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Correlation Functions in the Itzykson-Zuber Model

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Abstract. The n-point function for the integral over unitary matrices with Itzykson-Zuber measure is reduced to the integral over the Gelfand-Tzetlin table; the integrand (for generic n) is given by linear exponential times the rational function. For $n = 2$ and in some cases for $n > 2$ later in fast is the polynomial and this allows to give an explicit and simple expression for all 2-point and a set of n-point functions. For the most general n-point function a simple linear differential equation is constructed.

1. Introduction

In this letter I'll consider the following correlation function:

$$\begin{aligned}\langle g_{i_1 j_1} g^{+}_{k_1 l_1} \cdots g_{i_n j_n} g^{+}_{k_n l_n} \rangle \\ = \int [d\mu(g)] \exp[\mathrm{Tr}(gMg^{+}N)]\, g_{i_1 j_1} g^{+}_{k_1 j_1} \cdots g_{i_n j_n} g^{+}_{k_n l_n} \,. \end{aligned} \tag{1.1}$$

Here g is the N dimensional unitary matrix and M and N are Hermitian. The measure of integration is Haar measure. Without lack of generality we could assume that M and N are diagonal.

For the case of $n = 0$ (partition function) this integral was calculated by Harish-Chandra [1] and Itzykson and Zuber [2] a long time ago. Here we will use the method previously used in a similar problem in [3]; this simple algebraic method is known in the literature on representation theory since 1950 [4]. Let me mention that the main motivation to look on the integral (1.1) is related to investigation of the Kazakov-Migdal model [5]; also, this kind of integrals might be interesting for string theory related matrix models [6].

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I'll show that the integral (1.1) can be written in the form:

$$\langle g_{i_1 j_1} g^+_{k_1 l_1} \cdots g_{i_n j_n} g^+_{k_n l_n} \rangle = \frac{C_N}{\Delta(M)} \int \prod_{k=1}^{N-1} \prod_{i=1}^{N-k} dm_i^k R[m]$$

$$\times \exp\left[\sum_{k=1}^{N} \left(\sum_{i=1}^{N-k+1} m_i^{k-1} - \sum_{i=1}^{N-k} m_i^k \right) N_k \right], \quad (1.2)$$

where $\Delta(M)$ is the Vandermond determinant constructed from eigenvalues of matrix M, m_k^i are Gelfand-Tzetlin coordinates defining the convex body (see Fig. 1)

