Commun Math Phys 161, 51–61 (1994)

Communications in
Mathematical
Physics

Callias' Index Theorem, Elliptic Boundary Conditions, and Cutting and Gluing

Johan Råde
Department of Mathematics, Stanford University, Stanford, CA 94305, USA
Email: rade@math stanford.edu

Received: 5 February 1993/in revised form: 15 June 1993

Abstract: It is shown that elliptic boundary conditions play the same role in Callias' index theorem as spectral boundary conditions do in the Atiyah–Patodi–Singer index theorem. This is used to generalize Callias' index theorem to arbitrary complete spin-manifolds.

1. The Index Formula

Let X be a complete odd-dimensional smooth oriented spin-manifold, with complex spinor bundle S. Let V be a smooth Hermitian vector bundle over X, with a smooth unitary connection A and a smooth Hermitian endomorphism Φ. Let $\partial\!\!\!/_A$ denote the coupled Dirac operator acting on sections of $S \otimes V$. Form the operators

$$D = \partial\!\!\!/_A + i1 \otimes \Phi$$

and

$$D^* = \partial\!\!\!/_A - i1 \otimes \Phi$$

acting on sections of $S \otimes V$.

The index problem for such operators was first studied by C. Callias [C], who proved an index theorem in the case $X = \mathbb{R}^{2n+1}$, using traces of integral kernels. Callias' index theorem can also be derived from Fedosov's index theorem for elliptic operators on Euclidean space [F], see also Sect. 19.3 in [H2], as explained in [BS], see also [A1]. Callias' index theorem was generalized by N. Anghel [A2] to Dirac operators coupled to the trivial Hermitian vector bundle over manifolds with warped cylindrical ends, using the relative index theory of [GL]. In this paper we generalize this index theorem to Dirac operators on arbitrary complete oriented spin-manifolds coupled to arbitrary Hermitian vector bundles.

Let $\lambda(x)$ denote the smallest of the absolute values of the eigenvalues of $\Phi(x)$. Let $\hat{A}$ denote the $\hat{A}$-genus.

Theorem 1. *If there exists a compact region X_0 in X, whose boundary is a smooth hypersurface $Y_0 = \partial X_0$, and a constant $\lambda_0 > 0$, such that $\lambda(x) \geqq \lambda_0$ for $x \in X - X_0$, and $|\nabla_A \Phi(x)| \to 0$ as $x \to \infty$, then D and D^* have finite dimensional L^2-kernels, and*

$$\dim L^2\text{-}\ker D - \dim L^2\text{-}\ker D^* = -\int_{Y_0} \hat{A}(Y_0) \wedge \mathrm{ch}(V^+) = \int_{Y_0} \hat{A}(Y_0) \wedge \mathrm{ch}(V^-)\,,$$

where V^+ denotes the span of the positive eigenvectors of Φ, and V^- denotes the span of the negative eigenvectors of Φ.

Note that the right-hand side is equal to minus the index of the chiral Dirac operator on Y_0 coupled to V^+. Sign conventions are discussed at the beginning of Sect. 2.

In Sect. 2 we prove a cutting and gluing lemma, Proposition 2.3. This lemma states that if we cut the manifold X along a codimension one submanifold, and impose the right boundary conditions, then the index does not change. In Sect. 3 we use this cutting and gluing lemma to prove Theorem 1.

The condition $|\nabla_A \Phi(x)| \to 0$ can be weakened. Let $\pi^+ : V \to V$ denote orthogonal projection onto V^+. It suffices that $|\nabla_A \pi^+| < \varepsilon_n \lambda_0$ on the complement of a compact subset, that could be larger than X_0, see Sect. 2. The constant ε_n depends on n only.

The index can also be interpreted as a Fredholm index, as explained in Sect. 2. In particular, if Φ, the curvature of the connection A, and the scalar curvature of X are bounded, then D is a Fredholm operator $L^{2,1}(X, S \otimes V) \to L^2(X, S \otimes V)$, with index as above.

In [R] we use this index theorem to compute the index of the anti-selfduality complex coupled to a singular Yang–Mills connection.

2. A Cutting and Gluing Lemma

Let X be a complete odd-dimensional smooth oriented spin-manifold, with complex spinor bundle S, and Clifford multiplication γ: $\mathrm{Cliff}(T^*X) \otimes S \to S$. In this section, we allow X to have compact boundary, $Y = \partial X$. We also allow X to be non-compact, as long as it is complete. We orient the boundary of X as follows: if $e_1, \ldots, e_{2n}$ is an orthonormal frame on Y and v is the outward unit normal, then $v, e_1, \ldots, e_{2n}$ is an orthonormal frame on X.

We use the normalization $e_i^2 = -1$. Let $\omega_X = i^{n+1} e_1 \ldots e_{2n+1}$, where $e_1, \ldots, e_{2n+1}$ is an orthonormal frame on X. Then ω_X is a global section of $\mathrm{Cliff}(T^*X)$, $\omega_X^2 = 1$ and ω_X lies in the center of $\mathrm{Cliff}(T^*X)$. We use the normalization $\gamma(\omega_X) = 1$.

