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The Integrability Criterion in SU(2) Chern-Simons Gauge Theory

Georgios Daskalopoulos[1] **and Richard Wentworth**[2]*

[1] Department of Mathematics, Princeton University, Princeton, NJ 08544, USA, email: daskal@math.princeton.edu

[2] Department of Mathematics, Harvard University, One Oxford Street, Cambridge, MA 02138, USA, email: raw@math.harvard.edu

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Abstract. We prove that the multiplicity spaces appearing in Chern–Simons theory, as defined by Segal, vanish unless they are associated to integrable representations. This and other links with conformal field theory are examined.

1. Introduction

The purpose of this paper is to prove a conjecture attributed to Segal concerning the vanishing of certain multiplicity spaces appearing in the geometric quantization of SU(2) Chern–Simons gauge theory (see [Wi]). The result is closely related to the fact that only the *integrable* representations of the loop group of SU(2) play a role in the theory. The method used in this paper is the analytic description of certain moduli spaces of vector bundles developed in [D-W1].

Let us begin by describing the main result. Throughout the paper, let $\bar{\Sigma}$ denote a compact Riemann surface of genus $g > 3$; let p be a distinguished point of $\bar{\Sigma}$, $\Sigma = \bar{\Sigma} \setminus \{p\}$, and fix $\mathbf{G} = \mathrm{SU}(2)$. We shall denote by $\mathscr{A}_s$ the stable, smooth connections on a trivial $\mathbf{G}$-bundle over $\bar{\Sigma}$, and by $\Delta \to \mathscr{A}_s$ we shall mean the determinant line bundle. For any integer $k \geqq 0$, let $H^0(\mathscr{A}_s, \Delta^{\otimes k})$ denote the infinite dimensional space of holomorphic sections of $\Delta^{\otimes k}$. For λ a non-negative half-integer, let V_λ denote the irreducible representation of $\mathbf{G}$ of dimension $2\lambda + 1$. The complex gauge group $\mathscr{G}^{\mathbb{C}}$ acts on the determinant bundle, and also on V_λ via evaluation at the point p. Following Segal, we define the *space of states*

$$\mathscr{V}_\lambda = \mathrm{Hom}_{\mathscr{G}^{\mathbb{C}}}(V_\lambda^*, H^0(\mathscr{A}_s, \Delta^{\otimes k}))$$

(see [Ox]), where the homomorphisms are required to intertwine the actions of $\mathscr{G}^{\mathbb{C}}$. In Sect. 3 we shall prove the

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Main Theorem. *For $\lambda > k/2$, $\mathscr{V}_\lambda$ vanishes.*

This theorem was conjectured in [Wi] to be the manifestation in Chern–Simons theory of the integrability criterion for the Wess-Zumino-Witten model (cf. [G-Wi]). In the latter, the Ward identities show that the primary operators associated to representations of Kac–Moody algebras for level greater than k decouple from the theory; i.e. the space of conformal blocks should vanish for these insertions. In [Ox], Segal proposed the space $\mathscr{V}_\lambda$ as the Hilbert space appearing in Chern–Simons theory, and by the arguments in [Wi] this should be the same as the space of conformal blocks for WZW (see also [E-M-S-S, Gaw]). Our theorem is therefore a Chern–Simons version of the two-dimensional result.

Let us say a few words about the method of proof of the Main Theorem. Associated to λ and k we have a matrix

$$\alpha = \begin{pmatrix} \lambda/k & 0 \\ 0 & -\lambda/k \end{pmatrix}$$

and a space $\mathscr{M}_a$, the moduli space of parabolic stable bundles with weight λ/k at p (cf. [D-W1], Sect. 2.2). Strictly speaking, this interpretation only works for $a = \exp(2\pi i\alpha) \neq \pm I$, so care needs to be applied to these cases. Nevertheless, in [D-W1], Sect. 2.3, we showed that the spaces $\mathscr{M}_a$ form correspondence varieties between the moduli spaces $\mathscr{M}(2,0)$ and $\mathscr{M}(2,-1)$ of semistable vector bundles of rank 2 and degree 0 and -1, respectively. More precisely, there exist holomorphic maps p_0, p_1 from $\mathscr{M}_a$ to $\mathscr{M}(2,0)$ and $\mathscr{M}(2,-1)$, respectively, with generic fiber $\mathbb{P}^1$ in both cases (see Theorem 2.1). On the other hand, one of the main goals of [D-W1] was to construct on $\mathscr{M}_a$ a holomorphic line bundle $\mathscr{L}(k,\lambda)$ via the Chern–Simons functional whose space of holomorphic sections was isomorphic to $\mathscr{V}_\lambda$. The restriction of $\mathscr{L}(k,\lambda)$ on the fiber of p_0 is always a positive bundle, and indeed, this is what produces the representation V_λ. However, if $\lambda > k/2$, then $\mathscr{L}(k,\lambda)$ over the fiber of p_1 is negative, thus proving the vanishing of $\mathscr{V}_\lambda$.

