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Half-Sided Modular Inclusions of von-Neumann-Algebras

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Abstract. Let $\mathcal{N} \subset \mathcal{M}$ be von-Neumann-Algebras on a Hilbert space $\mathcal{H}$, Ω a common cyclic and separating vector. Denote $\Delta_{\mathcal{M}}, \Delta_{\mathcal{N}}$ resp. $J_{\mathcal{M}}, J_{\mathcal{N}}$ the associated modular operators and conjugations. Assume $\Delta_{\mathcal{M}}^{-it}\mathcal{N}\Delta_{\mathcal{M}}^{+it} \subset \mathcal{N}$ for $t \geqq 0$. We call such an inclusion half-sided modular. Then we prove the existence of a one-parameter unitary group $U(a)$ on $\mathcal{H}$, $a \in \mathbf{R}$, with generator $\frac{1}{2\pi}(\ln \Delta_{\mathcal{N}} - \ln \Delta_M) \geqq 0$ and relations

1. $\Delta_{\mathcal{M}}^{it} U(a) \Delta_{\mathcal{M}}^{-it} = \Delta_{\mathcal{N}}^{it} U(a) \Delta_{\mathcal{N}}^{-it} = U(e^{-2\pi t}a)$ for all $a, t \in \mathbf{R}$,
2. $J_{\mathcal{N}} J_{\mathcal{M}} = U(2)$,
3. $\Delta_{\mathcal{N}}^{it} = U(1)\Delta_{\mathcal{M}}^{it} U(-1)$ for all $t \in \mathbf{R}$
4. $\mathcal{N} = U(1)\mathcal{M}U(-1)$.

If $\mathcal{M}$ is a factor and Ω is also cyclic for $\mathcal{N}' \cap \mathcal{M}$, we show that $\mathcal{M}$ has to be of type III_1.

1. Introduction

In Algebraic Quantum Field Theory it is a long outstanding question, what physical meaning the Tomita–Takesaki modular objects have. The algbebraic approach of quantum field theory, as proposed by Haag and Kastler, see [6], is formulated in terms of nets of von-Neumann-algebras indexed by special open sets of the Minkowski space, forming the algebras of local observables. The Poincaré group acts covariantly on this net. One assumes a unique Poincaré invariant state ω on this net, the vacuum state, with the additional property: the spectrum of the representation of the translation subgroup in the associated GNS-Hilbert space (vacuum sector) lies in the forward light cone. Denote $\mathcal{H}$ the GNS Hilbert space,

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Ω the vector state of ω. The Reeh–Schlieder property guarantees that Ω is a common cyclic and separating vector for local algebras, see [6, 16]. Therefore one can apply the Tomita–Takesaki-Theory. A first result concerning the physical content of the objects was obtained by Bisognano and Wichmann [1], who were able to identify under physically reasonable assumptions the modular group of the local algebra to a wedge region

$$W = \{x = (x^0, x^1, \ldots, x^3) \in \mathbf{R}^{1,3}/|x^1| > |x^0| \text{ with arbitrary } x^2, x^3\} .$$

In this special case the modular group acts like the representation of special Lorentz-boosts

$$\Lambda(s) = \begin{pmatrix} \cosh(s) & \sinh(s) & 0 & 0 \\ \sinh(s) & \cosh(s) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{1}$$

i.e. as geometrical transformations on the net. The modular conjugation is found to be up to a rotation the physical PCT-conjugation.

In a conformal invariant field theory a similar result was obtained by Hislop and Longo, see [7], for regions

$$k_1 = \{x \in \mathbf{R}^{1,3}/|x^0| + |(x^1, x^2, x^3)| < 1\} .$$

It was this circle of ideas which lead Borchers [2] to consider the following setting:

Let $\mathscr{M}$ be a von-Neumann-algebra on $\mathscr{H}$, Ω cyclic and separating w.r.t. $\mathscr{M}$. One might think of $\mathscr{M}$ as a local algebra to a wedge region in Minkowski space, $\mathscr{H}$ the vacuum sector and Ω the vacuum state. Assume $U(a)$, $a \in \mathbf{R}$, to be a continuous unitary group on $\mathscr{H}$ with positive generator, leaving Ω fixed. This unitary group might be interpreted as a time-like translation group. Denote J, Δ the modular conjugation and operator to this setting. Then in a remarkable paper Borchers proved (see [2]).

Theorem 1 (*Borchers*). *If $U(a)MU(-a) \subset M$ for $a \geqq 0$ we get:*

1. $\Delta^{it}U(a)\Delta^{-it} = U(e^{-2\pi t}a)$ *for all* $t, a \in \mathbf{R}$.
2. $JU(a)J = U(-a)$.