The boundary Y is also a spin-manifold, and its spinor bundle and Clifford multiplication is the restriction of the spinor bundle S and Clifford multiplication γ on X. Now, Y is even-dimensional, so its spinor bundle splits into a positive and a negative spinor bundle,

$$S|_Y = S_+ \oplus S_-\,.$$

Let $\omega_Y = i^n e_1 \ldots e_{2n}$, where $e_i, \ldots, e_{2n}$ is an orthonormal frame on Y. Then ω_Y is a global section of $\mathrm{Cliff}(T^*Y)$, and $\omega_Y^2 = 1$. The positive and negative spinor bundles are the $+1$ and -1 eigenspaces of $\gamma(\omega_Y)$. By our orientation conventions, $\omega_X|_Y = iv\omega_Y$, and

$$1 = \gamma(\omega_X|_Y) = i\gamma(v)\gamma(\omega_Y)\,.$$

Hence the positive and negative spinor bundles are the $-i$ and $+i$ eigenspaces of $\gamma(\nu)$,

$$\begin{cases} \gamma(\nu)s_+ = -is_+ \\ \gamma(\nu)s_- = is_- \,. \end{cases} \tag{2.1}$$

Let $D = \partial\!\!\!/_A + i1 \otimes \Phi$ be an operator as in Theorem 1, except that we now allow X to have a compact boundary Y. For simplicity, we write $D = \partial\!\!\!/ + i\Phi$. It is an elliptic operator. It admits well known elliptic (coercive, Lopatinski type) boundary conditions, see for instance [S]. One such boundary condition is $s_+ = 0$, where $s|_Y = s_+ + s_-$ is the decomposition of s according to the splitting $(S \otimes V)|_Y = (S_+ \otimes V) \oplus (S_- \otimes V)$. The adjoint boundary condition is $s_- = 0$.

To verify that the boundary condition $s_+ = 0$ is elliptic, freeze the coefficients of the operators at a point on the boundary, to get the boundary value problem

$$\begin{cases} \partial\!\!\!/ s = 0 & \text{on } (-\infty, 0] \times \mathbb{R}^{2n} \\ s_+ = 0 & \text{on } \{0\} \times \mathbb{R}^{2n} \,. \end{cases}$$

One has to check that there are no bounded non-zero solutions of the form

$$s(t, x) = s(t)\exp(i\xi \cdot x)$$

with $\xi \neq 0$. This leads to a system of ordinary differential equations,

$$\begin{cases} \gamma(\nu)\partial_t s(t) + i\gamma(\xi)s(t) = 0 & \text{for } t \leqq 0 \\ s^+(0) = 0 \,, \end{cases}$$

where ν denotes the positive unit vector field on $(-\infty, 0]$. Extend the splitting into positive and negative spinors on $\{0\} \times \mathbb{R}^{2n}$ to $[0, \infty) \times \mathbb{R}^{2n}$. By (2.1), the system can then be written

$$\begin{cases} \partial_t s_+(t) - \gamma(\xi)s_-(t) = 0 & \text{for } t \leqq 0 \\ \partial_t s_+(t) + \gamma(\xi)s_-(t) = 0 & \text{for } t \leqq 0 \\ s_+(0) = 0 \,. \end{cases}$$

It follows that

$$\partial_t^2 s_+(t) = -\gamma^2(\xi)s_+(t) = |\xi|^2 s_+(t) \,.$$

If $s_+(0) = 0$ and $s_+(t)$ is bounded, then $s_+(t) = 0$. By the first equation, $\gamma(\xi)s_-(t) = 0$, so $s_-(t) = 0$, and $s(t) = 0$. We conclude that $s_+ = 0$ is an elliptic boundary condition for D. See for instance Sect. 10 in [H1] for more details on elliptic boundary value problems.

One can also form twisted boundary conditions. Let

$$V|_Y = V^+ \oplus V^-$$

be any splitting of $V|_Y$, not necessarily related to Φ. Then we can decompose s as $s = s^+ + s^-$ on Y according to the splitting $(S \otimes V)|_Y = (S \otimes V^+) \oplus (S \otimes V^-)$. We can also decompose s as $s = s_+^+ + s_+^- + s_-^+ + s_-^-$ according to the splitting

$$(S \otimes V)|_Y = (S_+ \otimes V^+) \oplus (S_+ \otimes V^-) \oplus (S_- \otimes V^+) \oplus (S_- \otimes V^-) \,.$$

Then

$$Bs = s_+^+ + s_-^- = 0 \quad \text{on } Y$$

is an elliptic boundary condition for D. The adjoint boundary condition is

$$B^*s = s_+^- + s_-^+ = 0 \quad \text{on } Y.$$

Sometimes we write B_Y or even $B_{X,Y}$ for B.

We need function spaces where D, and more generally, if $\partial X = Y \neq \emptyset$, $D \oplus B$, is Fredholm. Let η be a smooth compactly supported cut-off function on X that is equal to 1 near Y. Then we define $L_D^{2,1}(X, S \otimes V)$ as the completion of $C_0^\infty(X, S \otimes V)$ with respect to the norm

$$\|\eta \nabla s\|_{L^2(X)} + \|(1-\eta) Ds\|_{L^2(X)} + \|s\|_{L^2(X)}.$$

It follows from standard elliptic theory that this norm is equivalent to the norm

$$\|Ds\|_{L^2(X)} + \|Bs\|_{L^{2,1/2}(Y)} + \|s\|_{L^2(X)}.$$

(It is unavoidable to introduce Sobolev spaces with fractional derivatives. Functions in $L^{2,1}$ have boundary values in $L^{2,1/2}$.)