In Sect. 4 of the paper, we conclude with a few observations concerning the relationship between the moduli space point of view and the WZW model. In particular, we shall show that the correspondence varieties mentioned above interpolate between the SU(2) and SO(3) theories in a manner similar to the orbifold techniques of conformal field theory. In a later work, we shall treat the case of higher rank and multiple punctures [D-W2].

2. The Quantum Line Bundle

In this section we shall review the construction from [D-W1] of a line bundle whose space of holomorphic sections realizes the space of states $\mathscr{V}_\lambda$. This is essential in the proof of the vanishing theorem in the next section. Here we recall the notion of the moduli space of vector bundles with parabolic structure and the construction of a line bundle over this space. Moreover, we compute the Chern class of this line bundle in terms of an explicit set of generators for the second cohomology group of the moduli space.

Recall the notation $\bar{\Sigma}$, Σ, p, and $\mathbf{G}$ from the Introduction. Let E denote the trivial $\mathbf{G}$-bundle over Σ and let $\mathscr{M}_0$ denote the space of *based* equivalence classes of flat connections on E. According to [D-W1], Sect. 2.1, $\mathscr{M}_0$ is a smooth manifold diffeomorphic to $\mathbf{G}^{2g}$. Let

$$q : \mathscr{M}_0 \to \mathbf{G}$$

denote the map measuring the holonomy of based loops around the puncture p. Then q is a fibration away from the identity ([D-W1], Proposition 2.3). Let $\mathscr{F}_a$ denote the fiber of q over $a \in \mathbf{G}$. For $a \neq \pm I$, let $\mathbf{T}_a$ denote the normalizer of a in $\mathbf{G}$. $\mathbf{T}_a$ is clearly a maximal torus acting on $\mathscr{F}_a$, and the quotient space $\mathscr{F}_a/\mathbf{T}_a$ may be identified with the space of isomorphism classes of flat connections with holonomy about p conjugate to a (cf. [D-W1], Proposition 2.4).

It follows from the theorem of Mehta and Seshadri that the space $\mathscr{F}_a/\mathbf{T}_a$ admits naturally the structure of a complex manifold. Indeed, there is a natural identification of $\mathscr{F}_a/\mathbf{T}_a$ with the moduli space of parabolic stable bundles with weights determined by a (for a precise statement of the above theorem, see [D-W1], Sect. 2.2).

Next we fix a trivialization of the fibration

$$q : \mathscr{M}_0 \backslash q^{-1}(I) \to \mathbf{G} \backslash \{I\} .$$

This can be done, since for $\mathbf{G} = \mathrm{SU}(2)$, the space $\mathbf{G}\backslash\{I\}$ is $\mathbf{T}_a$-equivariantly contractible. Since q is a $\mathbf{G}$-map, this induces a $\mathbf{T}_a$-equivariant identification of $\mathscr{F}_a$ with $\mathscr{F}_{-I}$. Let

$$q_a : \mathscr{F}_{-I} \to \mathscr{F}_{-I}/\mathbf{T}_a, \quad q_{-I} : \mathscr{F}_{-I} \to \mathscr{F}_{-I}/\mathbf{G}$$

denote the obvious quotient maps. Then q_a and q_{-I} are principal $\mathbf{T}_a/\mathbb{Z}_2$, $\mathbf{G}/\mathbb{Z}_2 = \mathrm{SO}(3)$ bundles, respectively.