This theorem generalizes the results of Bisognano and Wichmann resp. Hislop and Longo considerably. Looking carefully at the proof a stronger version of the theorem can be seen to be true:

Theorem 2 (*Borchers*). *Let $U(a), a \in \mathbf{R}$ be a family of unitary operators leaving Ω fixed with the following properties:*

1. *$U(a)$ can be analytically continued to $\{z \in C/0 < \operatorname{Im} z < \pi\}$ with $\|U(a)\| \leqq 1$ for $a \in \{z \in \mathbf{C}/0 \leqq \operatorname{Im} z \leqq \pi\}$.*
2. $U^*(a) = U(a + i\pi) \quad \forall a \in \mathbf{R}$.
3. $U(a)\mathbf{M}U(-a) \subset \mathbf{M} \quad \forall a \in \mathbf{R}$.

Then one gets

a) $\Delta^{it}U(a)\Delta^{-it} = U(e^{-2\pi t}a) \ \forall t, a \in \mathbf{R}$.
b) $JU(a)J = U(a)^* \ a \in \mathbf{R}$.

Borchers' proof is rather difficult. He looks at matrix products

$$\langle \varphi, \Delta^{it} U(a) \Delta^{-it} \psi \rangle \quad \varphi, \psi \in D(\Delta) \cap D(\Delta^{-1}) .$$

Using the Tube Theorem he can enlarge the region of holomorphy in t and a. The Edge-of-the Wedge Theorem together with the assumptions on $U(a)$ and the modular properties of Δ^{it} are the other inputs in order to get a complex line in the domain of holomorphy. The estimates on $U(a)$ lead to a bounded holomorphic function on a line, i.e. a constant function, from which he concludes the result. For the proof the interested reader is referred to the beautiful original work of Borchers [2].

2. The Basic Result

The idea is to apply Borchers result to special inclusions. Let $\mathcal{N} \subset \mathcal{M}$ be von-Neumann-Algebras on a Hilbert space $\mathcal{H}$, Ω a common cyclic and separating vector in $\mathcal{H}$. Denote $\Delta_{\mathcal{M}}, \Delta_{\mathcal{N}}$, resp. $J_{\mathcal{M}}, J_{\mathcal{N}}$ the associated modular operators and conjugations. Assume $\Delta_{\mathcal{M}}^{-it} \mathcal{N} \Delta_{\mathcal{M}}^{+it} \subset \mathcal{N}$ for $t \geqq 0$.

From $\mathcal{N} \subset \mathcal{M}$ we conclude

$$\Delta_{\mathcal{M}}^{\frac{1}{2}} \leqq \Delta_{\mathcal{N}}^{\frac{1}{2}} \tag{2}$$

in the sense to quadratic forms, see [3] or the proof of Theorem 3 below. The log-function is operator monotone, see [12, p. 317 Ex. 51], and we get

$$\ln(\Delta_{\mathcal{N}}) - \ln(\Delta_{\mathcal{M}}) \geqq 0 \tag{3}$$

in the sense of quadratic forms. Assume now that $\ln(\Delta_{\mathcal{N}}) - \ln(\Delta_{\mathcal{M}})$ is essentially selfadjoint on $D(\ln(\Delta_{\mathcal{N}})) \cap D(\ln(\Delta_{\mathcal{M}}))$, i.e. we can apply the Trotter product formula, see [4]. We get

$$\exp(it(\ln(\Delta_{\mathcal{N}}) - \ln(\Delta_{\mathcal{M}}))) = s - \lim_{n \to \infty} (\Delta_{\mathcal{M}}^{\frac{-it}{n}} \Delta_{\mathcal{N}}^{\frac{+it}{n}})^n \tag{4}$$

from which we read off

$$\mathrm{Ad}(\exp(it(\ln(\Delta_{\mathcal{N}}) - \ln(\Delta_{\mathcal{M}}))))(\mathcal{N}) \subset \mathcal{N} \quad \text{for } t \geqq 0 . \tag{5}$$

($\mathrm{Ad}\, \Delta_{\mathcal{N}}^{\frac{it}{n}}(\mathcal{N}) \subset \mathcal{N}$ by modular theory, $\mathrm{Ad}\, \Delta_{\mathcal{M}}^{\frac{-it}{n}}(\mathcal{N}) \subset \mathcal{N}$ for $t \geqq 0$ by assumption.)

We can apply Borchers result (Theorem 1) to $\mathcal{N}$, Ω and $U(a) = \exp(ia(\ln(\Delta_{\mathcal{N}}) - \ln(\Delta_{\mathcal{M}})))$. Using a slightly different method we can avoid the assumption on essentially selfadjointness of $\ln(\Delta_{\mathcal{M}}) - \ln(\Delta_{\mathcal{N}})$.

