

Quantum Group Gauge Theory on Quantum Spaces

Thomasz Brzeziński^{*}, Shahn Majid^{**}

Department of Applied Mathematics and Theoretical Physics University of Cambridge,
CB3 9EW, U.K.

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Abstract: We construct quantum group-valued canonical connections on quantum homogeneous spaces, including a q -deformed Dirac monopole on the quantum sphere of Podles with quantum differential structure coming from the 3D calculus of Woronowicz on $SU_q(2)$. The construction is presented within the setting of a general theory of quantum principal bundles with quantum group (Hopf algebra) fibre, associated quantum vector bundles and connection one-forms. Both the base space (spacetime) and the total space are non-commutative algebras (quantum spaces).

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1. Introduction

Non-commutative geometry is based on the simple idea that in place of working with the points on a space or manifold M we may work equivalently with the algebra $C(M)$

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^{**} SERC Fellow and Drapers Fellow of Pembroke College, Cambridge

of functions on M . In this algebraic form we need not suppose that the algebra is commutative. A non-commutative algebra B when viewed as if it is the functions on some (non-existing) space is called a *quantum space*. The process of quantization in physics precisely turns the commutative algebra of observables of a classical system into a non-commutative one, hence the terminology.

Well-established in this programme are notions of integration, differential enveloping algebras (roughly speaking, differential forms), cohomology classes and Chern characters [4, 2, 7, 14]. Not only vector bundles but also $GL(n)$ frame bundles can be understood in this context [3]. This line of development can also be expected to have important applications in physics, see [6, 5] and also [10]. An important theme in these works is the use of non-commutative geometry to formulate some kind of generalization of gauge theory.

In contrast to this existing approach to non-commutative geometry, we would like to take here some steps towards developing a gauge theory in which a more fundamental role is played by quantum groups, appearing as the fibre of a quantum principal bundle and playing the role of structure group in the group of gauge transformations. Here quantum groups (Hopf algebras) are commonly accepted as the natural analogue in non-commutative geometry of a group. Moreover, nowadays a rich supply of true quantum groups (neither commutative nor dual to a commutative one) are known [9, 28, 12, 21]. Hence it seems an appropriate time to develop such a formalism. Most of the formalism needed is in fact relatively straightforward (and not incompatible with existing ideas in non-commutative geometry) and from this point of view perhaps the most significant part of the paper is the rich class of examples that we also provide. These examples are modelled on the principal bundles and canonical connections associated to suitable homogeneous spaces. We present the examples and some aspects of the formal setting in which they should be viewed.

We would like to mention at least two physical motivations for developing such a quantum-group gauge theory. The first is a formal interest in developing q -deformed versions of many constructions in physics. The introduction of such a parameter q may then be useful for example to regularise infinities that arise in the corresponding quantum field theory, which could appear now as poles in the q -plane [18]. After renormalizing (using identities from q -analysis) one could set $q = 1$. One may envisage other applications also in which q has a more physical meaning. The most popular quantum groups as in [9, 12] should be understood precisely as such q -deformations rather than arising literally from a process of physical quantization. The differential structure on quantum groups and certain quantum spaces are also well-understood from this deformation point of view and we shall need to make use of this when constructing examples.

The second and more standard motivation arises from the general indication that the small-scale structure of space time is not well-modelled by usual continuum geometry. At the Planck scale one may reasonably expect that our notion of geometry has to be modified to include quantum effects also. Non-commutative geometry clearly has the potential to do this, and this is surely one of the long-term motivations behind some of the serious attempts to develop it, such as [5]. It was also the motivation behind the introduction of the class of Hopf algebras in [21]. These (unlike the more familiar quantum groups) are genuinely the quantum algebras of observables of certain quantum systems. It is hoped that some of these various constructions can ultimately be combined with the quantum group gauge theory developed here.

An outline of the paper is as follows. In order to provide the context for our principal bundles we shall have to introduce a significant amount of formalism. Our

preliminary Sect. 2 begins by recalling the standard approach to quantum differential calculus. Given an algebra B (such as the quantum base space of the bundle) one can take as exterior algebra the universal differential envelope ΩB as in [4, 14]. One can also construct other differential calculi as quotients of it. The one-forms are denoted Γ_B .

The axioms and properties of Hopf algebras are recalled in Sect. 3 which then proceeds to give the most elementary version of the theory: the version in a local coordinate system valid for the case of trivial bundles. Gauge fields, curvature forms, sections, covariant derivatives and gauge transformation properties are defined in an obvious way that closely resembles formulae familiar to physicists for ordinary gauge fields. This section is also preliminary and serves to introduce several standard notions that will play an important role in the later sections, such as coactions, comodule algebras and the convolution product $*$. It also provides the local picture to which we feel any reasonable theory of principal and associated bundles should reduce in the trivial case.

An unusual feature encountered here even at the level of trivial bundles, is that the group of gauge transformations (which remains an ordinary group) does not consist only of algebra maps from A (the quantum group) to B (the base quantum space) as one might naively expect, but needs to be enlarged as soon as B is non-commutative.

In Sect. 4 we pass to the more abstract setting needed to handle non-trivial bundles. By definition these are algebras P (the total quantum space) on which A coacts with fixed-point subalgebra B . In addition, we need some condition corresponding to freeness of the action and an exactness condition to replace smoothness and dimension arguments in the classical situation. We do this in terms of a map \sim generating the fundamental vector fields on P induced by the coaction A . One can also formulate the “local triviality” of the situation in terms of the patching together of a collection of trivial bundles related to each other by gauge transformations. Other “purely quantum” possibilities also open up once the algebras are non-commutative.

In this abstract setting one works with a connection as a splitting of the tangent or cotangent space (in our case it is convenient to use the latter). A main (if tedious) task in any textbook on differential geometry is to relate this abstract definition of a connection to another definition as a connection one-form on P , and to show in the trivial case that this in turn implies the usual local picture of gauge fields relative to a choice of trivialization. This is the main result of Sect. 4. The general theory is further continued for associated vector bundle in Appendix A. Although relatively straightforward, there are a number of subtleties arising from the non-commutativity of the algebras and our propositions clarify and justify the various choices that are needed.

Since many readers may not be familiar with the necessary background in quantum differential calculi, we begin in Sect. 4 with the most accessible case of the universal differential envelope ΩP . We then come in the second half of the section to the non-universal calculi. We do not wish to claim that our formulation is the last word on this topic, but it is one that is general enough to include our current range of examples. It not only provides some kind of setting for the examples, but also provides for their local description via the propositions in this section and in Appendix A.

Finally we are in a position in Sect. 5 to construct our examples of quantum principal bundles and connections on them, based on quantum homogeneous spaces and their canonical connections. By quantum homogeneous space we mean a pair of quantum groups $P \rightarrow A$ (where the Hopf algebra surjection corresponds to the inclusion of the structure group as a subgroup in the classical case) subject to certain

conditions. For a connection one needs in the classical case that the subgroup is reductive – the analogue of this for our purposes is that we need to split the surjection by an Ad-covariant algebra map $i : A \hookrightarrow P$, at least locally.

The simplest non-trivial case is then examined in detail, with $A = k(S^1)$ and $P = SO_q(3)$. Here the base is the quantum sphere of Podleś [23] and the bundle is a quantum monopole bundle. The canonical connection is studied, and with the correct quantum-differential calculus (not the universal one) it recovers the standard $U(1)$ -Dirac monopole in the limit $q \rightarrow 1$. The differential calculus chosen for this example is inherited from the 3D one on $SU_q(2)$ introduced in [29]. It demonstrates the usefulness of the various conditions and results of the general theory of Sect. 4, and also connects ultimately with a local description as in Sect. 3.

Finally, because our formulae for abstract Hopf algebras may be a little unfamiliar, we collect together in Appendix B the various formulae in the case when A is a matrix quantum group. Here the convolution product $*$ corresponds to matrix multiplication.

Throughout the paper our algebras are assumed unital algebras over a field k of characteristic not 2. It is hoped that our algebraic formulation may be useful in purely algebraic work also, such as the introduction of new invariants of algebras and Hopf algebras based on gauge theory. In the other direction, the algebraic setting may be useful even in the classical case in the form of finite models of gauge theory – comparable to finite lattice models of gauge theory but preserving much more of the geometrical picture in an exact form. For example, the space of gauge fields relative to a given one could be some finite-dimensional space which could then be integrated over. For infinite systems of course one needs to work with operator algebras. Here we would like to note that all our constructions are fully compatible with $*$ -algebra structures placed on the algebras, and hence suitable for such a treatment. We will, however, have enough to do in the present paper at a purely algebraic level.

2. Preliminaries about Universal Differential Calculus

Here we recall some standard facts about differential calculus on an algebra. We refer to [4, 14] for further details.

The general notion is that of a \mathbb{Z}_2 -graded differential algebra, meaning an algebra \mathcal{E} equipped with \mathbb{Z}_2 -grading (denoted by ∂) and a linear operation $d : \mathcal{E} \rightarrow \mathcal{E}$ of degree 1, obeying the graded Leibniz rule and such that $d^2 = 0$. We will say that (Γ, d) is a *first order differential calculus* over an algebra A if $d : A \rightarrow \Gamma$ is a linear map obeying the Leibniz rule, Γ is a bimodule over A and every element of Γ is of the form $\sum_k a_k db_k$, where $a_k, b_k \in A$. To every first order differential calculus (Γ, d) over A one can associate a \mathbb{Z}_2 -graded differential algebra $(\Omega(A), d)$ in the following way. Firstly, one defines $\Omega^0(A) = A$ and

$$\Omega^n(A) \subset \Gamma \otimes_A \Gamma \otimes_A \dots \otimes_A \Gamma = \Gamma^{\otimes_A n}$$

for $n > 0$, as a set spanned by all elements:

$$(a_0, a_1, \dots, a_n) = a_0 \otimes_A da_1 \otimes_A \dots \otimes_A da_n \tag{1}$$

for any $a_0, a_1, \dots, a_n \in A$. One can then introduce the natural \mathbb{Z}_2 -grading, $\partial \omega_n = n(\text{mod } 2)$ and define $\Omega(A) = \bigoplus_{n=1}^{\infty} \Omega^n(A)$. The product of (a_0, \dots, a_n)

$\in \Omega^n(A)$ and $(a_{n+1}, \dots, a_{n+m}) \in \Omega^{m-1}(A)$ is given by

$$(a_0, \dots, a_n)(a_{n+1}, \dots, a_{m+n}) = \sum_{i=0}^n (-1)^i (a_0, \dots, a_{n-1-i}, a_{n-i} a_{n-i+1}, a_{n-i+2}, \dots, a_{n+m}), \tag{2}$$

and d is extended to the whole of $\Omega(A)$ by:

$$d(a_0, a_1, \dots, a_n) = (1, a_0, a_1, \dots, a_n), \\ d(1, a_0, a_1, \dots, a_n) = 0.$$

$\Omega(A)$ is therefore a free tensor algebra modulo relation (2). In some cases however one can consider ideals $I^n \subset \Omega^n(A)$ and define the exterior algebra of A associated to Γ_A by taking quotients $\Omega^n(A)/I^n$. Ideals I^n have to be compatible with the action of the differential d . In what follows we do not stress the difference between $\Omega(A)$ and suitable quotients of it.

It is known that every first order differential calculus on an algebra A can be obtained as the quotient of a universal differential calculus (A^2, d) . Here $A^2 = \ker \cdot$ (where $\cdot : A \otimes A \rightarrow A$ is the multiplication map in A) and $d : A \rightarrow A^2$ is defined by

$$da = 1 \otimes a - a \otimes 1. \tag{3}$$

This map d clearly obeys the Leibniz rule provided A^2 has the A -bimodule structure given by

$$c \left(\sum_k a_k \otimes b_k \right) = \sum_k ca_k \otimes b_k, \tag{4}$$

$$\left(\sum_k a_k \otimes b_k \right) c = \sum_k a_k \otimes b_k c \tag{5}$$

for any $\sum_k a_k \otimes b_k \in A^2$, $c \in A$. Furthermore, it is easy to see that every element of A^2 can be represented in the form $\sum_k a_k db_k$. In this way (A^2, d) is indeed a first order differential calculus over A as stated. The \mathbb{Z}_2 -graded differential algebra defined by (A^2, d) will be denoted by $(\Omega A, d)$ and called the *differential envelope* of A (cf. [14]). We have the following universality principle:

Proposition 2.1 ([14, 8]). *Let (Ξ, δ) be any differential algebra with unity, and A any algebra with unity. Any 0-degree homomorphism $\alpha : A \rightarrow \Xi$ can be lifted to a unique 0-degree homomorphism $\theta : \Omega A \rightarrow \Xi$ such that $\theta|_A = \alpha$ and $\theta \circ d = \delta \circ \alpha$.*

By the natural identification $A \otimes_A A \cong A$ one can easily prove by induction (see [4]) that

$$\Omega^n A = \{ \varrho \in A \otimes_k \dots \otimes_k A = A^{\otimes n+1} : \forall i \in \{1, \dots, n\}, \cdot_i \varrho = 0 \},$$

where

$$\cdot_i = \text{id} \otimes_k \text{id} \otimes_k \dots \otimes_k \cdot \otimes_k \dots \otimes_k \text{id}$$

(multiplication \cdot acting in the $i, i + 1^{\text{th}}$ place). Hence $\Omega^n A \subset A^{\otimes_k n+1}$. Notice that the description of $\Omega^n A$ is purely algebraic (i.e. it depends only on the properties of the multiplication in A). In particular, this means that if B is a subalgebra of A with $j : B \hookrightarrow A$ the inclusion map, then j can be extended as an inclusion $j : \Omega B \hookrightarrow \Omega A$.

Proposition 2.1 allows one to reconstruct any differential algebra $\Omega(A)$ as

$$\Omega^n(A) = \Omega^n A / N^n,$$

where $N^n \subset \Omega^n A$ are A -bimodules, $n = 1, 2, \dots$. If $B \subset A$ then we will take the differential structure $\Omega(B)$ as defined by the B -bimodules $N_B^n = N^n \cap \Omega^n B$. This assumption implies that the inclusion $j : B \hookrightarrow A$ extends to an inclusion $j : \Omega(B) \hookrightarrow \Omega(A)$, commuting with d .

3. Gauge Fields on Trivial Quantum Vector Bundles

In this second preliminary section we present the construction of trivial quantum vector bundles and gauge fields on them. This also serves to introduce the basic facts and constructions for Hopf algebras (quantum groups) which will be needed later. The role of the structure group is played by the quantum group or Hopf algebra and the roles of the base and fibre are played by algebras which can also be non-commutative (i.e. quantum spaces). In fact the definitions presented here are a special case of a general theory of quantum vector bundles which will be described later. Here we would like to emphasise instead the definition of quantum vector bundles from the point of view of gauge transformations. This gives a self-contained picture in which all fields live on the base. This point of view is closely related to physics and has proven to be very fruitful. Moreover, it provides the basic local theory to which our general abstract theory must reduce in the trivial case.

Let us recall that a Hopf algebra is an associative algebra A with unit equipped with a compatible coalgebra structure. This consists of algebra maps $\Delta : A \rightarrow A \otimes A$ (the comultiplication), $\varepsilon : A \rightarrow k$ (the counit) and a linear map $S : A \rightarrow A$ (the antipode) obeying the following axioms

1. $(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta,$
2. $(\varepsilon \otimes \text{id})\Delta = (\text{id} \otimes \varepsilon)\Delta = \text{id},$
3. $\cdot(S \otimes \text{id})\Delta = \cdot(\text{id} \otimes S)\Delta = \eta \circ \varepsilon.$

Here \cdot denotes multiplication in A and $\eta : k \rightarrow A$ is the unit map, i.e. $\eta(\lambda) = \lambda 1_A, \forall \lambda \in k$. We adopt Sweedler's sigma notation [26], namely $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$, for any $a \in A$.

If A is a Hopf algebra then we say that a vector space V is a *left A -comodule* if there exists a map $\varrho_L : V \rightarrow A \otimes V$ (a left coaction of A on V) such that

$$(\Delta \otimes \text{id})\varrho_L = (\text{id} \otimes \varrho_L)\varrho_L, \quad (\varepsilon \otimes \text{id})\varrho_L = \text{id}.$$

If V is an algebra and ϱ_L is an algebra map, i.e.

$$\varrho_L(ab) = \varrho_L(a)\varrho_L(b), \quad \varrho_L(1_V) = 1_A \otimes 1_V,$$

then we will say that V is a *left A -comodule algebra*. We will sometimes use the explicit notation $\varrho_L(v) = \sum v^{(1)} \otimes v^{(2)}$ for any $v \in V$.

Similarly we say that a vector space V is a *right A -comodule* if there exists a linear map $\varrho_R : V \rightarrow V \otimes A$ (a right coaction of A on V) such that

$$(\varrho_R \otimes \text{id})\varrho_R = (\text{id} \otimes \Delta)\varrho_R, \quad (\text{id} \otimes \varepsilon)\varrho_R = \text{id}.$$

If V is an algebra and ϱ_R is an algebra map then we say that V is a *right A -comodule algebra*.

Given a bialgebra A there is an opposite bialgebra A^{op} consisting of A with the opposite product. If A is a Hopf algebra with bijective antipode then S^{-1} makes A^{op} also into a Hopf algebra. When we come to the abstract theory of associated vector bundles we will need both A -comodule algebras and A^{op} -comodule algebras in order to make a quotient tensor product algebra by the coaction (a cotensor product).

To complete our preliminary remarks on Hopf algebras we recall the convolution product of linear maps on a Hopf algebra (or coalgebra) A . Let B be an algebra and $f_1, f_2 : A \rightarrow B$ be two linear maps. The *convolution* product of f_1 and f_2 (denoted by $g = f_1 * f_2$) is the linear map $g : A \rightarrow B$ given by $g(a) = \sum f_1(a_{(1)})f_2(a_{(2)})$ for any $a \in A$. The convolution product is associative and makes the set $\text{Lin}(A, B)$ into an algebra. Note that if B has a unit η_B (viewed as a map) then $f * (\eta_B \circ \varepsilon) = (\eta_B \circ \varepsilon) * f = f$, so that $\eta_B \circ \varepsilon$ is the identity in the convolution algebra $\text{Lin}(A, B)$. We say that a linear map $f : A \rightarrow B$ is *convolution invertible* if there exists a map $f^{-1} : A \rightarrow B$ such that $f^{-1} * f = f * f^{-1} = \eta_B \circ \varepsilon$. Similarly if V is a left A -comodule and $f_1 : A \rightarrow B, f_2 : V \rightarrow B$, then $(f_1 * f_2)(v) = \sum f_1(v^{(1)})f_2(v^{(2)})$ for any $v \in V$. Finally if Γ is any bimodule of B and $f_1 : A \rightarrow B, f_2 : V \rightarrow \Gamma$ we define $(f_1 * f_2)(v) = \sum f_1(v^{(1)})f_2(v^{(2)})$.

Now we are in a position to introduce the notion of a trivial (left) quantum vector bundle.

Definition 3.1. Let $(A, \Delta, \varepsilon, S)$ be a Hopf algebra. We say that $E(B, V, A)$ is a *trivial (left) quantum vector bundle* with base B , fibre V and structure group A if:

1. B is an algebra with unity;
2. (V, ϱ_L) is a left A -comodule algebra;
3. $E = V \otimes B$.

Let us note that E is a left A -comodule algebra. The coaction $\Delta_L : E \rightarrow A \otimes E$ is given by $\Delta_L = \varrho_L \otimes \text{id}$ and the multiplication $(v_1 \otimes b_1)(v_2 \otimes b_2) = v_1 v_2 \otimes b_1 b_2$ is the tensor product one.

A *quantum gauge transformation* of our trivial vector bundle $E(B, V, A)$ is then a convolution invertible map $\gamma : A \rightarrow B$ such that $\gamma(1) = 1$. We say that $\sigma : V \rightarrow B$ is a *section* of E if it transforms under the action of gauge transformation γ according to the law $\sigma \xrightarrow{\gamma} \sigma^\gamma = \gamma * \sigma$. A acts on V according to the left coaction ϱ_L . The set of sections of E will be denoted by $\Gamma(E)$. If $\Omega(B)$ is a differential algebra over B then we also consider n -form sections $\Gamma^n(E)$, the set of maps $V \rightarrow \Omega^n(B)$.