It follows from the Bochner–Weitzenböck formula,

$$\not{\partial}^2 s = \nabla^* \nabla s + \sum_{i,j} F_{ij} \gamma(e_i)\gamma(e_j) s + \frac{1}{4} Rs,$$

that if Φ, the curvature F of the connection A, and the scalar curvature R of X are bounded, then $L_D^{2,1}(X, S \otimes V) = L^{2,1}(X, S \otimes V)$.

Lemma 2.1. *There exists a constant $\varepsilon_n > 0$ with the following significance. If there exists $\lambda_0 > 0$ such that $\lambda(x) \geqq \lambda_0$ and $|\nabla_A \pi^+(x)| \leqq \varepsilon_n \lambda_0$ for all $x \in X$, and the boundary operator B is given by splitting $V = V^+ \oplus V^-$ into positive and negative eigenspaces of Φ, then $D \oplus B$ defines an invertible operator*

$$L_D^{2,1}(X, S \otimes V) \to L^2(X, S \otimes V) \oplus L^{2,1/2}(Y, (S_+ \otimes V^+) \oplus (S_- \otimes V^-)).$$

Proof. Let $s \in L_D^{2,1}(X)$. Recall that $\not{\partial} = \sum_i \gamma(e_i)\nabla_i$. Hence, by integration by parts and (2.1),

$$\int_X \not{\partial} s \cdot s \, d\mathrm{vol} = \int_X s \cdot \not{\partial} s \, d\mathrm{vol} + \int_Y \gamma(v) s \cdot s \, d\mathrm{vol}$$

$$= \int_X \overline{\not{\partial} s \cdot s} \, d\mathrm{vol} - \int_Y i(|s_+|^2 - |s_-|^2) \, d\mathrm{vol}.$$

(The integration by parts can be justified by multiplying by a sequence cut-off functions that exhaust X.) Take the imaginary part of this identity, to get

$$\mathrm{Im} \int_X \not{\partial} s \cdot s \, d\mathrm{vol} = -\frac{1}{2} \int_Y (|s_+|^2 - |s_-|^2) \, d\mathrm{vol}.$$

Hence

$$\mathrm{Im} \int_X Ds \cdot s \, d\mathrm{vol} = \int_X \Phi s \cdot s \, d\mathrm{vol} - \frac{1}{2} \int_Y (|s_+|^2 - |s_-|^2) \, d\mathrm{vol}.$$

Substitute s^+ for s, to get

$$\lambda_0 \int_X |s^+|^2 \, d\mathrm{vol} \leqq \int_X \Phi s^+ \cdot s^+ \, d\mathrm{vol} \leqq \mathrm{Im} \int_X Ds^+ \cdot s^+ \, d\mathrm{vol} + \frac{1}{2} \int_Y |s_+^+|^2 \, d\mathrm{vol}.$$

Similarly,

$$\lambda_0 \int_X |s^-|^2 \, d\text{vol} \leqq - \int_X \Phi s^- \cdot s^- \, d\text{vol} \leqq - \operatorname{Im} \int_X Ds^- \cdot s^- \, d\text{vol} + \frac{1}{2} \int_Y |s^-|^2 \, d\text{vol} .$$

It follows that

$$\begin{aligned} \lambda_0 \int_X |s|^2 \, d\text{vol} &\leqq \operatorname{Im} \int_X (Ds^+ \cdot s^+ - Ds^- \cdot s^-) \, d\text{vol} + \frac{1}{2} \int_Y |Bs|^2 \, d\text{vol} \\ &= \operatorname{Im} \int_X Ds \cdot (s^+ - s^-) \, d\text{vol} + \frac{1}{2} \int_Y |Bs|^2 \, d\text{vol} \\ &\quad + \int_X (Ds^+ \cdot s^- - Ds^- \cdot s^+) \, d\text{vol} . \end{aligned}$$

Now,

$$\begin{aligned} Ds^+ \cdot s^- &= \partial\!\!\!/ s^+ \cdot s^- + \Phi s^+ \cdot s^- \\ &= \sum_i \gamma(e_i) \nabla_i (\pi^+ s^+) \cdot s^- = \sum_i \gamma(e_i) (\nabla_i \pi^+) s^+ \cdot s^- , \end{aligned}$$

so

$$|Ds^+ \cdot s^-| \leqq c_n |\nabla \pi^+| \, |s^+| \, |s^-| \leqq c_n \varepsilon_n \lambda_0 |s|^2 .$$

The same bound applies to $Ds^- \cdot s^+$. Hence, if we choose $\varepsilon_n = (4c_n)^{-1}$, then

$$\lambda_0 \int_X |s|^2 \, d\text{vol} \leqq 2 \operatorname{Im} \int_X Ds \cdot (s^+ - s^-) \, d\text{vol} + \int_Y |Bs|^2 \, d\text{vol} .$$

By Cauchy's inequality,

$$\| s \|_{L^2(X)} \leqq 4\lambda_0^{-2} \| Ds \|_{L^2(X)} + 2\lambda_0^{-1} \| Bs \|_{L^2(Y)} .$$

Hence

$$\| s \|_{L_D^{2,1}(X)} \leqq (4\lambda_0^{-2} + 1) \| Ds \|_{L^2(X)} + (2\lambda_0^{-1} + 1) \| Bs \|_{L^{2,1/2}(Y)} .$$

This estimate shows that $D \oplus B$ has trivial kernel and closed range.

