Commun Math. Phys 159, 433–441 (1994)

Communications in
Mathematical
Physics

On the Distribution of Zeros of a Ruelle Zeta-Function

A. Eremenko[1]*, G. Levin[2], M. Sodin[3]

[1] Purdue University, West Lafayette, IN 47907, USA

[2] Institute of Mathematics, Hebrew University, Jerusalem 91904, Israel

[3] Institute of Low Temperature Physics and Engineering, Kharkov, 310164, Ukraine

Received: 14 January 1993/in revised form: 22 March 1993

Abstract: We study the limit distribution of zeros of a Ruelle ζ-function for the dynamical system $z \mapsto z^2 + c$ when c is real and $c \to -2-0$ and apply the results to the correlation functions of this dynamical system.

Consider the dynamical system defined by the complex polynomial map f_c: $z \mapsto z^2 + c$, where $c < -2$. We use the notions and results of the iteration theory of rational functions (see for example [5]). Denote by f_c^{*n} the n^{th} iterate of the function f_c. The Julia set $J(f_c)$ is a Cantor set on the real line. So in particular all finite periodic points are real. This system is expanding (hyperbolic) on its Julia set. When $c = -2$ the Julia set is the segment $[-2, 2]$ and the map $P = f_{-2}$ is not expanding anymore. We have the conjugation

$$P \circ \phi = \phi \circ Q\,, \tag{1}$$

where ϕ: $[0, 1] \to [-2, 2]$, $t \mapsto 2\cos \pi t$ and

$$Q = \begin{cases} t \mapsto 2t & 0 \leqq t \leqq 1/2\,, \\ t \mapsto 2-2t, & 1/2 \leqq t \leqq 1\,. \end{cases}$$

Remark that the chaotic dynamic of P on $[-2,2]$ was investigated by J. von Neuman and S. Ulam on one of the first computers.

We are going to study the dynamics of f_c, $c < -2$ when $c \to -2$ and then compare it with the behavior of the limit system P. The chaotic dynamics of f_c has to be described in probabilistic terms. This can be done by introducing an appropriate invariant probability measure σ_c on the Julia set. We will show that the rate of asymptotic decrease of correlation functions of the system (f_c, v_c) changes dramatically when we pass to the limit system as $c \to -2$.

Our tool is the Thermodynamic Formalism [12–15]. Let us introduce the main objects of this theory in our particular case. Consider the Fréchet space $C^\infty(U)$ of

* Supported by NSF grant DMS-9101798

infinitely differentiable functions defined in some real neighborhood U of the Julia set, such that $f_c^{-1}(U) \subset U$ and U does not contain the critical point of f_c. We define the Ruelle operator L_c acting on $C^\infty(U)$ by the formula

$$L_c g(x) = \sum_{\{y: f_c(y) = x\}} \frac{g(y)}{[f_c'(y)]^2}.$$

The weight $(f_c')^{-2}$ is strictly positive on U. According to Ruelle's extension of the Perron–Frobenius theorem L_c has a simple maximal positive eigenvalue $\lambda_0^{-1}(c)$ such that the moduli of all other eigenvalues are strictly less than $|\lambda_0^{-1}(c)|$. Let h_c and v_c denote the eigenvectors of L_c and the adjoint operator L_c^* respectively, corresponding to the eigenvalue $\lambda_0^{-1}(c)$ (h_c is a positive continuous function and v_c is a Borel measure). Then $\sigma_c = h_c v_c$ is an f_c-invariant ergodic probability measure on the Julia set, called "the Gibbs state, corresponding to the weight $(f_c')^{-2}$." The operator L_c can be also considered on the space A of functions analytic in a complex neighborhood of the Julia set. Namely, for every complex neighbourhood W of the Julia set such that $U \subset W, f_c^{-1}(W) \subset W$ and W does not contain the critical point of f_c, consider the Banach space $A(W)$ of functions analytic in W with the supremum norm. Then A is the union of all such $A(W)$. As the weight $(f_c')^{-2}$ is analytic, the spectrum and eigenfunctions of L_c in A are the same as in $C^\infty(U)$ (see [14], Corollary 3.3(i)). This fact allows us to use the explicit expressions for eigenfunctions found in [10] with the help of complex analysis. The following particular form of Ruelle's zeta-function is connected to the operator L_c:

$$\zeta_c(\lambda) = \exp\left(\sum_{m=1}^{\infty} \frac{\lambda^m}{m} \sum_{x \in \mathrm{Fix}(f_c^m)} \frac{1}{(f_c^{*m})'(x)} \right),$$

where $\mathrm{Fix}(f_c^{*m})$ is the set of fixed points of f_c^{*m}. (We choose the weight $\phi = (f_c')^{-1}$ in the definition of Ruelle ζ-function. See Sect. 8 of [14] and formula (3.3) with $\sigma = \infty$ in [10].) The function ζ_c can be expressed in terms of generalized Fredholm determinants ([14, Corollary 8.1]). In our particular case it coincides with the Fredholm determinant D_c of L_c [10]; this is an entire function of order zero and its zeros are reciprocal to the eigenvalues of L_c. There is an explicit formula found in [10] (see also [11]):

$$\zeta_c(\lambda) = D_c(\lambda) = 1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{2^n f_c(0) \ldots f_c^{*n}(0)}.$$

In the appendix we will give a short direct proof of the fact that the eigenvalues of L_c are reciprocal to the zeros of D_c.

Remark. Let us consider another extension of the operator L_c: $C^\infty(U) \to C^\infty(U)$ to the Fréchet space $C^\infty(W)$ of the C^∞-functions of two real variables u and v, $u + iv \in W$, given by the formula

$$L_c^{\mathbf{R}^2} g(x) = \sum_{\{y: f_c(y) = x\}} \frac{g(y)}{|f_c'(y)|^2}.$$

(Note that $|f_c'(x)|^2$ is the Jacobian of the map f_c': $\mathbf{R}^2 \to \mathbf{R}^2$ at the point x.) Then the eigenvalues and eigenfunctions of $L_c^{\mathbf{R}^2}$ coincide with those for L_c. Really, every eigenfunction of $L_c^{\mathbf{R}^2}$ restricted to $U = W \cap \mathbf{R}$ is an eigenfunction of L_c. Conversely, the eigenfunctions of L_c are analytic and, hence, belong to $C^\infty(W)$. In particular,

$(\lambda_0(c))^{-1}$ is the leading eigenvalue of the operator $L_c^{\mathbf{R}^2}$ and the value $\log \lambda_0(c)$ is the so-called "escape rate" [8].

One of the reasons why the study of eigenvalues of L_c is important is their connection to correlation functions. For any two continuous A and B on the Julia set define the correlation function $\rho_{c,A,B}$ by

$$\rho_{c,A,B}(m) = \sigma_c(A(f_c^{*m}) \cdot B) - \sigma_c(A) \cdot \sigma_c(B) ,$$

where $\sigma(A) = \int A d\sigma$. Let

$$S_{c,A,B}(z) = \sum_{m=0}^{\infty} \rho_{c,A,B}(m) z^m$$

be the corresponding generating function. *If A and B are infinitely differentiable on the Julia set then $S_{c,A,B}$ is meromorphic in* $\mathbf{C}$ *and its poles can be located only at the points $\lambda\lambda_0^{-1}$, where λ^{-1} runs over the eigenvalues of L_c other than λ_0* [14, Proposition 5.3].

1. First we investigate the limit distribution of eigenvalues of L_c or, which is equivalent, zeros of D_c. The following facts about distribution of zeros of D_c were established in [9]. For all $c < -2$ the zeros with moduli greater than 1000 are negative, and simple. There exists a constant $c_0 = -2.85\ldots$ such that for $c \leqq c_0$ all zeros of D_c are real. If $c < -2$ is close to -2 then there are non-real zeros and their number tends to infinity as c tends to -2.