To make these definitions more transparent let us consider their classical limit (see e.g. [13]). Let U be an open set on the base, G a Lie group, and suppose the vector space \mathbb{C}^n forms a representation of G . We can think of G concretely as a matrix group contained in $GL(n)$ and define, $A = C^\infty(G), V = C^\infty(\mathbb{C}^n)$ and $B = C^\infty(U)$. In a suitable algebraic context, A becomes a Hopf algebra and the algebra of functions on the trivial vector bundle $E = C^\infty(\mathbb{C}^n \times U)$ becomes $V \otimes B$. A section on the bundle $\mathbb{C}^n \times U$ is a vector valued function $s : U \rightarrow \mathbb{C}^n$ and a gauge transformation is a matrix valued function $g : U \rightarrow G$. Sections and gauge transformations give rise to algebra maps $\sigma : C^\infty(\mathbb{C}^n) = V \rightarrow B = C^\infty(U)$ and $\gamma : C^\infty(G) = A \rightarrow B = C^\infty(U)$ respectively, induced by pull-back. Moreover the gauge transformation g acting pointwise induces a transformation of sections $s \mapsto s^g$, which in components reads:

$$(s^g)^i(x) = g_j^i(x)s^j(x)$$

for all $x \in U$. This in turn gives rise to the transformation of σ , namely as

$$\sigma^\gamma(v^i) = \gamma(g_j^i)\sigma(v^j).$$

This explains our definition of quantum gauge transformations and sections of quantum vector bundles.

The next step in the construction of quantum group gauge theory consists of the definition of a covariant exterior derivative. To this end let us assume that (Γ_B, d) is a first order differential calculus over B and $\Omega(B)$ is the differential algebra induced by it. We say that a linear map $\nabla : \Gamma(E) \rightarrow \Gamma^1(E)$ is a *covariant exterior derivative* on the trivial quantum vector bundle E if for any quantum gauge transformation γ on E , there exists a map $\nabla^\gamma : \Gamma(E) \rightarrow \Gamma^1(E)$ such that for any section $\sigma \in \Gamma(E)$,

$$\nabla^\gamma \sigma^\gamma = \gamma * (\nabla \sigma). \tag{6}$$

In other words, $\nabla : \Gamma(E) \rightarrow \Gamma^1(E)$ is a covariant exterior derivative on E if ∇ transforms under a gauge transformation γ according to the rule:

$$\nabla \xrightarrow{\gamma} \nabla^\gamma = \gamma * \nabla \gamma^{-1} * . \tag{7}$$

Just as in the classical case we have the following:

Proposition 3.2. *Let $E(B, V, A)$ be a trivial quantum bundle. If a map $\beta : A \rightarrow \Gamma_B$ transforms by the quantum gauge transformation γ of E as*

$$\beta \xrightarrow{\gamma} \beta^\gamma = \gamma * \beta * \gamma^{-1} + \gamma * d(\gamma^{-1}) \tag{8}$$

then the map $\nabla : \Gamma(E) \rightarrow \Gamma^1(E)$ given by

$$\nabla = d + \beta * \tag{9}$$

is a covariant exterior derivative on E .

Proof. We have to check that the linear operation ∇ given by Eq. (9) transforms according to the rule (6). For any section $\sigma \in \Gamma(E)$ we have

$$\begin{aligned} \nabla^\gamma \sigma^\gamma &= d\sigma^\gamma + \beta^\gamma * \sigma^\gamma = d(\gamma * \sigma) + (\gamma * \beta * \gamma^{-1} + \gamma * d(\gamma^{-1})) * \gamma * \sigma \\ &= d\gamma * \sigma + \gamma * d\sigma + \gamma * \beta * \sigma - d\gamma * \sigma = \gamma * (\nabla \sigma). \end{aligned}$$

Hence ∇ transforms as a covariant derivative and the result follows. \square

A map $\beta : A \rightarrow \Gamma_B$ as in Proposition 3.2 is called a *connection one-form* on E or simply a connection on E (or quantum group gauge field). The transformation rule for connections implies the following:

Proposition 3.3. *Let $\gamma, \gamma' : A \rightarrow B$ be two gauge transformations on a trivial quantum vector bundle $E(B, V, A)$ and let $\beta : A \rightarrow \Gamma_B$ be a connection on E . Then*

$$(\beta^\gamma)^{\gamma'} = \beta^{\gamma' * \gamma}. \tag{10}$$

Proof. The proof is based on direct use of the rule (8), namely

$$\begin{aligned}
 (\beta^\gamma)^{\gamma'} &= \gamma' * \beta^\gamma * (\gamma')^{-1} + \gamma' * d(\gamma'^{-1}) \\
 &= \gamma' * \gamma * \beta * \gamma^{-1} * \gamma'^{-1} + \gamma' * \gamma * d(\gamma^{-1}) * \gamma'^{-1} + \gamma' * d(\gamma'^{-1}) \\
 &= \gamma' * \gamma * \beta * \gamma^{-1} * \gamma'^{-1} + \gamma' * \gamma * d((\gamma' * \gamma)^{-1}) - \gamma' * d(\gamma'^{-1}) + \gamma' * d(\gamma'^{-1}) \\
 &= \beta^{\gamma' * \gamma}. \quad \square
 \end{aligned}$$

To any connection β on a trivial quantum vector bundle $E(B, V, A)$ one can associate its curvature $F : A \rightarrow \Omega^2(B)$ defined as

$$F = d\beta + \beta * \beta. \tag{11}$$

Proposition 3.4. *Let $E(B, V, A)$ be a trivial quantum vector bundle. Let $\beta : A \rightarrow \Gamma_B$ be a connection one-form on E and $F : A \rightarrow \Omega^2(B)$ its curvature. Then we have:*

1. For any section $\sigma \in \Gamma(E)$

$$\nabla^2 \sigma = F * \sigma. \tag{12}$$

2. For any quantum gauge transformation γ of E

$$F^\gamma = \gamma * F * \gamma^{-1}. \tag{13}$$

3. The Bianchi identity

$$dF + \beta * F - F * \beta = 0. \tag{14}$$

Proof. 1. We have

$$\begin{aligned}
 \nabla^2 \sigma &= \nabla(d\sigma + \beta * \sigma) = d(\beta * \sigma) + \beta * d\sigma + \beta * \beta * \sigma \\
 &= (d\beta + \beta * \beta) * \sigma = F * \sigma.
 \end{aligned}$$

2. The transformation law for curvature follows immediately from the definition (11).

3. We compute:

$$\begin{aligned}
 dF &= d\beta * \beta - \beta * d\beta = d\beta * \beta + \beta * \beta * \beta - \beta * \beta * \beta - \beta * d\beta \\
 &= F * \beta - \beta * F. \quad \square
 \end{aligned}$$

As we can see, all the results obtained here are very similar to the classical ones except that the usual product of functions is replaced by the convolution product. In fact the convolution product appears also in the classical construction where it corresponds to group multiplication or the action of the group – but now instead of considering groups and representation spaces we consider algebras of functions on them. The main difference between classical and quantum vector bundles lies in the fact that if E is a noncommutative algebra and A is a quantum group, they cannot be interpreted as algebras of functions on an actual vector bundle and its structure group respectively.

In the construction above we have restricted ourselves to the consideration of left quantum vector bundles and structures related to them. But there is well established symmetry between left and right constructions. To conclude this section we summarize a version of the above results based on right quantum vector bundles.

Definition 3.5. Let $(A, \Delta, \varepsilon, S)$ be a Hopf algebra. We say that $E(B, V, A)$ is a *trivial (right) quantum vector bundle* with base B , fibre V and structure quantum group A if:

1. B is an algebra with unity;
2. (V, ϱ_R) is a right A^{op} -comodule algebra;
3. $E = B \otimes V$.

Then we have the following. The induced right coaction $\Delta_R : E \rightarrow E \otimes A$ of A on E is given by:

$$\Delta_R = \text{id} \otimes \varrho_R.$$

The gauge transformation of sections:

$$\sigma^\gamma = \sigma * \gamma. \tag{15}$$

The gauge transformation of covariant derivatives:

$$\nabla^\gamma \sigma^\gamma = (\nabla \sigma) * \gamma.$$

The gauge transformation of connection 1-forms β :

$$\beta^\gamma = \gamma^{-1} * \beta * \gamma + \gamma^{-1} * d\gamma.$$

Hence the covariant derivative acts on sections $\sigma \in \Gamma(E)$, as:

$$\nabla \sigma = d\sigma - \sigma * \beta, \tag{16}$$

and on the linear maps $\varrho \in \Gamma^n(E)$:

$$\nabla \varrho = d\varrho - (-1)^n \varrho * \beta. \tag{17}$$

Some properties of the curvature 2-form $F = d\beta + \beta * \beta$ are:

$$\begin{aligned} \nabla^2 \sigma &= -\sigma * F, \\ F^\gamma &= \gamma^{-1} * F * \gamma, \end{aligned}$$

and the Bianchi identity:

$$dF + \beta * F - F * \beta = 0.$$

Some of the relations above need more explanation. Although they look a little bit unusual, one can show that in fact the right-covariant construction provides the correct classical limit (as we will see in the next section). There are two facts which play a crucial role in this identification. First of all let us state the following elementary lemma:

Lemma 3.6. *Let A be a Hopf algebra and let (V, ϱ_R) be a right A^{op} -comodule algebra. Then V is the left A -comodule algebra (V, ϱ_L) with coaction given by*

$$\varrho_L = \tau(\text{id} \otimes S)\varrho_R,$$

where τ is the usual twist map.

Proof. This is an elementary exercise from the definitions above and the fact that for any Hopf algebra the antipode $S : A \rightarrow A$ is an antialgebra and anticoalgebra map. \square

Classically, a connection 1-form β is a Lie algebra-valued 1-form on the base. Here the Lie algebra is that of the classical gauge group G . We can view it as a subset of its universal enveloping Hopf algebra, and on this subset the antipode acts by -1 . In our dual picture it means that in the classical limit we have $\beta \circ S = -\beta$, where S is the antipode on A . Thus if we convert our right A^{op} - comodule algebra to a left A -comodule algebra by means of the above lemma (as is usually done) the “-” sign in (16) will be absorbed. This is why no “-” sign appears in the usual classical formulae for covariant derivatives. For general Hopf algebras the action of S is more complicated and this cancellation is not possible. Secondly, in the classical case the exterior algebra is graded-commutative so that β in Eq. (17) can be written on the left of ϱ , cancelling the factor depending on its degree. Again, this is not possible for a general quantum differential calculus. We note that the $(-1)^n$ is in any case an artifact of our writing d and ∇ acting from the left when, in our right-handed conventions, they act more simply from the right.

4. Quantum Principal Bundles and Connections on Them

In this section we give a general theory of quantum principal bundles. We first work in the universal differential envelope, and come to the case of a general differential calculus in the second subsection.

We begin with a brief outline of the classical theory of connections and fibre bundles, following [15] and emphasising the aspects that we shall generalise to the quantum case. Let M be a smooth manifold and G a Lie group. A principal bundle over M consists of a smooth manifold P and a smooth action of G on P such that G acts freely on P from the right, i.e. $P \times G \ni (u, a) \mapsto ua = R_a u \in P$ is an action and

$$P \times G \rightarrow P \times P, \quad (u, a) \mapsto (u, ua) \tag{18}$$

is an inclusion (freeness). Moreover, $M \cong P/G$ and the canonical projection $\pi : P \rightarrow M$ is a smooth map. We denote the principal bundle by $P(M, G)$ or simply by P . Locally $P \cong M \times G$. This means that if $U \subset M$ is an open set covered by one chart, then there exists a map $\phi_U : \pi^{-1}(U) \rightarrow G$ such that $\phi_U(ua) = \phi_U(u)a$ and such that the map $\pi^{-1}(U) \rightarrow U \times G$, defined by $u \mapsto (\pi(u), \phi(u))$ is an isomorphism.

For each $u \in P$ let $T_u P$ be the tangent space of P at u and G_u the subspace of $T_u P$ consisting of vectors tangent to the fibre through u . A connection Π in P is an assignment of a subspace Q_u of $T_u P$ to each $u \in P$ such that

$$T_u P = G_u \oplus Q_u \tag{19}$$

and $Q_{ua} = (R_a)_* Q_u$ for any $u \in P$ and $a \in G$. We call G_u the vertical subspace and Q_u the horizontal subspace of $T_u P$. Given a connection Π in P we define a 1-form ω on P with values in the Lie algebra \mathfrak{g} of G in the following way. Any $\xi \in \mathfrak{g}$ induces a *fundamental vector field* $\tilde{\xi}$ on P . Its value on a 1-form df is

$$(\tilde{\xi}, df)(u) = \left. \frac{d}{dt} \right|_0 f(u \exp t\xi), \tag{20}$$

i.e. it is the differential of the right action of G . Now for each $X \in T_u P$ we define $\omega(X)$ to be the unique $\xi \in \mathfrak{g}$ such that $\tilde{\xi}$ is equal to the vertical component of X . Clearly $\omega(X) = 0$ if and only if $X \in Q_u$.

Equivalently the connection 1-form ω is a \mathfrak{g} -valued 1-form on P such that $\omega(\tilde{\xi}) = \xi$ for any $\xi \in \mathfrak{g}$ and $(R_a)^*\omega = \text{ad}(a^{-1})\omega$, i.e. $\omega((R_a)_*X) = \text{ad}(a^{-1})\omega(X)$ for any $a \in G$ and any vector field X . Here ad denotes the adjoint representation of G in \mathfrak{g} . Given a connection 1-form the corresponding projection is recovered by $\Pi = \tilde{\omega}$.

4.1. The Case of Universal Differential Calculus

We now come to the quantum (non-commutative) case. The first ingredient is an algebra P analogous to the functions on the total space of the principal bundle. We require this to be a comodule algebra for a Hopf algebra A with right coaction $\Delta_R : P \rightarrow P \otimes A$. We assume that the action is free in the sense that the induced map $P \otimes P \rightarrow P \otimes A$ is a surjection. This is just the straightforward dualization of (18) and is quite standard, see for example [25]. We take the invariant subalgebra $B = P^A = \{u \in P \mid \Delta_R(u) = u \otimes 1\}$ for the algebra analogous to the functions on the base manifold. This is a subalgebra for if $u, v \in B$ then

$$\Delta_R(uv) = \Delta_R(u)\Delta_R(v) = (u \otimes 1)(v \otimes 1) = (uv) \otimes 1.$$

Hence $uv \in B$. There is a natural inclusion $j : B \hookrightarrow P$ which corresponds to the canonical projection π in the classical case.

Next, in place of working with tangent bundles etc., we work with forms. These serve also to specify the differential structure on P as recalled in Sect. 2. For now we develop the theory only with the differential structure given by the universal envelope ΩP . The necessary modifications for a general differential calculus will be given later. In the case of the universal envelope our right coaction Δ_R automatically extends to ΩP as a right A -comodule $\Delta_R : \Omega P \rightarrow \Omega P \otimes A$. One says that the differential calculus is *covariant* (cf. [27]). Explicitly, the coaction is given here by:

$$\Delta_R(u_0 du_1 \cdots du_n) = \sum u_0^{(1)} du_1^{(1)} \cdots du_n^{(1)} \otimes u_0^{(2)} u_1^{(2)} \cdots u_n^{(2)}, \tag{21}$$

where $u_0, \dots, u_n \in P$ and where we use an explicit notation for Δ_R on P .

Also automatically, the inclusion $j : B \hookrightarrow P$ extends to an inclusion $j : \Omega B \hookrightarrow \Omega P$. We will be especially interested in Γ_P the space of 1-forms on P . The natural P -subbimodule here is

$$\Gamma_{\text{hor}} = Pj(\Gamma_B)P \subseteq \Gamma_P, \tag{22}$$

where Γ_B is the space of 1-forms on B . Here we think of Γ_{hor} as analogous to the space of horizontal forms coming in the classical case by pull-back from the base. We say that a one-form $\alpha \in \Gamma_P$ is *horizontal* if $\alpha \in \Gamma_{\text{hor}}$. Obviously any $\beta \in \Gamma_B$ is by definition horizontal when viewed in Γ_{hor} via the canonical inclusion j .

Finally, we need the notion of a map $\tilde{}$ generating the fundamental vector fields for our coaction Δ_R . This appears in our dual formulation as a left P -module map

$$\tilde{} = (\cdot \otimes \text{id}) \circ (\text{id} \otimes \Delta_R)|_{P^2} : \Gamma_P \rightarrow P \otimes A. \tag{23}$$

Recall that by definition in the universal case Γ_P is the set $P^2 \subset P \otimes P$, where P^2 is the kernel of the product map. In explicit terms we have

$$\tilde{}(u dv) = \sum uv^{(1)} \otimes v^{(2)} - uv \otimes 1. \tag{24}$$

Because A coacts on P from the right, A^* acts on P from the left. The action of $\xi \in A^*$ is given by evaluation against the output of the coaction. Hence the left P -module map $\tilde{\xi} = (\text{id} \otimes \xi) \circ \sim : \Gamma_P \rightarrow P$ should be thought of as the “fundamental vector field” generated by the “infinitesimal” element $\xi - 1\varepsilon(\xi)$. Compare (20). It is also easy to see from these definitions that

$$\ker \tilde{} \supseteq \Gamma_{\text{hor}}. \tag{25}$$

This is because

$$\begin{aligned} \sim(u(dj(b))v) &= \sim(ud(j(b)v)) - \sim(uj(b)dv) \\ &= \sum u j(b)^{\bar{1}} v^{\bar{1}} \otimes j(b)^{\bar{2}} v^{\bar{2}} - \sum u j(b) v^{\bar{1}} \otimes v^{\bar{2}} = 0, \end{aligned}$$

where the first equality uses the Leibniz rule in Γ_P and second that P is a comodule algebra.

We are now ready to present the construction of quantum fibre bundles and connections on them.

Definition 4.1. We say that $P = P(B, A)$ is a *quantum principal bundle* with universal differential calculus, structure quantum group A and base B if:

1. A is a Hopf algebra.
2. (P, Δ_R) is a right A -comodule algebra.
3. $B = P^A = \{u \in P : \Delta_R u = u \otimes 1\}$.
4. $(\cdot \otimes \text{id}) (\text{id} \otimes \Delta_R) : P \otimes P \rightarrow P \otimes A$ is a surjection (freeness condition).
5. $\ker \tilde{} = \Gamma_{\text{hor}}$ (exactness condition for the differential envelope).

The last condition here needs some explanation. In the classical case smoothness and dimension considerations combine with freeness of the action to ensure that the quotient is a manifold and the fibre through a point u is a copy of our Lie group G . At the differential level the Lie algebra \mathfrak{g} of G is included in the vertical part of $T_u P$ by the map $\tilde{}$ that generates fundamental vector fields. Dimension arguments then imply that this map is an isomorphism of \mathfrak{g} with the vertical part of each $T_u P$. In our algebraic formulation we need to impose some kind of condition to replace this complex of ideas arising from the smoothness and dimension considerations. The one stated in the definition appears the most convenient for our formulation below. Other approaches are surely possible also. Roughly speaking, in place of dimension arguments we suppose directly that the image of the fundamental vector fields through each point spans all the vertical vectors through the point. Put another way in terms of forms, we suppose that the horizontal forms span all of the annihilator of the left-invariant vector fields. In dual form this leads to Condition 5 in the definition. We call it exactness because it states that the image of j fills out the kernel of $\tilde{}$. It is stated here for the case of the universal differential envelope on P .

We note also that this exactness condition is a kind of differential version of the idea of a Galois extension in algebra. Given Conditions 1–3 as above it is easy to see that the canonical map $(\cdot \otimes \text{id}) (\text{id} \otimes \Delta_R) : P \otimes P \rightarrow P \otimes A$ descends to a map $P \otimes_B P \rightarrow P \otimes A$ and $B \subset P$ is called a Galois extension if the map at this level is an isomorphism, see e.g. [25]. Surjectivity corresponds to our freeness condition and injectivity is sufficient to prove exactness in our sense. This is because $\tilde{}$ is the canonical map restricted to $P^2 \subset P \otimes P$. Hence an element of its kernel is also in the kernel of the canonical map and hence, in the Galois case, in the kernel of the projection $P \otimes P \rightarrow P \otimes_B P$. But the kernel of the restriction of this map

to P^2 can be identified with $Pj(B^2)P = \Gamma_{\text{hor}}$. On the other hand our geometrical condition is weaker and moreover, in a form that is suitable for generalisation later to non-universal differential calculi.

Example 4.2. Let A be a Hopf algebra and P an A -comodule algebra with invariant subalgebra B . Suppose that there exists a convolution invertible map $\Phi : A \rightarrow P$ such that

$$\Delta_R \circ \Phi = (\Phi \otimes \text{id}) \circ \Delta, \quad \Phi(1_A) = 1_P \tag{26}$$

(so Φ is an intertwiner for the right coaction). Then P is a quantum principal bundle. We call $P(B, A, \Phi)$ a trivial bundle with trivialization Φ .