Theorem 3. *Let $\mathcal{N} \subset \mathcal{M}$ be von-Neumann-Algebras acting on a Hilbert space $\mathcal{H}$, Ω a common cyclic and separating vector. Denote $\Delta_{\mathcal{M}}, \Delta_{\mathcal{N}}$* resp. *$J_{\mathcal{M}}, J_{\mathcal{N}}$ the related modular operators and conjugations. If*

$$\Delta_{\mathcal{M}}^{-it} \mathcal{N} \Delta_{\mathcal{M}}^{+it} \subset \mathcal{N} \quad \text{for all } t \geqq 0 \tag{6}$$

we get:

a) $\ln(\Delta_{\mathcal{N}}) - \ln(\Delta_{\mathcal{M}})$ *is essentially selfadjoint of* $D(\ln(\Delta_{\mathcal{N}})) \cap D(\ln(\Delta_{\mathcal{M}}))$.

Denote p the selfadjoint closure of $\frac{1}{2\pi}(\ln(\Delta_{\mathcal{N}}) - \ln(\Delta_{\mathcal{M}}))$, $U(a) := e^{iap}$. *Then*

b) $\Delta_{\mathcal{M}}^{it} U(a) \Delta_{\mathcal{M}}^{-it} = \Delta_{\mathcal{N}}^{it} U(a) \Delta_{\mathcal{N}}^{-it} = U(e^{-2\pi t}a)$ *for* $t, a \in \mathbf{R}$, (7)

c) $J_{\mathcal{M}} U(a) J_{\mathcal{M}} = J_{\mathcal{N}} U(a) J_{\mathcal{N}} = U(-a) \quad a \in \mathbf{R}$. (8)

Proof. First notice that

$$T(z) := \Delta_{\mathcal{M}}^{-iz} \Delta_{\mathcal{N}}^{iz} \tag{9}$$

can be analytically continued to $\{z \in \mathbf{C}/0 < \operatorname{Im} z < \frac{1}{2}\}$ with $\| T(z) \| \leqq 1$, see [3]. For the reader's convenience we will sketch this proof:

Let $A \in \mathcal{N}$ be entire analytic w.r.t. the modular group of $\mathcal{N}$. Then

$$\begin{aligned} T(0) A\Omega &= A\Omega\,, \\ T\left(\frac{i}{2}\right) A\Omega &= \Delta_{\mathcal{M}}^{\frac{1}{2}} \Delta_{\mathcal{N}}^{-\frac{1}{2}} A\Omega = J_{\mathcal{M}} J_{\mathcal{M}} \Delta_{\mathcal{M}}^{\frac{1}{2}} J_{\mathcal{N}} \Delta_{\mathcal{N}}^{\frac{1}{2}} J_{\mathcal{N}} \Omega \\ &= J_{\mathcal{M}} J_{\mathcal{N}} A\Omega\,. \end{aligned} \tag{10}$$

From this one concludes

$$\begin{aligned} \| T(t) \| &= \| \Delta_{\mathcal{M}}^{it} T(0) \Delta_{\mathcal{N}}^{-it} \| \leqq 1\,, \\ \left\| T\left(\frac{i}{2} + t\right) \right\| &= \left\| \Delta_{\mathcal{M}}^{it} T\left(\frac{i}{2}\right) \Delta_{\mathcal{N}}^{-it} \right\| \leqq 1 \end{aligned} \tag{11}$$

for $t \in \mathbf{R}$, and by the Hadamard–Three-Line theorem, see [13], the final assertion follows. Define

$$V(t) = \Delta_{\mathcal{N}}^{\frac{it}{2}} \Delta_{\mathcal{M}}^{-it} \Delta_{\mathcal{N}}^{\frac{it}{2}} = T(t)^* T(t), \quad t \in \mathbf{R}\,. \tag{12}$$

$V(t)$ is a unitary family which can analytically be continued to $\{z \in \mathbf{C}/ 0 < \operatorname{Im} t < 1\}$. Notice that

$$V^*(t) = V(-t) \tag{13}$$

and $\| V(t) \| \leqq 1$ for $t \in \{z \in \mathbf{C}/0 \leqq \operatorname{Im} z \leqq 1\}$.

Furthermore by assumption we have

$$\operatorname{Ad} V(t)(\mathcal{N}) \subset \mathcal{N}, \quad \text{for } t \geqq 0\,.$$

In order to apply Borchers result we make a variable transformation. The sinh-function maps $\{z \in \mathbf{C}/0 < \operatorname{Im} z < \pi\}$ biholomorphically onto the upper half plane $\{z \in \mathbf{C}/\operatorname{Im} z > 0\}$. The log-function maps this domain biholomorphically onto $\{z \in \mathbf{C}/0 < \operatorname{Im} z < \pi\}$. Therefore

$$z \mapsto \left(\frac{1}{\pi} \operatorname{arcsinh} \exp(z)\right) \tag{15}$$

maps $\{z \in \mathbf{C}/0 < \operatorname{Im} z < \pi\}$ biholomorphically onto $\{z \in \mathbf{C}/0 < \operatorname{Im} z < 1\}$.

