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Bohr-Sommerfeld Orbits in the Moduli Space of Flat Connections and the Verlinde Dimension Formula*

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Abstract. We show how the moduli space of flat $SU(2)$ connections on a two-manifold can be quantized in the real polarization of [15], using the methods of [6]. The dimension of the quantization, given by the number of integral fibres of the polarization, matches the Verlinde formula, which is known to give the dimension of the quantization of this space in a Kähler polarization.

1. Introduction

Let Σ^g be a (compact, oriented) two-manifold of genus g, and consider the moduli space $\bar{\mathscr{S}}_g$ of flat $SU(2)$ connections on Σ^g. This space contains a large open set $\mathscr{S}_g$ which is a symplectic manifold with symplectic form ω such that $2\pi i\omega$ is the curvature of a natural line bundle $\mathscr{L}$ on $\bar{\mathscr{S}}_g$. The quantization of this prequantum system has been the subject of much recent interest. Much of the mathematical work on this topic has concentrated on the Verlinde formula for the dimension of the quantization in a Kähler polarization.

In [15, 16] there was introduced a different approach to the quantization procedure, based on a real polarization of the space $\bar{\mathscr{S}}_g$. If (M, ω) is a compact symplectic manifold of dimension $2m$, a real polarization of M is a map $\pi: M \to B$ onto a manifold B of dimension m, such that $\omega|_{\pi^{-1}(b)} = 0$ for all $b \in B$. Under sufficiently strong hypotheses, a submanifold L appearing as a fibre $\pi^{-1}(b)$ must be a torus of dimension m, and the quantization procedure for a prequantum system over (M, ω) given by a line bundle $\mathscr{L} \to M$ with connection of curvature $2\pi i\omega$ is particularly simple. For there will be a finite number of fibres L_i of the

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polarization, the so-called Bohr-Sommerfeld fibres, such that the line bundle $\mathscr{L}$, restricted to L_i, possesses a (one-dimensional family of) global covariant constant sections. The quantization is then naturally isomorphic to the space of all such sections [12].

One motivation for the work of [15, 16] was the natural association of vectors in the quantization in a real polarization to Lagrangian submanifolds, which when applied to the case of the moduli space might be useful in trying to construct a topological quantum field theory. Some progress on that account will be discussed in a separate paper [9]. In this paper we turn our attention to a more careful study of the classical system, which enables us to obtain an explicit description of the Bohr-Sommerfeld fibres of the polarization of [15]. Most of these fibres are smooth Lagrangian tori, as in the smooth case. There are also some singular fibres; the role such fibres should play in quantization is somewhat unclear in general.

The real polarization of [15] is associated to a choice of $3g-3$ simple closed curves on Σ^g. These curves are obtained from a decomposition of Σ^g into copies of the two-sphere with three discs removed and marked points on each boundary component, also known as pants or trinions. If we choose a basepoint $*$ for Σ^g, and arcs connecting $*$ to each of the marked points on the boundaries of the trinions, we obtain a collection of $3g-3$ elements of the fundamental group of the surface Σ^g. The map $\pi:\bar{\mathscr{S}}_g \to B_g \subset \mathbb{R}^{3g-3}$ giving the polarization of [15] is obtained by assigning a flat connection representing a point $x \in \bar{\mathscr{S}}_g$ to the trace of the holonomy of the connection about each of the chosen curves. This is independent of the representative taken for the point x. The existence on the space $\bar{\mathscr{S}}_g$ of a real polarization reflects a geometric fact about the structure of $\bar{\mathscr{S}}_g$ as a symplectic manifold, which we exploit for other purposes in [9, 10].

Any trinion decomposition of the surface Σ^g gives rise to a trivalent graph obtained by associating a vertex to each trinion and an edge to each boundary circle (see Fig. 1). Then every fibre of our polarization is associated to a marked trivalent graph, where each edge is marked by the holonomy of the connections in the fibre about the appropriate curve in the surface. We shall see that the Bohr-Sommerfeld fibres of the polarization of the prequantum system associated to the

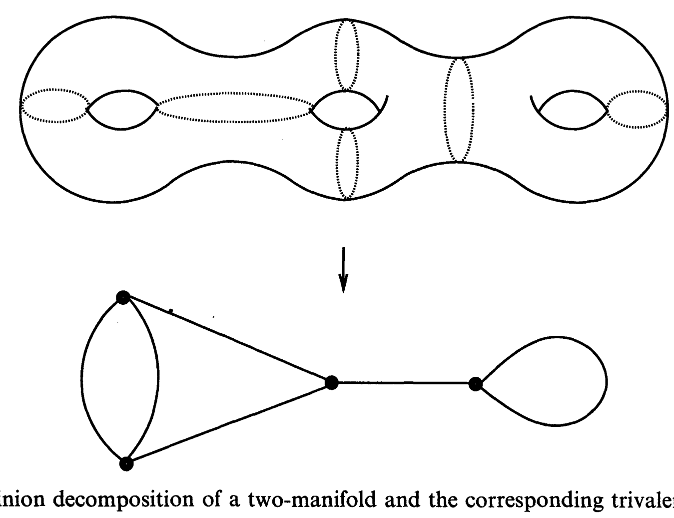

Fig. 1. A trinion decomposition of a two-manifold and the corresponding trivalent graph

line bundle $\mathscr{L}^k$ (for k a positive integer) are given by graphs marked with holonomies which correspond to k^{th} roots of unity. In order to count these fibres, we must check which markings of this type actually occur as holonomies of flat connections. The condition that such a marking occur is that there exist a flat connection on each trinion with the appropriate holonomy about each boundary curve; such a flat connection turns out to exist when the holonomies satisfy the quantum Clebsch-Gordan conditions (Eq. 8.2a–c). Thus Bohr-Sommerfeld fibres are associated to marked trivalent graphs with holonomies satisfying this relation at each vertex. We prove that the dimension of the quantization in our real polarization, given by the number of Bohr-Sommerfeld orbits, is given by the number of such marked graphs, which is called the *Verlinde dimension.* The Verlinde dimension is also now known to give the dimension of the quantization of the moduli space in a Kähler polarization.

This paper is organized as follows. In Sect. 2, we recall from [15] the construction of the real polarization of moduli space; the fibres of the polarization are given by level sets of certain functions on this space. The Hamiltonian vector fields of these functions will then define torus flows on $\bar{\mathscr{S}}_g$ preserving the polarization. In this formalism, the Lagrangian nature of the fibres is transparent, but their topological structure is somewhat obscure. As we will need to study the topology of the fibres, we include in this section an alternative description of the fibres of the polarization, related to the work of Witten [17]. In this description, we use trinion decompositions of two-manifolds to construct the moduli space of flat connections on a two-manifold Σ^g from the moduli space $\mathscr{M}(D)$ of flat connections on a trinion, by considering a collection of trinions whose union is Σ^g. A point in $\bar{\mathscr{S}}_g$ can then be specified by giving a point in $\mathscr{M}(D)^{3g-3}$ satisfying appropriate conditions, together with "gluing data" which specify how the corresponding flat connections on the trinions are to be put together to yield a flat connection on Σ^g. From this point of view the fibres of the polarization in question consist of all possible gluing data for a given point in $\mathscr{M}(D)^{3g-3}$. It is not immediate from this point of view that the fibres of the polarization are Lagrangian; as we mentioned above this fact is proved in [15].

This description of $\bar{\mathscr{S}}_g$ in terms of $\mathscr{M}(D)$ naturally occasions a careful study of the space $\mathscr{M}(D)$, which is none other than the space of representations of the fundamental group $\pi_1(D)$ of a trinion – that is, of the free group on two generators – in $SU(2)$. This is the topic of Sect. 3. At first glance it may seem that the study of the representations of the free group is fatuous. But the description of $\pi_1(D)$ as a free group on two generators is unnatural for us, as it treats the three punctures on the trinion differently; the loops about two of them are taken as generators of the fundamental group, and the third loop as the product of the first two. From our point of view we must treat all three on an equal footing. Any representation of $\pi_1(D)$ in $SU(2)$ then gives rise to three traces, corresponding to the three punctures; we wish to ascertain which combinations of values of the traces can actually occur. This is the content of Proposition 3.1; when the traces are restricted to values corresponding to holonomies which are k^{th} roots of unity, the quantum Clebsch-Gordan conditions will arise out of these very simple considerations.

Having described the moduli space in these terms we are ready to look for the Bohr-Sommerfeld orbits. The method we use to do this is the method of action variables, as applied in [6] to the case of flag manifolds and the representations of classical groups. The theorems standard in this subject cannot be applied directly to the space $\bar{\mathscr{S}}_g$ which is not a smooth manifold. However, by dissecting the proofs of these theorems we see that the methods of proof remain valid in the main. These are roughly as follows. The first crucial observation [3] is that *in the smooth case*, where a smooth symplectic manifold M fibres over a manifold B with fibres that are compact Lagrangian submanifolds L_α, the fibres L_α are of necessity tori. These tori are given by the level sets of Poisson commuting functions $f_i : M \to \mathbb{R}$. These functions may be adjusted so that the corresponding Hamiltonian vector fields v_{f_i} generate flows of period 1. The resulting functions are called *action variables.* Suppose then that the fibres of the polarization passing through two points $p, q \in M$ are Bohr-Sommerfeld fibres; then $f_i(p) - f_i(q) \in \mathbb{Z}$ for all i (see Theorem 4.4). The Bohr-Sommerfeld set can then be characterized if one Bohr-Sommerfeld point can be found, and a set of action variables constructed.

However, $\bar{\mathscr{S}}_g$ is not a smooth manifold, and the fibres of the polarization we are considering are not tori; they degenerate at certain points, corresponding, in terms of our trinion decompositions, to flat connections on the surface which restrict to an "abelian" flat connection on some trinion. Using the description of the polarization given in Sect. 2.3, we can nonetheless show that at the nondegenerate points the fibres of the polarization are still tori, and that these tori are covered by the flows given by the Poisson-commuting functions which defined the polarization. This is done in Sect. 5. Hence, for the generic orbits of the polarization the situation mirrors that of the smooth case.