By integration by parts, the orthogonal complement of the range of $D: L_D^{2,1}(X) \cap \ker B \to L^2(X)$ is the L^2-kernel of $D^* \oplus B^*$. The same way as above, one shows that the L^2-kernel of $D^* \oplus B^*$ is trivial. Hence $D: L_D^{2,1}(X) \cap \ker B \to L^2(X)$ has dense range. Hence $D \oplus B: L_D^{2,1}(X) \to L^2(X) \oplus L^{2,1/2}(Y)$ has dense range. Hence $D \oplus B$ is invertible. □

We can use Lemma 2.1 to prove a more general Fredholm result. Let ε_n be as in Lemma 2.1.

Lemma 2.2. *If there exists a compact subset X_0 of X and a constant $\lambda_0 > 0$, such that $\lambda(x) \geqq \lambda_0$ and $|\nabla_A \pi^+(x)| \leqq \varepsilon_n \lambda_0$ for all $x \in X - X_0$, then the operator*

$$D \oplus B: L_D^{2,1}(X, S \otimes V) \to L^2(X, S \otimes V) \oplus L^{2,1/2}(Y, (S_+ \otimes V^+) \oplus (S_- \otimes V^-))$$

is Fredholm. The Fredholm index of $D \oplus B$ is equal to the L^2-index

$$\dim L^2\text{-}\ker(D \oplus B) - \dim L^2\text{-}\ker(D^* \oplus B^*) .$$

Here B can be given by any splitting $V|_{\partial X} = V^+ \oplus V^-$.

Proof. After enlarging X_0, we may assume that it is bounded by a smooth hypersurface, and that $\partial X \subset \partial X_0$. Let $X_\infty = X - X_0$. Enlarge X_∞ slightly, so that $\{X_0, X_\infty\}$ is a good cover of X.

The splitting $V = V^+ \oplus V^-$ is only defined on ∂X. We extend the splitting to $\partial X_0 - \partial X$ in some arbitrary way. We extend the splitting to ∂X_∞ by letting V^+ and V^- be the span of the positive and negative eigenspaces of Φ respectively.

By standard elliptic theory $D_{X_0} \oplus B_{\partial X_0}$ has a parametrix P_0. By Lemma 2.1, $D_{X_\infty} \oplus B_{\partial X_\infty}$ has an inverse P_∞. It is then standard to construct a parametrix P for $D \oplus B$ using P_0, P_∞, and a partition of unity subordinate to the cover $\{X_0, X_\infty\}$.

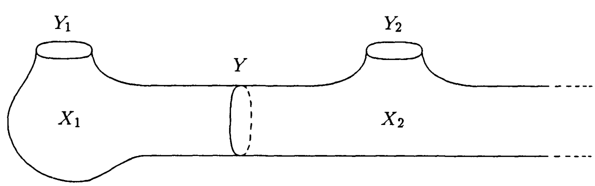
It is clear that the $L_D^{2,1}$-kernel of $D \oplus B$ is equal to the L^2-kernel of $D \oplus B$. The cokernel of $D \oplus B$ can be identified with the cokernel of $D\colon L_D^{2,1}(X) \cap \ker B \to L^2(X)$. Integration by parts shows that the orthogonal complement of the image of this operator is the L^2-kernel of $D^* \oplus B^*$. □

Now, let X be a manifold as above, that has been obtained by gluing two complete spin-manifolds X_1 and X_2, with boundaries $\partial X_1 = Y \cup Y_1$ and $\partial X_2 = \bar{Y} \cup Y_2$, along Y. (Here $\bar{Y}$ denotes Y with the opposite orientation.) We write $X = X_1 \cup_Y X_2$. Then $\partial X = Y_1 \cup Y_2$. We write D_X for the operator D on X, and D_{X_i} for its restriction to X_i.

Proposition 2.3.

$$\operatorname{index}(D_X \oplus B_{X, Y_1 \cup Y_2}) = \operatorname{index}(D_{X_1} \oplus B_{X_1, Y \cup Y_1}) + \operatorname{index}(D_{X_2} \oplus B_{X_2, \bar{Y} \cup Y_2}) .$$

Here B can be given by any splitting $V|_{Y \cup Y_1 \cup Y_2} = V^+ \oplus V^-$.



Proof. For simplicity, we assume that $Y_1 = Y_2 = \emptyset$. This is only to simplify the notation; the general case is handled the same way.

