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Bäcklund Transformation for Supersymmetric Self-Dual Theories for Semisimple Gauge Groups and a Hierarchy of A_1 Solutions

Ch. Devchand[1,*], **A. N. Leznov**[2]

[1] Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna, 141980 Russia
[2] Institute for High Energy Physics, Protvino, 142284 Russia

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Abstract: We present a Bäcklund transformation (a discrete symmetry transformation) for the self-duality equations for supersymmetric gauge theories in N-extended super-Minkowski space $\mathscr{M}^{4\,|\,4N}$ for an arbitrary semisimple gauge group. For the case of an A_1 gauge algebra we integrate the transformation starting with a given solution and iterating the process we construct a hierarchy of explicit solutions.

1. Introduction

The self-duality equations for supersymmetric gauge fields in superspace have a construction of solutions very similar to that for non-supersymmetric gauge fields. This is not surprising since, once the kinematic constraints have been solved, the dynamical equations for the superfield "prepotential" describing the theory have a form independent of the extension of superspace, holding for the non-supersymmetric case as well. The discrete symmetry transformation (Bäcklund transformation) for non-supersymmetric self-dual gauge fields taking values in an arbitrary semisimple Lie algebra recently presented [6] therefore actually holds, with appropriate replacement of ordinary fields by superfields, for the supersymmetric cases as well. The transformation, however, may be written in a simpler (lower order) form allowing more compact proofs and yielding the previous results for the non-supersymmetric ($N = 0$) self-duality equations as direct corollaries. We present the Bäcklund transformation for an arbitrary semisimple gauge group in Sect. 3 and present a simple expression for the change induced by the transformation in the topological charge density. Specialising in Sect. 4, to an A_1 gauge theory, we write an explicit form of the transformation for the gauge algebra components of the superfield prepotential. Starting from a given solution for the restriction of the gauge algebra to a solvable sub-algebra of A_1, the transformation is integrable, yielding a new solution

* Electronic address: devchand@theor.jinrc.dubna.su
Postal address: LTPh, JINR. P.O. Box 79, Head Post Office, 101000 Moscow, Russia

which spans the entire A_1 algebra. In fact, the Bäcklund transformation applied to the thus obtained solution can also be integrated; and this process can be iterated to construct a hierarchy of explicit solutions, which we present in Sect. 5. Even for the $N = 0$ case our solutions generalise those obtained in [2] from an explicit construction of the Atiyah-Ward ansätze.

2. The Dynamical Self-Duality Equations

In N-extended complexified super-Minkowski space $\mathscr{M}^{4\,|\,4N}$ of complex dimension $(4\,|\,4N)$ with coordinates $\{x^{\alpha\dot\beta}, \theta^\alpha_s, \bar\theta^{\dot\alpha t}\}$, where $\alpha, \dot\alpha$ are two-component spinor indices and $s, t = 1, \ldots, N$ are internal indices, the upper and lower ones referring to fundamental and conjugate representations respectively, the self-duality equations may be written in the form of the following super-curvature constraints in N-extended superspace (e.g. [10]):

$$\begin{aligned} \{\mathscr{D}_{\alpha s}, \mathscr{D}_{\beta t}\} &= 0\,, \\ \{\bar{\mathscr{D}}^{(s}_{\dot\alpha}, \bar{\mathscr{D}}^{t)}_{\dot\beta}\} &= 0\,, \\ \{\mathscr{D}_{\alpha s}, \bar{\mathscr{D}}^t_{\dot\beta}\} &= 2\delta^t_s \mathscr{D}_{\alpha\dot\beta}\,, \\ [\mathscr{D}_{\beta s}, \mathscr{D}_{\alpha\dot\alpha}] &= 0\,, \end{aligned} \tag{1}$$

where $\mathscr{D}_A \equiv \partial_A + A_A = \{\mathscr{D}_{\alpha\dot\beta} \equiv \sigma^\mu_{\alpha\dot\beta}\mathscr{D}_\mu, \mathscr{D}_{\alpha s}, \bar{\mathscr{D}}^t_{\dot\beta}\}$ are gauge-covariant derivatives and the supertranslation vector fields

$$\partial_A = \{\partial_{\alpha\dot\beta}, D_{\alpha s}, \bar D^t_{\dot\beta}\} \equiv \left\{\frac{\partial}{\partial x^{\alpha\dot\beta}}, \frac{\partial}{\partial \theta^{\alpha s}}, \frac{\partial}{\partial \bar\theta^{\dot\beta}_t} + 2\theta^{\alpha t}\partial_{\alpha\dot\beta}\right\},$$

realise the superalgebra

$$\begin{aligned} \{D_{\alpha s}, D_{\beta t}\} = 0 &= \{\bar D^s_{\dot\alpha}, \bar D^t_{\dot\beta}\}\,, \\ \{D_{\alpha s}, \bar D^t_{\dot\beta}\} &= 2\delta^t_s \partial_{\alpha\dot\beta}\,, \\ [D_{\beta s}, \partial_{\alpha\dot\beta}] = 0 &= [\bar D^t_{\dot\alpha}, \partial_{\alpha\dot\beta}]\,. \end{aligned}$$