Turning to the exceptional points, we see that, at such points, although the Hamiltonian flows of the Poisson-commuting functions no longer cover the fibre, they do cover a subspace of the fibre which generates the entire holonomy representation of the fundamental group of the fibre. This is demonstrated in Sect. 6, using again the topological description of the polarization. Hence, again, the study of the action variables will suffice to determine whether such a fibre is a Bohr-Sommerfeld fibre, in the sense that there exists a global covariant constant section of the line bundle $\mathscr{L}^k$ restricted to this fibre.

Hence we see that although $\bar{\mathscr{S}}_g$ is not a smooth manifold and our polarization of it is not a fibration, the Bohr-Sommerfeld orbits of our polarization may be described, as in the smooth case, by integer differences of functions we may as well call action variables. In order to classify the Bohr-Sommerfeld points it remains to find some known Bohr-Sommerfeld points at which these variables are known to take integer values. To do so we turn to (classical) Chern-Simons gauge theory, and yet another description of some fibres of our polarization. In this description, certain fibres of the polarization correspond to flat connections on the surface Σ^g which extend *as flat connections* to some three-manifold N^3 bounding Σ^g. The simplest example of such a fibre corresponds to the handlebody bounding Σ^g; it possesses a global covariant constant section of $\mathscr{L}$ given by a direct topological construction. It turns out that this is not sufficient for our purposes, as all the Hamiltonian flows degenerate at this fibre. We find other fibres of this type by using the branched cover construction of [8], which constructs fibres corresponding roughly to flat connections on Σ^g extending to the handlebody as connections with curvature concentrated on a link in the handlebody, or equivalently, to flat connections on a branched cover of the handlebody, branched over this link. These fibres possess global covariant constant sections of $\mathscr{L}^k$ by a similar topological construction, and for each Hamiltonian flow, we may construct a fibre of this type where the Hamiltonian flow will not degenerate. This construction occupies Sect. 7, which is the only part of this paper where Chern-Simons gauge theory, and indeed, the topology of three-manifolds, enters at all.

The fibres considered in Sect. 7, which we know to be Bohr-Sommerfeld fibres of the line bundle $\mathscr{L}^k$, correspond to connections whose holonomies about the boundaries of the trinions forming the surface Σ^g are conjugate to k^{th} roots of unity. The action variable construction considered above then shows that the Bohr-Sommerfeld fibres of $\mathscr{L}^k$ all satisfy this condition. They are, therefore, in view of the construction of Sect. 3, in one-to-one correspondence with trivalent graphs corresponding to the given surface, whose edges are labelled with k^{th} roots of unity, and whose labellings satisfy the quantum Clebsch-Gordan conditions at each vertex. The number of Bohr-Sommerfeld fibres is thus precisely given by the Verlinde formula, as stated in our final result (Theorem 8.3).

2. Real Polarization of the Moduli Space

In this section we will review some of the relevant material from [15] about the real polarization of the moduli space $\bar{\mathscr{S}}_g$. We give several characterizations of this polarization: the final characterization is related to the work of [17], and will be needed when we look for explicit generators of the fundamental group of the fibre.

2.1. The Moduli Space of Flat Connections

Let Σ^g be a compact, oriented two-manifold of genus g. The moduli space $\bar{\mathscr{S}}_g$ of flat $G=SU(2)$ connections on Σ^g has two convenient descriptions, between which we will alternate whenever necessary. The first more topological description of $\bar{\mathscr{S}}_g$ is as the set of conjugacy classes of representations of the fundamental group $\pi_1(\Sigma^g)$ into G. More explicitly, we choose for $\pi_1(\Sigma^g)$ the usual generators A_i, B_i for $i=1,\ldots,g$, satisfying the relation $\Pi A_i B_i A_i^{-1} B_i^{-1}=1$. Then $\operatorname{Hom}(\pi_1(\Sigma^g), G)$ is given by the set $\{a_i, b_i \in G\colon \Pi a_i b_i a_i^{-1} b_i^{-1}=1\}$. The group G acts on $\operatorname{Hom}(\pi_1(\Sigma^g), G)$ by simultaneous conjugation of the a_i and b_i, and we have $\bar{\mathscr{S}}_g=\operatorname{Hom}(\pi_1(\Sigma^g), G)/G$.

Alternatively, we can consider the space $\mathscr{A}_F$ of flat (smooth) connections on the trivial G bundle $P\to\Sigma^g$. Given a fixed trivialization $P=G\times\Sigma^g$ of P, the space $\mathscr{A}=\mathscr{A}(\Sigma^g)$ of connections on P may be identified with the space $\Omega^1(\Sigma^g)\otimes\mathfrak{g}$ of $\mathfrak{g}$-valued one forms on Σ^g, and $\mathscr{A}_F$ is then identified with the subset $\mathscr{A}_F\subset\mathscr{A}$ given by $\{A\in\mathscr{A}\colon F_A=dA+A\wedge A=0\}$. The gauge group $\mathscr{G}=\operatorname{Maps}(\Sigma^g, G)$ acts on $\mathscr{A}_F$, with a map $g\in\mathscr{G}$ taking $A\in\mathscr{A}_F$ to $A^g=g^{-1}Ag+g^{-1}dg$. Then $\bar{\mathscr{S}}_g=\mathscr{A}_F/\mathscr{G}$.

The variety $\bar{\mathscr{S}}_g$ contains an open set $\mathscr{S}_g$ corresponding to conjugacy classes of *irreducible* representations of $\pi_1(\Sigma^g)$, which is a symplectic manifold, with symplectic form ω described in [2]. Furthermore, in [11], there was constructed a line bundle $\mathscr{L}\to\bar{\mathscr{S}}_g$, with a connection ∇ with curvature equal to $2\pi i\omega$. Thus, setting aside for the moment our concern for the singularities of $\bar{\mathscr{S}}_g$, we see we have the usual prequantum data described in the introduction; we are given a symplectic manifold $(\mathscr{S}_g, \omega)$, and a line bundle $\mathscr{L}\to\mathscr{S}_g$ with connection whose curvature is the form $2\pi i\omega$. To produce a quantization we must polarize the space $\bar{\mathscr{S}}_g$.

2.2. Real Polarization of the Moduli Space

We recall from [15] the polarization of the moduli space $\bar{\mathscr{S}}_g$, obtained from some good functions on $\bar{\mathscr{S}}_g$, associated to closed curves on Σ^g. Let $C\subset\Sigma^g$ be a closed, oriented curve in Σ^g, and choose a basepoint $y \in C$. We define a function $\tilde{f}_C : \mathscr{A}_F \to \mathbb{R}$ by setting, for $A \in \mathscr{A}_F$,

$$\tilde{f}_C(A) = (1/2)\,\mathrm{Tr}\,\mathrm{hol}_C(A), \tag{2.1}$$

where by $\mathrm{hol}_C(A)$ we denote the holonomy of the connection A about the oriented curve C from y to y. The function $\tilde{f}_C$ then descends to a function $f_C : \bar{\mathscr{S}}_g \to \mathbb{R}$.

To obtain functions on $\bar{\mathscr{S}}_g$, we need some good closed curves on Σ^g. These are obtained from a trinion decomposition of the surface. A *trinion* (also called a *pair of pants*) is a copy of the two-holed disc

$$D = \{z \in \mathbb{C}: |z| \leqq 2\} - (\{z: |z-1| < 1/2\} \cup \{z: |z+1| < 1/2\}),$$

with marked points on the boundary components of D, and the standard orientation coming from $\mathbb{C}$. Suppose there is given a decomposition of Σ^g into a union of $2g-2$ trinions D_γ, $\gamma \in \Gamma = \{1, \ldots, 2g-2\}$ joined along their boundaries, with the marked points coinciding whenever two trinions have a nonempty intersection. Such a decomposition of Σ^g gives rise to a trivalent graph constructed from the trinion decomposition by associating a vertex to each trinion and an edge to each boundary circle (see Fig. 1). In any event a trinion decomposition of Σ^g provides us with a collection C_i, $i \in \mathscr{I} = \{1, \ldots, 3g-3\}$ of simple closed oriented curves with marked points on each curve, given by the boundary components of the trinions. We can then consider the functions $f_i : \bar{\mathscr{S}}_g \to \mathbb{R}$ defined by $f_i = f_{C_i}$, using the functions $f_C : \bar{\mathscr{S}}_g \to \mathbb{R}$ defined in Eq. 2.1 above.

The following is the main result of [15].

Theorem 2.1. *Let* $\mathbf{x} = (x_1, \ldots, x_{3g-3}) \in \mathbb{R}^{3g-3}$. *The set* $L_{\mathbf{x}} = \bigcap_i f_i^{-1}(x_i)$ *satisfies*

$$\omega|_{L_{\mathbf{x}}} = 0.$$

Furthermore, $L_{\mathbf{x}}$ *has dimension* $3g-3$ *for a generic point* $\mathbf{x}$ *in the image of* $\bar{\mathscr{S}}_g$ *under the* f_i.

It will be helpful to look at this as follows. Let $B_g \subset \mathbb{R}^{3g-3}$ be the image of the functions f_i; in other words

$$B_g = \{(f_1(\xi), \ldots, f_{3g-3}(\xi)) : \xi \in \bar{\mathscr{S}}_g\}.$$

Then the fibres of the map $\pi = (f_1, \ldots, f_{3g-3}) : \bar{\mathscr{S}}_g \to B_g$ foliate $\bar{\mathscr{S}}_g$ by isotropic subvarieties: the generic fibre is a Lagrangian subvariety.