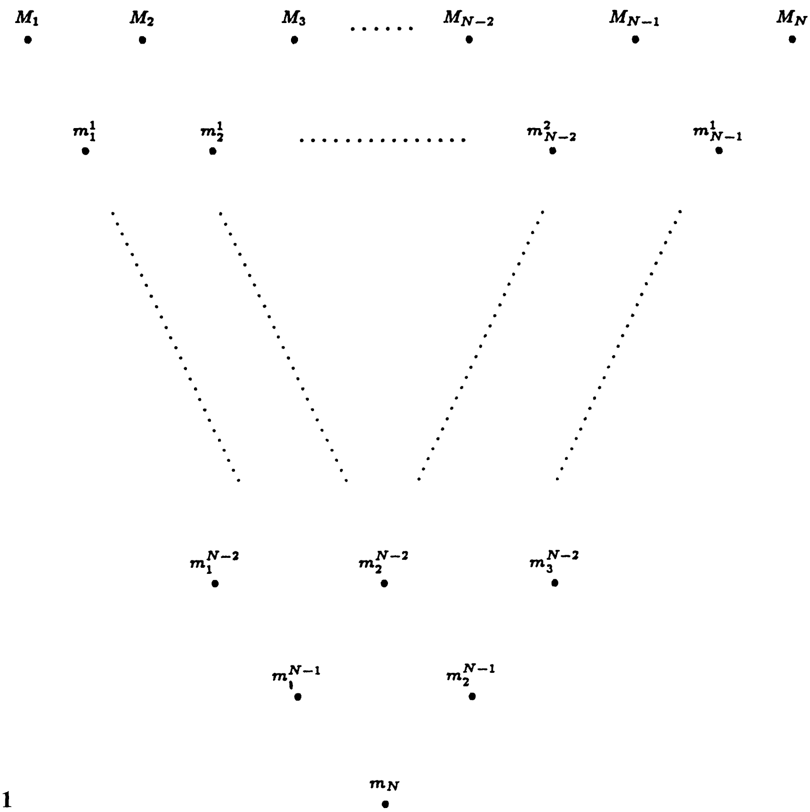

Fig. 1

$$m_k^i > m_k^{i+1} > m_{k+1}^i, \quad (1.3)$$

with m_k^0 being the eigenvalues of matrix M (we assume that they are ordered: $m_1^0 > m_2^0 > \ldots > m_N^0$), N_k are eigenvalues of matrix N, C is the number (see (2.12)) and $R[m]$ is given (in the general case) by the rational function on the GT table. This function will be described explicitly in Sect. 3 together with the linear differential equation for (1.2) (I'll give the corresponding algorithm).

$R[m]$ reduces to a simple polynomial in the case of $i_1 = i_2 = \ldots = i_n = 1$ and this allows to give the explicit formula (we denote by $\Delta(N_2, \ldots, N - N)$ the

Vandermond determinant for eigenvalues $N_2, N_3, \ldots, N_N$):

$$
\begin{aligned}
&\langle g_{1j_1} g^+_{k_1 1} g_{1j_2} g^+_{k_2 1} \cdots g_{1j_n} g^+_{k_n 1} \rangle \\
&= \delta_{j_1 k_1} \delta_{j_2 k_2} \ldots \delta_{j_n k_n} \frac{C_N}{\Delta(M)\Delta(N_2, \ldots, N_N)} \\
&\times \int \prod_{k=1}^{N-1} dm^1_k \frac{\prod_{l=1}^{N-1} (m^1_l - m^0_{j_1})}{\prod_{l \neq j_1} (m^0_l - m^0_{j_1})} \frac{\prod_{l=1}^{N-1} (m^1_l - m^0_{j_2})}{\prod_{l \neq j_2} (m^0_l - m^0_{j_2})} \cdots \frac{\prod_{l=1}^{N-1} (m^1_l - m^0_{j_n})}{\prod_{l \neq j_n} (m^0_l - m^0_{j_n})} \\
&\times \exp\left[\left(\sum_{k=1}^{N} m^0_k - \sum_{k=1}^{N-1} m^1_k \right) N_1 \right] \det[\exp(m^1_i N_j)] \,. \qquad (1.4)
\end{aligned}
$$

Here $i = 1, 2, \ldots, N-1$; $j = 2, 3, \ldots, N$. All nonzero 2-point functions $\langle g_{ik} g^+_{ki} \rangle$ and a set of multi-point functions $\langle g_{ij_1} g^+_{k_n i} \cdots g_{ij_n} g^+_{k_n i} \rangle$ are obtained from (1.4) by permutation of eigenvalues m^0_i after integration over m^1 in (1.4) [1]

2. Gelfand-Tzetlin Parametrization

Let us denote by X the combination that enters in IZ measure:

$$
X = gMg^+ \,.
$$

X is the element of the coadjoint orbit of the unitary group; this orbit is labeled by eigenvalues of $M: m^0_1 > \ldots > m^o_N$. We will now introduce the coordinates on the group and on G/H, with H being the Cartan subgroup defined by M, the so-called Gelfand-Tzetlin coordinates (for the reason why we call this coordinates GTC see [3]).

Let a_i be the basis in N dimensional complex plane, C^N, and e_i be fixed basis, in which the matrix M is diagonal with ordered eigenvalues m^o_i. Then we can write

$$
M = \sum_{i=1}^{N} m^0_i e_i e^+_i \,, \qquad X = \sum_{i=1}^{N} m^0_i a_i a^+_i \,, \qquad g_{ji} = (e_i, a_j) \,. \qquad (2.1)
$$

$a_i = ge_i$, $(,)$ is the scalar product: $(\alpha x, y) = \alpha^*(x, y)$; $(x, \alpha y) = \alpha(x, y)$. The Haar measure could be parametrized by the vectors a_i, $d\mu(g) = d\mu(a_1, \ldots, a_N)$. Then

$$
\begin{aligned}
d\mu(a_1, \ldots, a_N) &= \frac{(N-1)!}{(2\pi)^N} dt_1 \ldots dt_{N-1} d\theta_1 \ldots d\theta_N d\mu(a_2, \ldots, a_N) \,, \\
t_k &= |g_{1k}|^2 \,, \qquad k = 1, \ldots, N-1; \\
\theta_k &= \arg g_{1k} \,, \qquad k = 1, \ldots, N \,.
\end{aligned} \qquad (2.2)
$$