Using Jacobi identities, these equations for the superconnection A_A may easily be seen to imply the familiar self-duality relations

$$F_{\mu\nu} = \tfrac{1}{2}\varepsilon_{\mu\nu\varrho\sigma}F_{\varrho\sigma}$$

for the field strengths. A large number of the superconnection components are clearly pure-gauges, containing no dynamical information. In particular, in the supersymmetric form of the Yang gauge [13]

$$A^s_\alpha = 0 = A_{\dot 1 t} = A_{\alpha\dot 1}\,, \tag{2}$$

which is consistent with (1), the dynamical content of (1) is contained in the supersymmetric Yang equations [3]

$$A_{\dot 2 s} = g^{-1}\bar D_{\dot 2 s} g = \bar D_{\dot 1 s} f\,, \tag{3a}$$

where g is a matrix superfield in the gauge group and f takes values in the gauge algebra. Consistency conditions for these equations include all but the following residual constraints from (1):

$$F^s_{\alpha\dot{2}t} = 0\,, \quad \text{i.e.} \quad A_{\alpha\dot{2}} = \frac{1}{2N}\,\delta^s_t D_{\alpha s} D^t_{\dot{1}} f \tag{4}$$

and

$$F^s_{\alpha\beta\dot{2}} = D^s_\alpha A_{\beta\dot{2}} = 0\,. \tag{5}$$

Now if f is taken to be a chiral superfield satisfying $D_{\alpha s} f = 0$, (4) yields vector potentials

$$A_{\alpha\dot{2}} = g^{-1}\partial_{\alpha\dot{2}} g = \partial_{\alpha\dot{1}} f \tag{3b}$$

which are automatically chiral, i.e. satisfying (5). These are precisely the usual ($N = 0$) Yang equations [1, 9]. The only conditions for f which are not automatic are those arising from the Maurer-Cartan identities for g, viz.,

$$D^{(s}_{\dot{2}} D^{t)}_{\dot{1}} f + D^{(s}_{\dot{1}} D^{t)}_{\dot{1}} f = 0 \tag{6a}$$

from (3a) and

$$\partial_{\alpha\dot{\beta}}\partial^{\alpha\dot{\beta}} f = [\partial_{\alpha\dot{1}} f, \partial^\alpha_{\dot{1}} f] \tag{6b}$$

from (3b); the latter in Yang coordinates

$$x_{\alpha\dot{\beta}} = \begin{pmatrix} y & -\bar{z} \\ z & \bar{y} \end{pmatrix}$$

taking the familiar form

$$\Box f \equiv (\partial_y \partial_{\bar{y}} + \partial_z \partial_{\bar{z}}) f = [\partial_y f, \partial_z f]\,.$$

In fact, acting on (6a) by $D_{\alpha s} D^\alpha_t \equiv D_{1(s} D_{2t)}$ yields (6b), so we may think of the former as the equations of motion for the $N = 2, 3$ and 4 theories; and the latter as the equations of motion for the $N = 0$ and 1 cases. These equations are Euler-Lagrange equations for the functional [7]

$$\begin{aligned} \mathscr{L} &= \operatorname{tr} \frac{1}{2N(N+1)}\, D^{(s}_1 D^{t)}_2 \left(\tfrac{1}{2}\, D_{\dot{1}s} f D_{\dot{2}t} f + \tfrac{1}{3}\, f D_{\dot{1}s} f D_{\dot{1}t} f\right) \\ &= \operatorname{tr} \tfrac{1}{2} \left(\partial_{\alpha\dot{1}} f \partial^\alpha_{\dot{2}} f + \tfrac{1}{3}\, f \partial_{\alpha\dot{1}} f \partial^\alpha_{\dot{1}} f\right), \end{aligned} \tag{7}$$

the second expression following on performing the two odd differentiations for arbitrary N and using the chirality of f.

It is results for the dynamical form (6b) of the Yang equations advertised in [6] which we presently supersymmetrise. We shall present explicit proofs of the symmetry transformations mainly for the odd equation (6a), the statements in [6] for equation (6b) clearly following as corollaries.

3. Bäcklund Transformation for an Arbitrary Semisimple Gauge Group

We consider the system of equations

$$D_{\dot{2}}^{(s}D_{\dot{1}}^{t)}f + \{D_{\dot{1}}^{s}f, D_{\dot{1}}^{t}f\} = 0\,; \qquad D_{\alpha s}f = 0\,, \tag{8}$$

where f takes values in the algebra of a (semi-simple) gauge group. An auto-bäcklund transformation for these equations is given by the following system of equations for F and a matrix function S in the gauge group:

$$D_{\dot{1}}^{s}F = SD_{\dot{1}}^{s}\tilde{f}S^{-1} - D_{\dot{2}}^{s}S\,S^{-1}\,; \qquad D_{\alpha s}F = 0\,, \tag{9a}$$

where $\tilde{f} = rfr^{-1}$, the conjugation denoting an automorphism of the gauge algebra, which we shall take to be one which maps the maximal root to the minimal one. The proof that F satisfies (8) if f does is straightforward; consistency of (9a) requiring that

$$D_{\dot{2}}^{(s}D_{\dot{1}}^{t)}F + \{D_{\dot{1}}^{s}F, D_{\dot{1}}^{t}F\} = Sr(D_{\dot{2}}^{(s}D_{\dot{1}}^{t)}f + \{D_{\dot{1}}^{s}f, D_{\dot{1}}^{t}f\})r^{-1}S^{-1}\,.$$

Integrability of (9a) also requires that S satisfies the equation

$$D_{\dot{2}}^{(s}(S^{-1}D_{\dot{1}}^{t)}S) = \{S^{-1}D_{\dot{1}}^{(t}S, D_{\dot{1}}^{s)}\tilde{f}\} = 0\,. \tag{10a}$$

In order to have an explicit Bäcklund transformation, we need to insert a solution of this equation into (9a). An ansatz solving (10a) is given by

$$S^{-1}D_{\dot{1}}^{t}S = \frac{1}{f^{+}}\,[X_M^{+}, D_{\dot{1}}^{t}\tilde{f}] - D_{\dot{2}}^{t}\left(\frac{1}{f^{+}}\right)X_M^{+}\,, \tag{11a}$$

where X_M^{+} is the normalised algebra element corresponding to the maximal root and f^{+} is its coefficient in f. This ansatz is consistent in virtue of the equations of motion for f^{+}.