Use the identification between $\mathscr{F}_{-I}$ and $\mathscr{F}_a$, and the Mehta-Seshadri theorem to put complex structures on the spaces $\mathscr{F}_{-I}/\mathbf{T}_a \simeq \mathscr{M}_a$, $a \neq \pm I$. Also, let $\mathscr{M}(2,0)$ and $\mathscr{M}(2,-1)$ denote the moduli spaces of semistable vector bundles of rank 2, with fixed determinant of degree 0 and -1, respectively. In particular, in the former case we shall take the determinant to be trivial, in the latter we take it to be $\mathscr{O}(-p)$, where p is the puncture. The next theorem lies at the heart of our construction:

Theorem 2.1. (see [D-W1], Proposition 2.15) *The spaces $\mathscr{M}_a$ for $a \neq \pm I$ form a correspondence*

$$\begin{array}{ccc} & \mathscr{M}_a & \\ p_0 \swarrow & & \searrow p_1 \\ \mathscr{M}(2,0) & & \mathscr{M}(2,-1) \end{array} .$$

Moreover, the map p_0 is a holomorphic $\mathbb{P}^1$-bundle over $\mathscr{M}_s(2,0)$, the subvariety of stable points in $\mathscr{M}(2,0)$, and p_1 is a holomorphic $\mathbb{P}^1$-bundle over $\mathscr{M}(2,-1)$.

For the purpose of the next section, it is very important to fix an explicit set of generators for $H_2(\mathscr{M}_a, \mathbb{Z}) \simeq \pi_2(\mathscr{M}_a)$. Let $\bar{\beta}_1$ denote a choice of generator of

$$\pi_2(\mathscr{M}(2,-1)) \simeq H_2(\mathscr{M}(2,-1), \mathbb{Z}) \simeq \mathbb{Z}$$

such that the Poincaré dual $\overline{\beta_1}^*$ in $H^{6g-8}(\mathscr{M}(2,-1), \mathbb{Z})$ corresponds to the unique ample generator $\mathscr{L}_1$ of $\mathrm{Pic}(\mathscr{M}(2,-1)) \simeq \mathbb{Z}$. Choose $\hat{\beta}$ a generator of $\pi_2(\mathscr{F}_{-I})$ such that under the map q_{-I} we have $q_{-I*}\hat{\beta} = 2\bar{\beta}_1$. Also choose a generator $\hat{\gamma}$ of $\pi_1(\mathbf{T}_a/\mathbb{Z}_2) \simeq \mathbb{Z}$. Observe that $\hat{\beta}$ is uniquely determined, and $\hat{\gamma}$ is so up to sign.

From the long exact sequence in homotopy associated to the fibration q_a, we obtain

$$0 \to \pi_2(\mathscr{F}_{-I}) \xrightarrow{q_{a*}} \pi_2(\mathscr{M}_a) \xrightarrow{\partial} \pi_1(\mathbf{T}_a/\mathbb{Z}_2) \to 0 .$$

We define generators β, γ of $\pi_2(\mathscr{M}_a)$ as follows: first, let $\beta = q_{a*}(\hat{\beta})$. Let γ be any element of $\pi_2(\mathscr{M}_a) \simeq H_2(\mathscr{M}_a, \mathbb{Z})$ such that $\partial(\gamma) = \hat{\gamma}$ and under the Poincaré duality pairing $\langle \beta^*, \gamma \rangle = \langle \gamma^*, \beta \rangle = 0$. Again β is uniquely determined but γ is only determined up to sign. In order to fix the sign of γ we proceed as follows: let $p_0 : \mathscr{M}_a \to \mathscr{M}(2, 0)$ be the map of Theorem 2.1. Let $\iota_0 : \mathbb{P}^1 \to \mathscr{M}_a$ denote the inclusion of the fiber of p_0 over a point in $\mathscr{M}_s(2, 0)$. Let $[\mathbb{P}^1] \in H_2(\mathbb{P}^1, \mathbb{Z})$ denote the fundamental class of $\mathbb{P}^1$. Then

Lemma 2.2. *In $H_2(\mathscr{M}_a, \mathbb{Z}) \simeq \pi_2(\mathscr{M}_a)$, $\iota_{0*}[\mathbb{P}^1] = \pm 2\gamma$.*