To study the asymptotic distribution of complex zeros we introduce the probability measures μ_c which charge equally every zero whose modulus is less than 1000.

Theorem 1. *The measures μ_c tend weakly to the uniform distribution on the circle $\{\lambda: |\lambda| = 4\}$.*

Remarks. Notice that 4 is the radius of convergence of the series $D_{-2} = (4 - 2\lambda)/(4 - \lambda)$. Our proof is also applicable to the family of entire functions

$$H_a(z) = \sum_{n=0}^{\infty} \frac{z^n}{a^{2^n - 1}}, \quad a > 1 ,$$

whose distribution of zeros was studied by G.H. Hardy [6]. He proved that for fixed a all zeros with moduli greater than $r_0(a)$ are negative. (In fact $r_0(a)$ can be replaced by an absolute constant [9]). Our argument shows that the limit distribution of zeros of H_a when $a \to 1$ is the uniform distribution on the circle $\{z: |z| = 1\}$. Theorem 1 should be compared with the following theorem of Jentzsch and Szegö: *The limit distribution of zeros of partial sums of a power series $\sum a_k z^k$ is the uniform distribution on $\{z: |z| = 1\}$*, provided that $|a_k|^{1/k} \to 1$. Our proof is based on the same idea as Beurling's proof of the Jentzsch–Szegö theorem [3].

Proof. We assume that $-3 < c < -2$. It is convenient to introduce the variable $z = \lambda/2$ and set $F_c(z) = D_c(2z)$ and $r_n(c) = f_c^{*n}(0)$. Thus

$$F_c(z) = 1 + \sum_{n=1}^{\infty} \frac{z^n}{r_1(c) \ldots r_n(c)}$$

and

$$r_{n+1}(c) = r_n^2(c) + c, \quad r_1(c) = c, \quad c < -2 \,. \tag{2}$$

It is easy to see that all r_n, except r_1, are positive, the sequence (r_n) is increasing and $r_{n+1}(c)/r_n(c) \to \infty, n \to \infty, c < -2$. Denote by $k = k(c)$ the smallest natural k such that

$$\frac{r_{k+1}(c)}{r_k(c)} \geqq 36 \,. \tag{3}$$

It was proved in [9] that the number of zeros of F_c in any fixed disk $\{z: |z| < R\}$, $R > 1000$ is asymptotically equivalent to $k(c)$ when $c \to -2$. This fact also follows from the estimates below (formula (7) plus Rouché theorem).

Lemma 1. *If $k = k(c)$ is as defined above, then*

(i) $36 \leqq |r_k(c)| \leqq 1521 = 39^2$.
(ii) $k(c) \sim (\log|c+2|^{-1})/\log 4$, $c \to -2$.
(iii) $(1/k(c)) \log|r_1(c) \ldots r_{k(c)}(c)| \to \log 2$, $c \to -2$.

Proof. (i) From (3) we conclude that $k = k(c) > 1$. If $r_k(c) < 36$ then by (2) $r_{k+1}(c)/r_k(c) = r_k(c) + c/r_k(c) < r_k(c) < 36$, which contradicts the definition of k. This proves the left inequality in (i). Now assume that $|r_k| > 39^2$. Then in view of (2) we have $|r_{k-1}| > 39$ and we obtain $|r_k| = |r_{k-1}|^2 + c > |r_{k-1}|^2 - 3$ and $|r_k|/|r_{k-1}| > 36$, which contradicts the definition of k. This proves the right inequality in (i).