The functions $2f_i$ are traces of $SU(2)$ matrices; they can, therefore, be described as twice the cosine of angles θ_i. We define the *holonomy angle* θ_i of a connection A associated to the curve C_i by

$$\theta_i(A) = \cos^{-1} f_i(A), \tag{2.2}$$

where we take θ_i to lie in $[0, \pi]$, and the functions $f_i = f_{C_i}$ were defined in Eq. 2.1 above. We thus obtain a map

$$\underline{\theta} = (\theta_1, \ldots, \theta_{3g-3}) : \bar{\mathscr{S}}_g \to \mathbb{R}^{3g-3}. \tag{2.3}$$

Since the θ_j are constant on the fibres of π, they may also be viewed as functions on B_g. By abuse of notation, we shall also denote these functions by $\theta_i : B_g \to \mathbb{R}$.

The function θ_i is smooth on the open dense subset $U_i = \theta_i^{-1}((0, \pi))$ of $\bar{\mathscr{S}}_g$. Thus the Hamiltonian flows of all the θ_i are defined on $\bar{\mathscr{S}}_g^s = \bigcap_i U_i \subset \bar{\mathscr{S}}_g$. The Hamiltonian flows generated by the θ_i on $\bar{\mathscr{S}}_g^s$ are periodic with constant period, and so induce a torus action on $\mathcal{S}_g^s$. The map $\underline{\theta}=(\theta_1, \ldots, \theta_{3g-3}): \mathcal{S}_g^s \to \mathbb{R}^{3g-3}$ is the moment map for this torus action. We may, of course, think of $\underline{\theta}$ as a system of coordinates on B_g: we shall use this notation frequently in what follows.

2.3. Alternative Description of the Polarization

A different way of looking at the polarization $\pi: \bar{\mathcal{S}}_g \to B_g$ is related to the work of Witten [17, Sect. 4.5]. In this description the topology of the fibres of the polarization is transparent, though their symplectic structure as Lagrangian subvarieties is obscured. This alternative description thus nicely complements the description of [15] where the reverse is true.

The general idea of this description of the fibres is to characterize a fibre $\pi^{-1}(b)$ in terms of the gauge equivalence classes of the restrictions of the connections to the trinions D_γ. To do this, we pick a connection A on Σ^g whose gauge equivalence class is in $\pi^{-1}(b)$, and which satisfies certain good conditions.

Let $T \subset G$ be a fixed maximal torus, and let $\mathfrak{t} \subset \mathfrak{g}$ denote the corresponding subalgebra of $\mathfrak{g}$.

Definition 2.2. A connection A on Σ^g is said to be adapted to a trinion decomposition (a.t.d.) if there is a tubular neighbourhood $V_i \cong (-1,1) \times S^1$ of each boundary circle C_i ($i \in \mathcal{I}$), with coordinates on V_i given by (s, θ), $s \in (-1,1)$, $\theta \in S^1$, such that $A|_{V_i} = X_i d\theta$, where X_i is a constant element of $\mathfrak{t}$.

One may easily obtain the following

Lemma 2.3. *For all $y \in \pi^{-1}(b)$, there exists an a.t.d. connection A in the gauge equivalence class y.*

We now define certain subgroups of G corresponding to stabilizers of flat connections. Suppose A is an a.t.d. connection. Then the stabilizer of the action of the gauge group $\mathcal{G}(C_i) = \mathrm{Maps}(C_i, G)$ on $A|_{C_i}$ consists of *constant* maps, and so may be identified with a subgroup H_i of G. Under this identification, $H_i = G$ if $\theta_i(b) = 0$ or $\theta_i(b) = \pi$; otherwise, $H_i = T$.

We obtain a similar identification for the stabilizer J_γ of the restriction $A|_{D_\gamma}$ of the connection A to the trinion D_γ, under the action of $\mathcal{G}(D_\gamma) = \mathrm{Maps}(D_\gamma, G)$. This stabilizer also consists of constant maps, and $J_\gamma = Z(G) \simeq \mathbb{Z}_2$ if $A|_{D_\gamma}$ corresponds to an *irreducible* representation of $\pi_1(D_\gamma)$ into G, while $J_\gamma = T$ (resp. G) if the representation reduces to a representation into T [resp. into $Z(G)$]. The stabilizers J_γ, H_i depend only on the point $b \in B_g$, and not on the particular a.t.d. connection A whose gauge equivalence class lies in $\pi^{-1}(b)$.

We now describe the fibre $\pi^{-1}(b)$ in terms of a.t.d. connections. Suppose we are given one a.t.d. connection A whose gauge equivalence class $[A]$ is in $\pi^{-1}(b)$, and a collection of elements $\tau_i \in H_i$ ($i = 1, \ldots, 3g-3$). We define a map $\psi_A: \prod_i H_i \to \pi^{-1}(b)$ as follows. Given a set of elements $\tau = (\tau_i)_{i=1,\ldots,3g-3}$ in $\prod_i H_i$, we choose a collection of maps $\zeta_\gamma: D_\gamma \to G$ such that $\zeta_{\gamma(i)}, \zeta_{\gamma'(i)}$ are constant on a tubular neighbourhood of C_i, where C_i is the boundary circle bounding the trinions $D_{\gamma(i)}, D_{\gamma'(i)}$, and such that

$$\zeta_{\gamma(i)}|_{C_i} = \tau_i \zeta_{\gamma'(i)}|_{C_i}. \tag{2.4}$$

[The orientation of the trinion and that of the surface determine which trinion to call $\gamma(i)$ and which to call $\gamma'(i)$: we adopt the convention that the orientation of the surface is given by $v \wedge w$, where w is the tangent to the oriented boundary circle C_i and v is a tangent vector transverse to C_i and pointing *into* D_γ.] We define a connection A_τ on Σ^g by

$$A_\tau|_{D_\gamma} = A|_{D_\gamma}{}^{\zeta_\gamma}. \tag{2.5}$$

These connections agree on tubular neighbourhoods of the boundary circles C_i, and hence combine to give a connection A_τ on Σ^g. We then define $\psi_A(\tau) = [A_\tau]$; here $[A_\tau]$ denotes the point in $\bar{\mathscr{S}}_g$ corresponding to the gauge equivalence class of A_τ.

We now consider the question of when two such connections, corresponding to different elements of $\prod_i H_i$, are gauge equivalent. We have the following

Lemma 2.4. *If τ, τ' are two points in $\prod_i H_i$, then the connections A_τ and $A_{\tau'}$ are gauge equivalent if and only if there is a system of gauge transformations $\Phi_\gamma : D_\gamma \to G$ such that*
1. *$\Phi_\gamma \in J_\gamma$ for all γ. (In other words, the Φ_γ are constant maps taking their values in the subgroup J_γ.)*
2. *If the boundary circle C_i bounds the trinions $D_{\gamma(i)}, D_{\gamma'(i)}$, we have*

$$\Phi_{\gamma'(i)}|_{C_i} \cdot \tau_i = \tau_i' \cdot \Phi_{\gamma(i)}|_{C_i}. \tag{2.6}$$

We are now ready to complete the characterization of the fibre $\pi^{-1}(b)$:

Theorem 2.5. *The map $\psi_A : \prod_i H_i \to \pi^{-1}(b)$ is surjective. Moreover, the group $\mathscr{J} = \prod_\gamma J_\gamma$ has a natural action on $\prod_i H_i$, so that the fibre $\pi^{-1}(b)$ is given by $\prod_i H_i / \mathscr{J}$.*

Proof. To prove surjectivity it suffices to consider a collection of a.t.d. connections on the trinions with the desired holonomies; the a.t.d. condition then allows them to be combined to form a connection on the surface Σ^g with the given holonomies. It remains to construct the group actions. We define the action of an element $(\Phi_\gamma)_{\gamma=1,\dots,2g-2}$ by sending $(\tau_i)_{i=1,\dots,3g-3} \in \prod_i H_i$ to $(\Phi_{\gamma'(i)} \tau_i \Phi_{\gamma(i)}^{-1})_{i=1,\dots,3g-3}$. Then two connections $A_\tau, A_{\tau'}$ are gauge equivalent if and only if τ, τ' are equivalent under the action of $\mathscr{J}$, by the condition (2.6). □

3. The Moduli Space of Flat Connections on a Trinion

In Sect. 2.3 we showed how the moduli space $\bar{\mathscr{S}}_g$ of flat connections on Σ^g could be described in terms of the space $\mathscr{M}(D)$ of gauge equivalence classes of flat connections on a trinion, and how the real polarization of [15] arose naturally in the context of this description. In order to make the connection more explicit, we study the moduli space $\mathscr{M}(D)$ in more detail, and show how the functions θ_i defining the polarization behave on $\mathscr{M}(D)$.

The space $\mathscr{M}(D)$, just like its counterpart $\bar{\mathscr{S}}_g$, can be described as the quotient of the space $\mathrm{Hom}(\pi_1(D), G)$ by the conjugation action of G. Now $\pi_1(D)$ is the group generated by the homotopy classes $[C_1], [C_2], [C_3]$ of based loops C_1, C_2, C_3 corresponding to the three boundary components of the trinion D, with the relation $[C_1][C_2][C_3] = 1$. Corresponding to the three boundary components C_1, C_2, C_3 of the trinion D, we may therefore define three functions $\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3$ on $\mathrm{Hom}(\pi_1(D), G)$, given by

$$\tilde{\theta}_j(\varrho) = \cos^{-1}\left(\tfrac{1}{2}\mathrm{Tr}\,\varrho([C_j])\right).$$

These maps descend to maps $\theta_i: \mathcal{M}(D) \to [0, \pi]$, just as in the case of a closed surface Σ^g. Indeed, if we are given a trinion decomposition of a surface Σ^g, these functions agree with their counterparts defined on Σ^g, in that any representation of $\pi_1(\Sigma^g)$ restricts to a representation of $\pi_1(D_\gamma)$ for each trinion D_γ; under the map $\bar{\mathcal{S}}_g \to \mathcal{M}(D_\gamma)$ induced by this restriction, the functions θ_j defined on $\mathcal{M}(D)$ agree with the functions $\theta_{i_j(\gamma)}$ defined on $\bar{\mathcal{S}}_g$.