Proof. An elementary fact in the situation of the example is that the map

$$B \otimes A \rightarrow P, \quad b \otimes a \mapsto j(b)\Phi(a) \tag{27}$$

is an isomorphism of linear spaces. Explicitly the inverse is given by

$$u \mapsto \sum u^{(1)}\Phi^{-1}(u^{(2)}_{(1)}) \otimes u^{(2)}_{(2)}.$$

Using that Φ is an intertwiner and the properties of comodule algebras etc. as in Sect. 3 we observe that

$$\Delta_R \Phi^{-1}(a) = \Phi^{-1}(a_{(2)}) \otimes Sa_{(1)}, \tag{28}$$

after which it is clear that the image of our inverse map lies in $B \otimes A$. It is then easy to verify that it provides the necessary inversion.

From this it follows that the freeness and exactness Conditions 4 and 5 in Definition 4.1 are automatically satisfied in this case. For the first condition assume that $\sum u_k \otimes a^k \in P \otimes A$. Define an element $\varrho \in P \otimes P$ by

$$\varrho = \sum u_k \Phi^{-1}(a^k_{(1)}) \otimes \Phi(a^k_{(2)}).$$

Then

$$(\cdot \otimes \text{id})(\text{id} \otimes \Delta_R)(\varrho) = \sum u_k \Phi^{-1}(a^k_{(1)})\Phi(a^k_{(2)}) \otimes a^k_{(3)} = \sum u_k \otimes a^k.$$

The last equality follows from the intertwiner property (26). Hence the coaction is free.

For the exactness condition we have to show that $\ker \tilde{} = Pdj(B)P$, where d is the universal differential as recalled in Sect. 2 and we work with Γ_P as the subspace P^2 of $P \otimes P$. Now any element $\varrho \in \ker \tilde{}$ can be written as $\varrho = \sum_i u_i dv_i$ for $u_i, v_i \in P$.

But since Φ establishes an isomorphism between P and $B \otimes A$ we can write each $v_i = \sum_k j(b_i^k)\Phi(a_i^k)$ say. Applying $\tilde{}$ to ϱ in this form we deduce that

$$0 = \tilde{\varrho} = \sum_{i,k} (u_i j(b_i^k)\Phi(a_i^k_{(1)}) \otimes a_i^k_{(2)} - u_i j(b_i^k)\Phi(a_i^k) \otimes 1),$$

where we used that Φ is an intertwiner. Applying the map $(\Phi^{-1} \otimes \Phi) \circ \Delta$ to the second factor we obtain

$$0 = \sum_{i,k} (u_i j(b_i^k)\Phi(a_i^k_{(1)}) \otimes \Phi^{-1}(a_i^k_{(2)}) \otimes \Phi(a_i^k_{(3)}) - u_i j(b_i^k)\Phi(a_i^k) \otimes 1 \otimes 1).$$

Finally we multiply the first two factors to conclude that

$$0 = \sum_{i,k} (u_i j(b_i^k) \otimes \Phi(a_i^k) - u_i j(b_i^k) \Phi(a_i^k) \otimes 1) = \sum_{i,k} u_i j(b_i^k) d\Phi(a_i^k).$$

Hence using the Leibniz rule we have

$$\begin{aligned} \varrho &= \sum_{i,k} u_i d(j(b_i^k) \Phi(a_i^k)) = \sum_{i,k} (u_i (d(j(b_i^k)) \Phi(a_i^k) + u_i j(b_i^k) d\Phi(a_i^k)) \\ &= \sum_{i,k} u_i (dj(b_i^k)) \Phi(a_i^k), \end{aligned}$$

and hence manifestly lies in $Pdj(B)P$ as required. \square

Next in our dual formulation we define a *connection* Π on a quantum principal bundle P as an assignment of a left P -submodule $\Gamma_{\text{ver}} \subseteq \Gamma_P$ such that:

1. $\Gamma_P = \Gamma_{\text{hor}} \oplus \Gamma_{\text{ver}}$,
2. projection $\Pi : \Gamma_P \rightarrow \Gamma_{\text{ver}}$ is right invariant, i.e.

$$\Delta_R \Pi = (\Pi \otimes \text{id}) \Delta_R. \tag{29}$$

An element $\alpha \in \Gamma_{\text{ver}}$ is called a vertical form. If there exists a connection in P then any one-form $\alpha \in \Gamma_P$ can be uniquely written as a sum of horizontal and vertical forms.

We show now that every connection has a connection form. Notice first that the space $P \otimes \ker \varepsilon$ has a natural left P -module structure. Moreover there is a natural right coaction of A on $P \otimes \ker \varepsilon$ built up as follows. A coacts on P by Δ_R and A coacts on itself by the right adjoint coaction

$$\text{Ad}_R : A \rightarrow A \otimes A, \quad \text{Ad}_R(a) = \sum a_{(2)} \otimes (S a_{(1)}) a_{(3)}. \tag{30}$$

It is easy to see that this restricts to a coaction Ad_R on $\ker \varepsilon$ also. Hence we may define the right coaction $\Delta_R : P \otimes \ker \varepsilon \rightarrow P \otimes \ker \varepsilon \otimes A$ by

$$\Delta_R(u \otimes a) = \sum u^{(\bar{1})} \otimes a_{(2)} \otimes u^{(\bar{2})} (S a_{(1)}) a_{(3)}. \tag{31}$$

We will need the following

Lemma 4.3. *The map \sim intertwines right coactions on Γ_P and $P \otimes \ker \varepsilon$,*

$$\Delta_R \circ \sim = (\sim \otimes \text{id}) \Delta_R. \tag{32}$$

Proof. It is immediate from the form (24) that the image of \sim lies in $P \otimes \ker \varepsilon$. For any $\sum u_k \otimes v_k \in \Gamma_P$ we have

$$\begin{aligned} \Delta_R \left(\sum u_k \otimes v_k \right) &= \Delta_R \left(\sum u_k v_k^{(\bar{1})} \otimes v_k^{(\bar{2})} \right) \\ &= \sum u_k^{(\bar{1})} v_k^{(\bar{1})} \otimes v_k^{(\bar{2})}{}_{(3)} \otimes u_k^{(\bar{2})} v_k^{(\bar{2})}{}_{(1)} (S v_k^{(\bar{2})}{}_{(2)}) v_k^{(\bar{2})}{}_{(4)} \\ &= \sum u_k^{(\bar{1})} v_k^{(\bar{1})} \otimes v_k^{(\bar{2})}{}_{(1)} \otimes u_k^{(\bar{2})} v_k^{(\bar{2})}{}_{(2)}. \end{aligned}$$

On the other hand

$$\begin{aligned}
 (\sim \otimes \text{id})\Delta_R\left(\sum u_k \otimes v_k\right) &= (\sim \otimes \text{id})\left(\sum u_k^{(\bar{1})} \otimes v_k^{(\bar{1})} \otimes u_k^{(\bar{2})} v_k^{(\bar{2})}\right) \\
 &= \sum u_k^{(\bar{1})} v_k^{(\bar{1})} \otimes v_k^{(\bar{2})}_{(1)} \otimes u_k^{(\bar{2})} v_k^{(\bar{2})}_{(2)},
 \end{aligned}$$

as required. \square

The freeness and exactness conditions imply that the following sequence:

$$0 \rightarrow \Gamma_{\text{hor}} \xrightarrow{j} \Gamma_P \xrightarrow{\sim} P \otimes \ker \varepsilon \rightarrow 0 \tag{33}$$

is exact. The existence of the connection Π in P is now equivalent to the existence of a map $\sigma : P \otimes \ker \varepsilon \rightarrow \Gamma_P$ splitting the sequence (33), i.e. $\sim \circ \sigma = \text{id}$. Due to the fact that Π is a right-invariant left P -module map, the map σ has to be a right-invariant left P -module map. The projection Π is recovered as

$$\Pi = \sigma \circ \sim. \tag{34}$$

Now we define a map $\omega : A \rightarrow \Gamma_P$ by

$$\omega(a) = \sigma(1 \otimes (a - \varepsilon(a))). \tag{35}$$

We call this map the *connection form* of the connection Π .

Proposition 4.4. *Let P be a quantum principal bundle and Π a connection on it. Then the connection form $\omega : A \rightarrow \Gamma_P$ has the following properties:*

1. $\omega(1) = 0$,
2. $\sim \omega(a) = 1 \otimes a - 1 \otimes \varepsilon(a)$ for all $a \in A$,
3. $\Delta_R \circ \omega = (\omega \otimes \text{id}) \circ \text{Ad}_R$,

where Ad_R is the right adjoint coaction. Conversely, if ω is any linear map $\omega : A \rightarrow \Gamma_P$ obeying Conditions 1–3, then there is a unique connection Π ,

$$\Pi = \cdot \circ (\text{id} \otimes \omega) \circ \sim \tag{36}$$

such that ω is its connection 1-form.

Proof. Given Π we define $\omega(a) = \sigma(1 \otimes (a - \varepsilon(a)))$ as explained above. Then properties 1 and 2 follow immediately from the definition of ω .

Next we have to show that ω is Ad_R -covariant. We have

$$\begin{aligned}
 \Delta_R(\omega(a)) &= \Delta_R \sigma(1 \otimes (a - \varepsilon(a))) \\
 &= (\sigma \otimes \text{id})\Delta_R(1 \otimes (a - \varepsilon(a))) \\
 &= \sum (\sigma \otimes \text{id})(1 \otimes (a_{(2)} - \varepsilon(a_{(2)}))) \otimes (S a_{(1)}) a_{(3)} \\
 &= \sum \omega(a_{(2)}) \otimes (S a_{(1)}) a_{(3)}.
 \end{aligned}$$

From this it follows at once that ω obeys the equivariance Condition 3.

In the converse direction suppose that we are given a map ω obeying Conditions 1–3 and define $\sigma : P \otimes \ker \varepsilon \rightarrow \Gamma_P$ by $\sigma(u \otimes a) = u\omega(a)$ for $u \in P$ and $a \in \ker \varepsilon$. Then

$\tilde{\circ} \circ \sigma(u \otimes a) = u \tilde{\circ}(\omega(a)) = u \otimes a$ by the first condition on ω . Hence $\tilde{\circ} \circ \sigma = \text{id}$ and $\Pi = \sigma \circ \tilde{\circ}$ is a splitting $\Gamma_P = \Gamma_{\text{hor}} \oplus \Gamma_{\text{ver}}$ as required for a connection. Explicitly,

$$\begin{aligned} \Pi(u dv) &= \sigma \circ \tilde{\circ}(u dv) = \sum \sigma(uv^{(\bar{1})} \otimes (v^{(\bar{2})} - \varepsilon(v^{(\bar{2})}))) \\ &= \sum uv^{(\bar{1})}\omega(v^{(\bar{2})} - \varepsilon(v^{(\bar{2})})) = \cdot \circ (\text{id} \otimes \omega) \circ \tilde{\circ}(u dv), \end{aligned}$$

which is the form stated. Note that one can easily see directly that Π defined via (36) is a projection and $\ker \Pi \supseteq \Gamma_{\text{hor}}$, but its description in terms of a splitting as here is made possible by the exactness condition as explained above.

One can also see that σ as defined is equivariant if ω obeys Condition 3. From this it follows that Π is also. For a direct proof, if ω intertwines Δ_R and Ad_R , then

$$\begin{aligned} \Delta_R \Pi(u dv) &= \sum u^{(\bar{1})}v^{(\bar{1})(\bar{1})}\omega(v^{(\bar{2})})^{(\bar{1})} \otimes u^{(\bar{2})}v^{(\bar{1})(\bar{2})}\omega(v^{(\bar{2})})^{(\bar{2})} \\ &= \sum u^{(\bar{1})}v^{(\bar{1})}\omega(v^{(\bar{2})})^{(\bar{1})} \otimes u^{(\bar{2})}v^{(\bar{2})}{}_{(1)}\omega(v^{(\bar{2})})^{(\bar{2})} \\ &= \sum u^{(\bar{1})}v^{(\bar{1})}\omega(v^{(\bar{2})})_{(2)(2)} \otimes u^{(\bar{2})}v^{(\bar{2})}{}_{(1)}(Sv^{(\bar{2})})_{(2)(1)}v^{(\bar{2})}{}_{(2)(3)} \\ &= \sum u^{(\bar{1})}v^{(\bar{1})}\omega(v^{(\bar{2})})_{(1)} \otimes u^{(\bar{2})}v^{(\bar{2})}{}_{(2)} \\ &= \sum u^{(\bar{1})}v^{(\bar{1})(\bar{1})}\omega(v^{(\bar{1})(\bar{2})}) \otimes u^{(\bar{2})}v^{(\bar{2})} \\ &= \sum \Pi(u^{(\bar{1})}(dv^{(\bar{1})})) \otimes u^{(\bar{2})}v^{(\bar{2})} = (\Pi \otimes \text{id}) \circ \Delta_R(u dv), \end{aligned}$$

as required. We use that Δ_R is a comodule algebra, the intertwiner property of ω , the antipode axioms and finally that Δ_R is a comodule algebra again. \square

Condition 2 in the proposition is analogous to the classical condition that ω behaves like the Maurer-Cartan form when evaluated on fundamental vector fields. Condition 3 is analogous to its usual Ad-equivariance property. The proposition tells us how we can manufacture connections from connection one-forms.

Example 4.5. Let $P(B, A, \Phi)$ be the trivial quantum principal bundle in Example 4.2. There is a natural connection Π_{triv} given by the connection 1-form $\omega_{\text{triv}}(a) = \sum \Phi^{-1}(a_{(1)})d\Phi(a_{(2)})$. For this connection we have

$$\Gamma_{\text{ver}} = Pd\Phi(A) = \{u d\Phi(a) : u \in P, a \in A\}$$

and the splitting $\Gamma_P = \Gamma_{\text{hor}} \oplus \Gamma_{\text{ver}}$ is according to the Leibniz rule in Γ_P ,

$$ud(j(b)\Phi(a)) = u(dj(b)\Phi(a) + uj(b)d\Phi(a)) \in Pj(\Gamma_B)P \oplus Pd\Phi(A).$$

Proof. Firstly we compute

$$\begin{aligned} \tilde{\circ}\omega_{\text{triv}}(a) &= \sum \tilde{\circ}(\Phi^{-1}(a_{(1)})d\Phi(a_{(2)})) \\ &= \sum \Phi^{-1}(a_{(1)})\Phi(a_{(2)})^{(\bar{1})} \otimes \Phi(a_{(2)})^{(\bar{2})} - \varepsilon(a)1 \otimes 1 \\ &= 1 \otimes a - \varepsilon(a)1 \otimes 1, \end{aligned}$$

using right-invariance of the co-ordinate chart Φ .

Secondly we show that the map $\omega_{\text{triv}}(a)$ is an intertwiner between the adjoint coaction and the right coaction Δ_R of A on P . Using (28) we have

$$\begin{aligned} \Delta_R \omega_{\text{triv}}(a) &= \Delta_R \sum (\Phi^{-1}(a_{(1)})d\Phi(a_{(2)})) = \sum \Phi^{-1}(a_{(2)})d\Phi(a_{(3)}) \otimes (Sa_{(1)})a_{(4)} \\ &= (\omega_{\text{triv}} \otimes \text{id})\text{Ad}_R(a), \end{aligned}$$

as required. Obviously $\omega_{\text{triv}}(1) = 0$.

Hence by Proposition 4.4 we conclude that we have a connection Π_{triv} with ω_{triv} as its connection form. To compute Γ_{ver} we have

$$\begin{aligned} \Pi_{\text{triv}}(ud\Phi(a)) &= \sum u\Phi(a)^{\langle \bar{1} \rangle} \omega_{\text{triv}}(\Phi(a)^{\langle \bar{2} \rangle}) = \sum u\Phi(a_{(1)})\omega_{\text{triv}}(a_{(2)}) \\ &= \sum u\Phi(a_{(1)})\Phi^{-1}(a_{(2)})d\Phi(a_{(3)}) = ud\Phi(a), \end{aligned}$$

so that $Pd\Phi(A) \subseteq \text{image } \Pi_{\text{triv}}$. Next, applying Π_{triv} to a general element of Γ_P we have

$$\begin{aligned} \Pi_{\text{triv}}\left(\sum ud(j(b_i)\Phi(a_i))\right) &= \Pi_{\text{triv}}\left(\sum u(dj(b_i))\Phi(a_i) + \sum uj(b_i)d\Phi(a_i)\right) \\ &= \Pi_{\text{triv}}\left(\sum uj(b_i)d\Phi(a_i)\right) = \sum uj(b_i)d\Phi(a_i). \end{aligned}$$

The element $\sum u(dj(b_i))\Phi(a_i)$ here is manifestly horizontal and hence annihilated by Π_{triv} . This shows that $Pd\Phi(A) = \text{image } \Pi_{\text{triv}}$. \square

Thus every trivial bundle has a canonical trivial connection. More generally we have the following construction that gives the relationship between a connection 1-form ω as above and a connection 1-form β as defined in the previous section.

Proposition 4.6. *Let $\beta : A \rightarrow \Gamma_B$ be a linear map such that $\beta(1) = 0$. Then the map*

$$\omega(a) = \sum \Phi^{-1}(a_{(1)})j(\beta(a_{(2)}))\Phi(a_{(3)}) + \sum \Phi^{-1}(a_{(1)})d\Phi(a_{(2)}) \tag{37}$$

is a connection 1-form in the trivial principal bundle $P(B, A, \Phi)$ with trivialization Φ .

Proof. Note that the last part of (37) coincides with the connection 1-form ω_{triv} defined in Example 4.5. We have now

$$\begin{aligned} \Delta_R \omega(a) &= \sum (\Phi^{-1}(a_{(2)})j(\beta(a_{(3)}))\Phi(a_{(4)}) \otimes (Sa_{(1)})a_{(5)} \\ &\quad + \Phi^{-1}(a_{(2)})d\Phi(a_{(3)}) \otimes (Sa_{(1)})a_{(4)}) \\ &= ((\Phi^{-1} * (j \circ \beta) * \Phi + \omega_{\text{triv}}) \otimes \text{id})\text{Ad}_R(a). \end{aligned}$$

Hence ω is an intertwiner between Δ_R and Ad_R . Applying the map $\tilde{}$ to ω we see that the first part of the sum (37) is annihilated (because it is horizontal). From Example 4.5 we know that $\tilde{\omega}_{\text{triv}}(a) = 1 \otimes a - \varepsilon(a)1 \otimes 1$ for any $a \in A$, hence the same is true for ω . Hence by Proposition 4.4 we can define connection Π . \square

We note that in the case of a trivial bundle with connection and connection form ω as in the last proposition, one still has $P\Phi(\Gamma_A) \cong \Gamma_{\text{ver}}$. The isomorphism means that every form in Γ_A can be lifted to a form in Γ_P . The explicit formula is

$$ud\Phi(a) \mapsto \Pi(ud\Phi(a)) = \sum u\Phi(a_{(1)})\omega(a_{(2)}).$$

This follows from the same techniques as in the proof above.

These connections also provide covariant derivatives on horizontal pseudotensorial forms on P defined as the differential followed by horizontal projection. Moreover, these can be understood as sections of associated vector bundles etc. as in the classical theory. Details are given in the Appendix A and justify further our present formalism.

Next we come to the important notion of gauge transformation of principal bundles. Let $P(B, A, \Phi)$ be a trivial quantum principal bundle with trivialization $\Phi : A \rightarrow P$ and let $\gamma : A \rightarrow B$ be a convolution invertible linear map such that $\gamma(1) = 1$. We say that the map

$$\Phi^\gamma(a) = \sum j(\gamma(a_{(1)}))\Phi(a_{(2)}) = ((j \circ \gamma) * \Phi)(a) \tag{38}$$

is a gauge transformation of Φ .