As a corollary, since F, f and S are chiral, we clearly have that the system:

$$\partial_{\alpha\dot{1}}F = S\partial_{\alpha\dot{1}}\tilde{f}S^{-1} - \partial_{\alpha\dot{2}}S\,S^{-1}\,; \tag{9b}$$

$$S^{-1}\partial_{\alpha\dot{1}}S = \frac{1}{f^{+}}\,[X_M^{+}, \partial_{\alpha\dot{1}}\tilde{f}] - \partial_{\alpha\dot{2}}\left(\frac{1}{f^{+}}\right)X_M^{+} \tag{11b}$$

is a Bäcklund transformation for Eq. (6b); satisfying the consistency relations

$$\partial_{\alpha\dot{\beta}}\partial^{\alpha\dot{\beta}}F - [\partial_{\alpha\dot{1}}F, \partial_{\dot{1}}^{\alpha}F] = Sr(\partial_{\alpha\dot{\beta}}\partial^{\alpha\dot{\beta}}f - [\partial_{\alpha\dot{1}}f, \partial_{\dot{1}}^{\alpha}f])r^{-1}S^{-1}$$

and

$$\partial_{\alpha\dot{2}}(S^{-1}\partial_{\dot{1}}^{\alpha}S) + [S^{-1}\partial_{\alpha\dot{1}}S, \partial_{\dot{1}}^{\alpha}f] = 0\,. \tag{10b}$$

The topological charge density (or equivalently the Yang-Mills lagrangian density for self-dual fields) is given by the formula [1]

$$\begin{aligned} q(f) &= \tfrac{1}{8}\,\mathrm{tr}\,\varepsilon^{\mu\nu\varrho\sigma}F_{\mu\nu}F_{\varrho\sigma} \\ &= \mathrm{tr}(F_{\bar{y}z}F_{y\bar{z}} + F_{z\bar{z}}F_{y\bar{y}}) \\ &= \mathrm{tr}(\partial_z^2 f\partial_y^2 f - (\partial_z\partial_y f)^2)\,, \end{aligned}$$

using the Yang equations (3b).

Under the Bäcklund transformation (9, 11) *the topological charge density changes according to the formula*

$$q(F) = q(f) + \Box\Box \ln f^{+} . \tag{12}$$

Proof.

$$\begin{aligned} q(F) &= \tfrac{1}{2}\, \mathrm{tr}\, \partial_{\alpha 1} \partial^{\gamma}_{\dot{1}} F \partial^{\alpha}_{\dot{1}} \partial_{\gamma 1} F \\ &= \tfrac{1}{2}\, \mathrm{tr}\, \partial^{\alpha}_{\dot{1}} \partial^{\gamma}_{\dot{1}} (\partial_{\alpha \dot{1}} F \partial_{\gamma \dot{1}} F) \\ &= q(f) + \mathrm{tr}\, \partial^{\gamma}_{\dot{1}} (-\partial_{\alpha \dot{1}} \tilde{f} \partial^{\alpha}_{\dot{1}} (S^{-1} \partial_{\gamma \dot{2}} S)) + \tfrac{1}{2}\, \mathrm{tr}\, \partial^{\alpha}_{\dot{1}} \partial^{\gamma}_{\dot{1}} (\partial_{\alpha \dot{2}} S S^{-1} \partial_{\gamma \dot{2}} S S^{-1}) , \end{aligned} \tag{13}$$

using the Bäcklund transformation (9b) and the cyclic property of the trace. Now using the Maurer-Cartan identity

$$\partial^{\alpha}_{\dot{1}} (S^{-1} \partial_{\gamma \dot{2}} S) - \partial_{\gamma \dot{2}} (S^{-1} \partial^{\alpha}_{\dot{1}} S) + [S^{-1} \partial^{\alpha}_{\dot{1}} S, S^{-1} \partial_{\gamma \dot{2}} S] \equiv 0 ,$$

the second term on the right takes the form

$$\begin{aligned} &\mathrm{tr}\, \partial^{\gamma}_{\dot{1}} (-\partial_{\alpha \dot{1}} \tilde{f} \partial_{\gamma \dot{2}} (S^{-1} \partial^{\alpha}_{\dot{1}} S) + [\partial_{\alpha \dot{1}} \tilde{f}, S^{-1} \partial^{\alpha}_{\dot{1}} S] S^{-1} \partial_{\gamma \dot{2}} S) \\ &\quad = \mathrm{tr}\, \partial^{\gamma}_{\dot{1}} (-\partial_{\alpha \dot{1}} \tilde{f} \partial_{\gamma \dot{2}} (S^{-1} \partial^{\alpha}_{\dot{1}} S) + \partial_{\alpha \dot{2}} (S^{-1} \partial^{\alpha}_{\dot{1}} S) (S^{-1} \partial_{\gamma \dot{2}} S)) \end{aligned}$$