Our main result is the following:

Proposition 3.1. *The map $\hat{\theta} = (\theta_1, \theta_2, \theta_3): \mathcal{M}(D) \to [0, \pi]^3$ sends $\mathcal{M}(D)$ bijectively to the set satisfying the inequalities*

$$|\theta_1 - \theta_2| \leqq \theta_3 \leqq \min(\theta_1 + \theta_2, 2\pi - (\theta_1 + \theta_2)).$$

Proof. We wish to derive the condition on a triple of angles $(\theta_1, \theta_2, \theta_3)$ (with $0 \leqq \theta_i \leqq \pi$) needed for $(\theta_1, \theta_2, \theta_3)$ to arise as holonomy angles of some flat connection on the trinion. We use quaternionic notation

$$z + wj = \begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix}$$

(where $z, w \in \mathbb{C}$, $j^2 = -1$ and $zj = j\bar{z}$). We consider elements $g_i \in G$ such that g_i is conjugate to $e^{i\theta_i}$ ($i = 1, 2, 3$). We want to find the condition on θ_i in order for there to exist in these conjugacy classes solutions g_i of the equation

$$g_1 g_2 g_3 = 1.$$

By conjugation, we may assume without loss of generality that g_1 is of the form $e^{i\theta_1}$, $0 \leqq \theta_1 \leqq \pi$. We then have the freedom to conjugate g_2 and g_3 by an element of T. So we may assume g_2 is of the form

$$g_2 = z + wj, \qquad w \in \mathbb{R}^+, \tag{3.1}$$

in other words

$$g_2 = \cos\theta_2 + i \sin\theta_2 (\cos\beta - ij \sin\beta) \tag{3.2}$$

for some $\beta \in \mathbb{R}$. The condition that $g_1 g_2$ be conjugate to $e^{i\theta_3}$ is then

$$\mathrm{Re}\{e^{i\theta_1}(\cos\theta_2 + i \sin\theta_2(\cos\beta - ij \sin\beta))\} = \cos\theta_3$$

or

$$\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 \cos\beta = \cos\theta_3. \tag{3.3}$$

One may solve this for $\cos\beta$ if and only if

$$\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 \leqq \cos\theta_3,$$
$$\cos\theta_3 \leqq \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2$$

or

$$\cos(\theta_1 + \theta_2) \leqq \cos\theta_3 \leqq \cos(\theta_1 - \theta_2).$$

Noting that cos is a decreasing function on $[0, \pi]$, this becomes

$$|\theta_1 - \theta_2| \leqq \theta_3,$$
$$\theta_3 \leqq \theta_1 + \theta_2 \quad \text{if} \quad |\theta_1 + \theta_2| \leqq \pi,$$
$$\theta_3 \leqq 2\pi - (\theta_1 + \theta_2) \quad \text{if} \quad |\theta_1 + \theta_2| > \pi.$$

In other words, the condition is

$$|\theta_1-\theta_2|\leqq\theta_3\leqq\min\{\theta_1+\theta_2, 2\pi-(\theta_1+\theta_2)\} \tag{3.4}$$

as claimed. This equation defines a tetrahedron inscribed in the cube $0\leqq\theta_i\leqq\pi$.

To prove the map $\hat{\theta}$ is injective, we observe that we have chosen $\sin\theta_2\sin\beta\geqq 0$ in (3.1), and Eq. (3.3) determines $\sin\theta_2\cos\beta$ if $\sin\theta_1\neq 0$. Thus the equation

$$g_2=\cos\theta_2+i\sin\theta_2\cos\beta+j\sin\theta_2\sin\beta$$

for g_2 has a *unique* solution in terms of $\cos\theta_1, \cos\theta_2, \cos\theta_3$. This is true also for the degenerate case $\sin\theta_1=0$. Thus the functions θ_i can be considered as coordinates on the space of conjugacy classes of representations of the trinion fundamental group. □

The image of $\bar{\mathcal{S}}_g$ in $\mathbb{R}^{3g-3}$ under the maps $\theta_1, \ldots, \theta_{3g-3}$ is thus given by those values of θ_i satisfying the inequalities (3.4) on every trinion. This is because flat connections on trinions can be glued together to form a flat connection on the closed surface, provided the conjugacy classes of the holonomies around the boundary circles of adjacent trinions agree. This gluing process was discussed above in Theorem 2.5.

Thus we have the following

Proposition 3.2. *The image of $\bar{\mathcal{S}}_g$ under the maps $(\theta_1, \ldots, \theta_{3g-3})$ is the polyhedron defined by Eq.* (3.4) *for $\theta_1=\theta_{i_1(\gamma)}, \theta_2=\theta_{i_2(\gamma)}, \theta_3=\theta_{i_3(\gamma)}$ corresponding to every trinion D_γ, where $C_{i_1(\gamma)}, C_{i_2(\gamma)}, C_{i_3(\gamma)}$ are the boundary circles of D_γ.*

We recall the following fact from [5] (Lemma 2.1):

Proposition 3.3 [Guillemin-Sternberg]. (a) *The differential $(d\Phi)_x$ of the moment map Φ for the action of a torus $(S^1)^m$ on a symplectic manifold M of dimension $2m$ at x is a surjection into $\mathbb{R}^m$ if and only if the stabilizer at x is discrete.*
(b) *More generally, the codimension of the image of $(d\Phi)_x$ in $\mathbb{R}^m$ is equal to the dimension of the stabilizer of $x\in M$ under the torus action.*

We apply this to the case of $\bar{\mathcal{S}}_g$ as follows:

Corollary 3.4. *Suppose $x\in\pi(\bar{\mathcal{S}}_g^s)\subset B_g$. Then*
(a) *The Hamiltonian vector fields corresponding to the functions θ_i are linearly independent on the fibre $\pi^{-1}(x)$ if and only if x is a point where all the inequalities* (3.4) *are strict.*
(b) *More generally, the number of linearly independent Hamiltonian vector fields on the fibre $\pi^{-1}(x)$ is equal to $3g-3-s$, where s is the number of independent linear equations of the following type satisfied by $\underline{\theta}(x)$:*

$$\begin{aligned}
&\theta_{i_1(\gamma)}(x)+\theta_{i_2(\gamma)}(x)-\theta_{i_3(\gamma)}(x)=0,\\
&\theta_{i_2(\gamma)}(x)+\theta_{i_3(\gamma)}(x)-\theta_{i_1(\gamma)}(x)=0,\\
&\theta_{i_3(\gamma)}(x)+\theta_{i_1(\gamma)}(x)-\theta_{i_2(\gamma)}(x)=0,\\
&\theta_{i_1(\gamma)}(x)+\theta_{i_2(\gamma)}(x)+\theta_{i_3(\gamma)}(x)=2\pi.
\end{aligned} \tag{3.5}$$

These equations are the equalities corresponding to the inequalities (3.4).

We note also the following.

Lemma 3.5. *Let $x \in \mathscr{M}(D)$ and let $\theta_1(x), \theta_2(x), \theta_3(x)$ be the holonomy angles of x around the three boundary circles of D. Then x corresponds to a conjugacy class of reducible representations of the trinion fundamental group if and only if at least one of Eq.* (3.5) *is satisfied.*

This lemma motivates the following

Definition 3.6. A triple of angles $(\theta_1, \theta_2, \theta_3)$ $(\theta_i \in [0, \pi])$ will be called an interior triple if the point in $\mathscr{M}(D)$ with holonomy angles $(\theta_1, \theta_2, \theta_3)$ corresponds to a conjugacy class of irreducible representations of the trinion fundamental group.

For a generic fibre $\pi^{-1}(x)$, all triples $(\theta_{i_1(\gamma)}(x), \theta_{i_2(\gamma)}(x), \theta_{i_3(\gamma)}(x))$ corresponding to holonomies around the boundary circles of trinions D_γ are interior. Thus the generic fibre of π is a torus of dimension $3g-3$: this is a consequence of Theorem 2.5, as stated in the following proposition.

Proposition 3.7. *Let x be a point in B_g, and let A be a flat a.t.d. connection whose gauge equivalence class lies in $\pi^{-1}(x)$. Assume that for each trinion D_γ, the triple $(\theta_{i_1(\gamma)}(x), \theta_{i_2(\gamma)}(x), \theta_{i_3(\gamma)}(x))$ is interior. Then the fibre $\pi^{-1}(x)$ identifies with $T^{3g-3}/(\mathbb{Z}_2)^{2g-2}$ under the map ψ_A defined in Sect. 2.3.*

More generally, it will be convenient to focus our attention on good subsets of the base space B_g. The first good subset is the space $B_g^s = \pi(\mathscr{S}_g^s)$ defined by $B_g^s = \{x \in B_g \mid \theta_i(x) \in (0, \pi) \text{ for all } i\}$. The second good subset is the subspace $B_g^{s,\mathrm{ind}}$ of B_g^s consisting of those points $x \in B_g$ for which all the Hamiltonian vector fields are linearly independent [i.e., for which the triples $(\theta_{i_1(\gamma)}(x), \theta_{i_2(\gamma)}(x), \theta_{i_3(\gamma)}(x))$ corresponding to all $\gamma \in \Gamma$ are interior triples]. Both B_g^s and $B_g^{s,\mathrm{ind}}$ are open dense subsets of B_g.

4. Bohr-Sommerfeld Orbits and Quantization

We summarize the general method for quantizing using a real polarization. This leads to the consideration of a certain set of points in B_g, the *Bohr-Sommerfeld set*. The Bohr-Sommerfeld set in $\mathscr{S}_g$ is the main topic of this paper. We discuss the characterization of the Bohr-Sommerfeld points in terms of the values of an appropriate set of Hamiltonian functions with period 1 *(action variables)*. We show how this method can be developed into a theorem constructed to apply to the case of the moduli space $\bar{\mathscr{S}}_g$; this is Theorem 4.4.

4.1. Quantization in Real Polarizations

Our treatment is taken from [6], whither we refer the reader for more details.