(ii) Set $c = -2 - t$, $t > 0$. An easy induction gives

$$|r_n(c)| \geqq 2 + (4^{n-1} - 1)t, \quad n = 1, 2, \ldots \,. \tag{4}$$

To prove an inequality in the opposite direction we remark that $r_{n+1}(c) = [r_n(c)]^2 - 2 - t \leqq [r_n(c)]^2 - 2 = P(r_n(c))$, so

$$r_n(c) \leqq P^{*(n-1)}(r_1(c)) \leq P^{*(n-1)}(2+t) \,.$$

Using the semiconjugacy

$$2\cosh 2z = [2\cosh z]^2 - 2 = P(2\cosh z) \,,$$

(it is more convenient to use cosh rather than cos here) we obtain $r_n(c) \leqq 2\cosh(2^{n-1}y)$, where y is the smallest positive solution of the equation $2\cosh y = 2 + t$. There exists an absolute constant $C_0 = 30$ such that $2\cosh x \leqq C_0 x^2 + 2$ whenever $2\cosh x \leqq 1521$, $x \in \mathbf{R}$. Thus we obtain

$$r_n(c) \leqq 2 + 4^{n-1} C_0 t, \quad n = 1, 2, \ldots, k(c) \,. \tag{5}$$

The statement (ii) follows from (4) and (5).

(iii) From (ii) follows

$$t \leqq C_1 4^{-k} \,. \tag{6}$$

In view of (4), (5) and (6) we have

$$\left|\left(\frac{1}{k}\sum_{n=1}^{k}\log|r_n(c)|\right)-\log 2\right| \leqq \frac{1}{k}\sum_{n=1}^{k}\log(1+4^{n-1}C_0t)$$
$$\leqq \frac{1}{k}\sum_{n=1}^{k}C_0C_14^{n-k} \leqq \frac{1}{k}\sum_{n=0}^{\infty}C_0C_14^{-n}\to 0, \quad k\to\infty .$$

This finishes the proof of Lemma 1.

Denote by $A(t_1, t_2)$ the annulus $\{z: t_1 < |z| < t_2\}$ and set $A(c) = A(4r_k(c), 9r_k(c))$, where $k = k(c)$. Put $M_c(z) = z^k/(r_1(c)\ldots r_k(c))$. If $z \in A(c)$ we have

$$\left|1-\frac{F_c(z)}{M_c(z)}\right| \leqq \sum_{j=1}^{k}\frac{r_k\ldots r_{k-j+1}}{|z|^j}+\sum_{j=1}^{\infty}\frac{|z|^j}{r_{k+1}\ldots r_{k+j}}$$
$$\leqq \sum_{j=1}^{\infty}4^{-j}+\sum_{j=1}^{\infty}4^{-j}=\frac{2}{3}. \tag{7}$$

Thus if we denote $u_c(z) = (k(c))^{-1}\log|F_c(z)|$ then by (iii) of Lemma 1,

$$u_c(z) = (k(c)^{-1})\log|M_c(z)| + o(1) = \log|z/2| + o(1), \quad c\to -2, \tag{8}$$

uniformly when $z \in A(c)$. We are going to prove that

$$u_c(z)\to\log^+|z/2|, \quad |z|\leqq 324, \tag{9}$$

where the convergence holds in L^1 with respect to the Lebesgue measure (area) in $\{z: |z| \leqq 324\}$.

From the definition of $A(c)$ and Lemma 1, (i) follows that $A_c \subset A(144, 13689)$. So from any sequence $c_m \to -2$, $c_m < -2$ we can choose a subsequence (which we again denote by c_m) such that the annuli $A(c_m)$ contain a fixed annulus $A(q_1, q_2)$, $q_1 < q_2$, $q_2 > 324$. Then in view of (8) we have

$$u_{c_m}(z)\to\log|z/2| \quad \text{uniformly in } \bar{A}(q_1, q_2). \tag{10}$$

Furthermore we have

$$u_{c_m}(z)\to 0, \quad |z|<2, \tag{11}$$

(convergence in L^1 on compacts in $\{z: |z| < 2\}$), because $F_c(z)\to F_{-2}(z) = 1 - z/(2-z)$, $c \to -2$ uniformly on compacts in $\{z: |z| < 2\}$. Now we use the following fact (see for example [7], Theorem 4.1.9): *if a sequence of subharmonic functions u_m is bounded from above on $\{z: |z| = R\}$ and their values at the point 0 are bounded from below then there is a subsequence which converges in L^1 on every compact in $\{z: |z| < R\}$ to a subharmonic function u.* Applying this statement to our functions u_{c_m} and $R = q_2$, we obtain a subsequence (which we again denote by u_{c_m}) which converges to a subharmonic function u. This function u has the properties:

$$u(z) = 0, \quad |z| < 2 \tag{12}$$

and

$$u(z) = \log|z/2|, \quad q_1 < |z| < q_2, \tag{13}$$

which follows from (11) and (10) respectively. Remark that $u(z) \leqq 0$, $|z| = 2$. This follows from (12) and the following theorem of M. Brelot [4]: *if u is a subharmonic* *function and* $u(z_0) = a$ *then for every* $\varepsilon > 0$ *there exists a sequence of circles centered at* z_0 *and radii tending to zero such that* $u(z) \geqq a - \varepsilon$ *on these circles.* (It follows from the upper semi-continuity of u that $u(z) \geqq 0$, $|z| = 2$, but we do not need this.) Now $\log|z/2|$ is a harmonic majorant of u in the annulus $A(2, q_2)$, but $u(z) = \log|z/2|$ at some points in this annulus, for example for $|z| = q_1$. It follows from the Maximum Principle that $u(z) = \log^+|z/2|$, $|z| < q_2$.

Thus we have proved that from *every* sequence u_{c_m} we can select a subsequence tending to $\log^+|z/2|$. This means that (9) is true. In fact our proof shows that u_c converge to $\log^+|z/2|$ in L^1 on every compact in the plane. Now we conclude from the general results on convergence of subharmonic functions [1, 2, 7] that the Riesz measures μ_c of u_c converge weakly to the Riesz measure of u, which is the uniform measure on the circle $|\lambda| = 2|z| = 4$. This proves the theorem.

2. Now we consider the application of Theorem 1 to the dynamical system (f_c, σ_c), where σ_c is the Gibbs state defined in the introduction. We have

$$\zeta_c(\lambda) \to 1 - \frac{\lambda}{4 - \lambda}, \quad c \to -2\,,$$

uniformly on compacts in $\{\lambda\colon |\lambda| < 4\}$. So $\lambda_0(c) \to 2$ and

$$\inf\{\lambda\colon \zeta_c(\lambda) = 0,\ \lambda \neq \lambda_0\} \to 4, \quad c \to -2\,.$$

Thus by Theorem 1 and by Ruelle's theorem mentioned in the introduction we have the following asymptotic behavior of correlation functions:

$$\limsup_{m \to \infty} |\rho_{c,A,B}(m)|^{1/m} = r(c)\,,$$

where $r(c) \to 1/2$ as $c \to -2$.

We want to compare this result with the behavior of the limiting dynamical system when $c \to -2$. First we have to understand what the limit invariant measure is. Recall the conjugation (1). The Lebesgue measure l_1 on $[0, 1]$ is invariant with respect to Q thus its image $\sigma_{-2} = \phi_* l_1$ is invariant with respect to $P = f_{-2}$. The measure σ_{-2} is absolutely continuous with the density

$$\frac{1}{\pi\sqrt{4 - x^2}}$$

on the interval $[-2, 2]$.

Proposition 1. $\sigma_c \to \sigma_{-2}$ *weakly as* $c \to -2$.

Proof. We will use the explicit expressions for the eigenfunction h_c of L_c and for the Cauchy transform

$$H_c(z) = \int \frac{d\nu_c(x)}{x - z}$$

of the eigenmeasure ν_c of L_c^*, corresponding to the greatest eigenvalue λ_0^{-1} (see [16, 10]). Using the notation $r_n(c) = f_c^{*n}(0)$ we have

$$h_c(x) = \sum_{n=0}^{\infty} \frac{\lambda_0^n(c)}{2^n r_1(c) \ldots r_n(c)\,[r_{n+1}(c) - x]}$$

and

$$H_c(z) = \sum_{n=0}^{\infty} \frac{\lambda_0^n(c)}{2^n z f_c(z) \ldots f_c^{*n}(z)} .$$

The function $z \mapsto H_c(z)$ is holomorphic in the complement of the Julia set $J(f_c)$. We have

$$h_c(x) \to -\left(\frac{1}{2+x} + \frac{1}{2-x}\right), \quad c \to -2$$

in $\bar{\mathbf{C}} \setminus ((-\infty, -2] \cup [2, \infty))$ and

$$H_c(z) \to H_{-2}(z) = \sum_{n=0}^{\infty} \frac{1}{zP(z) \ldots P^{*n}(z)}, \quad c \to -2$$

in $\bar{\mathbf{C}} \setminus [-2, 2]$.