One can move to the new variables m^1_i from t_i in the following way: suppose m^1 is the eigenvalue of matrix PMP, where P is the projector on the subspace,

[1] The integral in (1.4) is easy to calculate, but I don't think that the integrated version looks simpler than this

orthogonal to the vector a_1. This means that if f is the corresponding eigenvector, $f = \sum_{k=1}^{N} b_k e_k$, one has

$$PMf = mf\,, \qquad Mf = m^1 f + b a_1\,, \qquad f = \sum_{k=1}^{N} b_k e_k\,.$$

Taking the scalar product of Mf with f_k and e_k we get

$$b_k = \frac{b g_{1k}}{m_k^0 - m^1}\,, \qquad b^2 = \sum_{k=1}^{N} \frac{|g_{1k}|^2}{(m_k^0 - m^1)^2}\,.$$

From the orthogonality condition, $(f, a_1) = 0$, one obtains the equation, which relates g_{1k} and eigenvalues m_k^1 by

$$\begin{gathered} \sum_{i=1}^{N} \frac{t_k}{m_k^0 - m^1} = 0\,, \qquad t_k = |g_{1k}|^2\,, \\ \sum_{k=1}^{N} t_k = 1\,. \end{gathered} \tag{2.3}$$

This equation describes one-to-one correspondence of coordinate system t_k and m_k^1; from it immediately follows that eigenvalues obey inequalities

$$m_i^0 > m_i^1 > m_{i+1}^0\,, \tag{2.4}$$

and these intervals are filled densely. Moreover after some simple algebra one finds relation

$$t_k = \frac{\prod\limits_{i=1}^{N-1} (m_i^1 - m_k^0)}{\prod\limits_{i \neq k} (m_i^0 - m_k^0)}\,. \tag{2.5}$$

For to move to coordinates m^1 we also need the Jacobian for integration measure. From the above expression it could be easily obtained and finally we have

$$\begin{aligned} J(t, m^1) &= \frac{\prod\limits_{k<i} (m_i^1 - m_k^1)}{\prod\limits_{l<p} (m_l^0 - m_p^0)}\,, \\ d\mu(a_1, \ldots, a_N) &= \frac{(N-1)!}{(2\pi)^N} J(t, m^1) \\ &\quad \times dm_1^1 \ldots dm_{N-1}^1 d\theta_1 \ldots d\theta_N d\mu(a_2, \ldots, a_N)\,. \end{aligned} \tag{2.6}$$

In addition one can show also that the following formula holds:

$$X_{11} = \sum_{i=1}^{N} g_{1k} m_k g_{k1}^{+} = \sum_{i=1}^{N} t_k m_k^0 = \sum_{i=1}^{N} m_1^0 - \sum_{i=1}^{N-1} m_i^1\,. \tag{2.7}$$

This completes the first step in descent procedure, which embeds the $N-1$ dimensional unitary group in the N dimensional one. After this step we have $N-1$ dimensional space spanned by orthonormal vectors $a_2, \ldots, a_N$, diagonal matrix

$M^1 = PMP$ with eigenvalues $m_1^1, \ldots, m_{N-1}^1$ and eigenvectors f. It is simple to continue the descend further; let me give the necessary expressions after the descent procedure is completed down to the level 0 [2] ($f_{j0}^0 = e_k, \alpha_{j0}^0 = 1$):

$$g_{lk} = (e_k, a_l) = \sum_1^{N-l} \sum_1^{N-2} \cdots \sum_1^{N-l+1} (e_k, f_{j_1}^1)(f_{j_1}^1, f_{j2}^2) \ldots (f_{j_{l-1}}^{l-1}, a_l)$$
$$= \sum_{[j_q]} \prod_{q=0}^{l-2} \frac{\alpha_{j_q}^q (f_{j_q}^q, a_{q+1})}{m_{j_q}^q - m_{j_{q+1}}^{q+1}} (f_{l-1}^{l-1}, a_l) , \tag{2.8}$$