Proposition 4.7. *If $P(B, A, \Phi)$ is a trivial quantum principal bundle as in Example 4.2 with trivialization Φ , then $P(B, A, \Phi^\gamma)$ is also a trivial quantum principal bundle with trivialization Φ^γ defined by (38).*

Proof. Note that since γ is a convolution invertible map, Φ^γ is also convolution invertible. Moreover, $\Phi^\gamma(1) = 1$. We need only to check that Φ^γ is an intertwiner. We have

$$\begin{aligned} \Delta_R \Phi^\gamma(a) &= \sum \Delta_R(j(\gamma(a_{(1)}))\Phi(a_{(2)})) = \sum j(\dot{\gamma}(a_{(1)}))\Phi(a_{(2)}) \otimes a_{(3)} \\ &= (\Phi^\gamma \otimes \text{id})\Delta(a). \end{aligned}$$

In the second equality we have used the intertwiner property of Φ and the fact that $j \circ \gamma(a)$ is in the invariant part of P . Hence Φ^γ is a trivialization of P . \square

The proposition gives the interpretation of gauge transformation as a change of local coordinates in P . It is clear that every trivialization of P can be obtained by a gauge transformation of a given trivialization Φ . Next we see that a gauge transformation induces a corresponding transformation of a connection 1-form β on our trivial quantum vector bundle. We have the following:

Proposition 4.8. *Let P, β and ω be as in Proposition 4.6. Let $\gamma : A \rightarrow B$ be a gauge transformation. The transformation $\beta \mapsto \beta^\gamma$,*

$$\beta^\gamma = \gamma^{-1} * \beta * \gamma + \gamma^{-1} * d\gamma \tag{39}$$

for fixed Φ induces a transformation $\omega \mapsto \omega^\gamma$ which can be understood as a gauge transformation $\Phi \rightarrow \Phi^\gamma$ for fixed β ,

$$\omega^\gamma = (\Phi^\gamma)^{-1} * j(\beta) * \Phi^\gamma + (\Phi^\gamma)^{-1} * d\Phi^\gamma.$$

Conversely, for fixed ω the change of trivialization Φ by a gauge transformation γ induces the transformation $\beta \mapsto \beta^{\gamma^{-1}}$, where

$$\beta^{\gamma^{-1}} = \gamma * \beta * \gamma^{-1} + \gamma * d\gamma^{-1}. \tag{40}$$

Proof. This follows by direct computation. The first statement is

$$\begin{aligned}\omega^\gamma &= \Phi^{-1} * j(\beta^\gamma) * \Phi + \Phi^{-1} * d\Phi \\ &= \Phi^{-1} * j(\gamma^{-1}) * j(\beta) * j(\gamma) * \Phi + \Phi^{-1} * j(\gamma^{-1}) * (dj(\gamma)) * \Phi + \Phi^{-1} * d\Phi \\ &= (\Phi^\gamma)^{-1} * j(\beta) * \Phi^\gamma + \Phi^{-1} * j(\gamma^{-1}) * d(j(\gamma) * \Phi) - \Phi^{-1} * d\Phi + \Phi^{-1} * d\Phi \\ &= (\Phi^\gamma)^{-1} * j(\beta) * \Phi^\gamma + (\Phi^\gamma)^{-1} * d\Phi^\gamma.\end{aligned}$$

Note that thanks to Proposition 4.6, ω^γ is a connection 1-form.

To prove the converse we have

$$\begin{aligned}\omega &= (\Phi^\gamma)^{-1} * j(\beta') * \Phi^\gamma + (\Phi^\gamma)^{-1} * d\Phi^\gamma \\ &= \Phi^{-1} * j(\gamma^{-1} * \beta' * \gamma) * \Phi + \Phi^{-1} * j(\gamma^{-1}) * d(j(\gamma) * \Phi) \\ &= \Phi^{-1} * j(\gamma^{-1} * \beta' * \gamma) * \Phi + \Phi^{-1} * (j(\gamma^{-1}) * dj(\gamma)) * \Phi + \Phi^{-1} * d\Phi.\end{aligned}$$

Comparing with Proposition 4.6 this means that β' necessarily obeys

$$\Phi^{-1} * j(\gamma^{-1} * \beta' * \gamma) * \Phi + \Phi^{-1} * j(\gamma^{-1} * d\gamma) * \Phi = \Phi^{-1} * j(\beta) * \Phi,$$

which is equivalent to $\beta' = \gamma * \beta * \gamma^{-1} + \gamma * d\gamma^{-1}$ by conjugating by Φ in the convolution algebra. Thus the effect of a gauge-transformation does not take us out of the class of connections of the form of Proposition 4.6 and the required transformation of β is uniquely determined. \square

In the same way the gauge transformations of quantum associated vector bundles and their sections are induced by a change of trivialization Φ . These details are included for completeness in Appendix A and tie up the present formulation precisely with the elementary local picture in Sect. 3.

Finally, now that we understand properly the notion of trivial bundles and their gauge transformation properties we are in a position to introduce the notion of a *locally trivial quantum bundle* as a collection of trivial bundles pasted together via gauge transformations. This is exactly in analogy with the usual definition of local trivializations of principal bundles except, of course, that we must work algebraically as in sheaf theory, and that by gauge transforms we mean the convolution by convolution-invertible maps as in Proposition 4.7. Thus, the most naive formulation of a locally-trivial principal bundle consists of the following data.

1. An index set $I = \{i, j, ij \dots\}$ to be thought of as labelling the members of an “open cover”, with analogous properties. There should be a partial ordering (corresponding to inclusion) and a product (corresponding to intersection) with $ij \leq i, j$. Indexed by this, we consider a collection of algebras P_i with maps $P \rightarrow P_i$ and $P_i \rightarrow P_j$ for $i \geq j$ (the *restriction maps*) and the equalizer

$$P \rightarrow \prod P_i \rightrightarrows \prod P_{ij}.$$

We mean here the usual picture in sheaf theory (see for example [1, Sect. 2.2]) so that if $u_i \in P_i$ are given such that their restrictions to each P_{ij} coincide then they are themselves the restriction of some $u \in P$. The algebras P_i are each A -comodule algebras (and the restriction maps are intertwiners), and $B_i = P_i^A$ are such that $B \rightarrow \prod B_i \rightrightarrows \prod B_{ij}$.

2. There are trivializations $\Phi_i : A \rightarrow P_i$ making $B_i \subseteq P_i$ trivial bundles.

3. There are convolution-invertible maps $\gamma_{ij} : A \rightarrow B_{ij}$ such that

$$\sum \gamma_{ij} * \gamma_{jk} = \gamma_{ik}, \quad \Phi_i = \sum \gamma_{ij} * \Phi_j,$$

where the maps are composed with the relevant restriction maps such that the results are maps $A \rightarrow B_{ijk}$ and $A \rightarrow P_{ij}$ respectively.

This is the most naive definition based on the transformation properties studied above. Note that in algebraic geometry, the ring of functions on the open set consisting of the space minus a number of points is achieved by inverting the points, i.e. by localization, and in this case the corresponding restriction maps are inclusions. While adequate to cover our examples in Sect. 5, it should be noted that this is not the only possible formulation. Also, the index set could have properties somewhat weaker than those of a classical open cover. It is expected that a rather bigger repertoire of non-commutative examples will be needed before the most suitable direction for a complete formulation can be determined.

4.2. The Case of General Quantum Differential Calculi

The theory above has been developed for simplicity in the case of the universal differential envelope on P . This made contact with the local picture of connections defined by one-forms on the base and gauge transformations as in Sect. 3. Now we give the further refinements needed for the non-universal case. We have to suppose differential structures on both P and A and suitable compatibility conditions between them. This refinement is needed to make contact with examples that truly deform the usual commutative differential calculus, such as our monopole example of Sect. 5.

We begin with a few words about the general theory of bicovariant differential calculi on quantum groups [29]. A bicovariant differential calculus on a quantum group A is a differential calculus (Γ_A, d) such that Γ_A is a left and right A -comodule and d is a comodule map, i.e.

$$\Delta d = (d \otimes \text{id})\Delta_R \quad \Delta d = (\text{id} \otimes d)\Delta_L,$$

where Δ_R and Δ_L are right and left coactions of A on Γ_A . If Γ_A is a left (right) A -comodule only and d is a comodule map then (Γ_A, d) is called a left-covariant (right-covariant) differential calculus on A . The universal differential calculus on A is an example of a bicovariant differential calculus. The coactions of A on A^2 are given by

$$\begin{aligned} \Delta_R^U &= (\text{id} \otimes \text{id} \otimes \cdot) \circ (\text{id} \otimes \tau \otimes \text{id}) \circ (\Delta \otimes \Delta), \\ \Delta_L^U &= (\cdot \otimes \text{id} \otimes \text{id}) \circ (\text{id} \otimes \tau \otimes \text{id}) \circ (\Delta \otimes \Delta). \end{aligned}$$

Every bicovariant differential calculus on A can be obtained from the universal one by taking a quotient $\Gamma_A = A^2/N_A$, where N_A is a sub-bimodule of A^2 such that

$$\Delta_R^U N_A \subset N_A \otimes A \quad \Delta_L^U N_A \subset A \otimes N_A.$$

Equally-well one can take a right ideal $M_A \subset \ker \varepsilon$ such that

$$\text{Ad}_R M_A \subset M_A \otimes A, \tag{41}$$

and define $N_A = \kappa(A \otimes M_A)$, where the map $\kappa : A \otimes A \rightarrow A \otimes A$, given by

$$\kappa(a \otimes a') = \sum a S a'_{(1)} \otimes a'_{(2)} \tag{42}$$

is a linear isomorphism. If the ideal does not obey (41) then the resulting calculus is only left-covariant. We will always assume that the differential structure on our quantum group A is bicovariant.

Next we come to the differential structure on the quantum principal bundle P . As explained in Sect. 2 it is sufficient to give the first order differential structure Γ_P as a quotient of the universal one, $\Gamma_P = P^2/N_P$, where N_P is a sub-bimodule of P^2 . We will always take Γ_P to be of this form.

For our first compatibility between these structures we need to suppose that the right coaction of A on P for our quantum principal bundle extends to a right coaction of A on Γ_P in a natural way. Recall that this was automatic in the universal case. A sufficient condition for the same formula (21) to project down to the non-universal case is clearly

$$\Delta_R N_P \subset N_P \otimes A.$$

Likewise we need that our map $\tilde{}$ generating the fundamental vector fields in (23) projects down to the non-universal case. It is easy to see that the relevant condition is

$$\tilde{(N_P)} \subset P \otimes M_A.$$

In this case we have a well-defined map $\tilde{} : \Gamma_P \rightarrow P \otimes \ker \varepsilon / M_A$ given by

$$\tilde{}(\varrho) = (\text{id} \otimes \pi_A) \circ \tilde{\varrho}_U, \tag{43}$$

where $\pi_{N_P} : P^2 \rightarrow \Gamma_P$ and $\pi_A : \ker \varepsilon \rightarrow \ker \varepsilon / M_A$ are the canonical projections and for $\varrho \in \Gamma_P$ we can take any representative $\varrho_U \in \pi_{N_P}^{-1}(\varrho)$. Note that the image of $\tilde{}$ in (23) is automatically in $\ker \varepsilon$ and we are relying on this now to project down to $\ker \varepsilon / M_A$. This time the corresponding vector field $\Gamma_P \rightarrow P$ is obtained by evaluation against an element of the dual of this.

Definition 4.9. We say that $P = P(B, A, N_P, M_A)$ is a *quantum principal bundle* with structure quantum group A and base B , and quantum differential calculi defined by N_P, M_A if:

1. A is a Hopf algebra.
2. (P, Δ_R) is a right A -comodule algebra.
3. $B = P^A = \{u \in P : \Delta_R u = u \otimes 1\}$.
4. $(\cdot \otimes \text{id})(\text{id} \otimes \Delta_R) : P \otimes P \rightarrow P \otimes A$ is a surjection (freeness condition).
5. $\Delta_R N_P \subset N_P \otimes A$ (right covariance of differential structure).
6. $\tilde{(N_P)} \subset P \otimes M_A$ (fundamental vector fields compatibility condition).
7. $\ker \tilde{} = \Gamma_{\text{hor}}$ (exactness condition).

Now we can define the notions of horizontal 1-forms, connections and connection 1-forms precisely as in the universal case. Thus a connection is an equivariant splitting of Γ_P . This time the freeness condition ensures in particular that

$$\text{Im } \tilde{} = P \otimes \ker \varepsilon / M_A.$$

Observe next that $\text{Ad}_R \ker \varepsilon \subset \ker \varepsilon \otimes A$. Since M_A is Ad_R -invariant (Eq. (41)) we have a right-adjoint coaction of A on $\ker \varepsilon / M_A$ by

$$\text{Ad}_R(\pi_A(a)) = \sum \pi_A(a_{(2)}) \otimes (S a_{(1)}) a_{(3)}, \tag{44}$$

where $a \in \ker \varepsilon$. Using the same methods as in Lemma 4.3 we prove that $\tilde{\sim}_{N_P}$ is an intertwiner. Finally, we have the exact sequence

$$0 \rightarrow \Gamma_{\text{hor}} \xrightarrow{j} \Gamma_P \xrightarrow{\tilde{\sim}_{N_P}} P \otimes \ker \varepsilon / M_A \rightarrow 0 \tag{45}$$

of left P -module maps. This sequence splits whenever there is a connection on Γ_P . If we denote the splitting (section) by σ_{N_P} , then we can define a connection 1-form by

$$\omega(a) = \sigma_{N_P}(1 \otimes \pi_A(a - \varepsilon(a))). \tag{46}$$

Now we can generalize Proposition 4.4.

Proposition 4.10. *Let $P(B, A, N_P, M_A)$ be a quantum principal bundle and Π a connection on it. Then its connection 1-form $\omega : A \rightarrow \Gamma_P$ has the following properties:*

1. $\omega(1) = 0$ and $\omega(M_A) = 0$,
2. $\tilde{\sim}_{N_P} \omega(a) = 1 \otimes \pi_A(a - \varepsilon(a))$ for all $a \in A$,
3. $\Delta_R \circ \omega = (\omega \otimes \text{id}) \circ \text{Ad}_R$,

where Ad_R is the right adjoint coaction. Conversely if ω is any linear map $\omega : A \rightarrow \Gamma_P$ obeying Conditions 1–3 then there is a unique connection Π ,

$$\Pi = \cdot \circ (\text{id} \otimes \omega) \circ \tilde{\sim}_{N_P} \tag{47}$$

such that ω is its connection one-form.

Proof. The proof for the most part follows just the same steps as the proof of Proposition 4.4 but at the quotient level. The map ω is extracted from the splitting defined by the connection and is Ad_R -covariant because σ_{N_P} is right invariant. In the converse direction suppose that we are given a map ω obeying Conditions 1–3. Condition 1 means that ω projects to a map $\ker \varepsilon / M_A \rightarrow \Gamma_P$ so that Π as stated is well-defined. Likewise $\sigma_{N_P} : P \otimes \ker \varepsilon / M_A \rightarrow \Gamma_P$ is well-defined by $\sigma_{N_P}(u \otimes a) = u\omega(a)$ for $u \in P$ and $a \in \ker \varepsilon / M_A$. Then $\tilde{\sim}_{N_P} \circ \sigma_{N_P}(u \otimes a) = \tilde{u}_{N_P}(\omega(a)) = u \otimes a$ by the second condition on ω . The remaining steps are likewise similar. \square

Example 4.11. Let $P(B, A, \Phi)$ be as in Example 4.2. If in addition the differential structures are such that $\Delta_R N_P \subset N_P \otimes A$ and

$$\tilde{\sim}(N_P) = P \otimes M_A,$$

then the remaining conditions in Definition 4.9 are automatically satisfied. We call this the *trivial principal bundle with trivialization Φ and general quantum differential calculus*.

Proof. The freeness condition is already proven in Example 4.2 and applies just as well here. For the exactness condition we also know that $\ker \tilde{\sim} = P(d_U j(B)P)$ (exactness in the universal calculus) from the proof there. Take $\varrho \in \ker \tilde{\sim}_{N_P}$ and choose a representative $\varrho_U \in \pi_{N_P}^{-1}(\varrho)$. From the definition of $\tilde{\sim}_{N_P}$ this means that $\tilde{\sim}(\varrho_U) \in P \otimes M_A$. By our stronger version of the fundamental vector fields compatibility condition as stated, we know that there exists $\varrho'_U \in N_P$ with $\tilde{\sim}\varrho'_U = \tilde{\sim}\varrho_U$. Hence by the exactness condition in the universal differential envelope, we conclude $\varrho_U - \varrho'_U \in P(d_U j(B)P)$. Since $\varrho = \pi_{N_P}(\varrho_U - \varrho'_U)$ we see that $\varrho \in P(dj(B)P) = \Gamma_{\text{hor}}$ as required. \square

The slightly stronger form of the fundamental vector fields compatibility condition (equality rather than merely an inclusion) certainly holds for usual trivial bundles with

commutative differential calculi, as well as for the trivial bundles (and also some non-trivial ones) constructed for general differential calculi in the next section, i.e. in all known examples. Hence it is natural to require it here for trivial bundles with general calculi. Clearly other formulations are also possible. If we use this formulation then we can also prove the existence of trivial connections on trivial bundles. These can be constructed as follows. Let $\{e^i \in \ker \varepsilon\}$ be such that $\{\pi_A(e^i)\}$ form a basis of $\ker \varepsilon/M_A$ and for any $a \in A$ write $\pi_A(a - \varepsilon(a)) = \sum_i c_i(a)\pi_A(e^i)$ say, with $c_i(a) \in k$.

Then

$$\omega(a) = \sum_i c_i(a)\Phi^{-1}(e^i_{(1)})d\Phi(e^i_{(2)})$$

is a connection with corresponding splitting according to the Leibniz rule (as for the trivial connection in Example 4.5) at least on the elements corresponding to the basis,

$$\Pi(uj(b)\Phi(e^i)) = uj(b)d\Phi(e^i), \quad (\text{id} - \Pi)(uj(b)\Phi(e^i)) = u(dj(b))\Phi(e^i).$$

We see here a significant complication caused by working with general quantum-differential calculi: unless Φ is required to obey further conditions the different choices of bases $\{e^i\}$ need not give the same connection ω . For example, a sufficient condition for uniqueness of the connection defined in this way is to assume that

$$\forall a \in M_A, \quad \sum \Phi^{-1}(a_{(1)}) \otimes \Phi(a_{(2)}) \in N_P, \quad (48)$$

in which case all choices of basis give $\omega(a) = \sum \Phi^{-1}(a_{(1)})d\Phi(a_{(2)})$. This condition is in turn implied in the commutative case by the condition that Φ is an algebra map. On the other hand for a quantum principal bundle we have already seen in Sect. 3 that one cannot assume that Φ is an algebra map because this is not closed under convolution, hence such a notion of trivial bundle could not be gauge transformed. Likewise, the above slightly weaker condition (48) is not closed under gauge transformation (i.e. if γ and Φ obey it then Φ^γ need not).

This is also the reason that we limit ourselves in Sect. 3 and Appendix A to the universal differential calculi. In fact, the general constructions in Sect. 3 are self-contained and can be verified for any algebras and differential calculi so long as we need only a local picture. For this picture to come by association to a geometrical theory of principal bundles we have to live with a certain amount of non-uniqueness or else impose further conditions. Likewise, the notion of patching together trivial bundles as outlined at the end of Sect. 4.1 can be refined according to further conditions on Φ and γ . In the examples to follow, based on homogeneous spaces, there is a natural such condition (see Proposition 5.7 below). On the other hand we feel that the right direction for a general formulation should be preceded by still further examples than these. We will not attempt this here.

5. Examples

In this section we come to the main task of the paper, which is to construct concrete examples of non-trivial quantum bundles and connections on them. This justifies the formalism developed in the last section and in Appendix A. We begin with a general development of quantum homogeneous spaces, both with universal and non-universal calculi. This includes the trivial frame bundle of S^3 in a non-commutative setting based on the quantum double Hopf algebra. We then give the full details of the

simplest non-trivial case of our construction where the homogeneous space is a q -deformed $S^2 = SO(3)/U(1)$ and the canonical connection on the associated bundle is a q -deformed Dirac monopole. This application is perhaps the main result of the paper and demonstrates in detail the various assumptions and propositions above and their smooth classical limit to the usual geometry as $q \rightarrow 1$.

5.1. Bundles on Quantum Homogeneous Spaces

We begin with the simplest example of all, namely with trivial base and connection given by the Maurer-Cartan form. This provides a useful warm-up for quantum homogeneous spaces as well as an instructive look at the content of our various axioms. We consider for our quantum principal bundle the base $B = k$, total space $P = A$ and the trivialization Φ given by the identity map. Recall here that every Hopf algebra coacts on itself by the right regular coaction provided by the coproduct Δ . We suppose also that the differential structure on P is taken to be the same as that on A .

Example 5.1. Let $P = A$ be a Hopf algebra equipped with the bicovariant differential calculus defined by an ideal $M_P = M_A$ in $\ker \varepsilon$. Let $\Delta_R = \Delta$ be the right regular coaction. Then $P(k, A, M_A)$ is a trivial quantum principal bundle in the sense of Example 4.11 with trivialization $\Phi = \text{id}$. The bundle is equipped with a trivial connection $\Pi = \text{id}$ with $\Gamma_{\text{ver}} = \Gamma_P$ and corresponding connection 1-form

$$\omega : A \rightarrow \Gamma_P, \quad \omega(a) = \sum (Sa_{(1)})da_{(2)}. \tag{49}$$

This is the Maurer-Cartan form on the Hopf algebra A .