Proof. Consider the long exact sequence in homotopy associated to the map p_0. As $\mathscr{M}_a \backslash p_0^{-1}(\mathscr{M}_s(2, 0))$ has complex codimension at least 2 (see [D-W1], Proposition 2.16, and recall that $g > 3$) we have

$$\begin{array}{ccccc} \pi_2(\mathbb{P}^1) & \xrightarrow{\iota_{0*}} & \pi_2(\mathscr{M}_a) & \xrightarrow{p_{0*}} & \pi_2(\mathscr{M}(2,0)) \to 0 \\ \| & & \| & & \| \\ H_2(\mathbb{P}^1, \mathbb{Z}) & & H_2(\mathscr{M}_a, \mathbb{Z}) & & H_2(\mathscr{M}(2,0), \mathbb{Z}) . \end{array}$$

According to [D-W1], Proposition 4.2, $p_{0*}(\beta)$ generates the free part of $\pi_2(\mathscr{M}(2, 0)) \simeq \mathbb{Z} \oplus \mathbb{Z}_2$. Thus, $\iota_{0*}[\mathbb{P}^1] = \pm 2\gamma$, proving our lemma.

Henceforth, we fix the sign of γ so that $\iota_{0*}[\mathbb{P}^1] = 2\gamma$. Thus, β and γ in $\pi_2(\mathscr{M}_a) \simeq H_2(\mathscr{M}_a, \mathbb{Z})$ are now *uniquely determined.*

We next restrict to a certain class of holonomies corresponding to representations of the group $\mathbf{G} = \mathrm{SU}(2)$. This correspondence was explained in detail in [D-W1], Sect. 6 for $\mathrm{SU}(n)$. We shall review this here for $n = 2$.

Representations of SU(2) are characterized by a non-negative half-integer λ, the *spin* of the representation. We also think of λ as a character of a maximal torus (or *weight*) $\mathbf{T} \subset \mathbf{G}$, $\lambda : \mathbf{T} \to U(1)$. The dimension of the representation space V_λ associated to the λ is $2\lambda + 1$. Assume that the weight λ is invariant under the center $\mathbb{Z}_2$ of $\mathbf{G}$. This corresponds to taking λ to be an integer. Let α be defined

$$\alpha = \begin{pmatrix} \lambda/k & 0 \\ 0 & -\lambda/k \end{pmatrix},$$

and let $a = \exp(2\pi i \alpha)$ denote the corresponding element in $\mathbf{G}$. For the next theorem, assume $a \neq \pm I$.

Theorem 2.3. *Let λ, α and a be as above. Then there is an hermitian line bundle $\mathscr{L}(k, \lambda)$ with connection over $\mathscr{M}_a$, constructed via the Chern–Simons functional, and such that*

$$c_1(\mathscr{L}(k, \lambda)) = k\beta + \lambda\gamma ,$$

where β, γ are the generators for $H^2(\mathscr{M}_a, \mathbb{Z}) \simeq H_2(\mathscr{M}_a, \mathbb{Z})$ described before.

Even when a is in the center, we can show the following

Theorem 2.4. *Let λ, α as before with* $\exp(2\pi i\alpha) = \pm I$. *Choose* $a \in \mathbf{G} \setminus \{\pm I\}$. *Then, there is a line bundle $\mathscr{L}(k, \lambda)$ with connection over $\mathscr{M}_a$, constructed via the Chern–Simons functional, and such that*

$$c_1(\mathscr{L}(k, \lambda)) = k\beta + \lambda\gamma ,$$

where β, γ are the generators for $H^2(\mathscr{M}_a, \mathbb{Z}) \simeq H_2(\mathscr{M}_a, \mathbb{Z})$ described before.

As we do not need the details behind the construction of the line bundle $\mathscr{L}(k, \lambda)$, we shall refer to [D-W1], Sects. 4.2 and 5 for further details. The importance of the line bundle $\mathscr{L}(k, \lambda)$ is indicated in the next theorem which states that $\mathscr{L}(k, \lambda)$ is a *quantum* line bundle on $\mathscr{M}_a$ whose space of holomorphic sections is isomorphic to $\mathscr{V}_\lambda$. More precisely,

Theorem 2.5. ([D-W1], Theorem 6.6) *Let λ be a weight invariant under the center and let α, a be associated to λ (and k) as before. Then*

$$\mathscr{V}_\lambda \simeq H^0(\mathscr{M}_a, \mathscr{L}(k, \lambda)) .$$

As a final remark, we note that since $-I$ acts trivially on the determinant bundle Λ, $\mathscr{V}_\lambda$ vanishes identically for λ *not* invariant under the center. This is built into the construction of the line bundle $\mathscr{L}(k, \lambda)$ (cf. [D-W1], Theorem 6.1).