Furthermore the real line is mapped onto the positive part of the real line, the elements with imaginary part $i\pi$ onto the negative part.

Let $\tilde{U}(a) := V\left(\frac{1}{\pi}\operatorname{arcsinh}(\exp(a))\right)$, $a \in \mathbf{R}$. We can apply Borchers result, i.e. Theorem 2, to $\tilde{U}(a)$, $\mathcal{N}$, Ω getting

$$\Delta_{\mathcal{N}}^{it}\tilde{U}(a)\Delta_{\mathcal{N}}^{-it} = \tilde{U}(e^{-2\pi t}a) \quad \text{for } t, a \in \mathbf{R}\,. \tag{16}$$

Rewriting this in terms of the modular operators we get

$$\Delta_{\mathcal{N}}^{i(t-\frac{\tilde{a}}{2}+\frac{e^{-2\pi t}\tilde{a}}{2})}\Delta_{\mathcal{M}}^{i\tilde{a}} = \Delta_{\mathcal{M}}^{ie^{-2\pi t}\tilde{a}}\Delta_{\mathcal{N}}^{i(t+\frac{\tilde{a}}{2}-\frac{e^{-2\pi t}\tilde{a}}{2})}\,, \tag{17}$$

with $\tilde{a} = \frac{1}{\pi}\operatorname{arcsinh}(\exp(a))$. This proves that $\Delta_{\mathcal{N}}^{it}, \Delta_{\mathcal{M}}^{is}, t, s \in \mathbf{R}$ generate a two dimensional unitary group. Instead of trying to identify the group by working out the above commutation relation directly we will use the following property: In the case of a unitary representation of a Lie group G on a Hilbert space, we always have a common core for the representation operators of the Lie algebra of G, see [8]. From this we get

a) $\ln(\Delta_{\mathcal{N}}) - \ln(\Delta_{\mathcal{M}})$ is essentially self adjoint on $D(\ln(\Delta_{\mathcal{M}})) \cap D(\ln(\Delta_{\mathcal{N}}))$. Therefore we can apply Borchers theorem for $\exp(-iap)$ as indicated above to get b) and c). The group generated by $\Delta_{\mathcal{N}}^{it}, \Delta_{\mathcal{M}}^{is}$ is now easily recognized to be the two-dimensional Poincaré-group. □

Exploiting the group structure of the 2-dimensional Poincaré group we get

Corollary 4. 1. $\Delta_{\mathcal{M}}^{it}\Delta_{\mathcal{N}}^{-it} = e^{i(-1+e^{-2\pi t})p} \quad t \in \mathbf{R}$.
2. $\Delta_{\mathcal{N}}^{it} = e^{ip}\Delta_{\mathcal{M}}^{it}e^{-ip}$.
3. $J_{\mathcal{M}}J_{\mathcal{N}} = e^{-i2p}$.

Proof. Let $g_n, g_m, g_{n,m}$ denote the Lie algebra elements belonging to the generators $\ln(\Delta_{\mathcal{N}})$, $\ln(\Delta_{\mathcal{M}})$ resp. p. From Theorem 3a) we know their commutation relations. Applying a Baker–Campbell–Hausdorff formula one gets a). The last statement is a specialization of a). Analytic continuation of a) to $t = \frac{-i}{2}$ leads to

$$e^{-i2p} = \Delta_{\mathcal{M}}^{\frac{1}{2}}\Delta_{\mathcal{N}}^{-\frac{1}{2}} = J_{\mathcal{M}}J_{\mathcal{N}}\,. \tag{18}$$

Therefore we get the announced relations. □

Remark. It is not difficult to see the following: let $\mathcal{M}$, $\mathcal{N}$, Ω be as in Theorem 3 but (6) replaced by

$$\Delta_{\mathcal{M}}^{it}\mathcal{N}\Delta_{\mathcal{M}}^{-it} \subset \mathcal{N} \quad \text{for all } t \geqq 0\,. \tag{19}$$

Then we get instead of b),

$$\text{b}')\ \Delta_{\mathcal{M}}^{it}U(a)\Delta_{\mathcal{M}}^{-it} = \Delta_{\mathcal{N}}^{it}U(a)\Delta_{\mathcal{N}}^{-it} = U(e^{+2\pi t}a) \quad \text{for } t, a \in \mathbf{R}\,. \tag{20}$$

Similarly in Corollary 4 1) has to be changed to

$$1')\ \Delta_{\mathcal{M}}^{it}\Delta_{\mathcal{N}}^{-it} = e^{i(-1+e^{2\pi t})p} \quad t \in \mathbf{R}\,.$$

The theorem suggests to give special names to such type of inclusions.