Proof. That this obeys Conditions 1–4 in Definition 4.9 is elementary. For Condition 3 we have only to note that because the coproduct has a counit ε , it follows that if $\Delta(b) = b \otimes 1$ then $b = \varepsilon(b)1$. Hence Condition 3 holds with $B = k$. The freeness Condition 4 follows because A has an antipode S so that $(\cdot \otimes \text{id})(\text{id} \otimes \Delta)(\sum aSb_{(1)} \otimes b_{(2)}) = \sum a \otimes b$. This is the content of the linear isomorphism κ in (42). Equivalently, the existence of the antipode S is precisely the requirement that $\Phi = \text{id}$ is convolution-invertible as needed in Example 4.2. Its is clearly also an intertwiner and hence a trivialization, from which both freeness and exactness follow from Example 4.2 in the universal case. In the non-universal case we note that the covariance Condition 5, $\Delta(N_A) \subset N_A \otimes A$, is just the condition that the differential calculus defined with ideal M_A and corresponding sub-bimodule N_A is left-covariant, as explained in Sect. 4.2. Finally, the equality $\widetilde{(N_A)} = A \otimes M_A$ follows using again the linear isomorphism $\kappa : A \otimes A \rightarrow A \otimes A$. In this case the exactness condition follows from Example 4.11.

Since $\Gamma_{\text{hor}} = 0$ there is a natural (trivial) connection Π in P , given by $\Pi(\varrho) = \varrho$ for any $\varrho \in \Gamma_P = \Gamma_A$. From Proposition 4.10 we know that it has a connection 1-form, which one can compute as shown. To also see directly that ω is covariant under the adjoint coaction Ad_R , we have

$$\begin{aligned} \Delta_R \omega(a) &= \sum (\Delta Sa_{(1)})(d \otimes \text{id})\Delta(a_{(2)}) = \sum Sa_{(2)}da_{(3)} \otimes (Sa_{(1)})a_{(4)} \\ &= (\omega \otimes \text{id})\text{Ad}_R(a) \end{aligned}$$

for any $a \in A$. Condition 2 in Proposition 4.10 holds from properties of the map κ in (42). It is related to the Ad_R -invariance of M_A arising from the assumption

that the differential calculus is bicovariant. Condition 1 is also easy. From another point of view, the trivialization Φ in this case obeys the condition (48) sufficient to define a basis-independent trivial connection $\Phi^{-1}(a_{(1)})d\Phi(a_{(2)})$, which is ω . These considerations are of course unnecessary for the universal differential calculus where $M_A = \{0\}$. \square

Thus the various points of view in the theory of Sect. 4 manifestly tie up in this example. Next let us assume that P is a quantum group such that there is an Hopf algebra projection $\pi : P \rightarrow A$. (This corresponds in the classical case to an inclusion of groups $G \subseteq P$ say). The right regular coaction of P on itself pushes out by π to a coaction $\Delta_R = (\text{id} \otimes \pi) \circ \Delta : P \rightarrow P \otimes A$ and we define the associated *quantum homogeneous space* as:

$$B = P^A \equiv \left\{ b \in P : \sum b_{(1)} \otimes \pi(b_{(2)}) = b \otimes 1 \right\}.$$

In the classical situation there is a principal bundle over the underlying classical homogeneous space. A theorem of Chevalley ensures that the bundle is locally trivial in the usual sense. Later we will give a criterion for patching in the quantum case, but for now we concentrate on the global properties expressed in Definitions 4.1 and 4.9. A useful sufficient condition for a bundle is

Lemma 5.2. *Let $\pi : P \rightarrow A$ be a Hopf algebra map and a surjection between two Hopf algebras A, P . Let Δ_R be the induced coaction by pushout of Δ and $B = P^A$. If π is such that*

$$\ker \pi \subset \cdot(\ker \pi|_B \otimes P),$$

then $P(B, A, \pi)$ is a quantum principal bundle in the sense of Definition 4.1 with the universal differential calculus. We say that π obeying this assumption is exact.

Proof. Since π is a surjection, freeness of the induced coaction Δ_R follows at once from freeness of the right coaction in the preceding example. We use that P is a Hopf algebra. In the universal case it remains to prove the exactness Condition 5 in Definition 4.1. This needs some condition on π and a convenient one for our applications is as stated. Note that $\pi = \varepsilon$ when restricted to the fixed subalgebra $j(B) \subset P$. Assuming the condition, let $\varrho \in P^2$. From the linear isomorphism $\kappa : P \otimes P \rightarrow P \otimes P$ in (42) applied to the Hopf algebra P we can write $\varrho = \sum \kappa(w^k \otimes u^k)$ for $u^k \in \ker \varepsilon$ and $w^k \in P$ with the latter set linearly independent. Then $\tilde{\varrho} = (\text{id} \otimes \pi) \circ \kappa^{-1} \varrho = \sum w^k \otimes \pi(u^k)$, and hence if $\varrho \in \ker \tilde{}$ we conclude that $\pi(u^k) = 0$. For each of these, we can write from our assumption on π that $u^k = \sum_i b^k_i v^k_i$, where $b^k_i \in \ker \varepsilon|_B$ and $v^k_i \in P$. Then

$$\begin{aligned} \varrho &= \sum w^k (S u^k_{(1)}) d u^k_{(2)} = \sum w^k (S v^k_{i(1)}) (S b^k_{i(1)}) d (b^k_{i(2)} v^k_{i(2)}) \\ &= \sum \varepsilon(b^k_i) w^k (S v^k_{i(1)}) d v^k_{i(2)} + w^k (S v^k_{i(1)}) (S b^k_{i(1)}) (d b^k_{i(2)}) v^k_{i(2)}, \end{aligned}$$

using the Leibniz rule in Γ_P . The first term vanishes by our assumption and the second term lies in Γ_{hor} . Hence $\ker \tilde{} = \Gamma_{\text{hor}}$ as required. \square

Next we come to the construction of connections. We recall for classical homogeneous spaces that in the compact semisimple case there is a canonical connection on the bundle. It is defined by an ad-invariant splitting of the Lie algebra $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{g}$ (provided by the Killing form). See [15]. Such a splitting can be viewed as inducing a coalgebra (but not usually algebra) map $U(\mathfrak{p}) \rightarrow U(\mathfrak{g})$ covering the inclusion

$U(p) \supseteq U(g)$ (the map sets m to zero). In our dual quantum group formulation then this means an algebra but not usually coalgebra map $i : A \rightarrow P$ which is Ad-covariant in a suitable sense and which obeys $\pi \circ i = \text{id}$. We assume this data now for our quantum homogeneous space.

Proposition 5.3. *Let $P(B, A, \pi)$ be quantum principal bundle over a homogeneous space and with universal differential structure. If there is an algebra map $i : A \rightarrow P$ such that $\pi \circ i = \text{id}$, $\varepsilon(i(a)) = \varepsilon(a)$ for any $a \in A$, and*

$$(\text{id} \otimes \pi)\text{Ad}_R i = (i \otimes \text{id})\text{Ad}_R,$$

then

$$\omega(a) = \sum Si(a)_{(1)} di(a)_{(2)}$$

is a connection 1-form. We call the corresponding Π from Proposition 4.4 the canonical connection on the quantum homogeneous space.

Proof. We have to check that ω obeys the assumptions of Proposition 4.4. First we prove that ω is Ad_R -covariant,

$$\begin{aligned} \Delta_R \omega(a) &= \sum Si(a)_{(2)} di(a)_{(3)} \otimes \pi(Si(a)_{(1)}) \pi \circ i(a)_{(4)} \\ &= \sum Si(a_{(2)})_{(1)} di(a_{(2)})_{(2)} \otimes (Sa_{(1)})_{(3)} \\ &= \sum \omega(a_{(2)}) \otimes (Sa_{(1)})_{(3)}, \end{aligned}$$

where in the second equality we used the fact that i is an intertwiner of $(\text{id} \otimes \pi)\text{Ad}_R$ on P and Ad_R on A as in the hypothesis.

Next we apply the map \sim to ω to obtain

$$\begin{aligned} \tilde{\omega}(a) &= \sum (Si(a)_{(1)})i(a)_{(2)} \otimes \pi(i(a)_{(3)}) - \sum (Si(a)_{(1)})i(a)_{(2)} \otimes 1 \\ &= 1 \otimes \pi(i(a)) - \varepsilon(a) \otimes 1 = 1 \otimes a - \varepsilon(a) \otimes 1. \end{aligned}$$

We now apply Proposition 4.4 to conclude the result. \square

Corollary 5.4. *Let $P \xrightleftharpoons[i]{\pi} A$ be a Hopf algebra projection, i.e. suppose that i is a Hopf algebra map and covered by π . This is an example of a quantum homogeneous space with universal differential calculus as in the preceding proposition. The bundle is trivial with trivialization given by i itself. The canonical connection ω above then coincides with the flat connection in Example 4.5.*

Proof. Because i is assumed to be a Hopf algebra map, and $\pi \circ i = \text{id}$, it is immediate that it is an intertwiner for Δ_R on A and P , and therefore defines a trivial bundle $P(B, A, i)$ from Example 4.2. One can also go through Lemma 5.2 which is satisfied in this trivial case. The map i is also covariant for Δ_L and hence Ad-invariant in the way required in Proposition 5.3. Hence we can apply that proposition to obtain a connection. We note that Hopf algebra projections of the type that we have assumed here are familiar in the theory of Hopf algebras [24, 16], where it is known that P here is necessarily isomorphic to a semidirect product, $B \rtimes A \cong P$. This is built on the linear space $B \otimes A$ with cross relations according to the action

$$a \triangleright b = \sum i(a_{(1)})bi(Sa_{(2)})$$

and gives the explicit structure of the trivial bundle in this case. \square

This corollary provides an important source of (trivial) quantum bundles.

Example 5.5. Let A be a finite-dimensional quasitriangular Hopf algebra in the sense of [9]. This means that it is equipped with an element $\mathcal{R} \in A \otimes A$ obeying some axioms. Let $P = D(A)$ be the quantum double of A as a Hopf algebra built on the linear space $A^* \otimes A$ [9]. It is known that there is a Hopf algebra projection [20]

$$D(A) \begin{matrix} \xrightarrow{\pi} \\ \xleftarrow{i} \end{matrix} A, \quad \pi(\phi \otimes a) = (S\phi \otimes \text{id})(\mathcal{R})a, \quad i(a) = 1 \otimes a.$$

Hence $P = D(A)$ is a trivial quantum principal bundle with structure quantum group A . It was also shown in [20] that we can identify the base $B = P^A$ as the algebra

$$B = A^*, \quad b \cdot c = \sum b_{(2)}c_{(3)} \langle \mathcal{R}, b_{(3)} \otimes Sc_{(1)} \rangle \langle \mathcal{R}, b_{(1)} \otimes c_{(2)} \rangle, \quad \forall b, c \in A^*,$$

where the right-hand side expresses the product of B in terms of that of A^* . The corresponding element of P is $j(b) = \sum b_{(1)} \langle \mathcal{R}^{(1)}, b_{(2)} \rangle \otimes \mathcal{R}^{(2)}$, where $\mathcal{R} = \sum \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}$.

Proof. We use here the conventions in which $D(A)$ has the tensor product comultiplication and a certain double-semidirect product algebra structure. The structure of B here is that of the braided group of function algebra type associated to the dual quantum group A^* [17]. Note that A here is of enveloping algebra type (a quasitriangular Hopf algebra) being regarded perversely as “functions” on some dual group. With this description of B the map $\theta : B \rtimes A \cong D(A)$ is [16, Prop. 4.1] (where A is denoted H) is,

$$\theta(b \otimes a) = \sum b_{(1)} \langle \mathcal{R}^{(1)}, b_{(2)} \rangle \otimes \mathcal{R}^{(2)} a = j(b)i(a)$$

for j as stated and as $(b \otimes a)(1 \otimes a') = b \otimes aa'$ for the product in $D(A)$. \square

The base of this bundle then is the algebra B introduced in [17] in another context. It is (in a certain sense) a braided-commutative Hopf algebra living in the braided category of A -modules. We do not discuss it further except to note that the example of B when $A = U_q(sl_2)$ is computed in [17] and called $BSL_q(2)$. Just as $SU_q(2)$ is some kind of quantum 3-sphere, $BSL_q(2)$ equipped with a suitable $*$ -algebra structure (which exists) can be called a *braided 3-sphere* [17]. This is the base for this case of the construction. Since $A = U_q(sl_2)$ is being regarded as of function algebra type, the “underlying” structure group in this case should be thought of as some kind of deformation of a dual of sl_2 . Of course, the algebras and Hopf algebras here are not finite-dimensional so appropriate care has to be taken to work with the correct generators.

The simplest case of the preceding construction is when $A = kG$ is the group algebra of a finite group G . This is quasitriangular with $\mathcal{R} = 1 \otimes 1$. In this case $D(G) = k(G)_{\text{Ad}} \rtimes kG$. Here $B = k(G)$ so that the base is classical, namely the discrete group G . The fiber on the other hand has structure group $kG \cong k(\widehat{G})$ in the case where G is Abelian. Here \widehat{G} is the character group of G and forms the classical structure group of our bundle. When G is non-Abelian there is no such group \widehat{G} . Instead, we can continue to do gauge theory with the non-commutative algebra kG in place of functions on \widehat{G} . This is a typical application of non-commutative geometry to groups.

We can also dualize the above construction to obtain a different bundle. This time we begin with a finite-dimensional quasitriangular Hopf algebra (H, \mathcal{R}) with

$\mathcal{R} \in H \otimes H$. $D(H)^*$ is the dual Hopf algebra of Drinfeld’s double. It has as algebra structure the tensor product algebra $H \otimes H^*$, but a doubly-twisted coalgebra structure. This works out [16, Appendix] as

$$\begin{aligned} \Delta(h \otimes a) &= \sum h_{(2)} \otimes (Sf^b_{(1)})a_{(1)}f^b_{(3)} \otimes e_b \otimes a_{(2)}\langle f^b_{(2)}, h_{(1)} \rangle, \\ \varepsilon(h \otimes a) &= \varepsilon(h)\varepsilon(a), \end{aligned}$$

where $\{e_b\}$ is a basis of H and $\{f^b\}$ a dual basis.

Example 5.6. Let H be a finite-dimensional quasitriangular Hopf algebra and $A = H^*$ its dual. Let $P = D(H)^*$ as described. Then

$$P \begin{matrix} \xrightarrow{\pi} \\ \xleftarrow{i} \end{matrix} A, \quad \pi(h \otimes a) = \varepsilon(h)a, \quad i(a) = \sum \langle a_{(1)}, \mathcal{R}^{(2)} \rangle S\mathcal{R}^{(1)} \otimes a_{(2)}$$

is a Hopf algebra projection as in the above corollary and hence defines a quantum principal bundle on a quantum homogeneous space. The base B can be identified as $B = H$ (as an algebra). The map j is then $j(b) = b \otimes 1$.

Proof. This is obtained by dualizing the preceding example in an elementary way. The maps π, i in the preceding example dualise to the maps i, π respectively now. The base B also has a braided-coalgebra structure (making it a braided group) though this need not concern us now. \square

Some examples of this dual quantum double have been studied in [22] as C^* -algebras, so many of the details here for an operator-algebraic treatment are already known. The double in the case when $H = U_q(sl_2)$ or more precisely, $A = SL_q(2)$ (with a suitable $*$ -structure) is called the quantum Lorentz group. Moreover, because H here is a factorizable quantum group one can show that $U_q(sl_2) \cong BSL_q(2)$ as algebras [16, Cor. 2.3] (for generic $q \neq 1$). Thus we see that the quantum Lorentz group is a trivial bundle with $SL_q(2)$ fiber and a base which is again our braided- S^3 . It seems reasonable to view this trivial bundle

$$P(BSL_q(2), SL_q(2)) \begin{matrix} \xrightarrow{\pi} \\ \xleftarrow{i} \end{matrix} SL_q(2)$$

with appropriate $*$ -structures as a kind of *frame bundle* for our braided- S^3 . The flat connection ω in this case should be thought of as the quantum spin-connection corresponding to its parallelization.

Finally, in the case when $H = kG$, the fiber is the classical (albeit, discrete) group G and the base is \widehat{G} in the Abelian case, viewed as a non-commutative space in the non-Abelian case.

This completes our construction at the level of universal differential calculus and some examples. The ones constructed via the corollary have trivial bundles and hence flat (and other) connections on them. Next, we come to the corresponding refinements for the non-universal case.

Proposition 5.7. *For (P, A, π) as in Lemma 5.2 we suppose further that P is equipped with a left-covariant differential structure generated by a right-ideal M_P , and A with a bicovariant one with ideal M_A . If*

1. $(\text{id} \otimes \pi)\text{Ad}_R(M_P) \subset M_P \otimes A$,
2. $M_A = \pi(M_P)$,

then $P(B, A, \pi, M_P, M_A)$ is a quantum principal bundle in the sense of Definition 4.9.

Proof. We have to prove Conditions 5–7 in Definition 4.9. The last of these builds on the exactness already proven in the universal case. First we prove covariance under A . Thus our first condition implies that for any $v \in M_P$ we have

$$\sum \kappa(1 \otimes v_{(2)}) \otimes \pi((Sv_{(1)})v_{(3)}) = \sum Sv_{(2)} \otimes v_{(3)} \otimes \pi((Sv_{(1)})v_{(4)}) \in N_P \otimes A,$$

where $(\text{id} \otimes \pi)\text{Ad}_R(v) = \sum v_{(2)} \otimes \pi((Sv_{(1)})v_{(3)})$ in an explicit notation. Consequently for any $u \in P$ we have

$$\sum u_{(1)}Sv_{(2)} \otimes v_{(3)} \otimes \pi(u_{(2)}(Sv_{(1)})v_{(4)}) \in N_P \otimes A.$$

Let $\varrho = \sum \kappa(u^k \otimes v^k) \in N_P$, where $u^k \in P$ and $v^k \in M_P$. Then

$$\begin{aligned} \Delta_R \varrho &= \Delta_R \left(\sum u^k Sv_{(1)}^k \otimes v_{(2)}^k \right) \\ &= \sum u_{(1)}^k Sv_{(2)}^k \otimes v_{(3)}^k \otimes \pi(u_{(2)}^k (Sv_{(1)}^k)v_{(4)}^k) \in N_P \otimes A \end{aligned}$$

as required for the covariance Condition 5 in Definition 4.9. Meanwhile, our second condition for M_A combined with the observation $\kappa^{-1}(N_P) = P \otimes M_P$ and $\tilde{\sim} = (\text{id} \otimes \pi)\kappa^{-1}$ gives the Condition 6 in Definition 4.9 for the projection of $\tilde{\sim}$ down to a map $\tilde{\sim}_{N_P}$. Finally, we need the exactness Condition 7 with respect to this map. We write any representative $\varrho_U \in P^2$ of $\varrho \in \ker \tilde{\sim}_{N_P}$ in the same way as in the proof of Lemma 5.2 and this time have $\sum w^k \otimes \pi_A \pi(u^k) = 0$ and hence $\pi(u^k) \in M_A$. Here π_A is the canonical projection to $\ker \varepsilon / M_A$ for the kernel of the counit of A . Then from our second condition on M_P we know there exist $u'^k \in M_P$ with $\pi(u^k - u'^k) = 0$. Moreover, $\varrho'_U = \sum \kappa(w^k \otimes (u^k - u'^k))$ has the same image ϱ in Γ_P but now lies in $\ker \tilde{\sim}$. Hence by Lemma 5.2 we conclude that $\varrho \in \Gamma_{\text{hor}}$. \square

Proposition 5.8. *Let $P(B, A, \pi, M_P, M_A)$ be a quantum principal bundle over the homogeneous space B equipped with a differential structure as in Proposition 5.7. If there is an algebra map $i : A \rightarrow P$ obeying the hypothesis of Proposition 5.3 and in addition*

$$i(M_A) \subset M_P,$$

then

$$\omega(a) = \sum Si(a)_{(1)} di(a)_{(2)} \tag{50}$$

defines a connection 1-form. We call the corresponding connection Π from Proposition 4.10 the canonical connection.