By Lemma 2.1, $D_{X_1} \oplus D_{X_2} \oplus B_{X_1, Y} \oplus B_{X_2, \bar{Y}}$ is a Fredholm operator

$$\begin{aligned} L_D^{2,1}(X_1, S \otimes V) \oplus L_D^{2,1}(X_2, S \otimes V) \to {} & L^2(X_1, S \otimes V) \oplus L^2(X_2, S \otimes V) \\ & \oplus L^{2,1/2}(Y, (S_+ \otimes V^+) \oplus (S_- \otimes V^-)) \\ & \oplus L^{2,1/2}(\bar{Y}, (S_+ \otimes V^+) \oplus (S_- \otimes V^-)) . \end{aligned}$$

If we identify Y and $\bar{Y}$, then

$$L^2(\bar{Y}, (S_+ \otimes V^+) \oplus (S_- \otimes V^-)) = L^2(Y, (S_- \otimes V^+) \oplus (S_+ \otimes V^-)) .$$

Hence the boundary operators $B_{X_1, Y}$ and $B_{X_2, \bar{Y}}$ can be subtracted to form one operator

$$B^{(0)}\colon L_D^{2,1}(X_1, S \otimes V) \oplus L_D^{2,1}(X_2, S \otimes V) \to L^{2,1/2}(Y, S \otimes V)$$

$$B^{(0)} = B_{X_1, Y} - B_{X_2, \bar{Y}} .$$

More generally, we define

$$B^{(\theta)}\colon L_D^{2,1}(X_1, S\otimes V)\oplus L_D^{2,1}(X_2, S\otimes V)\to L^{2,1/2}(Y, S\otimes V)$$

$$B^{(\theta)} = B_{X_1,Y} - B_{X_2,\bar{Y}} + \theta(B^*_{X_1,Y} - B^*_{X_2,\bar{Y}})\,.$$

The proposition follows from the following three facts.

a. The operators $D_{X_1}\oplus D_{X_2}\oplus B^{(\theta)}$ form a continuous (in the norm topology) family of Fredholm operators

$$L_D^{2,1}(X_1, S\otimes V)\oplus L_D^{2,1}(X_2, S\otimes V)$$
$$\to L^2(X_1, S\otimes V)\oplus L^2(X_2, S\otimes V)\oplus L^{2,1/2}(Y, S\otimes V)\,,$$

parametrized by $\theta\in\mathbb{R}$.

b. $\operatorname{index}(D_{X_1}\oplus D_{X_2}\oplus B^{(0)}) = \operatorname{index}(D_{X_1}\oplus B_{X_1,Y}) + \operatorname{index}(D_{X_2}\oplus B_{X_2,\bar{Y}})\,.$

c. $\operatorname{index}(D_{X_1}\oplus D_{X_2}\oplus B^{(1)}) = \operatorname{index}(D_X)$.

Proof of b. Essentially, $B^{(0)} = B_{X_1,Y}\oplus B_{X_2,\bar{Y}}$.

Proof of c. Let $s_1\oplus s_2\in L_D^{2,1}(X_1)\oplus L_D^{2,1}(X_2)$. Then $B^{(1)}(s_1\oplus s_2) = s_1|_Y - s_2|_{\bar{Y}}$. The sections s_1 and s_2 can be glued to form a section $s\in L_D^{2,1}(X)$ if and only if $B^{(1)}(s_1\oplus s_2) = 0$. Hence we have a commutative diagram with exact rows,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_D^{2,1}(X) & \longrightarrow & L_D^{2,1}(X_1)\oplus L_D^{2,1}(X_2) & \xrightarrow{B^{(1)}} & L^{2,1/2}(Y) & \longrightarrow & 0 \\ & & \downarrow D_X & & \downarrow D_{X_1}\oplus D_{X_2}\oplus B^{(1)} & & \downarrow & & \\ 0 & \longrightarrow & L^2(X) & \longrightarrow & L^2(X_1)\oplus L^2(X_2)\oplus L^{2,1/2}(Y) & \longrightarrow & L^{2,1/2}(Y) & \longrightarrow & 0\,. \end{array}$$

The claim follows from this diagram.

Proof of a. This is a boundary value problem where the boundary conditions mix different boundary components. Such boundary value problems are known as transmission problems. By identifying a neighborhood of Y in X_1 with a neighborhood of $\bar{Y}$ in $\bar{X}_2$ one gets a elliptic boundary condition in the usual sense. Thus the ellipticity (coerciveness, Lopatinski) condition is still meaningful, and ellipticity implies the existence of local parametrices near the boundary. See for instance [H1] p. 274.

Freeze the coefficients at a point on Y, to get the transmission problem

$$\begin{cases} \not{\partial} s_1 = 0 & \text{on } (-\infty, 0]\times\mathbb{R}^{2n} \\ \not{\partial} s_2 = 0 & \text{on } [0,\infty)\times\mathbb{R}^{2n} \\ (s_1)_+ - \theta\cdot(s_2)_+ = 0 & \text{on } \{0\}\times\mathbb{R}^{2n} \\ (s_2)_- - \theta\cdot(s_1)_- = 0 & \text{on } \{0\}\times\mathbb{R}^{2n}\,. \end{cases}$$

To verify ellipticity, one has to check that there are no bounded non-zero solutions of the form

$$\begin{cases} s_1(t,x) = s_1(t)\exp(i\xi\cdot x) & \text{for } t\leqq 0 \\ s_2(t,x) = s_2(t)\exp(i\xi\cdot x) & \text{for } t\geqq 0 \end{cases}$$

with $\xi \neq 0$. This leads to a system of ordinary differential equations,

$$\begin{cases} \gamma(v)\partial_t s_1(t) + i\gamma(\xi)s_1(t) = 0 & \text{for } t \leqq 0 \\ \gamma(v)\partial_t s_2(t) + i\gamma(\xi)s_2(t) = 0 & \text{for } t \geqq 0 \\ s_1(0)_+ - \theta s_2(0)_+ = 0 \\ s_2(0)_- - \theta s_1(0)_- = 0 \,. \end{cases}$$

It is not hard to check that there are no bounded non-zero solutions with $\xi \neq 0$.

