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Boundedness for Large $|x|$ of Suitable Weak Solutions of the Navier-Stokes Equations with Prescribed Velocity at Infinity

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Abstract. We consider time-dependent perturbations u of R. Finn's stationary PR-solution of the Navier-Stokes equations, which converges to a constant vector v_∞ as $|x| \to \infty$. For a given time interval $[\delta, T]$, we find a radius K such that u is essentially bounded on $[\delta, T] \times \{|x| \geq K\}$.

1. Introduction

We want to investigate the boundedness for large $|x|$ of weak solutions v of the Navier-Stokes system

$$
\begin{gathered}
v_t - \Delta v + (v \cdot \nabla) v + \nabla p = f\,, \\
\operatorname{div} v = 0 \quad \text{in} \quad [0, T] \times \Omega\,, \\
v(0, x) = v_0(x) \quad \text{for} \quad x \in \Omega\,, \\
v(t, x) = 0 \quad \text{for} \quad (t, x) \in [0, T] \times \partial\Omega\,, \\
v(t, x) \to v_\infty \quad \text{as} \quad |x| \to \infty\,, \qquad t \in [0, T]\,,
\end{gathered}
\tag{1}
$$

where Ω is a smooth exterior domain in $\mathbf{R}^3$, $\operatorname{div} f = 0$, $\operatorname{div} v_0 = 0$, $v_0|\partial\Omega = 0$, $v_0 \to v_\infty$ at infinity, $v_\infty \in \mathbf{R}^3$ is the prescribed constant velocity at infinity.

Most of the previous work concentrates on the case $v_\infty = 0$, where suitable weak solutions of (1) are known to become small in some average sense and bounded for large $|x| \cdot t$, if $f \to 0$ $(t, |x| \to \infty)$, see [CKN, MP, SW]. This means that singularities may occur only in a compact subset of $[0, \infty) \times \Omega$. Some important results are also surveyed in [W].

If $v_\infty \neq 0$ it is not apparent, whether a global weak solution to (1) will converge to a stationary solution as $t \to \infty$. This seems to happen in general only under some smallness assumptions on a corresponding stationary solution, see Miyakawa and Sohr [MS] and Masuda [MK].

In this note we will only assume the existence of a "reasonable" stationary solution $\overset{(0)}{v}$, we will not require any additional smallness. For the existence of $\overset{(0)}{v}$ we refer to

Finn's work [Fi], cf. also [Fa]. We construct a solution v to (1) as perturbation u of $\overset{(0)}{v}$.

As it may be expected, that u in general does not calm down, we consider an arbitrary bounded time interval $[0,T]$. We show that the boundedness criterion of Caffarelli, Kohn and Nirenberg [CKN, Proposition 1] can be carried over to weak solutions of a "perturbed Navier-Stokes system." For every $\delta \in (0,T)$ we construct a radius $K = K(\delta,T)$ such that u (and hence v) is essentially bounded on $[\delta,T] \times (\Omega \cap \{|x| \geq K\})$.

The problem is left open whether K may be chosen independent of T.

2. Preliminaries. Results

Most of the notation is adopted from [W]. In particular $H_q(\Omega)$ denotes the completion of $\{v \in C_0^\infty(\Omega)^3 : \operatorname{div} v = 0\}$ with respect to the L^q-norm, in a weak sense $H_q(\Omega)$ is the set of all divergence free L^q-vector-functions with zero normal component on $\partial\Omega$.

$H^{k,q}(\Omega)$, $H_0^{k,q}(\Omega)$ are the usual Sobolev spaces of functions with weak derivatives in L^q up to order k.

$(u \cdot \nabla)v := (u \cdot (\nabla v_1), u \cdot (\nabla v_2), u \cdot (\nabla v_3))$, "$\cdot$" denotes the scalar product in $\mathbf{R}^3$, $(u,v) := \int_\Omega u(\xi) \cdot v(\xi)\, d\xi$.