3. Proof of the Main Theorem

The proof of the vanishing theorem stated in the Introduction will occupy most of this section. We continue with all the notation of the previous section.

Lemma 3.1. $p_{1*}(\beta) = 2\bar{\beta}_1$.

Proof. Clearly, we have $q_{-I} = p_1 \circ q_a$, and so we obtain, by evaluating at $\hat{\beta} \in \pi_2(\mathscr{F}_{-I}) \simeq H_2(\mathscr{F}_{-I}, \mathbb{Z})$, that

$$q_{-I*}\hat{\beta} = p_{1*}(q_{a*}\hat{\beta}) .$$

But, by definition $q_{-I*}\hat{\beta} = 2\bar{\beta}_1$ and $q_{a*}\hat{\beta} = \beta$, hence the lemma follows.

Lemma 3.2. $p_{1*}(\gamma) = \bar{\beta}_1$.

Proof. Since $\pi_2(\mathscr{M}(2, -1))$ is generated by $\bar{\beta}_1$, it is enough to show that under the Poincaré pairing

$$\langle \bar{\beta}_1^*, p_{1*}\gamma \rangle = \langle p_1^* \bar{\beta}_1^*, \gamma \rangle = 1 .$$

However, in the case where the holonomy matrix is diag$(1/2, -1/2)$, then $p_1^*\bar{\beta}_1^*$ corresponds to the line bundle $\mathscr{L}(2, 1)$. Moreover, by the computation given in [D-W1], Theorem 4.12, $\langle \mathscr{L}(2, 1), \gamma \rangle = \pm 1$. On the other hand, if $\langle p_1^*\bar{\beta}_1^*, \gamma \rangle = -1$ then

$$\langle \iota_0^* p_1^* \bar{\beta}_1^*, \mathbb{P}^1 \rangle = \langle p_1^* \bar{\beta}_1^*, \iota_{0*}\mathbb{P}^1 \rangle = \langle p_1^* \bar{\beta}_1^*, 2\gamma \rangle = -2 ,$$

which contradicts the assumption that $\bar{\beta}_1^*$ is the ample generator of Pic$(\mathscr{M}(2, -1))$. This proves the lemma.

Lemma 3.3. *Let* $[\mathbb{P}^1]$ *denote the fundamental class of* $\mathbb{P}^1$ *in* $H_2(\mathbb{P}^1, \mathbb{Z}) \simeq \pi_2(\mathbb{P}^1)$ *and* ι_1 *the inclusion map into* $\mathscr{M}_a$. *Then* $\iota_{1*}[\mathbb{P}^1] = \beta - 2\gamma$.

Proof. Since $H_2(\mathbb{P}^1, \mathbb{Z}) \simeq \mathbb{Z}$, the image of ι_{1*} in $H_2(\mathscr{M}_a, \mathbb{Z})$ is generated by $\iota_{1*}[\mathbb{P}^1]$. On the other hand, by exactness, image $(\iota_{1*}) = \ker(p_{1*})$, and this is generated by $\beta - 2\gamma$, as is seen from Lemmas 3.1 and 3.2.

Proposition 3.4. $p_1^* \bar{\beta}_1^* = 2\beta^* + \gamma^*$ *in* $H^{6g-6}(\mathscr{M}_a, \mathbb{Z})$.

Proof. Since $H^{6g-6}(\mathscr{M}_a, \mathbb{Z})$ is generated freely by β^* and γ^*, we can write $p_1^* \bar{\beta}_1^* = m\beta^* + n\gamma^*$, where $m, n \in \mathbb{Z}$. Moreover, we know that $p_1 \circ \iota_1$ = constant map, so we obtain that

$$0 = \langle p_1^* \bar{\beta}_1^*, \iota_{1*}[\mathbb{P}^1] \rangle .$$

On the other hand, by substituting $p_1^* \bar{\beta}_1^* = m\beta^* + n\gamma^*$ and $\iota_{1*}[\mathbb{P}^1] = \beta - 2\gamma$, we obtain

$$0 = \langle m\beta^* + n\gamma^*, \beta - 2\gamma \rangle = m - 2n ,$$

since $\langle \beta^*, \gamma \rangle = \langle \gamma^*, \beta \rangle = 0$. Hence, $m = 2n$ and $p_1^* \bar{\beta}_1^* = n(2\beta^* + \gamma^*)$. However,