Hence there exists a parametrix on a neighborhood of Y. Then one can argue as in the proof of Lemma 2.2 to show that the operator is Fredholm. □

Remark 2.4. We have only considered the boundary operator B. However, Lemma 2.2 and Proposition 2.3 apply *mutatis mutandis* to B^*, for B^* is simply B with V^+ and V^- switched.

3. Proof of the Index Theorem

Proof of Theorem 1. By Lemma 2.2, the operator

$$D\colon L^{2,1}_{D}(X, S \otimes V) \to L^2(X, S \otimes V)$$

is Fredholm. Let $X_\infty = X - X_0$. Then $\partial X_\infty = \bar{Y}_0$. By Proposition 2.3,

$$\operatorname{index} D \equiv \operatorname{index} D_X = \operatorname{index}(D_{X_0} \oplus B_{Y_0}) + \operatorname{index}(D_{X_\infty} \oplus B_{\bar{Y}_0}) \,.$$

Now,

$$\operatorname{index}(D_{X_0} \oplus B_{Y_0}) = \operatorname{index}(\not{\partial}_{X_0} \oplus B_{Y_0}) \,,$$

for a lower order term Φ does not affect the index on a compact manifold X_0. Also, $|\nabla_A \pi^+| \leqq \lambda^{-1} |\nabla_A \Phi| \to 0$ on the ends of X. Hence we can deform the connection A smoothly on a compact set, so that $|\nabla_A \pi^+| \leqq \varepsilon_n \lambda_0$ on $X - X_0$. Then, by Lemma 2.1,

$$\operatorname{index}(D_{X_\infty} \oplus B_{\bar{Y}_0}) = 0 \,.$$

Hence

$$\operatorname{index} D = \operatorname{index}(\not{\partial}_{X_0} \oplus B_{Y_0}) \,. \tag{3.1}$$

Thus the index problem on X with L^2-boundary conditions is equivalent to an index problem on X_0 with elliptic boundary conditions. In other words, elliptic boundary conditions play the same role in Callias' index theorem as spectral boundary conditions do in the Atiyah–Patodi–Singer index theorem [APS].

There are many ways to compute the index of $D_{X_0} \oplus B_{Y_0}$. One could use the Atiyah–Bott index theorem [AB] for elliptic boundary value problems. One can also argue as follows. By Lemma 2.2,

$$\operatorname{index}(\not{\partial}_{X_0} \oplus B_{Y_0}) + \operatorname{index}(\not{\partial}_{X_0} \oplus B^*_{Y_0}) = 0 \,. \tag{3.2}$$

On the other hand, it follows from the Agranovich–Dynin difference theorem, [AD] Theorem 2, that

$$\operatorname{index}(\not{\partial}_{X_0} \oplus B_{Y_0}) - \operatorname{index}(\not{\partial}_{X_0} \oplus B^*_{Y_0}) = \operatorname{index}(\not{\partial}_{Y_0})^{+}_{-} + \operatorname{index}(\not{\partial}_{Y_0})^{-}_{+} \,, \tag{3.3}$$

where

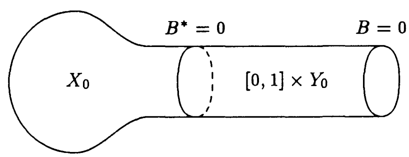
$$(\not{\partial}_{Y_0})^{\pm}_{+}: S_+ \otimes V^{\pm} \to S_- \otimes V^{\pm} ,$$
$$(\not{\partial}_{Y_0})^{\pm}_{-}: S_- \otimes V^{\pm} \to S_+ \otimes V^{\pm}$$

denote the chiral Dirac operators on Y_0 coupled to $V^{\pm}$.

This can also be seen as follows. We can assume that the metric on X_0 is a product metric on a collar $[-\delta, 0] \times Y_0$ of Y_0. We can assume that the bundle has a fixed product structure on $[-\delta, 0] \times Y_0$, that the connection A is the pullback of a connection on Y_0, and that this connection is reducible with respect to the splitting $V = V^+ \oplus V^-$. Form a smooth manifold $X_0 \cup_{Y_0} ([0, 1] \times Y_0)$ by gluing $Y_0 = \partial X_0$ to $\{0\} \times \bar{Y}_0$. Note that

$$\partial([0, 1] \times Y_0) = (\{0\} \times \bar{Y}_0) \cup (\{1\} \times Y_0) .$$

The bundle V and the connection A extend naturally to $X_0 \cup_{Y_0} ([0, 1] \times Y_0)$.



