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Instantons and Representations of an Associative Algebra

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Abstract. We give the correspondence between instantons on S^4 and some representations of an associative algebra. For the given structure group, we get simultaneous imbeddings to $\mathbb{C}^\infty$ (the inductive limit) of the moduli spaces for instantons on S^4 of all instanton numbers.

In this note we show that instantons on S^4 can be identified with some representations of an associative algebra.

Let A be the free algebra over $\mathbb{C}$ generated by two elements q, p. We define a new multiplication $*$ in A as follows:

$$f_1 * f_2 = f_1(pq - qp)f_2, \quad f_1, f_2 \in A .$$

Then $(A, *)$ is an associative algebra (with no unit), which is an extention of the Weyl algebra $\mathbb{C}\left[q, \frac{d}{dq}\right]$. We consider finite dimensional representations of $(A, *)$. Let W be the complex vector space of dimension l, and h be a linear map from A to End W. Then h induces a linear map $\tilde{h}$: $A \otimes W \to A^* \otimes W$ defined by

$$\langle \tilde{h}(f_1 \otimes w), f_2 \rangle = h(f_2 f_1)w, \quad f_1, f_2 \in A, \ w \in W .$$

We denote by $H(l, k)$ the set of all algebra homomorphisms $h: (A, *) \to \text{End } W$ such that the rank of $\tilde{h}$ is k. If h is an algebra homomorphism from $(A, *)$ to End W, then

$$h(f_1(pq - qp)f_2) = h(f_1)h(f_2) ,$$

so the linear map h is determined by $h(q^j p^i)$, $i, j \geqq 0$.

Let P be the principal $SU(l)$ bundle over $S^4 = \mathbb{R}^4 \cup \infty$ with $c_2 = k$, and $\tilde{M}(SU(l), k)$ be the framed moduli space for anti-self-dual (ASD) connections on P: $\{\text{ASD connections on } P\}/\mathscr{G}_\infty$, where $\mathscr{G}_\infty$ stands for the group of all gauge transformations on P fixing the points in the fiber over ∞. $\tilde{M}(SU(l), k)$ is a $4kl$-dimensional smooth manifold [1].

Our main result is the following:

Theorem 1. *The framed moduli space* $\tilde{M}(SU(l), k)$ *is diffeomorphic to* $H(l, k)$.

This gives an algebraic affine imbedding of $\bigsqcup_k \tilde{M}(SU(l), k)$ explicitly. We use Donaldson's theorem [1] to prove Theorem 1. In Sect. 1, we give a criterion in terms of linear algebra for the stability condition in Donaldson's theorem (Proposition 2). We prove Theorem 1 in Sect. 2.

1. Some Remarks on a Theorem of Donaldson

Let $X = \mathrm{Mat}(k, k; \mathbb{C}) \times \mathrm{Mat}(k, k; \mathbb{C}) \times \mathrm{Mat}(l, k; \mathbb{C}) \times \mathrm{Mat}(k, l; \mathbb{C})$. We define the action of $G = GL(k, \mathbb{C})$ on X as follows:

$$p \cdot (\alpha_1, \alpha_2, a, b) = (p\alpha_1 p^{-1}, p\alpha_2 p^{-1}, ap^{-1}, pb)$$

for $p \in G, (\alpha_1, \alpha_2, a, b) \in X$. We call a point x in X *stable* when the map $G \ni p \mapsto p \cdot x \in X$ is proper. We denote by X^s the set of all stable points in X. Let

$$\omega(\alpha_1, \alpha_2, a, b) = \mathrm{tr}(d\alpha_1 \wedge d\alpha_2 + db \wedge da) ,$$

$$\mu = \alpha_1\alpha_2 - \alpha_2\alpha_1 + ba .$$

The 2-form ω is a holomorphic symplectic structure on X. We can show by easy computation that

$$\omega(p\alpha_1 p^{-1}, p\alpha_2 p^{-1}, ap^{-1}, pb) = \omega(\alpha_1, \alpha_2, a, b) + \mathrm{tr}(p^{-1}dp \wedge d\mu) + \mathrm{tr}(p^{-1}dp \wedge p^{-1}dp \cdot \mu) .$$

This means that G-action on X preserves ω and that μ is the holomorphic moment map. (This is suggested to the author by H. Nakajima from the viewpoint of hyperkähler structure.)