Let (M, ω) be a compact connected symplectic manifold of dimension $2m$, and let $\mathscr{L} \to M$ be a line bundle with connection ∇ of curvature given by $2\pi i\omega$ (the prequantum line bundle with connection). A real polarization of M is a surjective map $\pi: M \to B$ onto a manifold B of dimension m, such that $\omega|_{\pi^{-1}(x)} = 0$ for every $x \in B$; for generic x, the fibre $L_x = \{\pi^{-1}(x)\}$ is a Lagrangian submanifold. The curvature of the line bundle $\mathscr{L}$, restricted to each fibre, is zero. Among the fibres will be a finite number on which the restriction of $\mathscr{L}$ will have a global covariant constant section. The fibres satisfying this *Bohr-Sommerfeld* condition are called the Bohr-Sommerfeld fibres of the polarization, and the points $x \in B$ in the image of the Bohr-Sommerfeld fibres are the *Bohr-Sommerfeld* points of the space B.

Assume in addition that the map $\pi: M \to B$ is a *fibration.* Then the relation between the Bohr-Sommerfeld set and quantization is the following. Let $\mathscr{J}_\pi$ denote the sheaf of local sections of $\mathscr{L}$ which are covariant constant along the fibres of π. The quantization of the polarized prequantum data corresponding to M, ω, $\mathscr{L}$, and ∇ is defined as the vector space

$$\mathscr{H} = \bigoplus_{i=0}^{2m} H^i(M, \mathscr{J}_\pi).$$

This cohomology can be computed by a theorem of Sniatycki [13]. Let $B_{\mathrm{bs}} \subset B$ denote the set of Bohr-Sommerfeld points of B; for $b \in B_{\mathrm{bs}}$, let S_b denote the (one-dimensional) space of global covariant constant sections of the restriction of $\mathscr{L}$ to $\pi^{-1}(b)$. Then Sniatycki's theorem declares that there is a natural isomorphism

$$\mathscr{H} \simeq \bigoplus_{b \in B_{\mathrm{bs}}} S_b.$$

Hence the quantization can be constructed from the sections of $\mathscr{L}$ over the Bohr-Sommerfeld fibres of the polarization.

Sniatycki's theorem does not apply to our case, since the moduli space $\bar{\mathscr{S}}_g$ is not a manifold and the map $\pi: \bar{\mathscr{S}}_g \to B_g$ is not a fibration. From our point of view, Sniatycki's result instead provides a motivation for considering the Bohr-Sommerfeld set. On the basis of Sniatycki's result, we would expect the Bohr-Sommerfeld set to correspond to a basis of a suitably defined quantum Hilbert space $\mathscr{H}$ corresponding to the real polarization. We shall, indeed, see that the number of points in the Bohr-Sommerfeld set is given by the Verlinde formula, which has recently been proved to give the dimension of the quantum Hilbert space $\mathscr{H}$ arising from a *Kähler* polarization.

4.2. *Alternative Characterization of the Bohr-Sommerfeld Fibres*

Let us then consider the Bohr-Sommerfeld fibres of a fibration $\pi: M \to B$. The set of Bohr-Sommerfeld fibres consists of those fibres $L_x = \pi^{-1}(x)$ of the polarization π for which the holonomies of the connection ∇ around a set of loops generating $\pi_1(L_x)$ are all equal to 1. Then a basis of loops is most conveniently obtained in terms of a basis of a certain lattice of functions H on B, the *period lattice.* For further discussion of the following material, see [3] and [6], particularly Theorems 2.4 and 2.6 of [6].

Let x be a point in B. Elements of the cotangent space T_x^*B define vertical vector fields along the fibre $\pi^{-1}(x)$: for $\alpha \in T_x^*B$, denote the associated vertical vector field by v_α. Denote by f_α the symplectic diffeomorphism of the fibre $\pi^{-1}(x)$ induced by the time 1 map of the flow along v_α.

Definition 4.1. The period lattice in T_x^*B is the set of α in T_x^*B such that the corresponding f_α are trivial.

This is a lattice of dimension m. One may show (see [3]) that in a sufficiently small neighbourhood $\mathcal{O}$ of any point $x \in B$, there is a set of functions H_λ $(\lambda \in \Lambda)$ on $\mathcal{O}$ forming a lattice Λ under addition, such that the period lattice at $x' \in \mathcal{O}$ is given by $\{(dH_\lambda)_{x'}\}_{\lambda \in \Lambda}$. (Under suitable hypotheses, the functions in the period lattice exist globally: see [3, Theorem 2.2].) By abuse of language, we shall use the term "period lattice" to designate this lattice of functions as well.

Let us denote a basis of the period lattice by $\tilde{\mu}_i \in C^\infty(B, \mathbb{R})$, and define $\mu_i = \tilde{\mu}_i \circ \pi \in C^\infty(M, \mathbb{R})$; then the Hamiltonian flows of the functions μ_i have period 1, and the fundamental group $\pi_1(L_x)$ is generated by the loops η_i which are the period 1 trajectories of the Hamiltonian flows of μ_i. These Hamiltonian flows generate a transitive action on the fibres of the polarization: in other words, these fibres are m dimensional tori. The functions $(\mu_1, \ldots, \mu_m)$ define a *moment map* for an action of the torus $(S^1)^m$ on M, preserving the Lagrangian fibration.

Remark. In classical mechanics the set of functions μ_i corresponding to a basis of the period lattice is known as a set of *action variables.*

The period lattice is important to us because, as we shall see below, the Bohr-Sommerfeld set is roughly characterized as the set of points where the functions in the period lattice take integer values. The precise statement that applies to the situation we shall consider is Theorem 4.4.

The material summarized in this section applies to the case when M is a smooth manifold and π is a smooth fibration. This is not the case for our moduli space situation: the purpose of the remainder of this section is to show how the methods described here generalize to our setting.

4.3. *Flows and Bohr-Sommerfeld Orbits*

The setting in which we wish to study quantization differs from the ideal setting described in the previous sections in several ways; for the map $\pi : \mathscr{S}_g \to B_g$ defining the polarization is not a fibration, and the moduli space $\mathscr{S}_g$ is not smooth. The fact that the map π is not a fibration will turn out to be our main concern, and will reflect itself in the fact that Hamiltonian flows of period one cannot be defined on all of $\mathscr{S}_g$. In this section we will characterize the Bohr-Sommerfeld orbits of a real polarization of a smooth manifold M of dimension $2m$ given by a map $\pi : M \to B$ onto a manifold B of dimension m, which is *not* a fibration. The result we obtain in Theorem 4.4 will apply to the moduli space case also.

Lemma 4.2. *Let M be a connected symplectic manifold of dimension $2m$ with a surjective map $\pi : M \to B$, where B has dimension m. Suppose y_0, y_1 are two points in M, and x_0, x_1 are their images in B. Suppose that H is a smooth function on B, and that $\mu = H \circ \pi$ is a Hamiltonian function (constant along the fibres of π) whose Hamiltonian flow has period 1. Let $\eta_t(y_n)$ $(n = 0, 1)$ be the closed loops arising from the Hamiltonian flow starting at y_0 and y_1. Then there is a map $\chi : I \times S^1 \to M$ such that $\chi(n, t) = \eta_t(y_n)$ $(n = 0, 1)$. Furthermore, the symplectic form pulls back under χ to a smooth two-form on $I \times S^1$.*

Proof. We take a smooth path $\lambda(s)$ in B between x_0 and x_1, and consider a lift to $\tilde{\lambda}(s) \subset M$ interpolating between y_0 and y_1. We may then define

$$\chi(s, t) = \phi_t(\tilde{\lambda}(s)),$$

where ϕ_t is the Hamiltonian flow of the function μ at time t. By definition the tangent vector $\chi_* \dfrac{\partial}{\partial t}$ is the Hamiltonian vector field X_μ. Thus the symplectic form on M pulls back to

$$\chi^*\omega=\omega_{\chi(s,t)}\left(\chi_*\frac{\partial}{\partial s},\chi_*\frac{\partial}{\partial t}\right)ds\wedge dt=\omega_{\chi(s,t)}\left(\chi_*\frac{\partial}{\partial s},X_\mu\right)ds\wedge dt$$
$$=(d\mu)\left(\chi_*\frac{\partial}{\partial s}\right)ds\wedge dt=\frac{\partial H(\lambda(s))}{\partial s}ds\wedge dt\,, \tag{4.1}$$

which is a smooth two-form on the cylinder. □

Using the existence of these cylinders constructed from Hamiltonian flows, we are able to relate the difference of holonomies of the line bundle $\mathscr{L}$ around loops given by the Hamiltonian flow with the difference of Hamiltonian functions at x_0 and x_1:

Proposition 4.3. *Suppose μ is a Hamiltonian function on a symplectic manifold M, whose Hamiltonian flow has period* 1. *Let $\chi: I\times S^1\to M$ be a one parameter family of integral curves of the Hamiltonian flow of μ, i.e., $\chi_*\frac{\partial}{\partial t}=X_\mu$ for $t\in S^1$, where X_μ denotes the Hamiltonian vector field associated to μ. Let $\mathscr{L}$ be a line bundle over M with connection ∇ of curvature $2\pi i\omega$. Then the function $\mu(\chi(s,t))$ depends only on s (since μ is constant along the orbits of the Hamiltonian flow), and therefore defines a function $H(s)=\mu(\chi(s,t))$. Then*

$$H(1)-H(0)=\frac{1}{2\pi i}(\log \mathrm{hol}_{\chi(1,t)}\nabla-\log \mathrm{hol}_{\chi(0,t)}\nabla)\,(\mathrm{mod}\,\mathbb{Z})\,.$$

(Here, $\mathrm{hol}_{\chi(n,t)}$ *denotes the holonomy of the connection ∇ around the closed loop $\chi(n,t)$.)*