$$(a_q, f_k^{q-1}) = \sqrt{t_k^q} \exp(i\theta_{qk}) , \qquad (f_i^q, f_k^{q-1}) = \alpha_i^q \frac{(a_q, f_k^{q-1})}{m_k^{q-1} - m_i^q} , \tag{2.9}$$

where by α and t we have denoted the following products:

$$(\alpha_i^q)^2 = - \frac{\prod_{j=1}^{N-q+1} (m_i^q - m_j^{q-1})}{\prod_{j \neq i}^{N-q} (m_i^q - m_j^q)} ,$$
$$t_i^q = \frac{\prod_{j=1}^{N-q} (m_j^q - m_i^{q-1})}{\prod_{j \neq i}^{N-q+1} (m_j^{q-1} - m_i^{q-1})} . \tag{2.10}$$

The variables m_i^k are (as it follows from the descent procedure, see (2.4)) from the Gelfand-Tzetlin table

$$m_i^k > m_i^{k+1} > m_{i+1}^k \tag{2.11}$$

and θ are just angle variables, $\theta_i^k = [0, 2\pi]$. Using all this one could check by straightforward calculation that

$$X_{kk} = \sum_{l=1}^N g_{kl} m_l^0 g_{lk}^+ = \sum_{l=1}^N m_l^0 |g_{kl}|^2 = \sum_{i=1}^{N-k+1} m_i^{k-1} - \sum_{i=1}^{N-k} m_i^k , \qquad X_{NN} = m^N ,$$
$$d\mu(g) = \frac{C_N}{(2\pi)^{\frac{(N+2)(N-1)}{2}}} \frac{1}{\Delta(M)} \prod_{k=1}^{N-1} \prod_{i=1}^{N-k} dm_i^k \prod_{k=1}^{N} \prod_{i=1}^{N-k+1} d\theta_i^k , \tag{2.12}$$
$$C_N = (N-1)!\,(N-2)! \ldots 2!1! .$$

The meaning of symbols f_k^i is simple; f.e. f_k^l forms the orthonormal basis in the space spanned by eigenvectors of matrix $M^l = \mathrm{diag}(m_k^l)$.

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[2] One can derive all these expressions using the procedure described above, or (after some modifications) they could be extracted from [4], see also [3]

3. Correlation Functions

Having at hand the expressions (2.8)–(2.10), (2.12) it is easy to prove the statements about (1.1), announced in the introduction, i.e. derive explicit formulas for all 2-point functions and for a set with $n > 2$ (1.4) and present the integral representation of the general n-point function in terms of the GT table (1.2). Also one could give the linear differential equation for the general n-point function. But first I will start with the 0-point function and will give the simple derivation of the IZ formula: [3]

$$\langle 1 \rangle = \int [d\mu(g)] \exp[\mathrm{tr}(gMg^{+}N)] . \tag{3.1}$$

The result follows immediately from the facts we have learned in the previous section (we need just (2.12)) and from two simple observations: (i) The integrand doesn't depend on angle variables θ, so it is the linear exponential on the GT table:

$$\langle 1 \rangle = \frac{C_N}{\Delta(M)} \int \prod_{\mathrm{GT}} [dm] \exp\left[\sum_{k=1}^{N} \left(\sum_{i=1}^{N-k+1} m_i^{k-1} - \sum_{i=1}^{N-k} m_i^{k} \right) N_k \right] . \tag{3.2}$$

(ii) The integration over the bottom coordinate M^N of

$$\exp([(m_1^{N-1} + m_2^{N-1}) - m^N] N_{N-1}) \exp(m_N N_N)$$

leads to

$$\frac{\det[\exp(m_i^{N-1} N_{N-j+1})]}{N_N - N_{N-1}} ,$$

and the variables m are from the second (counting from the bottom) line; $i, j = 1, 2$. Now, according to (3.2), we have to multiply this expression on the exponential

$$\exp\left(\sum_{i=1}^{3} m_i^{N-2} - \sum_{i=1}^{2} m_i^{N-1} \right) N_{N-2}$$

and integrate over m_i^{N-1}; once again we get the similar result:

$$\frac{\det[\exp(m_i^{N-2} N_{N-j+1})]}{\Delta(N_N, N_{N-1}, N_{N-2})} ,$$

with $i, j = 1, 2, 3$. Thus, using the induction method one easily proves that the following formula holds:

$$\int \prod_{i=1}^{l-1} [dm_i^{N-l+1}] \exp\left[\sum_{i=1}^{l} m_i^{N-l} - \sum_{i=1}^{l-1} m_i^{N-l+1} \right] N_{N-l+1}$$
$$\times \frac{\det[\exp(m_i^{N-l+1} N_{N-j+1})]}{\Delta(N_N, \ldots, N_{N-l+2})} = \frac{\det[\exp(m_p^{N-l} N_{N-q+1})]}{\Delta(N_N, \ldots, N_{N-l+1})} , \tag{3.3}$$

here $p, q = 1, \ldots, l$; $i, j = 1, \ldots, (l-1)$.