We start with a classical solution $(\overset{(0)}{v}, \overset{(0)}{p})$ of the stationary Navier-Stokes problem

$$
\begin{gathered}
-\Delta \overset{(0)}{v} + (\overset{(0)}{v} \cdot \nabla)\overset{(0)}{v} + \nabla \overset{(0)}{p} = 0\,, \\
\operatorname{div} \overset{(0)}{v} = 0 \quad \text{in} \quad \Omega\,, \\
\overset{(0)}{v}|\partial\Omega = 0 \qquad \overset{(0)}{v}(x) \to v_\infty \quad \text{uniformly as} \quad |x| \to \infty\,.
\end{gathered}
\tag{2}
$$

We require $\overset{(0)}{v}$ to satisfy

$$
\begin{gathered}
M_0 := \sup_{x\in\Omega} |x| \cdot |\overset{(0)}{v}(x) - v_\infty| < \infty\,, \qquad \nabla \overset{(0)}{v} \in L^3\,, \\
|\overset{(0)}{v}(x)| \leq M_1\,, \qquad |\nabla \overset{(0)}{v}(x)| \leq M_2 \quad \text{for all } x \in \Omega
\end{gathered}
\tag{3}
$$

with some constants M_0, M_1, M_2.

The existence of such a stationary solution $(\overset{(0)}{v}, \overset{(0)}{p})$ is ensured in Finn's article [Fi], if $|v_\infty|$ is not too large.

We construct a weak solution of (1) as perturbation of $\overset{(0)}{v}$, i.e. we look for $u := v - \overset{(0)}{v}$ as a weak solution of

$$
\begin{gathered}
u_t - \Delta u + (u \cdot \nabla)\overset{(0)}{v} + (\overset{(0)}{v} \cdot \nabla)u + (u \cdot \nabla)u + \nabla\hat{p} = f\,, \\
\operatorname{div} u = 0 \quad \text{in} \quad [0,T] \times \Omega\,, \\
u(0,x) = u_0(x) \quad \text{for} \quad x \in \Omega\,, \\
u(t,x) = 0 \quad \text{for} \quad (t,x) \in [0,T] \times \partial\Omega\,, \\
u(t,x) \to 0 \quad \text{as} \quad |x| \to \infty\,, \qquad t \in [0,T]\,.
\end{gathered}
\tag{4}
$$

Combining the methods of [W] and [MS, Sect. 5], where the additional lower order terms in (4) are treated, the existence of a weak solution to (4) with localized energy inequality can easily be shown. But in this inequality there are terms involving $\overset{(0)}{v}$ which cause some trouble.

Therefore we consider $w(t,x) := e^{-\lambda t}u(t,x)$, $\pi(t,x) := e^{-\lambda t}\hat{p}(t,x)$, $g(t,x) := e^{-\lambda t}f(t,x)$, $\lambda \geq 0$. $(u,\hat{p})$ solves (4) if and only if (w,π) solves

$$\begin{gathered} w_t - \Delta w + \lambda w + (w\cdot\nabla)\overset{(0)}{v} + (\overset{(0)}{v}\cdot\nabla)w + e^{\lambda t}(w\cdot\nabla)w + \nabla\pi = g\,, \\ \operatorname{div} w = 0 \quad \text{in} \quad [0,T]\times\Omega\,, \\ w(0,x) = u_0(x) \quad \text{for} \quad x\in\Omega\,, \\ w(t,x) = 0 \quad \text{for} \quad (t,x)\in[0,T]\times\partial\Omega\,, \\ w(t,x)\to 0 \quad \text{as} \quad |x|\to\infty\,, \qquad t\in[0,T]\,. \end{gathered} \tag{5}$$

For this problem we have the following existence theorem:

Theorem 1. *Let $T>0$, $u_0 \in H_2(\Omega)\cap H_{9/8}(\Omega)$, $g\in L^1((0,T),H_2(\Omega)\cap H_{9/8}(\Omega))\cap L^2_{\text{loc}}((0,T),H_2\cap H_{9/8})\cap L^2((0,T),H_2)$. Then we have:*
(i) *There is a weak solution w on $[0,T]\times\Omega$ to (5) in the following sense: $w:[0,T]\to H_2(\Omega)$ is weakly continuous, $w\in L^\infty((0,T),H_2(\Omega))\cap L^2((0,T),H_0^{1,2}(\Omega)^3)$,*

$$\begin{aligned} &-\int_0^T (w,\Phi_t)\,d\tau + \int_0^T (\nabla w,\nabla\Phi)\,d\tau + \lambda\int_0^T (w,\Phi)\,d\tau \\ &\quad + \int_0^T ((w\cdot\nabla)\overset{(0)}{v},\Phi)\,d\tau + \int_0^T ((\overset{(0)}{v}\cdot\nabla)w,\Phi)\,d\tau \\ &-\int_0^T e^{\lambda t}((w_i w_j)_{i,j},\nabla\Phi)\,d\tau = \int_0^T (g,\Phi)\,d\tau + (u_0,\Phi(0)) \end{aligned}$$