Definition 5. *Let $\mathcal{N} \subset \mathcal{M}$ be von-Neumann-algebras on a Hilbert space $\mathcal{H}$. Let Ω be a common cyclic and separating vector. Denote $\sigma_{\mathcal{M}}^t$ the modular group associated to*

$(\mathcal{M}, \Omega)$. *If* $\sigma_{\mathcal{M}}^{-t}(\mathcal{N}) \subset \mathcal{N}$ *for all* $t \geqq 0$ *or* $t \leqq 0$, *we call* $(\mathcal{N} \subset \mathcal{M}, \Omega)$ *a half-sided modular inclusion.*

In the next section we want to draw some conclusions from the result.

3. Some Conclusions

As a simple corollary we get the symmetry in the conditions in $\mathcal{N} \subset \mathcal{M}$ and $\mathcal{M}' \subset \mathcal{N}'$:

Corollary 6. *Let* $(\mathcal{N} \subset \mathcal{M}, \Omega)$ *be a half-sided modular inclusion,* $\Delta_{\mathcal{M}}, \Delta_{\mathcal{N}}$ *be the associated modular operators. Assume* $\Delta_{\mathcal{M}}^{-it} \mathcal{N} \Delta_{\mathcal{M}}^{it} \subset \mathcal{N}$ for $t \geqq 0$.

Then

$$U(t)\mathcal{M}U(-t) \subset \mathcal{M} \quad \text{for } t \geqq 0 \tag{21}$$

and

$$\Delta_{\mathcal{N}}^{it} \mathcal{M}' \Delta_{\mathcal{N}}^{-it} \subset \mathcal{M}' \quad \text{for } t \geqq 0 , \tag{22}$$

where $U(t) = \exp(it(\ln(\Delta_{\mathcal{N}}) - \ln(\Delta_{\mathcal{M}})))$.

Proof. By Theorem 3 we know

$$U(2)\mathcal{M}U(-2) = J_{\mathcal{N}} J_{\mathcal{M}} \mathcal{M} J_{\mathcal{M}} J_{\mathcal{N}} \subset \mathcal{N} \subset \mathcal{M} . \tag{23}$$

Using $\Delta_{\mathcal{M}}^{it} U(2) \Delta_{\mathcal{M}}^{-it} = U(2e^{-2\pi t})$ we immediately get (21). Applying the relation $\Delta_{\mathcal{N}}^{it} = \Delta_{\mathcal{M}}^{it} U(-e^{-2\pi t} + 1)$ from Corollary 4, a) leads to (22). □

Corollary 4 b) suggests $\mathcal{N} = U(1)\mathcal{M}U(-1)$. That this is really the case is the content of

Corollary 7. *Let* $\mathcal{N}, \mathcal{M}$ *be as in Corollary* 6.

Then $\mathcal{N} = U(1)\mathcal{M}U(-1)$ *with* $U(a) = \exp(ia(\ln(\Delta_{\mathcal{N}}) - \ln(\Delta_{\mathcal{M}})))$.

Proof. Let $A \in \mathcal{N}$. Then $\Delta_{\mathcal{N}}^{-it} A \Delta_{\mathcal{N}}^{+it} \in \mathcal{N}$ for all t and thereby

$$\Delta_{\mathcal{M}}^{it} \Delta_{\mathcal{N}}^{-it} A \Delta_{\mathcal{N}}^{it} \Delta_{\mathcal{M}}^{-it} \in \mathcal{M} \quad \text{for all } t \in \mathbf{R}. \tag{24}$$

By Corollary 4 we get

$$U(e^{-2\pi t} - 1) A U(1 - e^{2\pi t}) \in \mathcal{M} \quad \forall t \in \mathbf{R} . \tag{25}$$

Therefore

$$U(-1)\mathcal{N}U(1) \subset \mathcal{M} , \tag{26}$$

i.e.

$$\mathcal{N} \subset U(1)\mathcal{M}U(-1) . \tag{27}$$

Ω is a cyclic and separating vector for both algebras and their modular groups agree. From this follows equality, see [15]. □

For the next result we need some preparatory lemmatas. The aim is to show that in the case of factors $\mathcal{N}$ and $\mathcal{M}$ have to be of type III_1. It will be enough to prove the uniqueness of Ω as an invariant vector under $U(a)$, $a \in \mathbf{R}$.