in virtue of the integrability condition (10b); and inserting expression (11b) for $(S^{-1} \partial^{\alpha}_{\dot{1}} S)$ and using the property that $\mathrm{tr}(X^{+}_{M} \tilde{f}) = f^{+} \equiv e^{\beta}$, we obtain for this second term the expression

$$\mathrm{tr}\, \partial^{\gamma}_{\dot{1}} (2 \partial_{\alpha \dot{2}} \beta \partial_{\gamma \dot{2}} \beta e^{-\beta} \partial^{\alpha}_{\dot{1}} e^{\beta} - e^{-\beta} \partial_{\gamma \dot{2}} \Box e^{\beta} - \partial_{\gamma \dot{2}} \partial^{\alpha}_{\dot{2}} e^{-\beta} \partial_{\alpha \dot{1}} e^{\beta}) .$$

Finally, the last term in (13) is equal to

$$\tfrac{1}{2}\, \mathrm{tr}\, \partial^{\alpha}_{\dot{1}} \partial^{\gamma}_{\dot{1}} (\partial_{\alpha \dot{2}} S S^{-1} \partial_{\gamma \dot{2}} S S^{-1}) = \partial^{\alpha}_{\dot{1}} \partial^{\gamma}_{\dot{1}} (\partial_{\alpha \dot{2}} \beta \partial_{\gamma \dot{2}} \beta) ,$$

using (11b) and $\mathrm{tr}\, H^2 = 2$. Putting all the terms together yields the result (12). □

4. Explicit Bäcklund Transformation for an A_1 Gauge Theory

We now display the explicit form of the Bäcklund transformation for the A_1 case. The equations of motion (6a) for the coefficients of f in a Cartan-Weyl basis: $f = f^{+} X^{+} + f^{-} X^{-} + f^{0} H$, where $[H, X^{\pm}] = \pm 2 X^{\pm}$ and $[X^{+}, X^{-}] = H$ are

$$\begin{aligned} D^{(s}_{\dot{2}} D^{t)}_{\dot{1}} f^{0} + D^{(s}_{\dot{1}} f^{+} D^{t)}_{\dot{1}} f^{-} &= 0 , \\ D^{(s}_{\dot{2}} D^{t)}_{\dot{1}} f^{+} + 2 D^{(s}_{\dot{1}} f^{0} D^{t)}_{\dot{1}} f^{+} &= 0 , \\ D^{(s}_{\dot{2}} D^{t)}_{\dot{1}} f^{-} + 2 D^{(s}_{\dot{1}} f^{-} D^{t)}_{\dot{1}} f^{0} &= 0 , \end{aligned} \tag{14a}$$

where $f^{\pm}$ and f^{0} are chiral superfields; and Eqs. (6b) take the form

$$\begin{aligned} \Box f^{0} &= \partial_{[y} f^{+} \partial_{z]} f^{-} , \\ \Box f^{+} &= 2 \partial_{[y} f^{0} \partial_{z]} f^{+} , \\ \Box f^{-} &= 2 \partial_{[y} f^{-} \partial_{z]} f^{0} . \end{aligned} \tag{14b}$$

The ansatz (11) clearly means that S takes values in the solvable part of the algebra. Inserting the parametrisation $S = e^{\alpha_1 X^+} e^{\beta H}$ in (11a) and choosing the algebra automorphism r to be such that

$$rX^{\pm}r^{-1} = X^{\mp}, \qquad rHr^{-1} = -H,$$

yields $\beta = \ln f^+$ and the following equations for α_1:

$$D^t_{\dot 1}\alpha_1 = 2f^+ D^t_{\dot 1} f^0 + D^t_{\dot 2} f^+ ; \tag{15}$$

consistency clearly being guaranteed by the f^+ equation in (14a).

With $\beta = \ln f^+$ and using the Campbell-Hausdorff relations,

$$\begin{aligned} e^{-\beta H} X^{\pm} e^{\beta H} &= e^{\mp 2\beta} X^{\pm}, \\ e^{\alpha_1 X^+} H e^{-\alpha_1 X^+} &= H - 2\alpha_1 X^+, \\ e^{\alpha_1 X^+} X^- e^{-\alpha_1 X^+} &= X^- + \alpha_1 H - \alpha_1^2 X^+, \end{aligned} \tag{16}$$

the algebra components of (9a) may be written:

$$\begin{aligned} D^s_{\dot 1} F^- &= (f^+)^{-2} D^s_{\dot 1} f^+, \\ D^s_{\dot 1} F^0 &= -D^s_{\dot 1} f^0 - D^s_{\dot 2} \ln f^+ + (f^+)^{-2}\alpha_1 D^s_{\dot 1} f^+, \\ D^s_{\dot 1} F^+ &= (f^+)^2 D^s_{\dot 1} f^- - D^s_{\dot 2}\alpha_1 + \alpha_1 \left(D^s_{\dot 1}\left(\frac{\alpha_1}{f^+}\right) + D^s_{\dot 2} \ln f^+ \right). \end{aligned}$$

Now the second equation above together with (15) gives the relation $\alpha_1 = -f^+(F^0 - f^0)$, yielding the following explicit form of the Bäcklund transformation.