Consider the measure ν_{-2} on $[-2, 2]$ with the density $\sqrt{4 - x^2}$. We claim that $H_{-2}(z)$ is proportional to the Caushy transform of ν_{-2}. This follows from the fact that they both satisfy the same functional equation

$$H(z) - \frac{H(P(z))}{z} = \frac{\text{const}}{z}, \quad z \in \bar{\mathbf{C}} \setminus [-2, 2] .$$

Now Proposition 1 follows from the identity

$$\left(\frac{1}{2+x} + \frac{1}{2-x}\right)\sqrt{4 - x^2} = \frac{4}{\sqrt{4 - x^2}} .$$

So the dynamical system (P, σ_{-2}) is the limit of (f_c, σ_c) when $c \to -2$. We will show that the asymptotic behavior of correlations changes drastically when we pass to the limit as $c \to -2$.

Proposition 2. *Let A and B be holomorphic functions on $[-2, 2]$. Then there exists a constant $a = a(A, B) > 1$ such that*

$$\rho_{-2, A, B}(m) \sim a^{-2m}, \quad m \to \infty .$$

Proof. In view of Cauchy formula is enough to prove the proposition for the set of functions

$$A_z(x) = \frac{1}{z - x}, \quad x \in [-2, 2], \quad z \in \bar{\mathbf{C}} \setminus [-2, 2] .$$

After the pullback to the segment $[0, 1]$ via the conjugation (1) we have to consider the correlations

$$\rho_{A,B}(m) = l_1(A(Q^m) \cdot B) - l_1(A) \cdot l_1(B)$$

with A and B of the form

$$\frac{1}{z - 2\cos \pi t} .$$

If we introduce the operator

$$G\colon g(t) \mapsto \frac{1}{2} \sum_{y\colon Q(y)=t} g(y) = \frac{1}{2}(g(t/2) + g(1-t/2)) , \tag{14}$$

then

$$\rho_{A,B}(m) = l_1 (A \cdot G^m(B)) - l_1(A) \cdot l_1(B) . \tag{15}$$

Now we notice that

$$G\left(\frac{1}{z - 2\cos \pi t}\right) = \frac{P'(z)}{2(P(z) - 2\cos \pi t)} ,$$

which implies

$$G^m\left(\frac{1}{z - 2\cos \pi t}\right) = \frac{(P^{*m})'(z)}{2^m(P^{*m}(z) - 2\cos \pi t)} = S(z) + \frac{\cos \pi t + o(1)}{2^{m-1}(P^{*m}(z))^2} , \tag{16}$$

where S is a function depending only on z. Combining (15) and (16) we get the statement of Proposition 2.

Remark. The analyticity assumption in Proposition 2 is crucial. Indeed consider the operator G defined in (14) in the space of infinitely differentiable functions on $[0,1]$. Its eigenvalues are 4^{-m} $m = 0, 1, 2 \ldots$, and to each eigenvalue 4^{-m} corresponds one (up to a constant multiple) eigenfunction p_m which is a polynomial of degree $2m$. Now if A and B belong to the subspace of $L^2([0,1], l_1)$ generated by $\{p_m\colon m = 0, 1, 2, \ldots\}$ then we have

$$\rho_{A,B}(m) \sim \text{const.} 4^{-km}, \quad m \to \infty ,$$

where const $\neq 0$ and k depend on A and B.