Theorem (Donaldson [1]). *The framed moduli space* $\tilde{M}(SU(l), k)$ *is diffeomorphic to* $G \backslash \mu^{-1}(0) \cap X^s$.

So we deduce from geometric invariant theory [4] that $\tilde{M}(SU(l), k)$ is a non-singular quasiaffine algebraic variety. Theorem 1 gives an affine imbedding of $\tilde{M}(SU(l), k)$ explicitly and simultaneously for all k.

Donaldson gave a criterion for the stability in $\mu^{-1}(0)$:

Proposition (Donaldson [1]). *The point* $x = (\alpha_1, \alpha_2, a, b) \in \mu^{-1}(0)$ *is stable if and only if*

$$\mathrm{rank}\begin{pmatrix} \alpha_1 + z_1 \\ \alpha_2 + z_2 \\ a \end{pmatrix} = \mathrm{rank}(-\alpha_2 - z_2 \quad \alpha_1 + z_1 \quad b) = k \tag{1}$$

for all $z_1, z_2 \in \mathbb{C}$.

Here we seek a criterion for the stability in X.

Proposition 2. *For any point* $x = (\alpha_1, \alpha_2, a, b) \in \mu^{-1}(0)$, *the condition* (1) *is equivalent to the following*:

$$\bigcap_{f \in A} \mathrm{Ker}\, af(\alpha_1, \alpha_2) = 0, \quad \sum_{f \in A} \mathrm{Im} f(\alpha_1, \alpha_2) b = \mathbb{C}^k . \tag{2}$$

Proof. It is clear that (2) implies (1). Suppose that the vector space generated by the row vectors of $af(\alpha_1, \alpha_2)b (f \in A)$ is

$$\{(x, 0) \in \mathbb{C}^j \oplus \mathbb{C}^{k-j}\}, \quad j < k .$$

According to the splitting $\mathbb{C}^k = \mathbb{C}^j \oplus \mathbb{C}^{k-j}$ we set

$$\alpha_1 = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{13} & \alpha_{14} \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} \alpha_{21} & \alpha_{22} \\ \alpha_{23} & \alpha_{24} \end{pmatrix}, \quad a = (a' \quad 0) .$$

Then for any $f \in A$,

$$af(\alpha_1, \alpha_2) = (a'f(\alpha_{11}, \alpha_{21}) \quad 0) ,$$
$$a'f(\alpha_{11}, \alpha_{21})\alpha_{12} = 0 ,$$
$$a'f(\alpha_{11}, \alpha_{21})\alpha_{22} = 0 .$$

So we have $\alpha_{12} = \alpha_{22} = 0$, then

$$\alpha_1\alpha_2 - \alpha_2\alpha_1 + ba = \begin{pmatrix} * & 0 \\ * & \alpha_{14}\alpha_{24} - \alpha_{24}\alpha_{14} \end{pmatrix} .$$

This implies $\alpha_{14}\alpha_{24} = \alpha_{24}\alpha_{14}$. Thus there exists a nonzero common eigenvector $x' \in \mathbb{C}^{k-j}$ of α_{14}, α_{24}. Then $\begin{pmatrix} 0 \\ x' \end{pmatrix}$ is a nonzero common eigenvector of α_1, α_2 contained in Ker a. That contradicts with (1). It goes similarly in the case that the column vectors of $f(\alpha_1, \alpha_2)b$ $(f \in A)$ does not generate whole $\mathbb{C}^k$. □

2. The Proof of Theorem 1

First we give the map φ from $\tilde{M}(SU(l), k)$ to $H(l, k)$. Let

$$h(f) = \varphi(\alpha_1, \alpha_2, a, b)(f) = af(\alpha_1, \alpha_2)b$$

for $(\alpha_1, \alpha_2, a, b) \in \mu^{-1}(0) \cap X^s$. φ is G-invariant. Since $\mu(\alpha_1, \alpha_2, a, b) = 0$,

$$\begin{aligned} h(f_1 * f_2) &= h(f_1(pq - qp)f_2) \\ &= af_1(\alpha_1, \alpha_2)(\alpha_2\alpha_1 - \alpha_1\alpha_2)f_2(\alpha_1, \alpha_2)b \\ &= af_1(\alpha_1, \alpha_2)baf_2(\alpha_1, \alpha_2)b \\ &= h(f_1)h(f_2) . \end{aligned}$$