Proof. Now we show that the map ω satisfies the hypothesis of Proposition 4.10. First, $\omega(1) = 0$ because i is an algebra map. Let us denote by $\pi_{N_P} : P^2 \rightarrow \Gamma_P$ the canonical surjection. Then for any $a \in \ker \varepsilon_A$ we have

$$\begin{aligned} \omega(a) &= \sum Si(a)_{(1)} di(a)_{(2)} = \sum \pi_{N_P} (Si(a)_{(1)}(1 \otimes i(a)_{(2)} - i(a)_{(2)} \otimes 1)) \\ &= \sum \pi_{N_P} (Si(a)_{(1)} \otimes i(a)_{(2)}) = \pi_{N_P} \kappa(1 \otimes i(a)). \end{aligned} \tag{51}$$

If $a \in M_A$ then $i(a) \in M_P$, and $\kappa(1 \otimes i(a)) \in N_P$. Therefore $\omega(a) = 0$ for any $a \in M_A$. Similarly to the proof of Proposition 5.3 we can show that

$$\tilde{\sim}_{N_P} \circ \omega(a) = 1 \otimes \pi_A(a - \varepsilon(a)).$$

Finally the map ω is Ad_R -covariant by the same argument as in the proof of Proposition 5.3. Applying Proposition 4.10 we obtain the assertion. \square

It is obvious from this that if i is a Hopf algebra map then the bundle is trivial with trivialization $\Phi = i$ and the canonical connection is then the trivial one associated to this (here Φ is a Hopf algebra map and obeys the condition (48) so that there is a unique trivial connection). Rather more useful for us in the next section is a kind of “local” form of Proposition 5.8 as follows. We suppose for this that $P(B, A, \pi)$ is a locally trivial quantum principal bundle over the homogeneous space B in the sense that we are given one or more trivial bundles $P_k(B_k, A, \pi_k)$ of the type above and inclusions $P \rightarrow P_k$ etc. as at the end of Sect. 4.1, which we suppose now to be compatible with the π_k in the obvious sense.

Proposition 5.9. *Let $P(B, A, \pi)$ be a locally trivial bundle with trivial bundles $P_k(B_k, A, \Pi_k)$ as explained. Let $\{\omega^i\}$ denote a basis of left-invariant differential forms for Γ_P and assume that $\Gamma_{P_k} = P_k\{\omega^i\}$ is a basis for the differential structure on each $P_k(B_k, A, \Pi_k)$. In this situation, if for one of these $P_k(B_k, A, \Pi_k)$ there exists an Ad_R covariant map $i : A \hookrightarrow P_k$ such that $\pi_k \circ i = \text{id}$ on P_k then the map $\omega(a) = \sum Si(a)_{(1)} di(a)_{(2)}$ is globally defined on P and defines a connection Π .*

Proof. We have to show the map ω is defined globally. The rest of the proposition is deduced from Proposition 5.8. We represent $\omega(a)$ in the basis of the left-invariant one-forms $\{\omega^i\}$. Let $\chi_i \in P_k^*$ be such that [29]

$$du = \sum u_{(1)} \chi_i(u_{(2)}) \omega^i$$

for any $u \in P_k$. Using this representation we find

$$\begin{aligned} \omega(a) &= \sum (Si(a)_{(1)}) i(a)_{(2)} \chi_i(i(a)_{(3)}) \omega^i \\ &= \sum \chi_i(i(a)) \omega^i. \end{aligned} \tag{52}$$

Because $\chi_i(i(a))$ are defined for each a , and $\omega^i \in \Gamma_P$, the map ω is defined globally as having values in P . \square

5.2. Dirac Monopole Bundle and its Canonical Connection

We now come to the explicit construction of a non-trivial bundle by the general methods introduced above. This is a q -deformed analog of the usual Dirac $U(1)$ connection on S^2 obtained as the canonical connection in Proposition 5.8 with $P = SO_q(3)$ and a suitable differential calculus. The base in this case is a q -sphere in the sense of [23] and our construction has a smooth limit as $q \rightarrow 1$ to the usual Dirac monopole and its connection (with the usual classical differential calculus). This serves as an important check on our constructions, as well as providing a novel Hopf-algebraic derivation of this important configuration. We first construct the bundle for any suitable calculus (including the universal calculus as in Sect. 4.1) and then specialize to the 3D calculus of Woronowicz [29] for the computation of the connection.

For the standard construction of a monopole one works with S^2 as the homogeneous space $\text{Spin}(3)/\text{Spin}(2) = SU(2)/U(1)$. The canonical connection on this is the monopole of charge one. One can also take $S^2 = SO(3)/U(1)$, where the previous

$U(1)$ is a double cover of the new $U(1)$ and we arrive at a monopole of charge two. We will construct the quantum version of the second case, but will discuss both as far as possible. We begin by developing the classical theory in the algebraic setting above. Of course, we work with the functions on $SU(2)$ and $SO(3)$ rather than points themselves. Generating the functions on the former are the matrix co-ordinate functions $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, where $\alpha(X) = X^1_1$ etc. for a matrix $X \in SU(2)$. They obey the relations of commutativity and $\alpha\delta - \beta\gamma = 1$.

Next there is a canonical inclusion of $U(1)$ in $SU(2)$ along the diagonal. In algebraic terms this is given by a projection

$$\pi \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} Z^{\frac{1}{2}} & 0 \\ 0 & Z^{-\frac{1}{2}} \end{pmatrix},$$

where $A = k[Z^{\frac{1}{2}}, Z^{-\frac{1}{2}}]$ is the algebra of functions on $U(1)$. The matrix comultiplication on $SU(2)$ is $\Delta\alpha = \alpha \otimes \alpha + \beta \otimes \gamma$, etc. and this induces a coaction of $k[Z^{\frac{1}{2}}, Z^{-\frac{1}{2}}]$ via

$$\Delta_R \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha \otimes Z^{\frac{1}{2}} & \beta \otimes Z^{-\frac{1}{2}} \\ \gamma \otimes Z^{\frac{1}{2}} & \delta \otimes Z^{-\frac{1}{2}} \end{pmatrix}.$$

This extends to products as an algebra homomorphism (a comodule algebra) as required for the general theory. For example $\alpha\beta \mapsto \alpha\beta \otimes 1$, $\alpha\gamma \mapsto \alpha\gamma \otimes Z$, etc. From this it follows that the algebra of functions on the sphere is then the fixed-point subalgebra B of $SU(2)$ with generators

$$B = SU(2)^{k[Z^{\frac{1}{2}}, Z^{-\frac{1}{2}}]} = \langle 1, b_- = \alpha\beta, b_+ = \gamma\delta, b_3 = \alpha\delta \rangle$$

and $b_-b_+ = b_3(b_3 - 1)$.

Note that these algebras are $*$ -algebras. The relations $\alpha^* = \delta$, $\beta^* = -\gamma$ imply that $b_{\pm}^* = -b_{\mp}$ while $b_3^* = b_3$. Writing $b_{\pm} = \pm(x \pm iy)$ and $z = b_3 - \frac{1}{2}$ it is easy to see that the algebra B describes a sphere of radius $\frac{1}{2}$ in the usual Cartesian co-ordinates. Next, assuming that $b_3 \neq 0$, every remaining element of $SU(2)$ can be written uniquely in the form

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \sqrt{b_3} & \frac{b_-}{\sqrt{b_3}} \\ \frac{b_+}{\sqrt{b_3}} & \sqrt{b_3} \end{pmatrix} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

which gives one co-ordinate chart of $SU(2)$. The corresponding fibre co-ordinate function that returns the $U(1)$ group co-ordinate $e^{i\theta}$ is

$$\Phi_0(Z^{\frac{1}{2}}) = \sqrt{\delta^{-1}\alpha}, \quad \Phi_0(Z^{-\frac{1}{2}}) = \sqrt{\alpha^{-1}\delta}.$$

There is another co-ordinate chart that works when $1 - b_3 \neq 0$,

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \frac{b_-}{\sqrt{1-b_3}} & \sqrt{1-b_3} \\ -\sqrt{1-b_3} & \frac{-b_+}{\sqrt{1-b_3}} \end{pmatrix} \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}.$$

The corresponding fibre co-ordinate function is

$$\Phi_1(Z^{\frac{1}{2}}) = \sqrt{-\gamma\beta^{-1}}, \quad \Phi_1(Z^{-\frac{1}{2}}) = \sqrt{-\beta\gamma^{-1}}.$$

These can be used to give trivial bundles over the relevant patches. Over \mathbb{C} there is no problem with the square roots here. On the other hand they will be problematic in the general algebraic case and for this reason we pass now to the charge two setting with $SO(3)$.

To work with $SO(3)$ we note that because the relations of $SU(2)$ are either homogeneous or change degree by 2, there is an automorphism of the algebra of functions given by $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} -\alpha & -\beta \\ -\gamma & -\delta \end{pmatrix}$. The fixed point subalgebra under this automorphism is (the algebra of functions on) $SO(3)$ and consists precisely of expressions of even degree, i.e. is generated by $\langle 1, \alpha\beta, \alpha\gamma, \dots \rangle$ as a subalgebra of the functions on $SU(2)$. The same applies in the quantum case below. For the structure group one has to work with a different but isomorphic $U(1)$ to the one above. In our function algebra language one has to work with $A = k[Z, Z^{-1}]$ as a sub-Hopf algebra of the one above. Clearly the fixed subalgebra B in $SO(3)$ by this sub-Hopf algebra is just the same as the fixed subalgebra above. This is because the generators of the latter are already of even degree.

With this description of the function algebra of $SO(3)$ the corresponding co-ordinate chart for $b_3 \neq 1$ comes out now as

$$\Phi_0(Z) = \delta^{-1}\alpha, \quad \Phi_0(Z^{-1}) = \alpha^{-1}\delta,$$

and for $1 - b_3 \neq 0$ as

$$\Phi_1(Z) = -\gamma\beta^{-1}, \quad \Phi_1(Z^{-1}) = -\beta\gamma^{-1}.$$

The first gives a trivialization of the bundle $P_0 = SO(3)[\delta^{-1}\alpha, \alpha^{-1}\delta]$ over $B_0 = B[b_3^{-1}]$, and the second of the bundle $P_1 = SO(3)[\gamma\beta^{-1}, \beta\gamma^{-1}]$ over $B_1 = B[(1 - b_3)^{-1}]$, in both cases with structure Hopf algebra $k[Z, Z^{-1}]$. Note that we are restricting to functions in open sets $b_3 \neq 0$, etc. by means of localization. Finally, there is a bundle P_{01} over B_{01} obtained by making both localizations simultaneously. One may check that these are all trivial bundles (so $P_0 = B_0 k[Z, Z^{-1}]$, etc.) and that the maps are intertwiners for Δ_R and the right regular coaction of $k[Z, Z^{-1}]$ on itself. Finally, they paste-together correctly because the ratio

$$\gamma_{01}(Z) = \Phi_0(Z)\Phi_1(Z)^{-1} = -\delta^{-1}\alpha\beta\gamma^{-1} = -b_+^{-1}b_- = -((b_3 - 1)b_3)^{-1}b_-^2$$

lies in B_{01} as it should.

For the canonical connection on this bundle, we look for an Ad_R -covariant algebra map $i : k[Z, Z^{-1}] \rightarrow SO(3)$ to use in Proposition 5.8. Since the Ad_R action of $k[Z, Z^{-1}]$ on itself is trivial, $i(Z)$ must be a $(\text{id} \otimes \pi)\text{Ad}_R$ -invariant element of the function algebra $SO(3)$. Computing this gives that it must be a combination of α, δ .

We arrange $\pi \circ i = \text{id}$ if we take $i(Z) = \delta^{-1}\alpha$. Note that this does not exist globally, indeed it coincides with the co-ordinate chart Φ_0 . But from Proposition 5.9 we know that the resulting connection ω is globally defined provided the differential structures on P_0 and P are generated by the same ideal $M_P \subset \ker \varepsilon$. For now we proceed locally, concentrating on this co-ordinate chart. A further complication caused by this is that P_0 is only a formal Hopf algebra (the comultiplication $\Delta(\delta^{-1}\alpha)$ is a formal power-series). Again, this does not affect the answer.

Proposition 5.10. *Applying Proposition 5.8 to the bundle P_0 over B_0 , the map i , and the classical differential calculus, we find that the canonical connection*

$$\omega(Z) = \sum Si(Z)_{(1)} di(Z)_{(2)}$$

exists globally and equals the Dirac $U(1)$ monopole connection of charge two,

$$\omega(Z) = \begin{cases} \beta_0(Z) + \Phi_0^{-1}(Z)d\Phi_0(Z), & \beta_0(Z) = \frac{b_+db_- - b_-db_+}{b_3} \\ & = 2i \frac{(xdy - ydx)}{z + \frac{1}{2}} \\ \beta_1(Z) + \Phi_1^{-1}(Z)d\Phi_1(Z), & \beta_1(Z) = \frac{b_+db_- - b_-db_+}{b_3 - 1} \\ & = 2i \frac{(xdy - ydx)}{z - \frac{1}{2}}. \end{cases}$$

Proof. The formal proof that the ideals M_P etc. defining the usual commutative calculus obey the relevant conditions will follow immediately from Proposition 5.13 (by setting $q = 1$) so we do not give this separately here. It is however, quite instructive to compute ω from Proposition 5.8 and see that it gives (a new algebraic derivation of) the usual form. Namely, in our algebraic formalism the canonical connection from Proposition 5.8 at least in the stated patch is

$$\begin{aligned} \omega(Z) &= \sum Si(Z)_{(1)} di(Z)_{(2)} = \cdot(S \otimes d)\Delta i(Z) \\ &= \cdot(S \otimes d)\Delta(\delta^{-1}\alpha) = \cdot(S \otimes d)\frac{\alpha \otimes \alpha + \beta \otimes \gamma}{\delta \otimes \delta + \gamma \otimes \beta} \\ &= \cdot(S \otimes \text{id})\left(\frac{\alpha \otimes d\alpha + \beta \otimes d\gamma}{\delta \otimes \delta + \gamma \otimes \beta} - \frac{(\alpha \otimes \alpha + \beta \otimes \gamma)}{(\delta \otimes \delta + \gamma \otimes \beta)^2}(\delta \otimes d\delta + \gamma \otimes d\beta)\right) \\ &= \delta d\alpha - \beta d\gamma - \alpha d\delta + \gamma d\beta = 2(\delta d\alpha - \beta d\gamma), \end{aligned}$$

where we noted that the algebra and calculus are commutative and $\delta\alpha - \beta\gamma = 1$. The computation is done in the algebra of function on $SU(2)$. The result evidently exists globally in this form and can then be cast in the two forms stated. The Cartesian coordinates x, y, z were given above. Note that the two trivializations are connected by a gauge transformation $e^{2i\psi} = \frac{x + iy}{x - iy} = \frac{-b_+}{b_-}$, where ψ is the azimuthal angle. The charge one computation is similar but slightly more complicated because of the square-roots. \square

Now we consider the quantum case. We begin with the quantum group $SU_q(2)$. It has homogeneous non-commutation relations:

$$\begin{aligned} \alpha\beta &= q\beta\alpha, & \alpha\gamma &= q\gamma\alpha, & \alpha\delta &= \delta\alpha + (q - q^{-1})\beta\gamma, \\ \beta\gamma &= \gamma\beta, & \beta\delta &= q\delta\beta, & \gamma\delta &= q\delta\gamma \end{aligned}$$

and a determinant relation $\alpha\delta - q\beta\gamma = 1$. The $*$ -structure is $a^* = \delta, \beta^* = -q\gamma$. Of course, these are no longer functions but abstract elements of the algebra with analogous properties. We define $SO_q(3)$ in the same way as the even elements of this. For π and the resulting coactions we have Δ_R as above (unchanged). For generators b_{\pm}, b_3 of B we take the same expressions as above (unchanged) in terms of $\alpha, \beta, \gamma, \delta$. Their commutation relations inherited from $SU_q(2)$ are now non-trivial

$$\begin{aligned} b_3b_- &= (1 - q^{-2})b_- + q^{-2}b_-b_3, & b_3b_+ &= b_+(1 - q^2) + q^2b_+b_3, \\ b_3^2 &= b_3 + q^{-1}b_-b_+, & q^{-2}b_-b_+ &= q^2b_+b_- + (q^{-1} - q)(b_3 - 1), \end{aligned}$$

and the $*$ -algebra structure is $b_{\pm}^* = -q^{\mp 1}b_{\mp}$ and $b_3^* = b_3$. This B is a case of the quantum sphere S_q^2 of Podleś [23].

The expressions for Φ_i are unchanged (but note now that the order matters). We proceed for the $SO_q(3)$ case and localize by adjoining the same generators as before.

Proposition 5.11. *Let $P = SO_q(3), B = S_q^2$ as above. The localizations $P_0 = SO_q(3)[\delta^{-1}\alpha, \alpha^{-1}\delta]$ over $B_0 = S_q^2[b_3^{-1}]$, and $P_1 = SO_q(3)[\gamma\beta^{-1}, \beta\gamma^{-1}]$ over $B_1 = S_q^2[(1 - b_3)^{-1}]$ are trivial quantum principal bundles (with universal differential calculus and trivialisations Φ_i) and paste together in the double localization given by a trivial bundle P_{01} over B_{01} . We call P over B with these localizations the quantum monopole bundle. It is a quantum principal bundle in the sense of Definition 4.1.*

Proof. First we construct the nontrivial bundle $P(B, A, \pi)$ using the theory in Sect. 5.1. Since freeness is automatic because π is a surjection, we have only to show the exactness condition. To do this we use Lemma 5.2 where we have seen that it suffices to show that $\ker \pi \subset \cdot(\ker \pi|_B \otimes P)$. The only generators for which this is non-trivial may be written as follows

$$\begin{aligned} \beta &= q^{-1}b_- \delta - (b_3 - 1)\beta, \\ \gamma &= b_+ \alpha - q^{-2}(b_3 - 1)\gamma. \end{aligned}$$

Multiplying on the right by the generators gives the corresponding relations for elements of $SO_q(3)$. From this it is clear that every $u \in \ker \pi$ may be expressed as $u = \sum_i b_i v_i$, where $b_i \in \ker \pi|_B$ and $v_i \in P$, hence $\ker \pi \subset \cdot(\ker \pi|_B \otimes P)$. Using Lemma 5.2 we deduce that we have a quantum principal bundle (so far with the universal calculus). Moreover, we show that the each of the patches shown are trivial bundles and glue together by gauge transformations. Firstly, the coaction Δ_R extends to the localizations as an algebra homomorphism, and from this it is clear that Φ_i are intertwiners. Since $k[Z, Z^{-1}]$ is free they extend as algebra maps and are therefore necessarily convolution invertible. Hence each of the bundles is trivial from Example 4.2. Note that this implies that every element of P_i can be written uniquely in the

form $B_i k[Z, Z^{-1}]$ via the maps Φ_i . This comes out explicitly for P_0 as

$$\begin{aligned} \alpha^2 &= b_3 \Phi_0(Z), & \alpha\gamma &= qb_+ \Phi_0(Z), \\ \gamma^2 &= qb_+ b_3^{-1} b_+ \Phi_0(Z), & \beta\delta &= q^{-1} b_- \Phi_0(Z^{-1}), \\ \beta^2 &= q^{-3} b_3^{-1} b_-^2 \Phi_0(Z^{-1}), & \delta^2 &= (1 - q^{-2} + q^{-2} b_3) \Phi_0(Z^{-1}). \end{aligned}$$

From the commutation relations

$$\begin{aligned} \Phi_0(Z) b_3 &= (q^4 b_3 + (1 - q^4)) \Phi_0(Z), \\ \Phi_0(Z) b_- &= (q^4 b_- + q^2(1 - q^2)) \Phi_0(Z), \\ \Phi_0(Z) b_+ &= (q^4 b_+ + q^2(1 - q^2)) \Phi_0(Z), \end{aligned}$$

and linear independence arguments one can verify that all elements of P_0 can similarly be obtained in a unique way. For $P_1 \supset B_1$ one has

$$\begin{aligned} \alpha^2 &= q^2 b_-^2 (1 - b_3)^{-1} \Phi_1(Z), & \alpha\gamma &= -b_- \Phi_1(Z), \\ \gamma^2 &= q^{-1} (1 - b_3) \Phi_1(Z), & \beta\delta &= -b_+ \Phi_1(Z^{-1}), \\ \beta^2 &= q^{-1} (1 - b_3) \Phi_1(Z^{-1}), & \delta^2 &= b_+ (1 - b_3)^{-1} b_+ \Phi_1(Z^{-1}), \end{aligned}$$

and Φ_1 commutes with b_3, b_{\pm} .