[3] This is one more derivation of the well-known IZ formula [2], but I decided to include it here for completeness of the presentation; also this kind of derivation might be itself useful, i.e. for the large N limit

This completes the derivation of the IZ formula in our approach, based on GT parametrization:

$$\langle 1\rangle = \frac{C_N}{\Delta(M)\,\Delta(N)}\ \det[\exp(M_i N_j)]\,, \tag{3.4}$$

where we have denoted m_i^0 by M_i.

Let us now move to a discussion of correlation functions. First about general properties:

All nonzero correlation functions in (1.1) have $j_m = k_m$ and $(l_1, \ldots, l_n) = P(i_1, \ldots, i_n)$, where P is the element of permutation group; of course the "reversed" statement, when $i_m = l_m$ and $(k_1, \ldots, k_n) = P'(j_1, \ldots, j_n)$, is equivalent to the above.

To demonstrate this simple fact let us multiply the right-hand side in (1.1) by

$$1 = \frac{1}{(2\pi)^{2N}} \int [dH_1] \int [dH_2]\,,$$

where H_1, H_2 are from $U(1)^N$. Because the IZ measure is invariant under the transformation $g' = H_1 g H_2$ we can perform this transformation in the integrand; thus each operator inside the correlation function will be multiplied by H_1 from left and H_2 from the right (hermitian conjugates will be changed correspondingly). Because the integration over H_1, H_2 is over $U(1)$ angles we see that the integral will always be zero, except for the cases stated above.

The expressions derived at the end of Sect. 2 and the statement (3.3) immediately leads to the proof of (1.4). We have

$$\begin{aligned}
&\langle g_{1j_1} g^{+}_{k_1 1} g_{1j_2} g^{+}_{k_2 1} \cdots g_{1j_n} g^{+}_{k_n 1}\rangle \\
&= \frac{C_N}{\Delta(M)} \int \prod_{\mathrm{GT}} dm\, \frac{\prod_{l=1}^{N-1}(m_l^1 - m_{j_1}^0)}{\prod_{l\neq j_1}(m_l^0 - m_{j_1}^0)}\, \frac{\prod_{l=1}^{N-1}(m_l^1 - m_{j_2}^0)}{\prod_{l\neq j_2}(m_l^0 - m_{j_2}^0)} \cdots \frac{\prod_{l=1}^{N-1}(m_l^1 - m_{j_n}^0)}{\prod_{l\neq j_n}(m_l^0 - m_{j_n}^0)} \\
&\quad\times \exp\left[\sum_{k=1}^{N}\left(\sum_{j=1}^{N-k+1} m_i^{k-1} - \sum_{i=1}^{N-k} m_1^k\right) N_k\right] \\
&= \frac{C_N}{\Delta(M)\,\Delta(N_2, \ldots, N_N)} \int \prod_{k=1}^{N-1} dm_k^1\, \frac{\prod_{l=1}^{N-1}(m_l^1 - m_{j_1}^0)}{\prod_{l\neq j_1}(m_l^0 - m_{j_1}^0)} \\
&\quad\times \frac{\prod_{l=1}^{N-1}(m_l^1 - m_{j_2}^0)}{\prod_{l\neq j_2}(m_l^0 - m_{j_2}^0)} \cdots \frac{\prod_{l=1}^{N-1}(m_l^1 - m_{j_n}^0)}{\prod_{l\neq j_n}(m_l^0 - m_{j_n}^0)} \\
&\quad\times \exp\left[\left(\sum_{k=1}^{N} m_k^0 - \sum_{k=1}^{N-1} m_k^1\right) N_1\right] \det[\exp(m_i^1 N_j)]\,\delta_{j_1 k_1} \cdots \delta_{j_n k_n}\,.
\end{aligned} \tag{3.5}$$