First let me recall a result of R. Longo see [9, 10].[1]

[1] The author thanks R. Longo for pointing out an error in an earlier version of this work

Theorem 8. *Let $\mathcal{N} \subset \mathcal{M}$ be factors, Ω be a common cyclic and separating vector. Denote $\gamma := \mathrm{Ad}\, J_{\mathcal{N}} J_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ the canonical endomorphism to this situation, $J_{\mathcal{N}}, J_{\mathcal{M}}$ the modular conjugations to $(\mathcal{N}, \Omega)$,* resp. *$(\mathcal{M}, \Omega)$. If Ω is also cyclic for $\mathcal{N}' \cap \mathcal{M}$, it follows that*

$$\omega - \lim_{n \to \infty} \gamma^n(A) = \langle \Omega, A\Omega \rangle \quad A \in \mathcal{M} \quad n \in \mathbf{N}\,. \tag{28}$$

Proof. See [10, Chap. 4].

From this result we immediately conclude □

Corollary 9. *Let $\mathcal{N} \subset \mathcal{M}, \Omega$ as in Theorem 3 and also cyclic for $\mathcal{N}' \cap \mathcal{M}, \mathcal{M}$ a factor. Assume $\psi \in \mathcal{H}$ invariant under $U(a)$, $a \in \mathbf{R}$. Then*

$$\langle \psi, A\psi \rangle = \langle \Omega, A\Omega \rangle \langle \psi, \psi \rangle \quad \textit{for } A \in \mathcal{M}\,. \tag{29}$$

Proof. From Corollary 4 we get $J_{\mathcal{N}} J_{\mathcal{M}} = U(2)$. Then

$$\begin{aligned} \langle \psi, A\psi \rangle &= \langle U(-2n)\psi, AU(2n)^*\psi \rangle \quad \text{for all } n \in \mathbf{N} \\ &= \lim_{n \to \infty} \langle \psi, (\mathrm{Ad}\, J_{\mathcal{N}} J_{\mathcal{M}})^n(A)\psi \rangle \\ &= \langle \psi, \psi \rangle \langle \Omega, A\Omega \rangle \end{aligned} \tag{30}$$

by the result of Longo, see Theorem 8. □

To prove the uniqueness of the vector we will exploit the natural order structure of modular theory.

Denote $P^\natural(\mathcal{N}) := \{\Delta_{\mathcal{N}}^{\frac{1}{4}} A\Omega / A \in \mathcal{N}^+\}^-$ the standard cone to $(\mathcal{N}, \Omega)$, see [15]. We get

Lemma 10. *For $t \geqq 0$ $U(it)$ is positive* w.r.t. $P^\natural(\mathcal{N})$, i.e.

$$U(it) P^\natural(\mathcal{N}) \subset P^\natural(\mathcal{N})\,. \tag{31}$$

Proof. By the positivity of the generator we can analytically continue $U(a)$ to the upper half plane. Let $A, B \in \mathcal{N}^+$, we get

$$\langle \Delta_{\mathcal{N}}^{\frac{1}{4}} A\omega, U(it)\Delta_{\mathcal{N}}^{\frac{1}{4}} B\Omega \rangle = \left\langle A\Omega, \Delta_{\mathcal{N}}^{\frac{1}{4}} U\left(\frac{it}{2}\right) \Delta_{\mathcal{N}}^{-\frac{1}{4}} \Delta_{\mathcal{N}}^{\frac{1}{2}} \Delta_{\mathcal{N}}^{-\frac{1}{4}} U\left(\frac{it}{2}\right) \Delta_{\mathcal{N}}^{\frac{1}{4}} B\Omega \right\rangle\,. \tag{32}$$

From the commutation relations of Theorem 3 we conclude

$$= \left\langle A\Omega, U\left(-\frac{t}{2}\right) \Delta_{\mathcal{N}}^{\frac{1}{2}} U\left(\frac{t}{2}\right) B\Omega \right\rangle\,. \tag{33}$$

With $A \in \mathcal{N}^+$ we also have $U\left(\frac{t}{2}\right) A U\left(-\frac{t}{2}\right) \in \mathcal{N}^+$, i.e.

$$\langle \Delta_{\mathcal{N}}^{\frac{1}{4}} A\Omega, U(it)\Delta_{\mathcal{N}}^{\frac{1}{4}} B\Omega \rangle \geqq 0\,. \tag{34}$$

This proves the statement. □

As a simple application we can show

Corollary 11. *Let $\mathcal{M}$ be a factor, $\psi \in \mathcal{H}$ be $U(a)$-invariant. Then ψ is a multiple of Ω.*