$$\begin{aligned} \langle p_1^* \bar{\beta}_1^*, \beta \rangle &= \langle \bar{\beta}_1^*, p_{1*}\beta \rangle \\ &= \langle \bar{\beta}_1^*, 2\bar{\beta}_1 \rangle = 2 , \end{aligned}$$

$$\text{and} \quad \langle p_1^* \bar{\beta}_1^*, \beta \rangle = n\langle 2\beta^* + \gamma^*, \beta \rangle = 2n .$$

Therefore $n = 1$.

Corollary 3.5. *For* $\lambda = k/2$, *k even*,

$$\mathscr{V}_{k/2} \simeq H^0(\mathscr{M}_a, \mathscr{L}(k, \lambda)) = H^0(\mathscr{M}(2, -1), \mathscr{L}_1^{\otimes k/2}) .$$

Proof. By Theorem 2.4 and Proposition 3.4,

$$c_1(\mathscr{L}(k, \lambda)) = \frac{k}{2}(2\beta + \gamma) = c_1(p_1^* \mathscr{L}_1^{\otimes k/2}) .$$

Since $\mathscr{M}_a$ is compact and simply connected, Hodge theory and the exponential sequence show that $\mathscr{L}(k, \lambda)$ and $p_1^* \mathscr{L}_1^{\otimes k/2}$ are holomorphically equivalent. The Corollary follows immediately.

Proof of Main Theorem. Let λ be a non-negative half-integer, and let α, a be the corresponding holonomy matrices associated to λ as above. If λ is not in fact an integer, then by the remark following Theorem 2.5, there is nothing to show. We therefore assume λ is an integer. By Theorem 2.5 it suffices to show that $H^0(\mathscr{M}_a, \mathscr{L}(k, \lambda)) = 0$ for $\lambda > k/2$. According to Theorems 2.3 and 2.4,

$$\mathscr{L}(k, \lambda) = k\beta^* + \lambda\gamma^* .$$

Hence

$$\begin{aligned} \langle \iota_1^* \mathscr{L}(k, \lambda), \mathbb{P}^1 \rangle &= \langle \mathscr{L}(k, \lambda), \iota_{1*} \mathbb{P}^1 \rangle \\ &= \langle k\beta^* + \lambda\gamma^*, \beta - 2\gamma \rangle = k - 2\lambda . \end{aligned}$$

Thus if $\lambda > k/2$, the bundle $\iota_1^* \mathscr{L}(k, \lambda)$ on $\mathbb{P}^1$ is negative, and therefore has no non-zero sections. This implies $\mathscr{L}(k, \lambda)$ cannot have any either, and this proves the theorem.

4. Discussion

In this section, we would like to present an informal discussion of the results from the point of view of conformal field theory. More precisely, (i) we interpret Corollary 3.5 as a statement about orbifold models, (ii) we consider a particular selection rule, and finally (iii) we show how Corollary 3.5 plus the SU(2) theory produces the result of Thaddeus [Th].

It follows from Donaldson's version of the Narasimhan–Seshadri Theorem (cf. [Don], or [A-B], Sect. 6) that the moduli space of flat SO(3) bundles is equivalent to $\mathcal{M}(2, -1)$. More precisely, let $\hat{P}$ be an $SO(3) = SU(2)/\mathbb{Z}_2$ bundle over Σ with $w_2(\hat{P}) \neq I$, i.e. of non-trivial topological type. If we denote by $\hat{\mathcal{A}}$ the irreducible connections on $\hat{P}$, $\hat{\mathcal{A}}_F$ the subspace of flat connections, and $\hat{\mathcal{G}}$ the connected component to the identity of the group of automorphisms of $\hat{P}$, then we have that $\hat{\mathcal{A}}_F/\hat{\mathcal{G}}$ is diffeomorphic to $\mathcal{M}(2, -1)$.

Therefore, Kähler quantization of the space $\hat{\mathcal{A}}$ at level k, k even, reproduces the space $H^0(\mathcal{M}(2, -1), \mathcal{L}_1^{\otimes k/2})$ (cf. [Ax-DP-Wi]). The fact that k must be even is, in conformal field theory, a consequence of the requirement of modular invariance (cf. [Dj-Wi, M-S1]). In the context of this paper, the requirement is integrality of the symplectic form (see Theorem 4.12 of [D-W1]).