Here $i = 1, 2, \ldots, N-1$; $j = 2, \ldots, N$. As I have mentioned in the introduction the last integral is easy to calculate; it is clear from the integral representation that the result is the sum of quadratic exponentials of eigenvalues M_i, N_i times rational functions and it is regular when two eigenvalues of M coincide.[4] So we could permute the eigenvalues of M to obtain all the other correlation functions $\langle |g_{ik}|^2 \rangle$, as well as all the n-point functions with $i_1 = i_2 = \ldots = i_n$ (and others that are related to later by interchanging the matrix M with the matrix N) after the last integration in (3.5).

Now we will discuss the most general n-point function, $n > 2$, the one that couldn't be reduced to (3.5), and will prove (1.2). Using the general property discussed above we will order the correlation function (1.1) by the set of indices $(i_1, \ldots, i_n)$, $i_1 > i_2 > \ldots > i_n$. According to (2.8) we will draw two copies of the Yung type diagram: one for the set $(i_1, \ldots, i_n; j_1, \ldots, j_n)$, another for $(i_1, \ldots, i_n; k_1, \ldots, k_n)$ (see Fig. 2). Each line in the diagram has length i_{q+1}, and thus we write the numbers

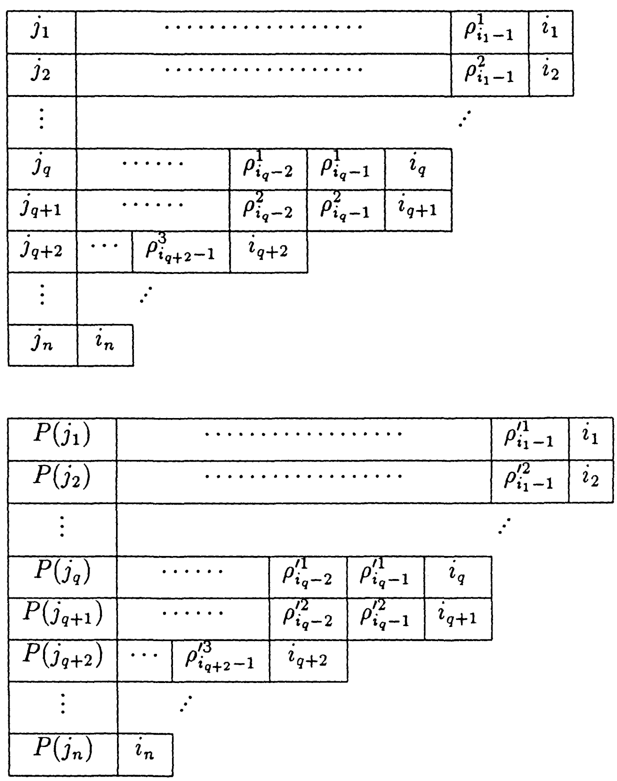

Fig. 2

$(i_1, \ldots, i_n)$ in the boxes at the right end of each line in each diagram. In the first diagram we put the numbers $(j_1, \ldots, j_n)$ in the left most boxes (again from top to bottom, but now the numbers aren't necessary ordered); equivalently in the second diagram we put in the same place numbers $(k_1, \ldots, k_n) = P(j_1, \ldots, j_n)$. So each line in the first diagram has i_q on the right and j_q on the left, at the same time the same line on the second diagram has i_q at the right and $P(j_q)$ at the left. Now we will fill all other boxes in the first diagram according to the following procedure:

[4] We have used the coordinates tied with the matrix M but we could use in the same integral the GT coordinates related to matrix N; the result shouldn't depend on this choice; note that in the GT coordinates for M the eigenvalues of N are not ordered

pick up all lines with length i_n (some of the numbers from set i might be equal, so we will have several lines with the same length) and put next to i_n the numbers $(\varrho^1_{i_n-1}, \ldots, \varrho^{A_{i_n}}_{i_n-1})$ from top to bottom with the requirement that