If $\{f^0, f^+, f^-\}$ *satisfy Eqs.* (14a, b), *then so do* $\{F^0, F^+, F^-\}$ *given by the following equations:*

$$F^- = -\frac{1}{f^+}, \tag{17a}$$

$$D^s_{\dot 1} F^0 = -D^s_{\dot 1} f^0 - D^s_{\dot 2} \ln f^+ - (F^0 - f^0) D^s_{\dot 1} \ln f^+, \tag{17b}$$

$$D^s_{\dot 1} F^+ = f^+\{(F^0 - f^0) D^s_{\dot 1}(F^0 - f^0) + f^+ D^s_{\dot 1} f^- + D^s_{\dot 2}(F^0 - f^0)\}. \tag{17c}$$

Proof. The transformation is manifestly true since it arises from the insertion of a consistent solution of (10) in (9). The direct verification however is not entirely straightforward. The first equation evidently implies

$$\begin{aligned} D^{(s}_{\dot 2} D^{t)}_{\dot 1} F^- &= -2(f^+)^{-3} D^{(s}_{\dot 2} f^+ D^{t)}_{\dot 1} f^+ + (f^+)^{-2} D^{(s}_{\dot 2} D^{t)}_{\dot 1} f^+ \\ &= -2(D^{(s}_{\dot 2} \ln f^+ + D^{(s}_{\dot 1} f^0) D^{t)}_{\dot 1} F^-, \quad \text{using (17a)}, \\ &= -2 D^{(s}_{\dot 1} F^0 D^{t)}_{\dot 1} F^- . \end{aligned}$$

The second equation, on using the equations of motion for f^0 and F^-, yields

$$\begin{aligned} D^{(s}_{\dot 2} D^{t)}_{\dot 1} F^0 = &-D^{(s}_{\dot 1} f^- D^{t)}_{\dot 1} f^+ - D^{(s}_{\dot 2}(F^0 - f^0) f^+ D^{t)}_{\dot 1} F^- - (F^0 - f^0) D^{(s}_{\dot 2} f^+ D^{t)}_{\dot 1} F^- \\ &- (F^0 - f^0) f^+ D^{(s}_{\dot 1} F^0 D^{t)}_{\dot 1} F^-, \quad \text{and using (17b)}, \\ = &-((f^+)^2 D^{(s}_{\dot 1} f^- + D^{(s}_{\dot 2}(F^+ - f^0) f^+ + (F^0 - f^0) D^{(s}_{\dot 1}(F^0 - f^0)) D^{t)}_{\dot 1} F^-, \\ = &-D^{(s}_{\dot 1} F^+ D^{t)}_{\dot 1} F^-, \quad \text{using (17c)}. \end{aligned}$$

Finally (17c) implies

$$\begin{aligned} D_{\dot{2}}^{(s} D_{\dot{1}}^{t)} F^{+} &= f^{+} D_{\dot{2}}^{(s}(F^{0}-f^{0}) D_{\dot{1}}^{t)}(F^{0}-f^{0}) \\ &+ D_{\dot{2}}^{(s} f^{+}\{(F^{0}-f^{0}) D_{\dot{1}}^{t)}(F^{0}-f^{0})+2 f^{+} D_{\dot{1}}^{t)} f^{-}+D_{\dot{2}}^{t)}(F^{0}-f^{0})\} \\ &+(F^{0}-f^{0}) f^{+} D_{\dot{2}}^{(s} D_{\dot{1}}^{t)}(F^{0}-f^{0})+(f^{+})^{2} D_{\dot{2}}^{(s} D_{\dot{1}}^{t)} f^{-} . \end{aligned}$$

Using (17c) the first term on the right is equal to

$$\{D_{\dot{1}}^{(s} F^{+}-(f^{+})^{2} D_{\dot{1}}^{(s} f^{-}\} D_{\dot{1}}^{t)}(F^{0}-f^{0}) ;$$

and using (17b, c) the second term may be written

$$-\{f^{+}(D_{\dot{1}}^{(s} f^{0}+D_{\dot{1}}^{(s} F^{0})+(F^{0}-f^{0}) D_{\dot{1}}^{(s} f^{+}\}\{(f^{+})^{-1} D_{\dot{1}}^{t)} F^{+} f^{+} D_{\dot{1}}^{t)} f^{-}\} .$$

Gathering all the terms and using the equations of motion for f^0, f^- and F^0, we obtain

$$D_{\dot{2}}^{(s} \bar{D}_{\dot{1}}^{t)} F^{+}=2 D_{\dot{1}}^{(s} F^{+} D_{\dot{1}}^{t)} F^{0} . \qquad \square$$

An equivalent proof of invariance of Eq. (6) under the transformation (17) is also implied by the following.

