exists, in the sense of distribution, a solution* $\mathbf{x} \in W_2^{2,1}(J_T)$ *of* (1.1) *with the conditions* $\mathbf{x}(s, 0) = \mathbf{x}_0(s)$, $\mathbf{x}_{ss}(\pm 1, t) = 0$.

Remark 4.1. For a closed vortex filament (IV) we obtain the same a priori estimates with (2.15) and the propositions similar to those in Sect. 3-B neglecting g and $g_\pm$. Therefore the assertion in Theorem 4.3 is also valid with $\mathbf{x}(s-1, t) = \mathbf{x}(s+1, t)$ instead of $\mathbf{x}_{ss}(\pm 1, t) = 0$.

Remark 4.2. If a priori estimates are independent of ε in the class $W_2^{4,2}$, we shall be able to prove that the solution is unique and classical in any case of (I)–(IV). However, it seems to be difficult to find such estimates.

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