Proof. Denote by $\mathscr{C}$ the image $\mathscr{C}=\chi(I\times S^1)\subset M$. Then $\mathscr{C}$ is a region over which the pullback of the line bundle $\mathscr{L}$ may be trivialized: choosing such a trivialization, we may represent the connection ∇ by a 1-form α on $\mathscr{C}$, such that $d\alpha=2\pi i\omega$. Thus

$$\log \mathrm{hol}_{\chi(1,t)}\nabla-\log \mathrm{hol}_{\chi(0,t)}\nabla=\int_{\chi(1,t)}\alpha-\int_{\chi(0,t)}\alpha=2\pi i\int_{\mathscr{C}}\omega\,. \tag{4.2}$$

Now consider the restriction of the symplectic form to $\mathscr{C}$. We have $\chi_*\frac{\partial}{\partial t}=(X_\mu)_{\chi(s,t)}$, so by (4.1), we see that

$$\int_{\mathscr{C}}\omega=\int_{I\times S^1}ds\,dt\,\omega\left(\chi_*\frac{\partial}{\partial s},\chi_*\frac{\partial}{\partial t}\right)=\int_0^1 ds\frac{\partial H}{\partial s}=H(1)-H(0)\,. \quad \square \tag{4.3}$$

We shall use Proposition 4.3 to find the Bohr-Sommerfeld points, by the following method. Let us assume we have a function $\mu=H\circ\pi$ whose Hamiltonian flow has period 1; denote by $\eta(y)$ the closed loop which is the integral curve of the Hamiltonian flow through a point $y\in M$. Suppose we want to find all points $x\in B$ for which there is a covariant constant section of $\mathscr{L}$ over $\eta(y)$. We may locate these points x by finding one such point (denoted x_μ) and using Lemma 4.2 to construct a cylinder whose (oriented) boundary is $-\eta(y_\mu)\cup\eta(y)$, where y,y_μ are points in $\pi^{-1}(x)$ [resp. $\pi^{-1}(x_\mu)$]. Then Proposition 4.3 tells us that there is a covariant constant section of $\mathscr{L}$ over $\eta(y)$ if and only if $H(x)-H(x_\mu)\in\mathbb{Z}$.

In order to locate the Bohr-Sommerfeld points, we must then find a set of functions $\tilde{\mu}_i$ on B such that the trajectories of the Hamiltonian flows of $\mu_i=\tilde{\mu}_i\circ\pi$

on M generate the entire fundamental group of each fibre. Given such a collection of flows, we will be able to characterize the Bohr-Sommerfeld points as given by integer values of the functions $\tilde{\mu}_i$ assuming that we can find, for each such function, a point x_i known *a priori* to be a Bohr-Sommerfeld point, and where $\tilde{\mu}_i(x_i) \in \mathbb{Z}$. Thus our characterization of the Bohr-Sommerfeld points is as follows:

Theorem 4.4. *Let (M, ω) be a connected symplectic manifold of dimension $2m$, and let $\pi: M \to B$ be a map onto a manifold B of dimension m such that $\omega|_{\pi^{-1}(x)} = 0$ for all $x \in B$. Let $\mathscr{L}$ be a line bundle over M with connection ∇ of curvature $2\pi i\omega$. Let $\mu_i = \tilde{\mu}_i \circ \pi$, $i = 1, \ldots, n$ (for $\tilde{\mu}_i : B \to \mathbb{R}$) be a set of Hamiltonian functions constant along the fibres $\pi^{-1}(x)$ and with Hamiltonian flows of period 1 with respect to the symplectic form ω. Let $X_i = \pi^{-1}(\pi(X_i))$ be a connected subset of M where μ_i is smooth. Denote by $\eta_i(y)$ the closed loop obtained from the Hamiltonian flow of μ_i through a point $y \in X_i$. Suppose that*

1. *(Flows generate the fundamental group): For all $x \in B$, the trajectories of those Hamiltonian flows corresponding to functions μ_i for which $x \in \pi(X_i)$ form a set of generators for the fundamental group of the fibre $\pi^{-1}(x)$.*
2. *(Existence of a priori Bohr-Sommerfeld points): For each i, there exists a point $x_i \in \pi(X_i)$ with $\tilde{\mu}_i(x_i) \in \mathbb{Z}$, such that for any $y \in \pi^{-1}(x_i)$, the line bundle $\mathscr{L}|_{\eta_i(y)}$ possesses a global covariant constant section.*
3. *Whenever $x \in B$ is a Bohr-Sommerfeld point, and $x \notin \pi(X_i)$, then $\tilde{\mu}_i(x) \in \mathbb{Z}$.*

Then the Bohr-Sommerfeld set $B_{\mathrm{bs}} \subset B$ is characterized as follows:

$$x \in B_{\mathrm{bs}} \quad \textit{if and only if} \quad \tilde{\mu}_i(x) \in \mathbb{Z}$$

for all i.

Remark. Condition (1) in Theorem 4.4 may be replaced by the following weaker condition:

(1′) *Let $x \in B$, and suppose $\tilde{\mu}_i(x) \in \mathbb{Z}$ for all i. The trajectories of the Hamiltonian flows corresponding to those functions μ_i for which $x \in \pi(X_i)$ generate the image of the entire fundamental group of $\pi^{-1}(x)$ under the holonomy representation associated to the connection ∇ on the line bundle $\mathscr{L}$.*

We wish to apply this result to the moduli space $\bar{\mathscr{S}}_g$, equipped with symplectic form $k\omega$ associated to the prequantum line bundle $\mathscr{L}^k$. This moduli space is not a smooth manifold; it consists of strata, corresponding to representations of the fundamental group of the surface Σ^g which are irreducible, or which reduce to the subgroups T or $Z(G)$ of G. The most straightforward approach would be to apply Theorem 4.4 stratum by stratum. This would require finding *a priori* Bohr-Sommerfeld points in each stratum. However, the proof of Theorem 4.4 shows that *a priori* Bohr-Sommerfeld points in one stratum can be used to fix the values of the action variables on the other strata, provided that smooth paths can be constructed connecting these points to any other point in the moduli space; then the smooth cylinders of Lemma 4.2 will still exist, allowing the proof of Theorem 4.4 to go through, exactly as in the smooth case. This is the version of the result of Theorem 4.4 which will be used for the moduli space.

Our plan is now apparent. Our Hamiltonian flows will be given by the functions θ_i defined in Sect. 2.2 (and by certain linear combinations of them). These flows will be defined on $\bar{\mathscr{S}}_g^s = \bigcap_i U_i$, where all the θ_i are smooth, and will be shown to generate the fundamental group of any fibre lying over a point of $\bar{\mathscr{S}}_g^s$; this is the result of Proposition 5.4. Those fibres of the polarization lying outside $\bar{\mathscr{S}}_g^s$ are treated in

Proposition 6.13, which applies the alternative form (1′) of condition (1) of Theorem 4.4. Finally, the *a priori* Bohr-Sommerfeld points [condition (2) of Theorem 4.4] are constructed in Sect. 7, using methods of Chern-Simons gauge theory. Section 7 is the only part of this paper where any is made of Chern-Simons theory, or indeed of three-manifold topology. The reader who is willing to accept on faith the existence of the *a priori* Bohr-Sommerfeld points will find that nothing else in this paper relies on the constructions of Sect. 7.

5. Torus Actions in $\bar{\mathcal{S}}_g$

The purpose of this paper is to apply the methods of quantization in a real polarization, studied in Sect. 4, to the symplectic variety $(\bar{\mathcal{S}}_g, k\omega)$; in other words, to look for the Bohr-Sommerfeld orbits of the line bundle $\mathcal{L}^k$. In this section we begin this process by studying torus flows defined in $\bar{\mathcal{S}}_g$, which will allow us to apply the cylinder construction of Theorem 4.4. These torus flows will come from the Hamiltonian flows corresponding to the functions θ_i defined in Sect. 2. We then determine a set of Hamiltonian functions generating the period lattice (by verifying that the Hamiltonian flows of these functions give a set of closed loops generating the fundamental groups of all fibres). One set of functions in the period lattice are, as one might expect, given by the functions

$$h_i = \frac{k}{\pi}\theta_i \quad \text{for} \quad i \in \mathcal{I};$$

however, there are additional generators

$$g_\gamma = \tfrac{1}{2}(h_{i_1(\gamma)} + h_{i_2(\gamma)} + h_{i_3(\gamma)}) \quad \text{for} \quad \gamma \in \Gamma .$$

Integer values of these functions will then correspond to the Bohr-Sommerfeld set, assuming, first, that we can find points known *a priori* be in the Bohr-Sommerfeld set [as in condition (2) of Theorem 4.4], and, second, that we can deal with the singularities of the torus actions. The former will be the topic of Sect. 7; the latter of Sect. 6.

5.1. Twist Flows

The construction of the torus flows in $\bar{\mathcal{S}}_g$ is due to Goldman [4]. In [4] it was shown that associated to every closed oriented curve $C \subset \Sigma^g$ there is an S^1 action $\Xi_t^C : U_C \to U_C$ defined on an open dense subset $U_C \subset \bar{\mathcal{S}}_g$, and called by Goldman a *twist flow*. To define twist flows, we first need the following auxiliary construction. For any conjugation invariant function $f: G \to \mathbb{R}$, we may define an associated G-equivariant function $F: G \to \mathfrak{g}$ by

$$\langle X, F(A)\rangle = df_A(X) = \frac{d}{dt} f(A \exp tX).$$

[Here, $\langle \cdot, \cdot \rangle$ denotes the basic inner product on the Lie algebra $\mathfrak{g}$, in other words $\langle X, Y\rangle = -\mathrm{Tr}(XY)$.]