We can continuously deform X_0 to $X_0 \cup_{Y_0} ([0, 1] \times Y_0)$ along with the operators, so

$$\mathrm{index}(\not{\partial}_{X_0} \oplus B_{Y_0}) = \mathrm{index}\,(\not{\partial}_{X_0 \cup_{Y_0}([0,1] \times Y_0)} \oplus B_{\{1\} \times Y_0}) .$$

By Proposition 2.3 and Remark 2.4,

$$\mathrm{index}(\not{\partial}_{X_0 \cup_{Y_0}([0,1] \times Y_0)} \oplus B_{\{1\} \times Y_0})$$
$$= \mathrm{index}(\not{\partial}_{X_0} \oplus B^*_{Y_0}) + \mathrm{index}(\not{\partial}_{[0,1] \times X_0} \oplus B^*_{\{0\} \times \bar{Y}_0} \oplus B_{\{1\} \times Y_0}) .$$

Hence,

$$\mathrm{index}(\not{\partial}_{X_0} \oplus B_{Y_0}) - \mathrm{index}(\not{\partial}_{X_0} \oplus B^*_{Y_0}) = \mathrm{index}(\not{\partial}_{[0,1] \times Y_0} \oplus B^*_{\{0\} \times \bar{Y}_0} \oplus B_{\{1\} \times Y_0}) .$$

We will now show that the kernel and cokernel of the operator on the right-hand side is given by harmonic spinors on Y_0. The splittings $S = S_+ \oplus S_-$ and $V = V^+ \oplus V^-$ on Y_0 extend naturally to $[0, 1] \times Y_0$. Note that with this notation, S_+ is the negative spinor bundle on $\{0\} \times \bar{Y}_0$ and vice versa. Separate variables on $[0, 1] \times Y_0$. Let t be the $[0, 1]$-coordinate. Let v denote the unit vector field in the t-direction; it is the natural extension of the outward unit normal vector field v on $\{1\} \times Y_0$. Then $\not{\partial}_{[0,1] \times Y_0} = \gamma(v)\partial_t + \not{\partial}_{Y_0}$. By (2.1), for $s = s^+_+ \oplus s^-_+ \oplus s^+_- \oplus s^+_-$, we have

$$\begin{aligned} \not{\partial} s = (\not{\partial}_{Y_0} s^{\pm}_{-} - i\partial_t s^{+}_{+}) \oplus (\not{\partial}_{Y_0} s^{-}_{-} - i\partial_t s^{-}_{+}) \\ \oplus (\not{\partial}_{Y_0} s^{+}_{+} + i\partial_t s^{+}_{-}) \oplus (\not{\partial}_{Y_0} s^{-}_{+} + i\partial_t s^{-}_{-}) . \end{aligned} \tag{3.4}$$

The boundary condition is tantamount to $s^+_+ = s^-_- = 0$ on $\partial([0, 1] \times Y_0)$. (On $\{0\} \times \bar{Y}_0$ we have imposed a B^*-condition, but S_+ is the negative spin bundle on $\{0\} \times \bar{Y}_0$ and vice versa.) Hence, if s^-_+ is a harmonic $S_+ \otimes V^-$-spinor on Y_0, then its pull back to $[0, 1] \times Y_0$ is in the kernel. The same holds for harmonic $S_- \otimes V^+$-spinors on Y_0. I claim that this gives the whole kernel.

To see this, assume that s is in the kernel. Then $\not{\partial}^2 s^+_+ = (\not{\partial}^2 s)^+_+ = 0$ on $[0, 1] \times Y_0$. Since $s^+_+ = 0$ on $\partial[0, 1] \times Y_0$, we can integrate by parts to get

$$0 = \int_{[0,1]\times Y_0} |\not{\partial}_{[0,1]\times Y_0} s^+_+|^2 \, d\mathrm{vol} = \int_{[0,1]\times Y_0} s^+_+ \cdot \not{\partial}^2_{[0,1]\times Y_0} s^+_+ \, d\mathrm{vol}$$

$$= \int_{[0,1]\times Y_0} s^+_+ \cdot (-\partial_t^2 + \not{\partial}^2_{Y_0}) s^+_+ \, d\mathrm{vol} = \int_{[0,1]\times Y_0} (|\partial_t s^+_+|^2 + |\not{\partial}_{Y_0} s^+_+|^2) \, d\mathrm{vol} .$$

Hence $\partial_t s^+_+ = 0$ on $[0, 1] \times Y_0$. By the boundary condition, $s^+_+ = 0$ on $\partial([0, 1] \times Y_0)$. Hence $s^+_+ = 0$ on $[0, 1] \times Y_0$. Similarly $s^-_- = 0$ on $[0, 1] \times Y_0$. It then follows from (3.4) that $\not{\partial}_{Y_0} s = 0$ and $\partial_t s = 0$, and the claim follows.