for every $\Phi(t,x)=\varphi(x)h(t)$, $\varphi\in H^{2,2}(\Omega)^3\cap H_0^{1,2}(\Omega)^3$, $\operatorname{div}\varphi = 0$, $h\in C^1([0,T],\mathbf{R})$, $h(T)=0$.
(ii) *w has the following additional properties:*

$$\begin{aligned} w \in &\bigcap_{\varepsilon:0<\varepsilon<T} L^{3/2}((\varepsilon,T),H^{2,9/8}(\Omega)^3\cap H_0^{1,9/8}(\Omega)^3\cap H_{9/8}(\Omega)) \\ &\cap L^{5/4}((\varepsilon,T),H^{2,5/4}(\Omega)^3\cap H_0^{1,5/4}(\Omega)^3\cap H_{5/4}(\Omega))\,, \\ w_t \in &\bigcap_{\varepsilon:0<\varepsilon<T} L^{3/2}((\varepsilon,T),L^{9/8}(\Omega)^3)\cap L^{5/4}((\varepsilon,T),L^{5/4}(\Omega)^3)\,. \end{aligned}$$

There is a mapping $\pi : (0,T) \to L^{15/7}(\Omega) \cap L^{9/5}(\Omega)$ *with*

$$\nabla \pi \in \bigcap_{\varepsilon : 0<\varepsilon<T} L^{5/4}((\varepsilon,T), L^{5/4}(\Omega)^3) \cap L^{3/2}((\varepsilon,T), L^{9/8}(\Omega)^3)\,,$$
$$\pi \in \bigcap_{\varepsilon : 0<\varepsilon<T} L^{5/4}((\varepsilon,T), L^{15/7}(\Omega)) \cap L^{3/2}((\varepsilon,T), L^{9/5}(\Omega))\,,$$

such that the Navier-Stokes type system (5) is fulfilled a.e. *in* $(0,T)\times\Omega$.
(iii) *If* $\lambda = M_2 + \frac{1}{2}M_1^2$ *(for* M_i *see (3)), we have the following localized energy inequality:*

$$\begin{aligned}\int_\Omega \Phi(t)\,|w(t)|^2\,d\xi + \int_s^t \int_\Omega \Phi|\nabla w|^2\,d\xi\,d\tau \\ \le \int_\Omega \Phi(s)\,|w(s)|^2\,d\xi + 2\int_s^t\int_\Omega (\Phi g)\cdot w\,d\xi\,d\tau \\ + \int_s^t\int_\Omega \{|w|^2(\Phi_t + \Delta\Phi) + \nabla\Phi\cdot[e^{\lambda\tau}|w|^2 w + 2\pi w]\}\,d\xi\,d\tau \end{aligned} \tag{6}$$

for all $\Phi \in C_0^2([0,T]\times\bar{\Omega})$, $\Phi \ge 0$, *for all* $t > 0$ *and almost all* $s \in (0,t]$.
Moreover for all $t > 0$, $s = 0$, *almost all* $s \in (0,t]$ *we have the generalized energy inequality*

$$\|w(t)\|^2_{H_2(\Omega)} + 2\int_s^t \|\nabla w(\tau)\|^2_{L^2(\Omega)}d\tau \le \|w(s)\|^2_{H_2(\Omega)} + 2\int_s^t (g,w)\,d\tau\,. \tag{7}$$

Proof. Imitating [W, Chap. II.4, II.5] and [MS, Sect. 5] we readily obtain part (i), (ii) and the following version of the localized energy inequality (6):

$$\begin{aligned}\int_\Omega \Phi(t)\,|w(t)|^2\,d\xi + 2\int_s^t\int_\Omega \Phi|\nabla w|^2\,d\xi\,d\tau \\ \le \int_\Omega \Phi(s)\,|w(s)|^2\,d\xi + 2\int_s^t\int_\Omega (\Phi g)\cdot w\,d\xi\,d\tau \\ - 2\int_s^t\int_\Omega (\phi w)\cdot[((w\cdot\nabla)\overset{(0)}{v}) + ((\overset{(0)}{v}\cdot\nabla)w)]\,d\xi\,d\tau - 2\lambda\int_s^t\int_\Omega \Phi|w|^2\,d\xi\,d\tau \\ + \int_s^t\int_\Omega\{|w|^2(\Phi_t+\Delta\Phi) + \nabla\Phi\cdot[e^{\lambda\tau}|w|^2 w + 2\pi w]\}\,d\xi\,d\tau\,. \end{aligned} \tag{8}$$