Appendix. Here we indicate a direct proof of the fact that the eigenvalues of L_c are reciprocal to the zeros of D_c, $c < -2$ (see also [11]). Let us look at the eigenvalues of the adjoint operator L_c^*. The dual space A^* is the space of functions g analytic in the complement of the Julia set $J(f_c)$ and equal to zero at infinity. To every such function corresponds a linear functional given by

$$h \mapsto \frac{1}{2\pi i} \int gh ,$$

where the integral is taken along some contour surrounding $J(f_c)$. Now a change of the variable in this integral shows that λ^{-1} is an eigenvalue iff for every function h holomorphic in a neighborhood of $J(f_c)$,

$$\int \left(g - \lambda \frac{g \circ f_c}{f_c'}\right) h = 0 .$$

Thus $w = g - \lambda g \circ f_c / f_c'$ is holomorphic on J_c. It is also holomorphic in $\bar{\mathbf{C}} \backslash (J(f_c) \cup \{0\})$ because $f_c'(z) = 2z$. We conclude that $w(z) = \text{const}/z$ and after the normalization of g we get the functional equation

$$g(z) = \frac{\lambda}{2z} g(f_c(z)) + \frac{1}{z} ,$$

from which follows that

$$g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{\lambda^n}{2^n z f_c(z) \ldots f_c^{*n}(z)} .$$

Now g is holomorphic at 0 so the residue of the series in the right side should vanish, that is

$$D_c(\lambda) = 1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{2^n f_c(0) \ldots f^{*n}(0)} = 0 .$$

We thank the referee for his valuable comments.

References

1 Azarin, V.: On the asymptotic behavior of subharmonic functions of finite order. Math. USSR Sbornik **36**, 135 (1980)
2. Anderson, J , Baernstein, A.: The size of the set on which a meromorphic function is large Proc. London Math. Soc. **36**, 518–539 (1978)
3 Beurling, A.: Some remarks on entire functions of exponential type Collected Works, v. **1**, Boston: Birkhäuser 1989, p 386
4. Brelot, M : On topologies and boundaries in potential theory Lect. Notes Math. **175**, Berlin, Heidelberg, New York: Springer 1971
5. Eremenko, A , Lyubich, M.: The dynamics of analytic transformations Leningrad Math J. **1**, 563–634 (1990)
6. Hardy, G.H : On the zeroes of a class of integral functions. Mess. Math. **34**, 97–101 (1905); Collected Papers, v. **IV**, pp. 95–99
7 Hörmander, L.: The analysis of linear partial differential operators, Vol. **1** Berlin, Heidelberg, New York: Springer 1983
8 Kadanoff, L., Tang, C.: Escape from strange repellers. Proc. Nat. Acad. Sci. USA **81**, 1276–1279 (1984)
9. Levin, G.: On Mayer's conjecture and zeros of entire functions. Preprint 17 (1991/92), Hebrew University, Jerusalem, 1992
10. Levin, G., Sodin, M , Yuditskii, P.: A Ruelle operator for a real Julia set. Commun. Math. Phys **141**, 119–132 (1991)
11. Levin, G., Sodin, M., Yuditskii, P : Ruelle operators with rational weights for Julia sets. To appear in J. d' Analyse Math.
12 Ruelle, D.: Zeta-Functions for Expanding Maps and Anosov Flows. Invent. Math. **34**, 231–242 (1976)
13 Ruelle, D.: Repellers for real analytic maps. Ergod. Th. & Dynam. Sys. **2**, 99–107 (1982)
14 Ruelle, D.: The thermodynamic formalism for expanding maps Commun Math. Phys **125**, 239–262 (1989)
15. Ruelle, D.: Spectral properties of a class of operators associated with conformal maps in two dimension. Commun. Math. Phys. **144**, 537–556 (1992)
16. Sodin, M , Yuditskii, P : The limit-periodic finite-difference operator on $l^2(\mathbf{Z})$ associated with iterations of quadratic polynomials J Stat Phys **60**, 853–873 (1990)

Communicated by J.-P. Eckmann