By a similar argument the double localization P_{01} is a trivial quantum bundle over the double localization B_{01} . There are two trivialisations of P_{01} , one is related to Φ_0 while the second to Φ_1 . They are both intertwiners and convolution invertible. To give the unique decomposition explicitly it suffices to show that $\gamma\beta^{-1}$ and $\beta\gamma^{-1}$ can be represented in terms of elements of B_{01} and map Φ_0 or equivalently that $\delta^{-1}\alpha$ and $\alpha^{-1}\delta$ can be represented in terms of B_{01} and map Φ_1 . This comes out as,

$$\begin{aligned} \gamma\beta^{-1} &= -q^2(1 - b_3)^{-1} b_+ b_3^{-1} b_+ \Phi_0(Z), \\ \beta\gamma^{-1} &= -q^{-2}(1 - b_3)^{-1} b_3^{-1} b_-^2 \Phi_0(Z^{-1}), \end{aligned}$$

and

$$\begin{aligned} \delta^{-1}\alpha &= q^2 b_3^{-1} b_-^2 (1 - b_3)^{-1} \Phi_1(Z), \\ \alpha^{-1}\delta &= q^{-2} b_+ b_3^{-1} b_+ (1 - b_3)^{-1} \Phi_1(Z^{-1}). \end{aligned}$$

Finally, these two trivialisations of P_{01} are equivalent via the gauge transformation

$$\gamma_{01}(Z) = \Phi_0(Z) \Phi_1(Z)^{-1}$$

(see Proposition 4.7), because

$$\begin{aligned} \gamma_{01}(Z) &= -q^{-1} b_+^{-1} b_- = -q^2 b_3^{-1} b_-^2 (b_3 - 1)^{-1} \in B_{01}. \\ \gamma_{01}(Z^{-1}) &= q^{-2} b_+ b_3^{-1} b_+ (1 - b_3)^{-1} \in B_{01}. \end{aligned} \quad \square$$

Thus we have a quantum principal bundle (with universal calculus) and a local trivialization for it. Next, the argument that the Ad_R -covariant function i must be a combination of α, δ etc. goes through unchanged and so we can consider $i(Z) = \delta^{-1}\alpha$ as before. In principle we can proceed formally with the corresponding canonical connection ω as above, but note that because the universal differential calculus has no commutation relations between functions and forms on P , there is no way to cancel inverses arising in ω from δ^{-1} as was the case in Proposition 5.10. One can proceed in the universal case only on the basis of formal power-series.

Now we come to the details for a non-universal differential calculus, where we will be able to compute the canonical ω from Proposition 5.9 in closed form.

We take for Γ_P the left-covariant differential calculus on $SO_q(3)$ inherited from the left-covariant 3D differential calculus on $SU_q(2)$ in [28]. As $q \rightarrow 1$ this tends to the usual commutative differential calculus in which forms and functions commute. For convenience we work in $SU_q(2)$ and afterwards restrict to the relevant subalgebra. The relevant ideal $M_P \in SU_q(2)$ for generic q is generated by six elements

$$\begin{aligned} &\delta + q^2\alpha - (1 + q^2), \quad \gamma^2, \quad \beta\gamma, \\ &\beta^2, \quad (\alpha - 1)\gamma, \quad (\alpha - 1)\beta. \end{aligned}$$

We choose the basis of the space of the left-invariant 1-forms on P to be

$$\omega^0 = \pi_{N_P} \kappa(1 \otimes \beta), \quad \omega^1 = \pi_{N_P} \kappa(1 \otimes (\alpha - 1)), \quad \omega^2 = -q^{-1} \pi_{N_P} \kappa(1 \otimes \gamma).$$

Explicitly

$$\begin{aligned} \omega^0 &= \delta d\beta - q^{-1} \beta d\delta, \\ \omega^1 &= \delta d\alpha - q^{-1} \beta d\gamma, \\ \omega^2 &= \gamma d\alpha - q^{-1} \alpha d\gamma. \end{aligned} \tag{53}$$

We have the following commutation relations between ω^i , $i = 0, 1, 2$ and the generators of $SU_q(2)$,

$$\begin{aligned} \omega^0\alpha &= q^{-1}\alpha\omega^0, \quad \omega^0\beta = q\beta\omega^0, \\ \omega^1\alpha &= q^{-2}\alpha\omega^1, \quad \omega^1\beta = q^2\beta\omega^1, \\ \omega^2\alpha &= q^{-1}\alpha\omega^2, \quad \omega^2\beta = q\beta\omega^2. \end{aligned}$$

The remaining relations can be obtained by the replacement $\alpha \rightarrow \gamma$, $\beta \rightarrow \delta$. The relation between exterior differential d and basic one-forms ω^i is given by

$$d\alpha = \alpha\omega^1 - q\beta\omega^2, \quad d\beta = \alpha\omega^0 - q^2\beta\omega^1, \tag{55}$$

and similarly with α replaced by γ and β replaced by δ . Restricted to $SO_q(3)$ and projected down to $U(1)$ this gives the ideal M_A generated by

$$Z^{-1} + q^4 Z - (1 + q^4).$$

Obviously this ideal is Ad_R -invariant, hence the resulting calculus is bicovariant as required. The commutation relation in Γ_A reads

$$Z dZ = q^4 dZZ. \tag{56}$$

One has to check that the 3D calculus fulfills in this way the various requirements in Proposition 5.7 so that we have a quantum homogeneous bundle in the sense of the general theory developed in earlier sections.

Proposition 5.12. *Let $P = SO_q(3)$ and $A = k[Z, Z^{-1}]$ with projection π be the data as above for the quantum monopole bundle but equipped now with M_P and the induced M_A for the 3D differential calculus. Then $P(B, A, \pi, M_P, M_A)$ is a quantum principal bundle on $B = S_q^2$ in the sense of Proposition 5.7.*

Proof. By the direct computation one easily finds that $(\text{id} \otimes \pi) \text{Ad}_R(M_P) \subset M_P \otimes A$. Explicitly

$$\begin{aligned} (\text{id} \otimes \pi) \text{Ad}_R(\delta + q^2\alpha - (1 + q^2)) &= (\delta + q^2\alpha - (1 + q^2)) \otimes 1, \\ (\text{id} \otimes \pi) \text{Ad}_R(\gamma^2) &= \gamma^2 \otimes Z^2, \\ (\text{id} \otimes \pi) \text{Ad}_R(\beta^2) &= \beta^2 \otimes Z^{-2}, \\ (\text{id} \otimes \pi) \text{Ad}_R(\beta\gamma) &= \beta\gamma \otimes 1, \\ (\text{id} \otimes \pi) \text{Ad}_R((\alpha - 1)\gamma) &= (\alpha - 1)\gamma \otimes Z, \\ (\text{id} \otimes \pi) \text{Ad}_R((\alpha - 1)\beta) &= (\alpha - 1)\beta \otimes Z^{-1}. \end{aligned}$$

Moreover $M_A = \pi(M_P)$ by definition. Hence the hypothesis of Proposition 5.7 is satisfied and the assertion follows. \square

Proposition 5.13. *The map*

$$\omega(a) = \sum Si(a)_{(1)} di(a)_{(2)}$$

is a connection 1-form on the quantum monopole bundle for the 3D calculus in Proposition 5.12. In terms of one forms ω^i it can be written explicitly as

$$\omega(f(Z)) = [2]_{q^{-2}} D_{q^{-4}} f(Z) |_{Z=1} \omega^1, \tag{57}$$

where we used the standard notation $[n]_x = \frac{x^n - 1}{x - 1}$, $f(Z)$ represents a general element of A understood as a Laurent series in variable Z , and D_x is the Jackson's derivative labelled by x , i.e.

$$D_x(f(Z)) = \frac{f(xZ) - f(Z)}{(x - 1)Z}. \tag{58}$$

Proof. We show that $i(M_A) \subset M_P$. From Proposition 5.8 we then deduce that ω is a connection 1-form. First we notice that $\delta^2 + q^4\alpha^2 - (1 + q^4) \in M_P$. Next, applying i to the generator of M_A we find

$$\begin{aligned} i(Z^{-1} + q^4Z - (1 + q^4)) &= \alpha^{-1}\delta + q^4\delta^{-1}\alpha - (1 + q^4) \\ &= \alpha^{-1}\alpha\delta\delta - q\alpha^{-1}\beta\gamma\delta + q^4\delta^{-1}\delta\alpha\alpha \\ &\quad - q^3\delta^{-1}\beta\gamma\alpha - (1 + q^4) \\ &= \delta^2 + q^4\alpha^2 \\ &\quad - (1 + q^4) - \beta\gamma(q^{-1}\alpha^{-1}\delta + q^5\delta^{-1}\alpha) \in M_P. \end{aligned}$$

According to Proposition 5.8, ω is a connection 1-form and hence there is a map $\sigma_{N_P} : P \otimes \ker \varepsilon / M_A \rightarrow \Gamma_P$ such that

$$\omega(a) = \sigma_{N_P}(1 \otimes \pi_A(a - \varepsilon(a))). \tag{59}$$

Using the definition of the ideal M_A it is easy to compute

$$\pi_A(f(Z) - f(1)) = D_{q^{-4}} f(Z) |_{Z=1} \pi_A(Z - 1). \tag{60}$$

Hence

$$\omega(f(Z)) = D_{q^{-4}} f(Z) |_{Z=1} \omega(Z). \tag{61}$$

Now it remains to compute $\omega(Z)$ explicitly. First we notice that

$$\omega^1 = ([2]_{q^{-2}})^{-1} \pi_{N_P} \kappa(1 \otimes (\delta^{-1} \alpha - 1)). \tag{62}$$

This follows from the fact that

$$0 \sim \delta + q\alpha - (1 + q^2) \sim \delta\alpha + q^2\alpha^2 - (1 + q^2)\alpha \sim 1 + q^2\alpha^2 - (1 + q^2)\alpha,$$

and that

$$\delta^{-1}\alpha = \delta^{-1}\delta\alpha^2 - q^{-3}\beta\gamma\delta^{-1}\alpha \sim \alpha^2.$$

The symbol \sim means that we identify two elements of $\ker \varepsilon_P$ if they differ by an element in M_P , and we used that

$$\alpha\delta = 1 + q\beta\gamma \sim 1 \sim \delta\alpha.$$

On the other hand we know that ω is given by (51). For $a = Z$ we find

$$\begin{aligned} \omega(Z) &= \pi_{N_P} \kappa(1 \otimes i(Z - 1)) = \pi_{N_P} \kappa(1 \otimes (\delta^{-1}\alpha - 1)) \\ &= [2]_{q^{-2}} \omega^1 = [2]_{q^{-2}} (\delta d\alpha - q^{-1} \beta d\gamma). \end{aligned}$$

Hence finally,

$$\omega(f(Z)) = [2]_{q^{-2}} D_{q^{-4}} f(Z) |_{Z=1} \omega^1$$

as stated. \square

We observe that ω admits the following local representation (compare Proposition 5.10)

$$\omega(Z) = \begin{cases} \Phi_0^{-1}(Z)\beta_0(Z)\Phi_0(Z) + \Phi_0^{-1}(Z)d\Phi_0(Z), & \beta_0(Z) = qb_3^{-1}(q^2b_+db_- \\ & \quad - q^{-2}b_-db_+ - \lambda db_3) \\ \Phi_1^{-1}(Z)\beta_1(Z)\Phi_1(Z) + \Phi_1^{-1}(Z)d\Phi_1(Z), & \beta_1(Z) = q(b_3 - 1)^{-1}(q^2b_+db_- \\ & \quad - q^{-2}b_-db_+ - \lambda db_3), \end{cases}$$

where $\lambda = q - q^{-1}$.

This completes our treatment of the charge two monopole. To conclude we discuss the situation for the connection 1-form corresponding to the charge one monopole as discussed in the classical situation. Firstly, there is no problem to construct the bundle $P(B, A, \pi, M_P, M_A)$ with $P = SU_q(2)$, $A = U(1)$, $B = S_q^2$, π , M_P as before and M_A generated by $Z^{-\frac{1}{2}} + q^2Z^{\frac{1}{2}} - (1 + q^2)$. We have already done the relevant computations. On the other hand, to define local trivializations of $P(B, A, \pi, M_P, M_A)$ and eventually the map i one has to formally adjoin the square roots $\sqrt{\delta^{-1}\alpha}$, $\sqrt{\alpha^{-1}\delta}$ to P . Assuming this, one can define the map $i : A \rightarrow P_0$ by

$$i(Z^{\frac{1}{2}}) = \sqrt{\delta^{-1}\alpha}, \quad i(Z^{-\frac{1}{2}}) = \sqrt{\alpha^{-1}\delta}$$

and argue that $i(M_A) \subset M_P$. We have

$$\begin{aligned} i(Z^{-\frac{1}{2}} + q^2Z^{\frac{1}{2}} - (1 + q^2)) &= \sqrt{\alpha^{-1}\delta} + q^2\sqrt{\delta^{-1}\alpha} - (1 + q^2) \\ &= \sqrt{\alpha^{-1}\alpha\delta^2 - q\alpha^{-1}\beta\gamma\delta} + q^2\sqrt{\delta^{-1}\delta\alpha^2 - q^{-1}\delta^{-1}\beta\gamma\alpha} - (1 + q^2) \\ &= \sqrt{(1 - q^{-1}\beta\gamma(\delta\alpha)^{-1})\delta^2} + q^2\sqrt{(1 - q\beta\gamma(\alpha\delta)^{-1})\alpha^2} - (1 + q^2) \\ &= \delta + q^2\alpha - (1 + q^2) - \sum_{n=1}^{\infty} c_n(\beta\gamma)^n\delta - q^2 \sum_{n=1}^{\infty} d_n(\beta\gamma)^n\alpha \in M_P, \end{aligned}$$

where c_n and d_n are coefficients. For this reason the computation of the charge one monopole is formal.

Proceeding formally we next apply Proposition 5.9 and deduce that there is a canonical connection in the bundle $P(B, A, \pi, M_P, M_A)$. We can compute its connection 1-form explicitly, using the same methods as before. First we notice that

$$\pi_A(f(Z^{\frac{1}{2}}) - 1) = D_{q^{-2}}f(Z^{\frac{1}{2}}) \Big|_{Z^{\frac{1}{2}}=1} \pi_A(Z^{\frac{1}{2}} - 1).$$

Hence from the definition of the connection 1-form we deduce that

$$\omega(f(Z^{\frac{1}{2}})) = D_{q^{-2}}f(Z^{\frac{1}{2}}) \Big|_{Z^{\frac{1}{2}}=1} \omega(Z^{\frac{1}{2}}).$$

Finally we notice that

$$\sqrt{\delta^{-1}\alpha} = \alpha - \sum_{n=1}^{\infty} d_n(\beta\gamma)^n \alpha \sim \alpha,$$

so that

$$\begin{aligned} \omega(Z^{\frac{1}{2}}) &= \pi_{N_P} \kappa(1 \otimes i(Z^{\frac{1}{2}} - 1)) \\ &= \pi_{N_P} \kappa(1 \otimes (\alpha - 1)) = \omega^1. \end{aligned}$$

Therefore

$$\omega(f(Z^{\frac{1}{2}})) = D_{q^{-2}}f(Z^{\frac{1}{2}}) \Big|_{Z^{\frac{1}{2}}=1} \omega^1. \tag{63}$$

Comparing this result with (57) we see that the quantum integer $[2]_{q^{-2}}$ has a natural interpretation as the q -monopole charge. Note also that the power appearing in the expression for i corresponds to the winding number in the classical situation, which is the topological interpretation of the monopole charge. A corresponding picture in the quantum case, as well as the construction of higher monopole charges, are interesting directions for further work.

In addition, it is hoped to give some concrete applications of this construction along the lines sketched in the introduction. For example we note that non-trivial superselection sectors for quantum mechanics on S_q^2 have recently been detected in [11], and it would be interesting to try to relate them to our quantum monopole bundle. Moreover, our constructions are not tied to this example and with suitable projections and inclusions can be used for other quantum groups and their canonical connections just as well. For example, a natural next goal would be the construction of a q -deformed instanton based on these techniques. The first problems for this are quantum-group theoretical (one needs the analogous quantum groups and their inclusions), and will be attempted elsewhere.

A. Appendix: Quantum Associated Vector Bundles

In this appendix we develop the non-commutative analogue of the following classical theory. This is needed to tie our theory in Sect. 4.1 to the local picture in Sect. 3.

Let $P(M, G)$ be a usual principal bundle and let V be a vector space and ϱ a representation of G on V . Any V -valued form ϕ on P such that

$$(R_a^* \phi)(X) \equiv \phi((R_a)_* X) = \varrho(a^{-1})\phi(x) \tag{64}$$

is called a pseudotensorial form on P . A pseudotensorial form ϕ on P is said to be tensorial if it vanishes on horizontal vectors (it corresponds to the section of a bundle associated to P). If ϕ is a tensorial form then we can define covariant derivative on ϕ by

$$D\phi = d\phi \circ (\text{id} - \sim \circ \omega),$$

i.e.

$$D_X \phi = i_X d\phi - i_{\sim_{\omega(X)}} d\phi = i_X d\phi + \varrho(\omega(X))\phi,$$

where ω is a connection 1-form, X is a vector field and i denotes interior product (evaluation).

For any principal bundle $P(M, G)$ and vector space V on which G acts, we can define the associated vector bundle $E(M, V, G)$ with fibre V . Let ϱ be the representation of G on V and define the equivalence relation \sim on $P \times V$ given by $(u, v) \sim (ua, \varrho(a^{-1})v)$. The total space E of the bundle $E(M, V, G)$ associated to P is the quotient of $P \times V$ by the relation \sim . In local coordinates:

$$E \cong (M \times G) \times_G V \cong M \times (G \times_G V) \cong M \times V.$$

We now develop the quantum picture, working for simplicity in the case of universal differential calculus. Let $P(B, A)$ be a quantum principal bundle as defined in Definition 4.1 and let Π be a connection in the principal bundle P . We define horizontal n -forms on P to be elements of the set $\Omega^n P_{\text{hor}} = Pj(\Gamma_B)Pj(\Gamma_B)P \cdots Pj(\Gamma_B)P$ (n times). The space of all horizontal forms will be denoted by Ω_{hor} . We say that a form $\alpha \in \Omega P$ is *strongly horizontal* if $\alpha \in j(\Omega B)P$. We write $\Omega P_{\text{shor}} \equiv j(\Omega B)P$. Note that $\Omega P_{\text{shor}} \subset \Omega P_{\text{hor}}$.

Proposition A.1. *If the bundle $P(B, A)$ has a connection Π , then the map*

$$h(u_0 du_1 \dots du_n) = u_0(\text{id} - \Pi)(du_1)(\text{id} - \Pi)(du_2) \dots (\text{id} - \Pi)(du_n), \tag{65}$$

where $u_0, \dots, u_n \in P$, is a linear projection of ΩP onto ΩP_{hor} . Moreover,

$$\Delta_R h = (h \otimes \text{id})\Delta_R. \tag{66}$$

Proof. It is easy to see that the map h is well-defined as stated. It is a projection because every $(\text{id} - \Pi)$ is a projection and $h(\Omega^n P) = \Omega^n P_{\text{hor}}$ as $(\text{id} - \Pi)(\Gamma_P) = Pj(\Gamma_B)P$. Finally Eq. (66) can be checked directly as

$$\begin{aligned} \Delta_R h(u_0 du_1 \dots du_n) &= \Delta_R(u_0(\text{id} - \Pi)(du_1) \dots (\text{id} - \Pi)(du_n)) \\ &= \Delta_R(u_0)\Delta_R((\text{id} - \Pi)(du_1)) \dots \Delta_R((\text{id} - \Pi)(du_n)) \\ &= \sum u_0^{(\bar{1})}(\text{id} - \Pi)du_1^{(\bar{1})} \dots (\text{id} - \Pi)du_n^{(\bar{1})} \\ &\quad \otimes u_0^{(\bar{2})}u_1^{(\bar{2})} \dots u_n^{(\bar{2})} \\ &= (h \otimes \text{id})\Delta_R(u_0 du_1 \dots du_n). \end{aligned}$$

Here the third equality uses covariance of the universal envelope ΩP and invariance of the connection Π (see (21) and (29)). \square

Let (V, ϱ_R) be a right A^{op} -comodule algebra, and let $\phi : V \rightarrow \Omega P$ be a linear map. We say that ϕ is a *pseudotensorial form* on P if

$$\Delta_R \phi = (\phi \otimes \text{id})\varrho_R. \tag{67}$$

A map $\phi : V \rightarrow \Omega P$ is called a *tensorial form* on P (*strongly tensorial form* on P) if it is pseudotensorial and for any $v \in V$, $\phi(v)$ is horizontal (resp. strongly horizontal) (compare Eq. (64)).

Lemma A.2. *Let $\phi : V \rightarrow \Omega P$ be a tensorial form on a quantum principal bundle $P(B, A)$ with connection Π . Then $d\phi : V \rightarrow \Omega P$ is pseudotensorial.*

Proof. To prove the lemma we need only note that

$$\Delta_R(d\phi) = (d \otimes \text{id})\Delta_R \phi = (d \otimes \text{id})(\phi \otimes \text{id})\varrho_R = (d\phi \otimes \text{id})\varrho_R. \quad \square$$

The map

$$D = hd \tag{68}$$

is called the *exterior covariant derivative* in P . Here D sends tensorial forms into tensorial forms (since the projection Π is right invariant).