Using the Cauchy-Schwarz inequality, $\Phi \geq 0$, the bounds for $\overset{(0)}{v}$ and $\nabla\overset{(0)}{v}$ we conclude

$$\left| -2\int_s^t\int_\Omega (\Phi w)\cdot((w\cdot\nabla)\overset{(0)}{v})\,d\xi\,d\tau\right| \leq 2M_2\int_s^t\int_\Omega \Phi|w|^2\,d\xi\,d\tau\,,$$

$$\begin{aligned}\left| -2\int_s^t\int_\Omega (\Phi w)\cdot((\overset{(0)}{v}\cdot\nabla)w)\,d\xi\,d\tau\right|&\\ \leq 2M_1\int_s^t\int_\Omega (\sqrt{\Phi}\,|w|)\cdot(\sqrt{\Phi}\,|\nabla w|)\,d\xi\,d\tau&\\ \leq 2M_1\int_s^t\int_\Omega \left(\frac{M_1}{2}\,\Phi|w|^2+\frac{1}{2M_1}\,\Phi|\nabla w|^2\right)d\xi\,d\tau&\\ =\int_s^t\int_\Omega \Phi|\nabla w|^2\,d\xi\,d\tau + M_1^2\int_s^t\int_\Omega \Phi|w|^2\,d\xi\,d\tau\,.&\end{aligned}$$

These two estimates are inserted into (8), $\int_s^t\int_\Omega \Phi|\nabla w|^2\,d\xi\,d\tau$ is subtracted on both sides. Taking notice of $2M_2+M_1^2-2\lambda=0$ we arrive at (6).

To obtain (7) we can argue slightly differently: Integration by parts yields

$$\begin{aligned}&\int_s^t\int_\Omega (\Phi w)\cdot((\overset{(0)}{v}\cdot\nabla)w)\,d\xi\,d\tau\\ &\quad=-\int_s^t\int_\Omega [(\Phi w)\cdot((\overset{(0)}{v}\cdot\nabla)w)+|w|^2(\overset{(0)}{v}\cdot\nabla\Phi)]\,d\xi\,d\tau\,.\end{aligned}$$

We simply mimic Chap. II.5 of [W] (Φ approximates the constant 1) and deduce the generalized energy inequality (7). □

Now we can give our partial regularity result for w, note that u and w differ on bounded time intervals only by a bounded factor.

Theorem 2. *Let T, u_0, g, w, π, λ be as described in Theorem 1. Assume additionally $g\in\bigcap_{0<\varepsilon<T} L^q((\varepsilon,T),L^q(\Omega)^3)$ for some $q>\frac{5}{2}$. Let $\delta\in(0,T)$.*

Then there exist numbers $K=K(\delta,T,w,\pi,M_0,M_1,M_2)$ and $L=L(\delta,T,M_1,M_2)$ such that

$$|w(t,x)|\leq L$$

for almost all $(t,x)\in[\delta,T]\times\Omega$ with $|x|\geq K$.

The proof is based upon a generalization of a boundedness criterion of Caffarelli, Kohn, Nirenberg, see [CKN, Proposition 1], which we will develop in the following section.

3. A Boundedness Criterion

Throughout this chapter let the assumptions of Theorem 2 be satisfied. We remark that all the integrability properties needed below can be derived by means of interpolation inequalities, see the proof of Theorem 2.

Parabolic cylinders will play an important rôle in the following:

$$Q_R(t,x) := \{(\tau,\xi) \in \mathbf{R} \times \mathbf{R}^3 : t - R^2 < \tau < t, |x-\xi| < R\}.$$

Lemma. *Let* $Q_R(t_0,x_0) \subset [\tilde\delta, T] \times \Omega$ *for some* $\tilde\delta > 0$. *There are constants* $\varepsilon_1 = \varepsilon_1(M_1, M_2, T)$, $\varepsilon_2 = \varepsilon_2(M_1, M_2, T, q)$, $L = L(M_1, M_2, T)$, *such that the validity of*

$$R^{-2} \iint_{Q_R(t_0,x_0)} (|w|^3 + |w|\cdot|\pi|)\,d\xi\,d\tau + R^{-13/4} \int_{t_0-R^2}^{t_0} \left(\int_{|\xi-x_0|<R} |\pi|\,d\xi \right)^{5/4} d\tau \le \varepsilon_1, \tag{9}$$

$$R^{3q-5} \iint_{Q_R(t_0,x_0)} |g|^q\,d\xi\,d\tau \le \varepsilon_2. \tag{10}$$

$$|\overset{(0)}{v}(x) - v_\infty| \le \varepsilon_1 R^{-1} \quad \textit{for} \quad |x-x_0| < R \tag{11}$$

implies

$$|w(t,x)| \le L \cdot R^{-1} \tag{12}$$

almost everywhere in $Q_{R/2}(t_0,x_0)$.