To summarize,

$$\ker(\not{\partial}_{[0,1]\times Y_0} \oplus B^*_{\{0\}\times \bar{Y}_0} \oplus B_{\{1\}\times Y_0}) \cong \ker(\not{\partial}_{Y_0})^-_+ \oplus \ker(\not{\partial}_{Y_0})^+_- .$$

Similarly,

$$\ker(\not{\partial}_{[0,1]\times Y_0} \oplus B_{\{0\}\times \bar{Y}_0} \oplus B^*_{\{1\}\times Y_0}) \cong \ker(\not{\partial}_{Y_0})^+_+ \oplus \ker(\not{\partial}_{Y_0})^-_- .$$

Hence,

$$\begin{aligned}
&\mathrm{index}(\not{\partial}_{[0,1]\times Y_0} \oplus B^*_{[0,1]\times X_0,\{0\}\times \bar{Y}_0} \oplus B_{[0,1]\times X_0,\{1\}\times Y_0}) \\
&\quad = \dim \ker(\not{\partial}_{[0,1]\times Y_0} \oplus B^*_{\{0\}\times \bar{Y}_0} \oplus B_{\{1\}\times Y_0}) \\
&\qquad - \dim \ker(\not{\partial}_{[0,1]\times Y_0} \oplus B_{\{0\}\times \bar{Y}_0} \oplus B^*_{\{1\}\times \bar{Y}_0}) \\
&\quad = \dim \ker(\not{\partial}_{Y_0})^-_+ + \dim \ker(\not{\partial}_{Y_0})^+_- - \dim \ker(\not{\partial}_{Y_0})^+_+ - \dim \ker(\not{\partial}_{Y_0})^-_- \\
&\quad = \mathrm{index}(\not{\partial}_{Y_0})^-_+ + \mathrm{index}(\not{\partial}_{Y_0})^+_- ,
\end{aligned}$$

and we have proven (3.3).

By the Atiyah–Singer index theorem,

$$\begin{cases} \mathrm{index}(\not{\partial}_{Y_0})^+_+ = -\,\mathrm{index}(\not{\partial}_{Y_0})^+_- = \int_{Y_0} \hat{A}(Y_0) \wedge \mathrm{ch}(V^+) \\ \mathrm{index}(\not{\partial}_{Y_0})^-_+ = -\,\mathrm{index}(\not{\partial}_{Y_0})^-_- = \int_{Y_0} (\hat{A}(Y_0) \wedge \mathrm{ch}(V^-) . \end{cases} \tag{3.5}$$

Finally, the forms $\hat{A}(Y_0)$ and $\mathrm{ch}(V)$ extend as closed forms across X_0. By Stokes' theorem

$$\int_{Y_0} \hat{A}(Y_0) \wedge (\mathrm{ch}(V^+) + \mathrm{ch}(V^-)) = \int_{Y_0} \hat{A}(Y_0) \wedge \mathrm{ch}(V) = 0 . \tag{3.6}$$

The theorem now follows from (3.1)–(3.3), (3.5), and (3.6). □

Acknowledgements. The author wishes to thank Dan Freed, Lars Hörmander, Rafe Mazzeo, Tom Mrowka and Karen Uhlenbeck for interesting discussions and valuable comments.

Note. Theorem 1 has been obtained independently by N. Anghel [A3].

References

[A1] Anghel, N.: Remark on Callias' index theorem. Rep. Math. Phys. **28**, 1–6 (1988)
[A2] Anghel, N.: L^2-index formulae for perturbed Dirac operators. Commun. Math. Phys. **128**, 77–97 (1990)

[A3] Anghel, N.: The index of Callias-type operators. Preprint, University of North Texas

[AD] Agranovich, M.S , Dynin, A.S : General boundary value problems for elliptic systems in an n-dimensional domain. Dokl. Akad. Nauk SSSR Ser. Mat. **146**, 511–514 (1962) [English transl.: Soviet Math. Dokl. **3**, 1323–1327 (1962)

[AB] Atiyah, M F , Bott, R : The index theorem for manifolds with boundary. In: Differential Analysis (Bombay 1964). Oxford: Oxford University Press, 1964, pp. 175–186

[APS] Atiyah, M.F., Patodi, V.K , Singer, I.M.: Spectral asymmetry and Riemannian geometry I Math. Proc. Camb. Phil. Soc. **77**, 43–69 (1975)

[BS] Bott, R., Seeley, R : Some remarks on the paper of Callias. Commun. Math. Phys. **62**, 236–245 (1978)

[C] Callias, C.: Axial anomalies and index theorems on open spaces. Commun Math. Phys. **62**, 213–234 (1978)

[F] Fedosov, B.V.: Analytic formulas for the index of elliptic operators. Trudy Mosk Mat. Obšč. **30**, 159–241 (1974) [English transl.: Trans. Mosc. Math. Soc. **30**, 159–240 (1974)

[GL] Gromov, M., Lawson, H.B.: Positive scalar curvature and the Dirac operator. Publ. Math IHES **58**, 83–196 (1983)

[H1] Hörmander, L.: Linear Partial Differential Operators. Grundl. math. Wiss. vol. **116**, Berlin, Heidelberg, New York: Springer 1964

[H2] Hörmander, L.: The Analysis of Linear Partial Differential Operators III. Grundl. math. Wiss vol **274**, Berlin, Heidelberg, New York: Springer 1985

[R] Råde, J.: Singular Yang–Mills fields II. Global theory. Preprint, Stanford University 1993

[S] Singer, I.M.: The η-invariant and the index. In: S.-T. Yau (ed.) Mathematical Aspects of String Theory (San Diego 1986). Singapore: World Scientific, 1987, pp. 239–258

Communicated by G. Felder