$$1 < \varrho_{i_n-1} < N - i_n + 1 .$$

A_{i_n} is the number of lines of length i_n. We continue this procedure before reaching the point with $i_n - p = i_{n-1}$, where p counts the boxes from right to left on the top line of the diagram. After this the height of each column is larger, so we will fill the next column with numbers $(\varrho^1_{i_{n-1}-1}, \ldots, \varrho^{A_{i_{n-1}}}_{i_n-1})$, with $A_{i_{n-1}}$ being the number of lines of length i_n plus the number of lines of length i_{n-1}. This procedure we will continue to the end, when all the boxes in Yung diagram are filled. The numbers ϱ should satisfy the inequality

$$1 < \varrho^A_{q-1} < N - q + 1 .$$

Let us call this diagram $Y^{ij}(\varrho)$. We will repeat the same procedure with the second diagram, assigning the numbers ϱ' to empty boxes and defining $Y^{i,P(j)}(\varrho')$.

These diagrams are just convenient representations of the products

$$g_{i_1 j_1} \cdots g_{i_n j_n} , \tag{3.6}$$

$$g^+_{P(j_1) i_1} \cdots g^+_{P(j_n) i_n} , \tag{3.7}$$

when they are viewed in terms of the sum in (2.8). For to proceed further we have to assign to each box in the diagram the factor (except the boxes $i_1, \ldots, i_n$)

$$\frac{\alpha^q_{\varrho_q}(f^q_{\varrho_q}, a_{q+1})}{m^q_{\varrho_q} - m^{q+1}_{\varrho_q+1}} \tag{3.8}$$

(for $q+1 = l$, where $l+1$ is the length of given row, we have to replace (3.8) just by $(f^q_{\varrho_q}, a_{q+1})$) and then multiply over q in each line; we should do the same with the second diagram, but assigning to the boxes the complex conjugate of (3.8). So, for each diagram we have assigned the particular product dictated by the above procedure and the diagram. Let us call these factors $F^{ij}(Y)$ and $F^{iP(j)}(Y')$. Finally, to obtain the factor R coming from insertion of (3.6) and (3.7) in the IZ integral we have to sum over all possible diagrams the objects $F(Y)$ and $F(Y')$ ((with fixed $(i_1, \ldots, i_N); (j_1, \ldots, j_N); P$) and then multiply:

$$R = \sum_{YY'} F(Y)\, F(Y') . \tag{3.9}$$

The described procedure looks very complicated but the result of integration over angle variables makes it simpler. Let us take the product of F_1 and F_2 with some particular set of ϱ and ϱ' and integrate over angle variables with measure (2.5). Because we already know that the dependence on angles is coming only from the pre-exponent in (1.1) (see (2.5) and (3.2)), this is the only angle integral we have to calculate. From the other side the angle dependence in (3.8) is due to (2.10), thus we have the simple angle integral to calculate: fix the column (let us say l^{th}) in both diagrams and denote by Θ the sum over this column of the angles $\theta_{\varrho_l, l}$; denote by

Θ' the same sum for the second diagram. The angle integral that enters in (1.1) for this column is just

$$\int \prod_{\mathrm{row}[l]} (d\theta) \exp(\Theta - \Theta'),$$

which is zero if the set ϱ' on the second diagram in a given column (l^{th} in this case) is not related to set of ϱ in this first one by the permutation

$$(\varrho_l'^1, \ldots, \varrho_l'^{A_l}) = P(\varrho_l^1, \ldots, \varrho_l^{A_l}).$$

Later we relate the numbers in the second diagram to numbers in the first by a simple law: we should consider only those pair of diagrams, when numbers in the fixed column of the second are related to numbers in the same column of the first diagram by some element of permutation group, so for each diagram Y we have the set of mappings τ defined by the above law: $Y' = \tau(Y)$. Thus we have for R after integration over all angle variables:

$$R_0 = R_0[m] = \sum_{Y,\tau} F_0^{(i,j)}[Y] F_0^{(i,P[j])}[\tau(Y)]. \tag{3.10}$$