Proof. We can use some parts of the proof of Proposition 1 of [CKN] with only minor changes, these parts will not be repeated here but only referred to.

Step 1. We shift (t_0,x_0) to the origin $(0,0)$ and scale the cylinder Q_R:
For $(t,x) \in Q_1(0,0)$, let

$$\begin{aligned} \tilde w(t,x) &:= Rw(t_0+R^2t, x_0+Rx), \qquad \tilde v(x) := R\overset{(0)}{v}(x_0+Rx) \\ \tilde\pi(t,x) &:= R^2\pi(t_0+R^2t, x_0+Rx), \qquad \tilde g(t,x) := R^3 g(t_0+R^2t, x_0+Rx). \end{aligned} \tag{13}$$

Then $(\tilde w, \tilde\pi)$ is a weak solution of the differential equations

$$\begin{aligned} &\tilde w_t - \Delta\tilde w + \tilde\lambda\tilde w + (\tilde w\cdot\nabla)\tilde v + (\tilde v\cdot\nabla)\tilde w + e^{\tilde\lambda t}e^{\lambda t_0}(\tilde w\cdot\nabla)\tilde w + \nabla\tilde\pi = \tilde g, \\ &\operatorname{div}\tilde w = \operatorname{div}\tilde v = 0, \end{aligned} \tag{14}$$

where $\tilde\lambda = \lambda R^2$. Moreover for every $\Phi \in C_0^2((-1,0] \times B_1(0))$, $\Phi \ge 0$, and every $t \in (-1,0]$ the localized energy inequality holds:

$$\int_{B_1(0)} \Phi(t,\xi)\,|\tilde w(t,\xi)|^2\,d\xi + \int_{-1}^{t}\int_{B_1(0)} \Phi(\tau,\xi)\,|\nabla\tilde w(\tau,\xi)|^2\,d\xi\,d\tau$$

$$\le \int_{-1}^{t}\int_{B_1(0)} \{(2\Phi\tilde g)\cdot\tilde w + |\tilde w|^2(\Phi_t + \Delta\Phi) + \nabla\Phi\cdot[e^{\tilde\lambda\tau}e^{\lambda t_0}|\tilde w|^2\tilde w + 2\tilde\pi\tilde w]\}\,d\xi\,d\tau. \tag{15}$$

The smallness conditions (9)–(11) now read as follows:

$$\iint_{Q_1(0,0)} (|\tilde{w}|^3 + |\tilde{w}| \cdot |\tilde{\pi}|)\, d\xi\, d\tau + \int_{-1}^{0} \left(\int_{B_1(0)} |\tilde{\pi}|\, d\xi \right)^{5/4} d\tau \leq \varepsilon_1 , \tag{16}$$

$$\iint_{Q_1(0,0)} |\tilde{g}|^q\, d\xi\, d\tau \leq \varepsilon_2 , \tag{17}$$

$$|\tilde{v}(x) - Rv_\infty| \leq \varepsilon_1 , \tag{18}$$

where ε_1, ε_2 have to be determined below. W.l.o.g. we assume $\varepsilon_1, \varepsilon_2 \leq 1$. We remark that $e^{\lambda t_0} \leq C$, $C = C(M_1, M_2, T)$.

Step 2. Let $(s, a) \in Q_{1/2}(0, 0)$ be an arbitrary point, $Q^n := Q_{r_n}(s, a)$, $r_n := 2^{-n}$. We will prove inductively:

Claim ($\mathrm{A_n}$):

$$\sup_{s - r_n^2 < \tau \leq s} \frac{1}{r_n^3} \int_{|\xi - a| < r_n} |\tilde{w}(\tau, \xi)|^2\, d\xi + \frac{1}{r_n^3} \iint_{Q^n} |\nabla \tilde{w}|^2\, d\xi\, d\tau \leq C_0 \varepsilon_1^{2/3} , \qquad n \geq 2 .$$

with a constant $C_0 = C_0(M_1, M_2, T)$ which does not depend on either n or $\tilde{w}$.