The lagrangian density whose variation yields Eqs. (14b),

$$\mathscr{L}_{A_1}(f)=\partial_{\alpha \dot{2}} f^{0} \partial_{\dot{1}}^{\alpha} f^{0}+\partial_{\alpha \dot{2}} f^{+} \partial_{\dot{1}}^{\alpha} f^{-}-f^{+} \partial_{\alpha \dot{1}} f^{0} \partial_{\dot{1}}^{\alpha} f^{-}+f^{0} \partial_{\alpha \dot{1}} f^{+} \partial_{\dot{1}}^{\alpha} f^{-},$$

equivalent to (7) *up to a divergence, changes under the Bäcklund transformation* (17) *by a total divergence, namely,*

$$\begin{aligned} \mathscr{L}_{A_1}(F)-\mathscr{L}_{A_1}(f) &= \partial_{\alpha \dot{2}}\big(F^{0} \partial_{\dot{1}}^{\alpha}(F^{0}-f^{0})+F^{+} \partial_{\dot{1}}^{\alpha} F^{-} \\ &+\tfrac{1}{2} \ln f^{+} \partial_{\dot{1}}^{\alpha}(F^{0}-f^{0})^{2}-\ln f^{+} \partial_{\dot{2}}^{\alpha}(F^{0}-f^{0})\big) \\ &+\partial_{\alpha \dot{1}}\Big(-F^{+} F^{-} \partial_{\dot{1}}^{\alpha} F^{0}-\tfrac{1}{2} f^{0} \partial_{\dot{1}}^{\alpha}(F^{0}-f^{0})^{2} \\ &-\frac{\ln f^{+}}{3} \partial_{\dot{1}}^{\alpha}(F^{0}-f^{0})^{3}-\frac{\ln f^{+}}{2} \partial_{\dot{2}}^{\alpha}(F^{0}-f^{0})^{2}\Big) . \end{aligned}$$

Proof. Using the equations of motion (14b),

$$\mathscr{L}_{A_1}(F)=-F^{-} \partial_{\alpha \dot{1}} F^{0} \partial_{\dot{1}}^{\alpha} F^{+},$$

up to divergence terms. Now inserting (17a) and the even equations equivalent to (17b, c)

$$\begin{aligned} & \partial_{\alpha \dot{1}} F^{0}=-\partial_{\alpha \dot{1}} f^{0}-\partial_{\alpha \dot{2}} \ln f^{+}-(F^{0}-f^{0}) \partial_{\alpha \dot{1}} \ln f^{+}, \\ & \partial_{\alpha \dot{1}} F^{+}=f^{+}\{(F^{0}-f^{0}) \partial_{\alpha \dot{1}}(F^{0}-f^{0})+f^{+} \partial_{\alpha \dot{1}} f^{-}+\partial_{\alpha \dot{2}}(F^{0}-f^{0})\}, \end{aligned}$$

immediately yields the result. $\square$

5. Explicit Integration of the A_1 Theory

We now construct a hierarchy of explicit solutions $f_n = \{f_n^0, f_n^+, f_n^-\}$ to the A_1 theory by iterating the transformation (17) starting from the solution $f_0 = \{\tau, \alpha_0, 0\}$ corresponding to a restriction of the gauge algebra to a solvable subalgebra, with τ, α_0 satisfying the system of equations

$$\begin{aligned} D_{\dot 2}^{(s} D_{1}^{t)} \tau &= 0\,, \\ D_{\dot 2}^{(s} D_{1}^{t)} \alpha_0 + 2 D_{\dot 1}^{(s} \tau D_{\dot 1}^{t)} \alpha_0 &= 0\,. \end{aligned} \tag{18}$$

The general solution (τ, α_0) to this system clearly depends on four arbitrary functions (two each for τ and α_0). A restricted example of a solution depending on two arbitrary functions is given by

$$\begin{aligned} \tau &= \int d\lambda\, \varrho(x_{\alpha \dot 1} + \lambda x_{\alpha \dot 2}, \bar\theta_{\imath 1} + \lambda \bar\theta_{\imath \dot 1}, \lambda)\,, \\ \alpha_0 &= \int d\lambda\, \sigma(x_{\alpha \dot 1} + \lambda x_{\alpha \dot 2}, \bar\theta_{\imath \dot 1} + \lambda \bar\theta_{\imath 1}, \lambda) e^{\left\{ \int \frac{d\lambda'}{\lambda - \lambda'} \varrho(x_{\alpha \dot 1} + \lambda' x_{\alpha \dot 2}, \bar\theta_{\imath 1} + \lambda' \bar\theta_{\imath 1}, \lambda') \right\}}\,, \end{aligned} \tag{19}$$

where ϱ, σ are arbitrary functions. Now, taking $f = f_0$, the Bäcklund transformation (9) can be easily integrated, using a superpotential α_2 satisfying

$$D_{\dot 1}^{t} \alpha_2 = 2\alpha_1 D_{\dot 1}^{t} \tau + D_{\dot 2}^{t} \alpha_1\,, \tag{20}$$

to yield a new solution $F = f_1$ with components

$$\begin{aligned} f_1^- &= -\frac{1}{\alpha_0}\,, \\ f_1^0 &= \tau - \frac{\alpha_1}{\alpha_0}\,, \\ f_1^+ &= \frac{\alpha_1^2}{\alpha_0} - \alpha_2\,. \end{aligned} \tag{21}$$

In turn, taking $f = f_1$, the Bäcklund transformation is yet again integrable and the process can be iterated, the transformation being integrable for all the $f = f_n$ produced by iteration from the above f_0. The integrations use a succession of superpotentials α_n satisfying the relations

$$D_{\dot 1}^{t} \alpha_{n+1} = 2\alpha_n D_{\dot 1}^{t} \tau + D_{\dot 2}^{t} \alpha_n\,. \tag{22}$$