For the function $f(A)$, the associated function is

$$F_c(A) = \tfrac{1}{4}(A - A^{-1}).$$

In Sect. 2, we also find it useful to study functions on $\bar{\mathscr{S}}_g$ related to the invariant function $\theta: G \to \mathfrak{g}$ defined by

$$\theta(A) = \cos^{-1}(f(A)),$$

where the value of $\cos^{-1}$ is chosen to lie between 0 and π. We observe thus that if $A = \mathrm{diag}(e^{i\theta}, e^{-i\theta})$,[1] we have $F_c(A) = \frac{1}{2}\mathrm{diag}(i \sin\theta, -i\sin\theta)$. Then the function F_θ associated to the invariant function θ (where the values of θ are taken to be in $[0, \pi]$) is

$$F_\theta = -\frac{1}{\sin\theta} F_c \tag{5.1}$$

(as $d\cos\theta = -\sin\theta d\theta$). Thus we get

$$F_\theta(A) = \tfrac{1}{2}\mathrm{diag}(-i, i) \tag{5.2}$$

for $A = \mathrm{diag}(e^{i\theta}, e^{-i\theta})$ and $\theta \in (0, \pi)$. For $B = gAg^{-1}$ and A of the above form, we define

$$F_\theta(B) = \tfrac{1}{2} g\, \mathrm{diag}(-i, i) g^{-1}.$$

For the values $\theta = 0$, π (corresponding to $A = \pm 1$), F_θ is undefined.

We now give the definition of twist flows. To do so we must first construct an action of S^1 on an open dense subset of $\mathrm{Hom}(\pi_1(\Sigma^g), G)$. Given a simple closed curve C in Σ^g, and a basepoint $*$, we may choose an arc δ joining $*$ to a point $x \in C$; then $[C] = [\delta \circ C \circ \delta^{-1}] \in \pi_1(\Sigma^g, *)$ is canonically defined up to conjugation by $\pi_1(\Sigma^g, *)$. We may then define open dense sets $\tilde{U}_C = \{\phi \in \mathrm{Hom}(\pi_1(\Sigma^g), G) \mid \phi([C]) \neq \pm 1\}$, and $U_C \subset \mathscr{S}_g$ is defined as the image of $\tilde{U}_C$ in $\mathscr{S}_g$. Then we define $\zeta_t^C(\phi) \in G$, for $t \in \mathbb{R}$ and $\phi \in \tilde{U}_C$, by[2]

$$\zeta_t^C(\phi) = \exp 4\pi^2 \{t F_\theta(\phi([C]))\}, \tag{5.3}$$

where $F_\theta: G \to \mathfrak{g}$ is the G-equivariant function defined above.

In defining the twist flow, we will deal first with the case where C is a *nonseparating curve* in Σ^g, i.e., $\Sigma^g - C$ is connected. Suppose $C \subset \Sigma^g$ is a simple closed oriented curve in Σ^g which does not separate Σ^g. There exists another (oriented) simple closed curve $B \subset \Sigma^g$ which intersects C once transversely with positive intersection number. The fundamental group $\pi_1(\Sigma^g)$ is then generated by the two subgroups $\pi_1(\Sigma^g - C)$ and $\langle [B] \rangle$, with the relation $[B] A_+ [B]^{-1} A_-^{-1}$, where A_+, A_- are the elements of $\pi_1(\Sigma^g - C)$ whose image in $\pi_1(\Sigma^g)$ is $[C]$.

We then define a flow on $\tilde{U}_C$ by the map $\tilde{\Xi}_t^C: \tilde{U}_C \to \tilde{U}_C$ given by

$$\begin{aligned} \tilde{\Xi}_t^C(\phi)(\alpha) &= \phi(\alpha) \quad \text{for} \quad \alpha \in \pi_1(\Sigma^g - C), \\ \tilde{\Xi}_t^C(\phi)([B]) &= \phi([B]) \zeta_t^C(\phi). \end{aligned} \tag{5.4}$$

---

[1] We denote by $\mathrm{diag}(a, b)$ the matrix $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

[2] We normalize the symplectic form $\tilde{\omega}$ on $\mathscr{A}$ as follows:

$$\tilde{\omega}(a, b) = \frac{1}{4\pi^2} \int_\Sigma \mathrm{Tr}(a \wedge b),$$

where $a, b \in T\mathscr{A}|_A = \Omega^1(\Sigma^g) \otimes \mathfrak{g}$; this differs from Goldman's normalization, which omits the factor $4\pi^2$. In our normalization, $\tilde{\omega}$ gives rise to a class in the *integer* cohomology group $H^2(\mathscr{S}_g, \mathbb{Z})$. The discrepancy between our normalization and Goldman's explains the difference in normalization between our formulas and those given in [4]

There is a similar formula if C is a *separating curve*; let Σ_1, Σ_2 be the two components of $\Sigma^g - C$. The fundamental group of $\pi_1(\Sigma^g)$ is then generated by $\pi_1(\Sigma_1)$ and $\pi_1(\Sigma_2)$, amalgamated over the subgroup generated by $[C]$. We define a flow on $\mathrm{Hom}(\pi_1(\Sigma^g), G)$ by the map $\tilde{\Xi}_t^C : \tilde{U}_C \to \tilde{U}_C$ given by

$$\begin{aligned} \tilde{\Xi}_t^C(\phi)(\alpha) &= \phi(\alpha) \quad \text{for} \quad \alpha \in \pi_1(\Sigma_1), \\ \tilde{\Xi}_t^C(\phi)(\alpha) &= \zeta_t^C(\phi)\phi(\alpha)\zeta_t^C(\phi)^{-1} \quad \text{for} \quad \alpha \in \pi_1(\Sigma_2). \end{aligned} \tag{5.5}$$

The following is a summary of the results of Theorems 1.10, 4.3, 4.5, and 4.7 of [4].

Theorem 5.1 [Goldman]. *The flow $\tilde{\Xi}_t^C$ on $\tilde{U}_C$ covers the Hamiltonian flow on U_C associated to the function $\theta_C : \bar{\mathcal{S}}_g \to \mathbb{R}$ defined in* (2.2).

It is now apparent from (5.2), (5.3), (5.4), and (5.5) that the Hamiltonian flow associated to the function $f = \theta_C$ has period $1/\pi$ if C is a nonseparating curve, and period $1/2\pi$ if C is a separating curve. More generally, if we multiply the symplectic form by $k \in \mathbb{Z}$, we see that the functions $k\theta_C/\pi$ (if C is nonseparating) and $k\theta_C/2\pi$ (if C is separating) have Hamiltonian flows with period 1. Corollary 3.4 gives the condition for these Hamiltonian vector fields to be linearly independent.

A trinion decomposition of Σ^g determines a set $\{C_i\}$ of $3g-3$ closed oriented curves on Σ^g. Hence it gives rise to a collection of $3g-3$ flows given by $\tilde{\Xi}_t^{C_i}$ on $\tilde{U}_{C_i}$, and corresponding flows we denote by $\Xi_t^{C_i}$ on $U_{C_i} = U_i = \theta_i^{-1}((0, \pi))$. Since the Hamiltonian functions θ_i are constant on the fibres of π, their Hamiltonian flows also preserve the polarization given by $\pi : \bar{\mathcal{S}}_g \to B_g$. Further, the functions $f_i = \cos\theta_i$ (and hence also the θ_i) Poisson commute: this property was shown in [4] and by a different method in [15], and was used to construct the polarization π. Hence the flows generated by the θ_i commute.

In summary, then, we have the following:

Proposition 5.2. *Let C_i, $i = 1, \ldots, 3g-3$ be the curves defined by the boundary circles of a trinion decomposition for Σ^g. Then there exist $3g-3$ functions h_i defined by $h_i = k\theta_{C_i}/\pi : \bar{\mathcal{S}}_g \to [0, k]$. The Hamiltonian flow of h_i is defined on the open dense subset $U_i \subset \bar{\mathcal{S}}_g$; it has period 1 if C_i is a nonseparating curve, and period 1/2 if C_i is a separating curve. The functions h_i, h_j Poisson commute on $U_i \cap U_j$. Furthermore, the corresponding Hamiltonian vector fields are linearly independent whenever the inequalities* (3.4) *corresponding to all trinions are strict inequalities.* □

5.2. The Period Lattice in $\bar{\mathcal{S}}_g$

We have seen above that the Bohr-Sommerfeld set is characterized by integer values of a set of functions generating the period lattice. We must, however, point out a subtlety arising in finding such a set of generators. Certainly, our functions $h_i = k\theta_i/\pi$ are in the lattice, but they do not form a set of generators. There is an additional set of flows of period 1, namely those given by the functions

$$g_\gamma = (h_{i_1(\gamma)} + h_{i_2(\gamma)} + h_{i_3(\gamma)})/2, \quad \gamma \in \Gamma, \tag{5.6}$$

and defined on the dense open subset $U_\gamma = U_{i_1(\gamma)} \cap U_{i_2(\gamma)} \cap U_{i_3(\gamma)}$ of $\bar{\mathcal{S}}_g$; here $i_1(\gamma)$, $i_2(\gamma)$, $i_3(\gamma)$ label the boundary components of a trinion D_γ. To see that the flows associated to these functions have period 1, it will be necessary to understand the relationship between the Goldman flows and the description of the fibration in Sect. 2.3 in terms of connections over trinions.

Proposition 5.3. *Let A be a flat a.t.d. connection on Σ^g whose holonomy (starting at a point $(*,1)\in\Sigma^g\times G$) is a representation $\phi:\pi_1(\Sigma^g,*)\to G$ (where $*$ is the basepoint of the fundamental group). Let C_j be a boundary circle in a trinion decomposition, and suppose that the holonomy angle θ_j corresponding to ϕ is in $(0,\pi)$. Then the map $S^1\to\mathscr{S}_g$ given by the Goldman twist flow $\Xi_t^{C_j}(\phi)$ in the fibre $\pi^{-1}(x)$ containing the conjugacy class of ϕ is the same as the map ψ_A embedding S^1 as the copy of T corresponding to the boundary circle C_j (given in Theorem 2.5): that is,*

$$\Xi_t^{C_j}(\phi)=\psi_A(\tau_j(t)),$$

where $\tau_j(t)\in\prod_i H_i$ is given by

$$(\tau_j(t))_l=1 \qquad (l\neq j),$$
$$(\tau_j(t))_j=e^{2\pi^2 it}.$$

Proof. For simplicity we treat the case when the boundary loop C_j is nonseparating. The proof when C_j is separating is similar.