Claim ($\mathrm{B_n}$):

$$r_n^{-22/5} \iint_{Q^n} |\tilde{w}| \cdot |\tilde{\pi}(\tau, \xi) - \bar{\tilde{\pi}}_n(\tau)|\, d\xi\, d\tau + r_n^{-5} \iint_{Q^n} |\tilde{w}|^3\, d\xi\, d\tau \leq \varepsilon_1^{2/3} , \qquad n \geq 3 ,$$

where $\bar{\tilde{\pi}}(\tau) := \frac{1}{|B_{r_n}(a)|} \int_{B_{r_n}(a)} \tilde{\pi}(\tau, \xi)\, d\xi$.

From ($\mathrm{A_n}$) it follows $r_n^{-3} \int_{B_{r_n}(a)} |\tilde{w}(s, \xi)|^2\, d\xi \leq C_0$ and further $|\tilde{w}(s, a)|^2 \leq C_0$, if (s, a) is a Lebesgue-point for $|\tilde{w}|^2$, i.e. almost everyhwere in $Q_{1/2}(0, 0)$.

To prove ($\mathrm{A_2}$), we choose a smooth function $\Phi \geq 0$, $\Phi = 1$ in Q^2 and $\Phi = 0$ outside Q^1 and see from (15), that the left-hand side of ($\mathrm{A_2}$) is bounded by

$$C \iint_{Q^1} \{|\tilde{g}| \cdot |\tilde{w}| + |\tilde{w}|^2 + |\tilde{w}|^3 + |\tilde{\pi}| \cdot |\tilde{w}|\}\, d\xi\, d\tau .$$

If ε_1, ε_2 are small enough, this is at most $C_0 \varepsilon_1^{2/3}$ by Hölder's inequality, (16) and (17).

Step 3. ($\mathrm{A_k}$), $2 \leq k \leq n$, *implies* ($\mathrm{B_{n+1}}$), *if* $n \geq 2$.

We use the following Sobolev and interpolation inequality (see [CKN], Lemma 3.1.):

$$\iint_{Q^n} |\tilde{w}|^3\, d\xi\, d\tau \le C r_n^{1/2} \Bigg\{ \Bigg(\sup_{s-r_n^2<\tau\le s} \int_{|\xi-a|<r_n} |\tilde{w}(\tau,\xi)|^2\, d\xi \Bigg)^{3/2} + \Bigg(\sup_{s-r_n^2<\tau\le s} \int_{|\xi-a|<r_n} |\tilde{w}(\tau,\xi)|^2\, d\xi \Bigg)^{3/4} \Bigg(\iint_{Q^n} |\nabla \tilde{w}|^2\, d\xi\, d\tau \Bigg)^{3/4} \Bigg\}.$$

From this and the inductive hypothesis (A_n) it follows:

$$r_{n+1}^{-5} \iint_{Q^{n+1}} |\tilde{w}|^3\, d\xi\, d\tau \le C r_n^{-5} \iint_{Q^n} |\tilde{w}|^3\, d\xi\, d\tau \le C^* \varepsilon_1 . \tag{19}$$

If ε_1 is so small, that

$$C^* \varepsilon_1^{1/3} \le \tfrac{1}{2}$$

is satisfied, then

$$r_{n+1}^{-5} \iint_{Q^{n+1}} |\tilde{w}|^3\, d\xi\, d\tau \le \tfrac{1}{2}\, \varepsilon_1^{2/3} . \tag{20}$$

The second term in (B_{n+1}) causes more trouble. In Step 4 we will give a sketch of proof for the following estimate:

$$\begin{aligned}
&\iint_{Q^{n+1}} |\tilde{w}| \cdot |\tilde{\pi} - \bar{\bar{\pi}}_{n+1}|\, d\xi\, d\tau \\
&\le C \Bigg(\iint_{Q^{n+1}} |\tilde{w}|^3\, d\xi\, d\tau \Bigg)^{1/3} \\
&\quad \times \Bigg\{ \Bigg(\iint_{Q^n} |\tilde{w}|^3\, d\xi\, d\tau \Bigg)^{2/3} + \varepsilon_1 r_{n+1}^{5/3} \Bigg(\iint_{Q^n} |\tilde{w}|^3\, d\xi\, d\tau \Bigg)^{1/3} \Bigg\} \\
&\quad + C r_{n+1}^{13/3} \Bigg(\iint_{Q^{n+1}} |\tilde{w}|^3\, d\xi\, d\tau \Bigg)^{1/3} \\
&\quad \times \sup_{s-r_{n+1}^2<\tau\le s} \Bigg(\int_{r_n<|\xi-a|<r_2} \frac{|\tilde{w}|^2 + \varepsilon_1 |\tilde{w}|}{|\xi-a|^4}\, d\xi \Bigg) \\
&\quad + C r_{n+1}^{3} \Bigg(\iint_{Q^{n+1}} |\tilde{w}|^3\, d\xi\, d\tau \Bigg)^{1/3} \\
&\quad \times \Bigg\{ \Bigg(\iint_{Q^2} |\tilde{w}|^3\, d\xi\, d\tau \Bigg)^{2/3} + \varepsilon_1 \Bigg(\iint_{Q^2} |\tilde{w}|^3\, d\xi\, d\tau \Bigg)^{1/3} \Bigg\}
\end{aligned}$$

$$+ Cr_{n+1}^{14/5} \left(\sup_{s-r_{n+1}^2 < \tau \le s} \int_{|\xi-a|<r_{n+1}} |\tilde{w}(\tau,\xi)|^2 \, d\xi \right)^{1/5}$$

$$\times \left(\iint_{Q^{n+1}} |\tilde{w}|^3 \, d\xi \, d\tau \right)^{1/5} \left(\int_{s-1/16}^{s} \left(\int_{|\xi-a|<1/4} |\tilde{\pi}| \, d\xi \right)^{5/4} d\tau \right)^{4/5}$$

$$=: \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} . \tag{21}$$

These four terms are estimated using (16), (19) and ($\mathrm{A_k}$) (note that $\varepsilon_i \le 1$):

$$\mathrm{I} \le C(C\varepsilon_1 r_n^5)^{1/3} \{ (C\varepsilon_1 r_n^5)^{2/3} + \varepsilon_1 r_n^{5/3} (C\varepsilon_1 r_n^5)^{1/3} \} \le C\varepsilon_1 r_n^5 ,$$

$$\mathrm{II} \le Cr_n^{13/3} (C\varepsilon_1 r_n^5)^{1/3} \times \sum_{k=2}^{n-1} \sup_{s-r_k^2<\tau\le s} \left\{ r_k^{-4} \int_{r_{k+1}<|\xi-a|<r_k} |\tilde{w}|^2 \, d\xi \right.$$

$$\left. + C\varepsilon_1 r_k^{3/2-4} \left(\int_{r_{k+1}<|\xi-a|<r_k} |\tilde{w}|^2 \, d\xi \right)^{1/2} \right\}$$

$$\le C\varepsilon_1^{1/3} r_n^6 \left(\sum_{k=2}^{n-1} r_k^{-1} \right) \varepsilon_1^{2/3} \le C\varepsilon_1 r_n^5 ,$$

$$\mathrm{III} \le Cr_n^3 (C\varepsilon_1 r_n^5)^{1/3} (\varepsilon_1^{2/3} + \varepsilon_1^{4/3}) \le C\varepsilon_1 r_n^{14/3} ,$$

$$\mathrm{IV} \le Cr_n^{14/5} (Cr_n^3 \varepsilon_1^{2/3})^{1/5} (C\varepsilon_1 r_n^5)^{1/5} \varepsilon_1^{4/5} \le C\varepsilon_1 r_n^{22/5} .$$

Collecting terms we obtain from (21):

$$r_{n+1}^{-22/5} \iint_{Q^{n+1}} |\tilde{w}| \cdot |\tilde{\pi} - \bar{\tilde{\pi}}_{n+1}| \, d\xi \, d\tau \le C\varepsilon_1 r_n^{-22/5} (2r_n^5 + r_n^{14/3} + r_n^{22/5}) \le C^{**} \varepsilon_1 .$$

We require now ε_1 to be small enough to satisfy

$$C^{**} \varepsilon_1^{1/3} \le \tfrac{1}{2}$$

and conclude:

$$r_{n+1}^{-22/5} \iint_{Q^{n+1}} |\tilde{w}| \cdot |\tilde{\pi} - \bar{\tilde{\pi}}_{n+1}| \, d\xi \, d\tau \le \tfrac{1}{2} \, \varepsilon_1^{2/3} . \tag{22}$$

From (20) and (22) we obtain ($\mathrm{B_{n+1}}$).