For the case of the special initial solution (19), these equations have solutions

$$\alpha_n = \int d\lambda\, \lambda^n \sigma(x_{\alpha \dot 1} + \lambda x_{\alpha \dot 2}, \bar\theta_{\imath 1} + \lambda \bar\theta_{\imath \dot 1}, \lambda) e^{\int \frac{d\lambda'}{\lambda - \lambda'} \varrho(x_{\alpha 1} + \lambda' x_{\alpha 2}, \bar\theta_{\imath 1} + \lambda' \bar\theta_{\imath 1}, \lambda')}\,. \tag{23}$$

The thus obtained solutions may be represented compactly in terms of the determinant of a symmetry recursive matrix M^{n+1}

$$D_{n+1} \equiv \det M^{n+1} \equiv \det \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{n+1} \\ \alpha_2 & \alpha_3 & \alpha_4 & \cdots & \alpha_{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{n-1} & \alpha_n & \alpha_{n+1} & \cdots & \alpha_{2n-1} \\ \alpha_n & \alpha_{n+1} & \alpha_{n+2} & \cdots & \alpha_{2n} \end{pmatrix} \tag{24a}$$

and of one of its minors

$$\dot{D}_n \equiv \det \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_n \\ \alpha_2 & \alpha_3 & \alpha_4 & \cdots & \alpha_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{n-2} & \alpha_{n-1} & \alpha_n & \cdots & \alpha_{2n-3} \\ \alpha_n & \alpha_{n+1} & \alpha_{n+2} & \cdots & \alpha_{2n-1} \end{pmatrix}, \tag{24b}$$

where the dot represents a derivation which shifts the index on α by one, i.e. $\dot{f}(\{\alpha_n\}) \equiv \sum_l \alpha_{l+1} \frac{\partial}{\partial \alpha_l} f(\{\alpha_n\})$. Evidently, $\dot{D}_n \equiv -\tilde{M}^{(n+1)}_{n,n+1}$, where $\tilde{M}$ denotes the matrix adjugate to M.

The recursive Bäcklund transformation

$$f^-_{n+1} = -\frac{1}{f^+_n} \tag{25a}$$

$$D^s_{\dot{1}} f^0_{n+1} = -D^s_{\dot{1}} f^0_n - D^s_{\dot{2}} \ln f^+_n - (f^0_{n+1} - f^0_n) D^s_{\dot{1}} \ln f^+_n \tag{25b}$$

$$D^s_{\dot{1}} f^+_{n+1} = f^+_n \{(f^0_{n+1} - f^0_n) D^t_{\dot{1}}(f^0_{n+1} - f^0_n) + f^+_n D^s_{\dot{1}} f^-_n + D^t_{\dot{2}}(f^0_{n+1} - f^0_n)\}, \tag{25c}$$

on iteration of the solution $\{f^0_0 = \tau, f^+_0 = \alpha_0, f^-_0 = 0\}$, *yields, on integration, the hierarchy of solutions*

$$f^-_n = (-1)^n \frac{D_{n-1}}{D_n}, \tag{26a}$$

$$f^0_n = \tau - \frac{\dot{D}_n}{D_n}, \tag{26b}$$

$$f^+_n = (-1)^n \frac{D_{n+1}}{D_n}, \tag{26c}$$

where $D_0 = 1, D_{-1} = 0, \dot{D}_0 = 0$ *and for* $n \geq 1$, D_n *and* $\dot{D}_n$ *are the determinants* (24).

Proof. Inserting (26) in (25b), we obtain the equations

$$D^s_{\dot{1}}\left(\frac{\dot{D}_{n+1}}{D_n}\right) + \left(\frac{D_{n+1}}{D_n}\right)^2 D^s_{\dot{1}}\left(\frac{\dot{D}_n}{D_{n+1}}\right) - (D^s_{\dot{2}} + 2D^s_{\dot{1}}\tau) + \left(\frac{D_{n+1}}{D_n}\right) = 0, \tag{27}$$

which may be proven following the proof of an analogous equation in Appendix B of [2]. In virtue of (22), Eqs. (27) are equivalent to the algebraic equations

$$\begin{aligned} &\dot{D}_n \frac{\partial}{\partial \alpha_l} D_{n+1} - D_{n+1} \frac{\partial}{\partial \alpha_l} \dot{D}_n + \dot{D}_{n+1} \frac{\partial}{\partial \alpha_l} D_n - D_n \frac{\partial}{\partial \alpha_l} \dot{D}_{n+1} \\ &+ D_n \frac{\partial}{\partial \alpha_{l-1}} D_{n+1} - D_{n+1} \frac{\partial}{\partial \alpha_{l-1}} D_n = 0. \end{aligned} \tag{28}$$

For any l, (28) contains three sets of determinants of the form appearing in the following identity for determinants due to Jacobi:

$$\begin{aligned}
&\det A^{(n-1)} \det \begin{pmatrix} a & \alpha_i & b \\ \beta_i & A^{(n-1)} & \gamma_i \\ c & \delta_i & d \end{pmatrix} \\
&= \det \begin{pmatrix} a & \alpha_i \\ \beta_i & A^{(n-1)} \end{pmatrix} \det \begin{pmatrix} A^{(n-1)} & \gamma_i \\ \delta_i & d \end{pmatrix} \\
&- \det \begin{pmatrix} \alpha_i & b \\ A^{(n-1)} & \gamma_i \end{pmatrix} \det \begin{pmatrix} \beta_i & A^{(n-1)} \\ c & \delta_i \end{pmatrix} ,
\end{aligned} \tag{29}$$

where $A^{(n-1)}$ is any $(n-1)$-dimensional matrix and $i = 1, \ldots, (n-1)$; and (28) may easily be seen to be a consequence of this identity. The solution (26c) follows from (27) and the recursion relation