We give $i: \mathbb{C}^k \to A^* \otimes \mathbb{C}^l$, $j: A \otimes \mathbb{C}^l \to \mathbb{C}^k$ by

$$\langle i(v), f \rangle = af(\alpha_1, \alpha_2)v ,$$
$$j(f \otimes w) = f(\alpha_1, \alpha_2)bw$$

for $f \in A$, $v \in V$, $w \in W$. Then we have $\tilde{h} = i \circ j$. Proposition 2 implies that i is injective and that j is surjective, so rank $\tilde{h} = k$. Therefore $h \in H(l, k)$.

On the other hand, the inverse $\psi: H(l, k) \to \tilde{M}(SU(l), k)$ is defined as follows. For $h' \in H(l, k)$, we set $V = \operatorname{Im} \tilde{h} \cong \mathbb{C}^k$. Let

$$\tilde{h}' = i' \circ j', \quad \begin{aligned} &i': V \to A^* \otimes W , \\ &j': A \otimes W \to V . \end{aligned}$$

For $f \in A$ we define $\langle f| \in \mathrm{Hom}(V, W)$, $|f\rangle \in \mathrm{Hom}(W, V)$ by

$$\langle f|(v) = \langle i'(v), f\rangle, \quad v \in V,$$

$$|f\rangle(w) = j'(f \otimes w), \quad w \in W.$$

We set $a' = \langle 1|$, $b' = |1\rangle$. The multiplications by q, p in A induce linear maps $\alpha_1', \alpha_2' \in \mathrm{End}\, V$ respectively:

$$\alpha_1'|f\rangle = |qf\rangle, \quad \alpha_2'|f\rangle = |pf\rangle$$

for $f \in A$. If $|f\rangle = 0$, then $h(f'f) = 0$ for all $f' \in A$. So $\alpha_1', \alpha_2' \in \mathrm{End}\, V$ are well-defined. We get

$$\psi(h') = (\alpha_1', \alpha_2', a', b') \in X$$

by fixing the basis of V, W. Since

$$\bigcap_{f \in A} \mathrm{Ker}\; a' f(\alpha_1', \alpha_2') = \bigcap_{f \in A} \mathrm{Ker}\,\langle f| = 0,$$

$$\sum_{f \in A} \mathrm{Im}\, f(\alpha_1', \alpha_2') b' = \sum_{f \in A} \mathrm{Im}\, |f\rangle = V,$$

we deduce from Proposition 2 that $\psi(h')$ is stable. Since $h'\colon (A, *) \to \mathrm{End}\, W$ is an algebra homomorphism, we have

$$\begin{aligned}\langle f_1|\alpha_1'\alpha_2' - \alpha_2'\alpha_1' + b'a'|f_2\rangle &= h'(f_1(qp - pq)f_2) + \langle f_1|1\rangle\langle 1|f_2\rangle \\ &= -h'(f_1 * f_2) + h'(f_1)h'(f_2) \\ &= 0.\end{aligned}$$

Therefore $\psi(h') \in G\backslash\mu^{-1}(0) \cap X^s$.

If $(\alpha_1', \alpha_2', a', b') = \psi(h')$,

$$\begin{aligned}a' f(\alpha_1', \alpha_2') b' &= \langle 1|f(\alpha_1', \alpha_2')|1\rangle \\ &= \langle 1|f\rangle \\ &= h'(f).\end{aligned}$$

Hence $\varphi \circ \psi(h') = h'$.

If $h' = \varphi(\alpha_1, \alpha_2, a, b)$, we can take $i' = i$, $j' = j$ by the stability. Then

$$\langle f| = af(\alpha_1, \alpha_2), \quad |f\rangle = f(\alpha_1, \alpha_2)b.$$

This implies that

$$\langle 1| = a, \quad |1\rangle = b,$$

$$|qf\rangle = \alpha_1 f(\alpha_1, \alpha_2)b = \alpha_1|f\rangle,$$

$$|pf\rangle = \alpha_2 f(\alpha_1, \alpha_2)b = \alpha_2|f\rangle.$$

Hence $\psi \circ \varphi = \mathrm{id}$. □

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