Proof. From $U(a)\psi = \psi$ for all $a \in \mathbf{R}$ we get

$$U(a)J_{\mathcal{N}}\psi = J_{\mathcal{N}}U(-a)\psi = J_{\mathcal{N}}\psi\,, \tag{35}$$

that is, $J_{\mathcal{N}}\psi$ also $U(a)$ invariant. Denote

$$\psi_1 = \psi + J_{\mathcal{N}}\psi, \quad \psi_2 = i\psi - iJ_{\mathcal{N}}\psi\,. \tag{36}$$

Both vectors are $U(a)$-invariant and by construction $J_{\mathcal{N}}$-invariant. We can therefore uniquely decompose $\psi_{1/2}$ into

$$\psi_{1/2} = \xi_{1/2}^{+} - \xi_{1/2}^{-}$$

with $\xi_{1/2}^{+/-} \in P^{\natural}(\mathcal{N})$, $\xi_{1/2}^{+} \perp \xi_{1/2}^{-}$, see [15]. For $t \geqq 0$ we can estimate

$$\begin{aligned}\langle(\xi_{1/2}^{+} + \xi_{1/2}^{-}), U(it)(\xi_{1/2}^{+} + \xi_{1/2}^{-})\rangle &= \langle \xi_{1/2}^{+} - \xi_{1/2}^{-}, U(it)(\xi_{1/2}^{+} - \xi_{1/2}^{-})\rangle \\ &\quad + 2(\langle \xi_{1/2}^{-}, U(it)\xi_{1/2}^{+}\rangle + \langle \xi_{1/2}^{+}, U(it)\xi_{1/2}^{-}\rangle) \\ &\geqq \langle \psi_{1/2}, U(it)\psi_{1/2}\rangle \end{aligned} \tag{37}$$

by the positivity of $U(it)$ w.r.t. $P^{\natural}(\mathcal{N})$, Lemma 10. But from the positivity of the generator of $U(a)$ we know $\|U(it)\| \leqq 1$ for $t \geqq 0$, i.e.

$$\|U(it)(\xi_{1/2}^{+} + \xi_{1/2}^{-})\| = \|\xi_{1/2}^{+} + \xi_{1/2}^{-}\| \tag{38}$$

from which we conclude

$$\xi_{1/2}^{+} + \xi_{1/2}^{-} \quad U(a)\text{-inv.}\,, \tag{39}$$

and therefore

$$\xi_{1/2}^{+/-} \quad U(a)\text{-inv.}\,. \tag{40}$$

But $\xi_{1/2}^{+/-} \in P^{\natural}(\mathcal{N})$ by the very definition. From Corollary 9 we get that

$$\langle \xi_{1/2}^{+/-}, A\xi_{1/2}^{+/-}\rangle = \langle \Omega, A\Omega\rangle\langle \xi_{1/2}^{+/-}, \xi_{1/2}^{+/-}\rangle\,. \tag{41}$$

Now the vector representation of states in $P^{\natural}(\mathcal{N})$ is unique, see [15], from which we conclude $\xi_{1/2}^{+/-}$ is a multiple of Ω and therefore the final proof. □

Collecting the results we get

Theorem 12. *Let $(\mathcal{N} \subset \mathcal{M}, \Omega)$ be a half-sided modular inclusion of von-Neumann algebras on a Hilbert space $\mathcal{H}$. If $\mathcal{N} \neq \mathcal{M}$, $\mathcal{M}$ a factor, $\mathcal{M}$ has to be of type III_1.*

Proof. Applying Theorem 3 and Corollary 11 we get a unitary group $U(a)$, $a \in \mathbf{R}$ with a unique U-invariant vector Ω and positive generator. Furthermore we have

$$U(a)\mathcal{M}U(-a) \subset \mathcal{M} \quad \text{for } a \geqq 0\,. \tag{42}$$

From $\mathcal{N} \neq \mathcal{M}$ we easily get that

$$U(a)\mathcal{M}U(-a) \not\subset \mathcal{M} \quad \text{for } a < 0\,. \tag{43}$$

For such a situation Longo showed in [11] that $\mathcal{M}$ has to be of type III_1. □

Remark. Using Corollary 11 one can prove uniqueness of the vacuum for models in Algebraic Quantum Field Theory. For example consider the von-Neumann-Algebra $\mathcal{M}$ of observables localized in a wedge region. Assume $\mathcal{M}$ to be a factor. Let $U(a)$, $a \in \mathbf{R}$ be the unitary representation of timelike translations into the wedge region, $U(a)\mathcal{M}U(-a) \subset \mathcal{M}$ for $a \geqq 0$. U is assumed to have positive generator. If Ω is a cyclic and separating vector for $\mathcal{M}$, U-invariant, we conclude with the help of Corollary 11 that it is the unique U-invariant vector. Notice that we do not use any localization property or asymptotic abelianness for this argument. To get the conclusion one applies Borchers result [2], see Theorem 1, to that situation. Then define $\mathcal{N} := U(1)\mathcal{M}U(-1)$. It is easy to see that $\mathcal{N} \subset \mathcal{M}$ fulfills the conditions of Corollary 11.