We can now define the notion of a quantum vector bundle associated to a quantum principal bundle $P(B, A)$.

Definition A.3. Let $P(B, A)$ be a quantum principal bundle and let V be a right A^{op} comodule algebra with coaction $\varrho_R : V \rightarrow V \otimes A$. The space $P \otimes V$ is naturally endowed with a right A -comodule structure $\Delta_E : P \otimes V \rightarrow P \otimes V \otimes A$ given by

$$\Delta_E(u \otimes v) = \sum u^{(\bar{1})} \otimes v^{(\bar{1})} \otimes u^{(\bar{2})}v^{(\bar{2})}$$

for any $u \in P$ and $v \in V$. We say that the space

$$E = (P \otimes V)^A = \{u \otimes v \in P \otimes V : \Delta_E(u \otimes v) = u \otimes v \otimes 1\}$$

is a *quantum vector bundle associated to P* over B with structure group A and standard fibre V . We denote it by $E = E(B, V, A)$.

Lemma A.4. 1. E is a subalgebra of $P \otimes V$.

2. B is a subalgebra of E .

Proof. To prove the first assertion let us take $u_1 \otimes v_1, u_2 \otimes v_2 \in E$. Then we have

$$\begin{aligned} \Delta_E(u_1 u_2 \otimes v_1 v_2) &= \sum u_1^{(\bar{1})} u_2^{(\bar{1})} \otimes v_1^{(\bar{1})} v_2^{(\bar{1})} \otimes u_1^{(\bar{2})} u_2^{(\bar{2})} v_1^{(\bar{2})} v_2^{(\bar{2})} \\ &= \sum (u_1^{(\bar{1})} \otimes v_1^{(\bar{1})} \otimes u_1^{(\bar{2})})(u_2^{(\bar{1})} \otimes v_2^{(\bar{1})} \otimes u_2^{(\bar{2})} v_2^{(\bar{2})})(1 \otimes 1 \otimes v_1^{(\bar{2})}) \\ &= \sum (u_1^{(\bar{1})} \otimes v_1^{(\bar{1})} \otimes u_1^{(\bar{2})})(u_2 \otimes v_2 \otimes 1)(1 \otimes 1 \otimes v_1^{(\bar{2})}) \\ &= u_1 u_2 \otimes v_1 v_2 \otimes 1. \end{aligned}$$

Hence $(u_1 \otimes v_1)(u_2 \otimes v_2) \in E$, and E is a subalgebra of $P \otimes V$ as stated. To prove the second statement of the lemma let us observe that there is a map $j_E : B \hookrightarrow P \otimes V$ defined by $j_E(b) = b \otimes 1_V$ for any $b \in B$ and $j_E(b) \in E$ since

$$\Delta_E j_E(b) = \Delta_E(b \otimes 1_V) = b \otimes 1_V \otimes 1_A = j_E(b) \otimes 1_A.$$

This proves the lemma. \square

Let $E(B, V, A)$ be a quantum vector bundle associated to $P(B, A)$. We say that a map $s : E \rightarrow B$ is a *cross-section* of E if:

$$s \circ j_E = \text{id}. \tag{69}$$

Proposition A.5. *Let $\phi : V \rightarrow P$ be a pseudotensorial 0-form on P such that $\phi(1_V) = 1_P$. Then the map $s : E \rightarrow B$ given by*

$$s = \cdot(\text{id}_P \otimes \phi)|_E \tag{70}$$

is a cross-section of E .

Proof. First we show that s takes its values in B . Take $u \otimes v \in E$, where $u \in P$, $v \in V$. By the definition of E ,

$$\Delta_E(u \otimes v) = u \otimes v \otimes 1.$$

Hence

$$\begin{aligned} \Delta_R s(u \otimes v) &= \Delta_R(\cdot(\text{id}_P \otimes \phi)(u \otimes v)) = \sum u^{(\bar{1})} \phi(v^{(\bar{1})}) \otimes u^{(\bar{2})} v^{(\bar{2})} \\ &= (\cdot \text{id}_A)(\text{id}_P \otimes \phi \otimes \text{id}_A) \Delta_E(u \otimes v) = u \phi(v) \otimes 1. \end{aligned}$$

Thus $s(x) \in B$ for any $x \in E$. Next we show that s is a cross-section of E . We have

$$s \circ j_E(b) = s(j(b) \otimes 1) = j(b) \phi(1) = j(b) = b$$

for any $b \in B$. The last equality is a consequence of the fact that the inclusion j is just the identity on B . \square

Let us assume now that we have a trivial bundle $P(B, A, \Phi)$ as defined in Example 4.2 and moreover that our Hopf algebra A has bijective antipode. Then the map $\Phi : A \hookrightarrow P$ induces naturally a map $\Phi_E : V \hookrightarrow E$, given by

$$\Phi_E(v) = \sum \Phi(S^{-1}v^{(\bar{2})}) \otimes v^{(\bar{1})}$$

for any $v \in V$. This map obviously takes its values in $P \otimes V$. We want to show that $\Phi_E(v) \in E$ for any $v \in V$. We have

$$\begin{aligned} \Delta_E \Phi_E(v) &= \sum \Delta_E(\Phi(S^{-1}v^{(\bar{2})}) \otimes v^{(\bar{1})}) \\ &= \sum \Phi(S^{-1}v^{(\bar{2})})^{(\bar{1})} \otimes v^{(\bar{1})(\bar{1})} \otimes \Phi(S^{-1}v^{(\bar{2})})^{(\bar{2})} v^{(\bar{1})(\bar{2})}, \end{aligned}$$

but since Φ is an intertwiner of Δ_R and Δ , we obtain

$$\begin{aligned} \Delta_E \Phi_E(v) &= \sum \Phi(S^{-1}v^{(\bar{2})})_{(3)} \otimes v^{(\bar{1})} \otimes (S^{-1}v^{(\bar{2})})_{(2)} v^{(\bar{1})}_{(1)} \\ &= \sum \Phi(S^{-1}v^{(\bar{2})}) \otimes v^{(\bar{1})} \otimes 1 = \Phi_E(v) \otimes 1. \end{aligned}$$

Hence $\Phi_E(v) \in E$ for any $v \in V$. Notice also that $\Phi_E(1_V) = 1_E$ because of the second of Eqs. (26).

Moreover, using an analogous proof to that in Example 4.2 we see that the map

$$\theta : B \otimes V \rightarrow E, \quad \theta(b \otimes v) = j_E(b) \Phi_E(v) \tag{71}$$

is an isomorphism of vector spaces. Explicitly, the required inverse map is

$$\theta^{-1}(u \otimes v) = \sum u^{(\bar{1})} \Phi^{-1}(u^{(\bar{2})}) \otimes v = \sum u \Phi^{-1}(S^{-1}v^{(\bar{2})}) \otimes v^{(\bar{1})}, \tag{72}$$

where the second form follows since $u \otimes v$ lies in $E = (P \otimes V)^A$. Accordingly, we call E in this case a *trivial associated vector bundle* and Φ_E its *trivialization*.

Proposition A.6. *Let $E(B, V, A)$ be the trivial vector bundle associated to a trivial quantum principal bundle $P(B, A, \Phi)$ as explained. If $s : E \rightarrow B$ is a cross-section of E then the map $\phi : V \rightarrow P$*

$$\phi(v) = \sum j \circ s \circ \Phi_E(v^{(\bar{1})}) \Phi(v^{(\bar{2})}) \tag{73}$$

is a tensorial 0-form on P .

Proof. We need to show that $\phi : V \rightarrow P$ defined by (73) is an intertwiner between the coaction Δ_R and the corepresentation $\varrho_R : V \rightarrow V \otimes A$. Using (26) we obtain

$$\begin{aligned} \Delta_R \phi(v) &= \sum (j \circ s \circ \Phi_E(v^{(\bar{1})}) \otimes 1) (\Phi(v^{(\bar{2})})_{(1)} \otimes v^{(\bar{2})}_{(2)}) \\ &= \sum j \circ s \circ \Phi_E(v^{(\bar{1})}) \Phi(v^{(\bar{2})})_{(1)} \otimes v^{(\bar{2})}_{(2)} \\ &= \sum \phi(v^{(\bar{1})}) \otimes v^{(\bar{2})}. \end{aligned} \quad \square$$

We now look at the description of quantum bundles in local coordinates. For this we restrict ourselves from now on to trivial bundles. We would like to show how the general theory developed above reduces to the theory described in Sect. 3 (where the bundles considered were all trivial). The gauge transformations encountered there will appear now as transformations of the local description.

Proposition A.7. *Let $P(B, A, \Phi)$ be a trivial quantum principal bundle. Let (V, ϱ_R) be a right A^{op} -comodule algebra and let $\sigma : V \rightarrow \Omega B$ be any linear map. Then the map $\phi : V \rightarrow \Omega P$ given by*

$$\phi(v) = \sum (j \circ \sigma)(v^{(\bar{1})}) \Phi(v^{(\bar{2})}) \tag{74}$$

is a pseudotensorial form on P . Conversely, if $\phi : V \rightarrow \Omega P$ is a strongly tensorial form on P then

$$\sigma(v) = \sum \phi(v^{(\bar{1})}) \Phi^{-1}(v^{(\bar{2})})$$

defines a linear map $\sigma : V \rightarrow \Omega B$ which reproduces ϕ according to (74).

Proof. To prove the first assertion we have to check that ϕ as defined is an intertwiner. We have

$$\begin{aligned} \Delta_R \phi(v) &= \sum \Delta_R(j \circ \sigma(v^{(\bar{1})})) \Delta_R \Phi(v^{(\bar{2})}) \\ &= \sum (j \circ \sigma(v^{(\bar{1})}) \otimes 1) (\Phi(v^{(\bar{2})})_{(1)} \otimes v^{(\bar{2})}_{(2)}) \\ &= \sum j \circ \sigma(v^{(\bar{1})}) \Phi(v^{(\bar{2})})_{(1)} \otimes v^{(\bar{2})}_{(2)} = (\phi \otimes \text{id}) \varrho_R. \end{aligned}$$

Conversely, we need to prove that $\sigma(v) \in \Omega B$ for any $v \in V$. But $\sigma(v)$ is strongly horizontal since $\phi(v)$ is strongly horizontal, i.e., $\sigma(v) \in j(\Omega B)P$. Moreover,

$$\begin{aligned} \Delta_R \sigma(v) &= \sum (\phi(v^{(\bar{1})}) \otimes v^{(\bar{2})}_{(1)}) (\Phi^{-1}(v^{(\bar{2})})_{(3)}) \otimes S v^{(\bar{2})}_{(2)} \\ &= \sum \phi(v^{(\bar{1})}) \Phi^{-1}(v^{(\bar{2})}) \otimes 1. \end{aligned}$$

Therefore $\sigma(v)$ is invariant, and since ΩB contains any invariant subset of $j(\Omega B)P$, we conclude that $\sigma(v) \in \Omega B$. Finally, using the fact that j is the identity on ΩB we obtain

$$\sum j \circ \sigma(v^{(1)})\Phi(v^{(2)}) = \sum \phi(v^{(1)})\Phi^{-1}(v^{(2)}_{(1)})\Phi(v^{(2)}_{(2)}) = \phi(v). \quad \square$$

Composing Proposition A.6 with Proposition A.7 we obtain:

Corollary A.8. *Let $E(B, V, A)$ be the trivial quantum vector bundle associated to a trivial quantum principal bundle $P(B, A, \Phi)$. Then any map $\sigma : V \rightarrow B$ such that $\sigma(1_V) = 1_B$ induces a cross-section $s : E \rightarrow B$. Conversely any cross-section s of E induces a map $\sigma : V \rightarrow B$.*

Proof. This follows from the above, but a direct proof is also instructive. Namely, we consider the trivialization $\Phi_E : V \rightarrow E$ and use the isomorphism θ in (71). It is evident that $\theta^{-1}(j_E(b)) = b \otimes 1$. Let $\sigma : V \rightarrow B$ be any map such that $\sigma(1_V) = 1_B$ and let $s = \cdot(\text{id} \otimes \sigma) \circ \theta^{-1}$. Obviously $s : E \rightarrow B$. Moreover

$$s \circ j_E(b) = \cdot(\text{id} \otimes \sigma)\theta^{-1}(j_E(b)) = \cdot(\text{id} \otimes \sigma)(b \otimes 1) = b.$$

Thus s is a section on E . Conversely if s is any section of E then we define $\sigma = s \circ \Phi_E$. \square

Now we consider gauge transformations as defined by a change in trivialization. Such a gauge transformation γ also changes the coordinates in the quantum vector bundle $E(B, V, A)$ associated to P , inducing a transformation of sections of E , where the latter are identified with maps $\sigma : V \rightarrow B$ by Corollary A.8.

Proposition A.9. *Let $P(B, A, \Phi)$ be a trivial quantum principal bundle and (V, ϱ_R) a right A^{op} -comodule algebra. Let $\sigma : V \rightarrow B$ be a map defining a tensorial 0-form ϕ by Proposition A.7, and let $\gamma : A \rightarrow B$ be a gauge transformation. Then the transformation $\sigma \mapsto \sigma^\gamma = \sigma * \gamma$ for a fixed trivialization Φ induces a gauge transformation $\phi \mapsto \phi^\gamma$. This can also be understood as a transformation of Φ with fixed σ ,*

$$\phi^\gamma = j(\sigma) * \Phi^\gamma.$$

Conversely if ϕ is a fixed tensorial 0-form on P and the map $\sigma : V \rightarrow B$ is obtained from ϕ by Proposition A.7, then a gauge transformation of the trivialization $\Phi \mapsto \Phi^\gamma$ induces a transformation of the local description

$$\sigma \mapsto \sigma^{\gamma^{-1}} = \sigma * \gamma^{-1}.$$

Proof. This is by direct computation using the fact that j is an algebra map. The first statement is

$$\phi^\gamma \equiv j(\sigma^\gamma) * \Phi = j(\sigma * \gamma) * \Phi = j(\sigma) * \Phi^\gamma.$$

For the converse let us observe that $(\Phi^\gamma)^{-1} = \Phi^{-1} * j(\gamma^{-1})$. Then

$$\sigma^{\gamma^{-1}} = \phi * (\Phi^\gamma)^{-1} = \phi * \Phi^{-1} * j(\gamma^{-1}) = \sigma * \gamma^{-1}$$

because j is the identity map on B . \square

The first part of the proposition represents the active point of view on gauge transformations of principal bundles, while the second represents the passive point of view. From the latter point of view, gauge transformations are automorphisms of the bundle P .

Let us note that the transformation law for a map σ (from the active point of view), is exactly the same as that given in Eq. (15) in Sect. 3.

Let us finally compute an explicit formula for the covariant derivative in the case of trivial bundles (to compare it with (16) and (17)). Thanks to Proposition A.7 we know the form of any strongly tensorial form on P . We can define a linear operator ∇ in the space of maps $\sigma : V \rightarrow \Omega B$ by means of

$$D\phi = j(\nabla\sigma) * \Phi, \tag{75}$$

where ϕ is a strongly tensorial form and σ is a map decomposing ϕ according to (74). We have:

Lemma A.10. *Let $P(B, A, \Phi)$ be a trivial quantum principal bundle with differential structure given by ΩP . Let ω given by (37) define a connection in P . Then for any $\sigma : V \rightarrow \Omega^n B$ we have*

$$\nabla\sigma = d\sigma - (-1)^n \sigma * \beta. \tag{76}$$

Proof. Using the definition of the covariant derivative D in Eq. (68) we compute

$$\begin{aligned} D(j(\sigma) * \Phi) &= h(j(d\sigma) * \Phi + (-1)^n j(\sigma) * d\Phi) \\ &= j(d\sigma) * \Phi + (-1)^n j(\sigma) * d\Phi - (-1)^n j(\sigma) * \Pi(d\Phi) \\ &= j(d\sigma) * \Phi + (-1)^n j(\sigma) * d\Phi - (-1)^n j(\sigma) * \beta * \Phi - (-1)^n j(\sigma) * d\Phi \\ &= j(d\sigma - (-1)^n \sigma * \beta) * \Phi \end{aligned}$$

as required. \square

Thus we have obtained from the abstract theory the local picture quoted at the end of Sect. 3, at least for the universal calculus.

B. Appendix: Quantum Matrix Case of the Local Picture

Here we collect some results concerning trivial quantum vector bundles in the case when the structure quantum group is of matrix type. Let A be such a quantum group generated by the matrix $t = (t^i_j)_{i,j=1}^n$ obeying some commutation relations (see [12]). There is a natural comultiplication in A given by matrix multiplication (we assume summation over repeated indices), namely $\Delta t^i_j = t^i_k \otimes t^k_j$. The counit is $\varepsilon t^i_j = \delta^i_j$. For example, we can begin with the matrix bialgebra $A(R)$ defined by the solution R of Yang-Baxter equation:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$

Here $R \in \text{End}(k^n \otimes k^n)$ and $R_{12} = R \otimes I$, etc., where k is our field (such as $k = \mathbb{C}$). The commutation relations of $A(R)$ are given by the equation

$$Rt_1t_2 = t_2t_1R$$

and in nice cases lead to Hopf algebras A after quotienting $A(R)$ by suitable “determinant-type” relations.

We can also obtain examples of suitable fibres from the same matrix R by setting $V = Z(R)$, the Zamolodchikov algebra generated by the set $v = (v^i)_{i=1}^n$, obeying the relations and left $A(R)$ -coaction

$$Rv_1v_2 = \lambda v_2v_1, \quad \varrho_L v^i = t^i_j \otimes v^j,$$

where $\lambda \in k^*$ is a parameter. One can easily check that $Z(R)$ is indeed a left $A(R)$ -comodule algebra with coaction ϱ_L . It was explicitly done in [19, Sect. 6.3.2] in these conventions. We suppose this quotients also to a coaction of A .

If B is any algebra with unit we define the trivial left quantum vector bundle $E(B, Z(R), A)$ as in Sect. 3 and we keep the formalism of that section. Adopting the shorthand

$$\sigma^i \equiv \sigma(v^i), \quad (\sigma^\gamma)^i \equiv \sigma^\gamma(v^i), \\ \beta_j^i \equiv \beta(t^i_j), \quad (\beta^\gamma)^i_j \equiv \beta^\gamma(t^i_j), \quad F^i_j \equiv F(t^i_j), \quad \gamma^i_j \equiv \gamma(t^i_j),$$

we have the following formulae:

$$(\sigma^\gamma)^i = \gamma^i_j \sigma^j, \\ (\beta^\gamma)^i_j = \gamma^i_k \beta^k_l (\gamma^{-1})^l_j + \gamma^i_k d(\gamma^{-1})^k_j, \\ \nabla \sigma^i = d\sigma^i + \beta^i_j \sigma^j, \\ F^i_j = d\beta^i_j + \beta^i_k \beta^k_j, \\ \nabla^2 \sigma^i = F^i_j \sigma^j, \\ dF^i_j + \beta^i_k F^k_j - F^i_k \beta^k_j = 0.$$

This describes a matrix example of our quantum-group gauge theory in the left-handed conventions that appeared in the main part of Sect. 3.

Now consider $V = \bar{Z}(R)$, where $\bar{Z}(R)$ is an algebra generated by the set $w = (w_i)_{i=1}^n$ modulo the following relations and right $A(R)$ -coaction:

$$w_1 w_2 R = \lambda w_2 w_1, \quad \varrho_R w_i = w_j \otimes t^j_i,$$

where, as previously, $\lambda \in k^*$. One can easily check that $\bar{Z}(R)$ is a right $A(R)^{\text{op}}$ -comodule algebra with ϱ_R as stated. We suppose it quotients also to a coaction of A .

If B is any algebra with unit then $E(B, \bar{Z}(R), A)$ is a trivial right quantum vector bundle. Adopting the shorthand

$$\sigma_i \equiv \sigma(w_i), \quad (\sigma^\gamma)_i \equiv \sigma^\gamma(w_i),$$

we now have the following formulae:

$$(\sigma^\gamma)_j = \sigma_i \gamma^i_j, \\ (\beta^\gamma)^i_j = (\gamma^{-1})^i_k \beta^k_l \gamma^l_j + (\gamma^{-1})^i_k d\gamma^k_j, \\ \nabla \sigma_j = d\sigma_j - \sigma_i \beta^i_j, \\ F^i_j = d\beta^i_j + \beta^i_k \beta^k_j, \\ \nabla^2 \sigma_j = -\sigma_i F^i_j, \\ dF^i_j + \beta^i_k F^k_j - F^i_k \beta^k_j = 0.$$

This describes a matrix example of our quantum-group gauge theory in the right-handed conventions that appeared at the end of Sect. 3.

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