Here the subscript 0 means that we have to put to zero all angles in (3.8); after this R becomes the function on the GT table, $R_0[m]$. Now it is easy to see that $R_0[m]$ is the rational function. Simply after applying the mapping τ to Y there are only moduli squares of (2.9) and (2.10) left in (3.10); this means that $R_0[m]$ is rational. On this we complete the construction of R Going back to (1.1) in the parametrization (2.12) we have (for nonzero correlators):

$$\begin{aligned}
&\langle g_{i_1 j_1} g^{+}_{i_1 P(j_1)} \cdots g_{i_n j_n} g^{+}_{i_n P(j_n)} \rangle \\
&= \frac{C_N}{\Delta(M)} \int \prod_{\mathrm{GT}} dm R_0[m] \exp\left[\sum_{k=0}^{N} X_{kk} N_k\right] \\
&= \frac{C_N}{\Delta(M)} \int \prod_{k=1}^{N-1} \prod_{i=1}^{N-k} dm_i^k R_0[m] \\
&\quad \times \exp\left[\sum_{k=1}^{N-1} \left(\sum_{i=1}^{N-k+1} m_i^{k-1} - \sum_{i=1}^{N-k} m_i^k \right) N_k \right],
\end{aligned} \tag{3.11}$$

and here the indices for $R[m]$ are arranged according for those in the right-hand side. Thus we have derived the formula announced in the introduction.

The integrand in (3.11) still looks complicated, (even having the explicit algorithm of its construction), except the one when $i_1 = i_2 = \ldots = i_n$; but it seems that it could be simplified using algebraic identities for particular rational functions that enter in R_0 [5]. It would also be interesting to have some geometric interpretation of the above construction. At the moment the best we could extract from (3.11) in the general case

[5] This should be viewed as a conjecture based on knowledge of the n-point function (3.5); i.e. in our basis the expression for $\langle |g_{ik}|^2 \rangle$ has a similar structure as the general correlation function in (3.11) and at the same time the answer obtained from the one with $i = 1$ by permutation of eigenvalues of M is simple

is the following: first we could extract from $R_0[m]$ two polynomials P_1 and P_2 by defining

$$R_0[m] = \frac{P_1[m]}{P_2[m]} \,. \tag{3.12}$$

Equation (3.12) uniquely defines P_1, P_2; there is a simple formula for P_2 in the case of the general n-point function P_2,

$$P_2 = \prod_{[q]} \prod_{i \neq j} (m_i^q - m_j^{q-1})^{a_q} \prod_{k \neq l} (m_k^q - m_l^q)^{b_q} \prod_{p \neq t} (m_p^{q-1} - m_t^{q-1})^{c_q} \,, \tag{3.13}$$

where as set $[q]$ and the numbers a_q, b_q, c_q are fixed by the particular correlator under consideration. Then consider the more general than (3.11) integral: introduce for each point on the GT table the corresponding "dual" coordinate z_i^k and define I by

$$I(z) = \int \prod_{\mathrm{GT}} dm R[m] \exp \left[\sum_{j=1}^{N-1} \sum_{i=1}^{N-k} z_i^k m_i^k \right] . \tag{3.14}$$

From $I(z)$ integral (3.11) is obtained by the requirement that all coordinates on the "z" table in a given row k are equal to $N_k - N_{k-1}$; from the other side I obeys the linear differential equation

$$P_2(\partial_z) I = P_1(\partial_z) I_0 \,, \qquad I_0 = I(R = 1) \,, \tag{3.15}$$

so that the right-hand side in (3.15) is a known function. Thus, some particular solution of (3.15) for the values of z, that I have defined previously, should coincide with (3.11) and therefore with (1.1).

4. Conclusion

One should note that the expressions (1.4), (1.2) are very simple. This simplicity of course should be related to the fact that the integral (1.1) actually is not over the unitary group G, but rather over its left/right coset. We have used only part of the symmetries when we tied the coordinate system to the matrix M (and not to the matrix N). Using the fact that the result shouldn't depend on this choice, additional identities could be obtained. It might be possible that the latter allows us to get all nontrivial n-point functions for $n > 2$ from those that we have calculated (see (1.4)).

The simplicity of (1.4), (1.2) might be helpful for the applications, mainly for the large N limit; the latter is important both in string theory and the Kazakov-Migdal model. In this respect I would like to note the elegance of (3.3); it might lead to an interesting equation in the large N limit. These results might also be helpful in the statistical models, discussed in [7]. Migdal has derived an explicit formula for the general 2-point function in leading order of the large N approximation in his approach based on the Riemann-Hilbert method [8]. I known from Morozov [9] that he has conjectured a simple formula for the 2-point function that looks similar to the one obtained from (1.4) after integration over m^1 in the case of the 2-point function. It would be interesting to find the geometric approach that could explain in more invariant way the results obtained in the IZ model both for finite and large N.

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