All these statements on the inclusion of $\mathcal{N} \subset \mathcal{M}$ depend on the state Ω. The natural question arises under what general conditions we can find to a given inclusion $\mathcal{N} \subset \mathcal{M}$ a common cyclic and separating vector Ω with $\sigma_{\mathcal{M}}^{-t}(\mathcal{N}) \subset \mathcal{N}$ for all $t \geqq 0$, $\sigma_{\mathcal{M}}^{t}$ the modular group of $(\mathcal{M}, \Omega)$. Let us restrict to the case of factors.

Next let me recall the following definitions:

An inclusion $\mathcal{N} \subset \mathcal{M}$ is called split iff there exists a type I factor N in between, i.e. $\mathcal{N} \subset N \subset \mathcal{M}$. A vector Ω is called standard iff Ω is cyclic and separating for $\mathcal{N}$, $\mathcal{M}$ and $\mathcal{N}' \cap \mathcal{M}$, see [5].

Assume now $\mathcal{N} \subset \mathcal{M}$ to be factors of type III_1. If $\mathcal{N} \subset \mathcal{M}$ is split it follows that $\mathcal{N}' \cap \mathcal{M}$ is too a type III_1 factor, see [5]. We get as an easy application of the above results

Lemma 13. *Let $\mathcal{N} \subset \mathcal{M}$, $\mathcal{N}' \cap \mathcal{M}$ be factors of type III_1, Ω a standard vector. If $\sigma_{\mathcal{M}}^{-t}(\mathcal{N}) \subset \mathcal{N}$ for all $t \geqq 0$, $\sigma_{\mathcal{M}}^{t}$ the modular group of $(\mathcal{M}, \Omega)$, the inclusion cannot be split.*

Proof. Assume $\mathcal{N} \subset \mathcal{M}$ to be split. Denote $J_{\mathcal{N}' \cap \mathcal{M}}$ the modular conjugation of $(\mathcal{N}' \cap \mathcal{M}, \Omega)$. By the results of Doplicher and Longo [5, Th. 4.1.]

$$N := J_{\mathcal{N}' \cap \mathcal{M}}(\mathcal{M}) J_{\mathcal{N}' \cap \mathcal{M}} \cap \mathcal{M} \tag{44}$$

has to be of type I, Ω a cyclic and separating vector for this algebra N. By the very assumption we get

$$\sigma_{\mathcal{M}}^{-t}(\mathcal{N}') \subset \mathcal{N}' \quad \text{for } t \leqq 0 \tag{45}$$

and therefore

$$\sigma_{\mathcal{M}}^{-t}(\mathcal{N}' \cap \mathcal{M}) \subset \mathcal{N}' \cap \mathcal{M} \quad \text{for } t \leqq 0 \,. \tag{46}$$

From this we conclude by Theorem 3 the existence of a unitary group $\tilde{U}(a) = \exp(ia(\ln(\Delta_{\mathcal{N}' \cap M}) - \ln(\Delta_{\mathcal{M}})))$ with the special properties listed there. Especially we get

$$J_{\mathcal{N}' \cap \mathcal{M}} = \tilde{U}(2) J_{\mathcal{M}} \,. \tag{47}$$

We rewrite

$$N = J_{\mathcal{N}' \cap \mathcal{M}}(\mathcal{M}) J_{\mathcal{N}' \cap \mathcal{M}} \cap \mathcal{M} = \tilde{U}(2)\mathcal{M}'\tilde{U}(-2) \cap \mathcal{M} \,. \tag{48}$$

Using the commutation relation we conclude

$$\sigma_{\mathcal{M}}^{-t}(N) \subset N \quad \text{for } t \leqq 0 \,. \tag{49}$$

From Theorem 12 we get a contradiction to N being of type I. □

4. Final Remarks

We showed in this article that half-sided modular inclusions carry a strikingly rich structure. In a sloppy way they lie between Jones inclusions and split inclusions. In the former case one has a faithful conditional expectation from $\mathscr{M}$ onto $\mathscr{N}$ w.r.t. a state ω. By Takesaki's theorem, see [11], this is equivalent to $\sigma^t_{\mathscr{M}}(\mathscr{N}) \subset \mathscr{N}$ for all $t \in \mathbf{R}$, $\sigma^t_{\mathscr{M}}$ the modular group of $(\mathscr{M}, \omega)$. On the other hand Lemma 13 shows that in the standard case the position of $\mathscr{N}$ in $\mathscr{M}$ is too narrow to $\mathscr{M}$ to be interpolated by a type I factor.

We will continue our investigations on such types of inclusions in the near future.

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