$$D_{n+1} D_{n-1} = \ddot{D}_n D_n - \dot{D}_n \dot{D}_n , \tag{30}$$

which is a special case of the identity (29). The right side of (25c), on insertion of (26) and using the Jacobi relation (30) to eliminate D_{n-1} yields

$$\begin{aligned}
&\frac{D_{n+1}^2}{D_n^2} \dot{D}_n \left(\frac{D^s_{\dot{1}} \dot{D}_n}{D_n} - \frac{\dot{D}_n D^s_{\dot{1}} D_n}{D_n^2} - \frac{D^s_{\dot{1}} \dot{D}_{n+1}}{D_{n+1}} + \frac{\dot{D}_{n+1} D^s_{\dot{1}} D_{n+1}}{D_{n+1}} \right) \\
&- \frac{\dot{D}_{n+1} D_{n+1}}{D_n} \left(\frac{D^s_{\dot{1}} \dot{D}_n}{D_n} - \frac{\dot{D}_n D^s_{\dot{1}} D_n}{D_n^2} - \frac{D^s_{\dot{1}} \dot{D}_{n+1}}{D_{n+1}} + \frac{\dot{D}_{n+1} D^s_{\dot{1}} D_{n+1}}{D_{n+1}} \right) \\
&+ \frac{D_{n+1}^2}{D_n^2} \left(D^s_{\dot{1}} \ddot{D}_n D_n - \frac{D^s_{\dot{1}} \dot{D}_n^2}{D_n} \right. \\
&\left. - \frac{\ddot{D}_n D^s_{\dot{1}} D_{n+1}}{D_{n+1}} + \frac{(\dot{D}_n)^2 D^s_{\dot{1}} D_{n+1}}{D_n D_{n+1}} + \frac{(\dot{D}_n)^2 D^s_{\dot{1}} D_n}{D_n^2} \right) + \frac{D_{n+1}^2}{D_n} D^s_{\dot{2}} \left(\frac{\dot{D}_n}{D_n} - \frac{\dot{D}_{n+1}}{D_{n+1}} \right) ,
\end{aligned}$$

which on using (27) together with its derivation

$$\begin{aligned}
&(D_{n+1} D^s_{\dot{2}} D_n - D_n D^s_{\dot{2}} D_{n+1})^{\cdot} \\
&= \ddot{D}_{n+1} D^s_{\dot{1}} D_n - D_n D^s_{\dot{1}} \ddot{D}_{n+1} - D_{n+1} D^s_{\dot{1}} \ddot{D}_n + \ddot{D}_n D^s_{\dot{1}} D_{n+1}
\end{aligned}$$

reduces to

$$\begin{aligned}
&\frac{D^s_{\dot{1}} D_n}{D_n^2} (\ddot{D}_{n+1} D_{n+1} - \dot{D}_{n+1} \dot{D}_{n+1}) - \frac{D_{n+1} D^s_{\dot{1}} \ddot{D}_{n+1}}{D_n} + \frac{D^s_{\dot{1}} (\dot{D}_{n+1}^2)}{D_n} \\
&- \frac{\dot{D}_{n+1}^2 D^s_{\dot{1}} D_{n+1}}{D_n D_{n+1}} = - D^s_{\dot{1}} \left(\frac{D_{n+2}}{D_{n+1}} \right)
\end{aligned}$$

in virtue of (30), proving (26c). □

6. Remarks

a) Our Bäcklund transformations correspond to discrete symmetry transformations of the self-duality equations. In fact, there exists an infinite dimensional discrete group of symmetry transformations.

b) Our solutions are not the general solutions of Eqs. (6) since they do not depend on sufficiently many arbitrary functions; e.g. for the A_1 gauge theories, they depend on only four arbitrary functions (the general solutions for τ and α_0 depending on two arbitrary functions each) rather than six. It remains to be seen whether the harmonic superspace formulation of super-self-duality recently presented [5] can be used to obtain more general solutions. Our superfield f, in harmonic space, is the non-analytic superfield prepotential V^{--} discussed in [5]; and as shown there, it may be used not only to reconstruct the spinor and vector potentials of the theory, but also to directly construct the superfield strengths.
c) Following Witten's idea [12] of intermingling self-dual and anti-self-dual data to obtain the full (non-self-dual) $N = 3$ equations, (6) and (7) were used in [4] to construct a lagrangian and conservation laws for the full $N = 3$ theory. We expect the constructions of the present paper to be of relevance for the full $N = 3$ theory.
d) Equations (6, 7) have also recently appeared in relation to $N = 2, 4$ superstring theories [11], whose degrees of freedom, it has been argued, describe $N = 4$ super self-dual gauge theories. The relevance of our discrete symmetries and solution hierarchy for string physics remains to be investigated.
e) Apart from the harmonic superspace construction mentioned above, there is an alternative construction of solutions using an auxiliary space. Consider the first order integro-differential equation in the two auxiliary parameters λ and g:

$$\frac{\partial \varphi(\lambda, g)}{\partial g} = \int \frac{d\lambda'}{\lambda - \lambda'} [\varphi(\lambda), \varphi(\lambda')] .$$

In terms of solutions $\varphi(\lambda, g)$ of this equation,

$$f = \int_0^1 dg \int d\lambda \, \varphi(\lambda, g)$$

solves (6) [8].

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