On the other hand, it was argued in [E-M-S-S] that quantizing $\hat{\mathcal{A}}$ should be equivalent to the Wess–Zumino–Witten model based on the group SO(3). The WZW model based on $SO(3) = SU(2)/\mathbb{Z}_2$ may be treated using orbifold techniques. The *twisted sector* of the theory, which corresponds to non-trivial SO(3) bundles, may be generated from the SU(2) theory by insertion of a *twist field* ψ. For $k = 0$ mod 4, this field has (chiral) spin $k/2$ (cf. [G-Wi, M-S1]). Hence, the space of conformal blocks is, according to Segal, isomorphic to $\mathcal{V}_{k/2}$. Thus, Corollary 3.5 is nothing but the statement that the conformal blocks for the twisted sector of the SO(3) theory correspond to the conformal blocks for the SU(2) theory, twisted by ψ.

Let $\mathcal{H}(g; \phi_{n_1}, \ldots, \phi_{n_l})$ denote the Friedan–Shenker bundle associated to the primary fields ϕ_{n_j} inserted on a compact Riemann surface of genus g. The integers n_j correspond to twice the spin of the associated representation. Let S_{mn} be defined

$$S_{mn} = \left(\frac{2}{k+2}\right)^{1/2} \sin\left[\frac{(m+1)(n+1)\pi}{k+2}\right] \quad 0 \leqq m, n \leqq k .$$

Then the Verlinde dimension is (see [V, M-S2])

$$\operatorname{rank} \mathcal{H}(g; \phi_{n_1}, \ldots, \phi_{n_l}) = \sum_{m=0}^{k} (S_{m0})^{-2(g-1)} \frac{S_{mn_1}}{S_{m0}} \cdots \frac{S_{mn_l}}{S_{m0}} .$$

Let us denote by μ the spectral flow $\mu(m) = k - m$ (cf. [M-S1]). Then it is easy to see that

$$S_{\mu(m)n} = (-1)^n S_{mn} , \tag{4.1}$$

and so if $F = \sum_{j=1}^{l} n_j$,

$$\sum_{m=0}^{k} (S_{m0})^{-2(g-1)} \frac{S_{mn_1}}{S_{m0}} \cdots \frac{S_{mn_l}}{S_{m0}} = (-1)^F \sum_{m=0}^{k} (S_{m0})^{-2(g-1)} \frac{S_{mn_1}}{S_{m0}} \cdots \frac{S_{mn_l}}{S_{m0}} .$$

From this we obtain the familiar selection rule that $\mathscr{H}(g; \phi_{n_1}, \ldots, \phi_{n_l})$ is zero unless F is even[1]. In particular, the only 1-point contributions come from integral spin. In Theorem 6.1 of [D-W1], we showed that integral spin was necessary for the very construction of the quantum line bundle $\mathscr{L}(k, \lambda)$. This in turn was a consequence of the requirement of invariance under the center of the gauge group (see also the remark following Theorem 2.5 above). In [D-W2], we shall show that the more general selection rule, i.e. for multiple insertions, has a similar interpretation.

For one insertion of spin $k/2$, k even,

$$S_{mk} = S_{m\mu(0)} = (-1)^m S_{m0}$$

by (4.1), so

$$\operatorname{rank} \mathscr{H}(g; \phi_k) = \sum_{m=0}^{k} (-1)^m (S_{m0})^{-2(g-1)} .$$

By Corollary 3.5, the predicted dimension for the space of holomorphic sections is therefore

$$\begin{aligned}\dim H^0(\mathscr{M}(2, -1), \mathscr{L}^{\otimes k/2}) &= \dim \mathscr{V}_{k/2} = \operatorname{rank} \mathscr{H}(g; \phi_k) \\ &= \left(\frac{k+2}{2}\right)^{g-1} \sum_{m=0}^{k} (-1)^m \left(\sin\left[\frac{(m+1)\pi}{k+2}\right]\right)^{-2(g-1)}\end{aligned}$$

This was the result conjectured in [Th] and proven recently in [Sz]. In light of Corollary 3.5, we see that the formula follows immediately from the SU(2) theory and Segal's definition of the space of conformal blocks.

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