Step 4. We give a sketch of proof for (21). Applying div to (14), we see that $\tilde{\pi}$ is a weak solution of

$$\Delta \tilde{\pi} = - \sum_{i,j=1}^{3} [2\tilde{w}_{x_i}^{(j)} \tilde{v}_{x_j}^{(i)} + e^{\tilde{\lambda} t} e^{\lambda t_0} \tilde{w}_{x_i}^{(j)} \tilde{w}_{x_j}^{(i)}]$$

$$= - \sum_{i,j=1}^{3} [2\tilde{w}_{x_i}^{(j)} (\tilde{v} - Rv_\infty)_{x_j}^{(i)} + e^{\tilde{\lambda} t} e^{\lambda t_0} \tilde{w}_{x_i}^{(j)} \tilde{w}_{x_j}^{(i)}]$$

$$= - \sum_{i,j=1}^{3} \frac{\partial^2}{\partial x_i \partial x_j} [2\tilde{w}^{(j)} (\tilde{v} - Rv_\infty)^{(i)} + e^{\tilde{\lambda} t} e^{\lambda t_0} \tilde{w}^{(j)} \tilde{w}^{(i)}] . \tag{23}$$

Now we can proceed in exactly the same way as in [CKN, Lemma 3.2]: localization of (23), integral representation for $\tilde{\pi}$, Calderon-Zygmund theorem, etc.

The only change lies in the additional term $\tilde{w}^{(j)}(\tilde{v} - Rv_\infty)^{(i)}$, which is treated by using the smallness condition (18): $|\tilde{v} - Rv_\infty| \le \varepsilon_1$ on $B_1(0)$. The time dependent factor $e^{\tilde{\lambda}t}e^{\lambda t_0}$ is estimated by a constant $C = C(M_1, M_2, T)$.

Step 5. $(\mathrm{B_k})$, $3 \le k \le n$, *implies* $(\mathrm{A_n})$, *if* $n \ge 3$.

This may be proved by copying the corresponding part of [CKN, pp. 792–795].

We have to insert a regularized fundamental solution of the backward heat equation into the energy inequality (15). We remark that our energy inequality (15) differs from that one used in [CKN] only by bounded factors. □

Proof of Theorem 2. We follow the proof of [W, Theorem III.2.1], so we only give the estimates without calculations.

Let K so be large, that $\{|x| \ge K - \sqrt{\delta}\} \subset \Omega$. Let $0 < R^2 < \frac{\delta}{2}$ fixed, $0 < \delta < t < T$, $|x| \ge K$, then is $Q_R(t,x) \subset \left[\frac{\delta}{2}, T\right] \times \Omega$. We have:

$$\iint\limits_{Q_R} |w|^3 \, d\xi \, d\tau \le CR^{1/2} \|w\|^3_{L^{10/3}(Q_R)}$$

$$\le CR^{1/2} \Big(\sup_{t-R^2<\tau\le t} \|w(\tau)\|_{L^2(B_R)} \Big)^{6/5} \cdot \|\nabla w\|^{9/5}_{L^2((t-R^2,t),L^2(B_R))},$$

$$\iint\limits_{Q_R} |w| \cdot |\pi| \, d\xi \, d\tau \le CR^{1/6} \|w\|_{L^{10/3}(Q_R)} \cdot \|\pi\|_{L^{3/2}(Q_R)}$$

$$\le CR^{1/2} \|w\|_{L^{10/3}(Q_R)} \cdot \|\pi\|_{L^{3/2}((t-R^2,t),L^{9/5}(B_R))},$$

$$\int\limits_{t-R^2}^{t} \Bigg(\int\limits_{|\xi-x|<R} |\pi| \, d\xi \Bigg)^{5/4} d\tau \le CR^2 \|\pi\|^{5/4}_{L^{3/2}((t-R^2,t),L^{9/5}(B_R))}.$$

Using the integrability properties of w, π, g, we conclude that the integrals in (9), (10) become uniformly small on $Q_R(t,x)$ for all $t \in [\delta, T]$ and fixed R, if $|x| \to \infty$, i.e. (9), (10) are fulfilled if $|x| \ge K$ for a constant $K = K(\delta, T, w, \pi, g, M_1, M_2)$. If K is sufficiently large, we deduce from (3):

$$|\overset{(0)}{v} - v_\infty| \le \frac{\varepsilon_1}{R} \quad \text{on} \quad B_R(x)$$

for $|x| > K$.

The conclusion of the lemma yields the statement of Theorem 2.

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