The Goldman flow (5.4) is given in terms of the evolution in t of the holonomy (around certain curves in Σ^g) of a one parameter family of flat connections with parameter t. In accordance with Sect. 2.3, we consider the fibre $\pi^{-1}(x)$ to consist of the family of gauge equivalence classes of flat connections A_τ corresponding (under the map ψ_A of Theorem 2.5) to elements $\tau\in\prod_l H_l$. We examine the evolution in t of the holonomies of this family of flat connections as one component τ_j [given by $e^{2\pi^2 it}\in U(1)=T$] varies in S^1.

Let β be a curve that intersects C_j once transversely. We want to compute the holonomy of a connection A_τ around β. We take the basepoint to be a point $*$ near C_j inside the trinion γ_n, as in Fig. 2. As shown in the figure, the curve β decomposes as

$$\beta=\lambda_1\circ\lambda_2\circ\ldots\circ\lambda_n,$$

where the λ_i, $i=1,\ldots,n$, are arcs each of which lies within one trinion γ_i. The parallel transport with respect to the flat connection $A(\theta_{i_1}(\gamma),\theta_{i_2}(\gamma),\theta_{i_3}(\gamma))$ along a (non-closed) curve λ within a trinion γ is well-defined as an element in G, since we

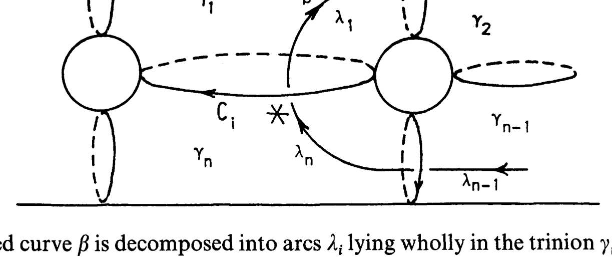

Fig. 2. The closed curve β is decomposed into arcs λ_i lying wholly in the trinion γ_i, as in the proof of Proposition 5.3

are working with a fixed trivialization of the principal bundle $G \times \Sigma^g \to \Sigma^g$. With respect to this choice of trivialization, parallel transport along an arc λ_r using the connection A corresponds to left multiplication by an element $\varrho_r \in G$. Likewise, parallel transport along λ_r using A_τ is left multiplication by $\varrho_r \tau_{i_r}$, where C_{i_r} is the boundary circle shared by γ_r and γ_{r-1}.

Since

$$\beta = \lambda_1 \circ \lambda_2 \circ \ldots \circ \lambda_n ,$$

we have

$$\mathrm{hol}_\beta A_\tau = \varrho_n \tau_{i_n} \ldots \tau_{i_2} \varrho_1 \tau_{i_1} . \tag{5.7}$$

We also see that if α is a closed loop not intersecting $C_j = C_{i_1}$, then $\mathrm{hol}_\alpha A_\tau$ is given by an expression similar to (5.7), but one which does not involve $\tau_{i_1} = \tau_j$. So a variation in τ_j does not change the holonomy around such loops α.

The set of possible τ_j is the image $\psi_A(1 \times \ldots \times H_j \times \ldots \times 1)$ of the group $H_j = T$ under $\psi_A : \prod_j H_j \to \pi^{-1}(x)$ (see Theorem 2.5). We may now compare (5.7) with the formula (5.4) for the Goldman flow, to see that the image of H_j indeed corresponds to the closed loop given by the Goldman twist flow $\Xi_t^{C_j}$ for $0 \leqq t \leqq 1/\pi$. □

5.3. The Fundamental Group of the Fibre

We now confirm that

Proposition 5.4. *Let $x \in B_g^{s,\mathrm{ind}}$. The fundamental group of the fibre $\pi^{-1}(x)$ is generated by the closed loops given by the period 1 flows of h_i and g_γ.*

(Recall $B_g^{s,\mathrm{ind}}$ was defined at the end of Sect. 3.)

Proof. We have shown that $\pi^{-1}(x)$ is isomorphic to $T^{3g-3}/(\mathbb{Z}_2)^{2g-2}$ (Proposition 3.7). This is the same as $\mathbb{R}^{3g-3}/\Lambda$, where Λ is the lattice generated by the usual basis vectors e_i $(i = 1, \ldots, 3g-3)$ for $\mathbb{R}^{3g-3}$ and by $f_\gamma = \frac{1}{2}(e_{i_1(\gamma)} + e_{i_2(\gamma)} + e_{i_3(\gamma)})$, $\gamma \in \Gamma$. We have already (Proposition 5.3) identified the loop te_i $(0 \leqq t \leqq 1)$ with the loop arising from the time 1 Hamiltonian flow of h_i. Thus by definition of $g_\gamma = \frac{1}{2}(h_{i_1(\gamma)} + h_{i_2(\gamma)} + h_{i_3(\gamma)})$, the loop tf_γ arises from the time 1 Hamiltonian flow of g_γ. □

To find the Bohr-Sommerfeld orbits it suffices to verify that there are points $x_i, x_\gamma \in B_g$ for which we can construct covariant constant sections of $\mathscr{L}^k$ over the loops in $\pi^{-1}(x_i), \pi^{-1}(x_\gamma)$ arising from the Hamiltonian flows of h_i and g_γ, and for which the functions h_i and g_γ take integral values on $\pi^{-1}(x_i)$ and $\pi^{-1}(x_\gamma)$, respectively. This will be the result of Proposition 7.1. Once we have achieved that result, we see that the hypotheses of Theorem 4.4 are satisfied, and hence that the Bohr-Sommerfeld points are those points in B_g for which h_i and g_γ have integer values. In other words, we have

Proposition 5.5. *Let x be a point in $B_g^{s,\mathrm{ind}}$. Then x is a Bohr-Sommerfeld point if and only if, for any $y \in \pi^{-1}(x)$,*

$$h_i(y) = l_i \in \mathbb{Z} , \tag{5.8}$$

where l_i is even if C_i is a separating loop; and

$$g_\gamma(y)=\tfrac{1}{2}\{l_{i_1(\gamma)}+l_{i_2(\gamma)}+l_{i_3(\gamma)}\}\in\mathbb{Z} \tag{5.9}$$

for all boundary circles C_i and trinions D_γ.

Finally, we shall need the following observation characterizing the subspace $\mathcal{T}_g$ of reducible connections:

Proposition 5.6. *Suppose $y\in\bar{\mathcal{S}}_g^s$ (as defined at the end of Sect. 2.2). Then y corresponds to a conjugacy class of reducible representations of $\pi_1(\Sigma^g)$ if and only if, for each trinion D_γ, one of Eqs. (3.5) is satisfied.*

Proof. The "only if" part is obvious. For the "if" part, Eqs. (3.5) guarantee that the flat connection A_γ representing y on the trinion D_γ is abelian, i.e., it takes values in $\mathfrak{t}$. Moreover, as none of the θ_i are 0 or π, the stabilizer groups H_j from Sect. 2.3 are copies of T. Thus also the gauge transformations ζ_γ from Sect. 2.3 may be chosen to have values in T. Hence the restrictions of the flat connection A_τ to the trinions D_γ, $A_\tau|_\gamma=A_\gamma^{\zeta_\gamma}$, all take values in $\mathfrak{t}$: in other words, A_τ is reducible. □

Corollary 5.7. *For $y\in\mathcal{T}_g\cap\bar{\mathcal{S}}_g^s$, the Hamiltonian flow of h_i at y preserves the subspace $\mathcal{T}_g$ of reducible connections.*

Proof. $\mathcal{T}_g$ is defined in a neighbourhood of y by the vanishing of a set of functions $f_\gamma, \gamma=1,\dots,2g-2$, where for each γ, either f_γ is $\theta_{i_1(\gamma)}-\theta_{i_2(\gamma)}+\theta_{i_3(\gamma)}$ (or a permutation of i_1,i_2,i_3 in this), or else $f_\gamma=\theta_{i_1(\gamma)}+\theta_{i_2(\gamma)}+\theta_{i_3(\gamma)}-2\pi$. But then h_i Poisson commutes with all the f_γ, so that its Hamiltonian flow preserves their zero locus $\mathcal{T}_g$. □

6. Singular Bohr-Sommerfeld Fibres

In this section we consider those nongeneric points $x\in\bar{\mathcal{S}}_g$ where some of the Hamiltonian flows degenerate. In this case the fibres containing x may not be tori, but by Theorem 2.5 have the form of products of tori with factors of G, $G/Z(G)$, and G/T. In this section we study the holonomy of the line bundle $\mathcal{L}^k$ on these fibres and show that naïve extrapolation of the Bohr-Sommerfeld rule derived for fibres which are tori suffices to characterize the singular Bohr-Sommerfeld fibres also.

The fibres may degenerate for two reasons. First, if some θ_i take values 0 or π on a fibre, the Hamiltonian flow $\Xi_t^{C_i}$ is not defined on that fibre. This is because the function θ_i is continuous but not smooth at these points. We may see this behaviour explicitly from the fact that Goldman's function $F_\theta: G\to\mathfrak{g}$ defined by (5.1) is constant with the value $\pm\operatorname{diag}(i/2, =i/2)$ on each component of $T-\{\pm 1\}$ but takes a different value on each maximal torus, determined by the requirement that the function F_θ must be equivariant under the action of G by conjugation. It thus cannot be defined on the central elements, which belong to every maximal torus.

Second, even if all the flows are defined, there will be some values of θ_i for which the flows degenerate, i.e., the $3g-3$ Hamiltonian vector fields are no longer linearly independent. This phenomenon is well known in the study of global actions of the torus T^m on a compact symplectic manifold M of dimension $2m$ [1, 5]. In this situation, the image of the moment map is a convex polyhedron cut out by hyperplanes in $\mathbb{R}^m$, and the flows degenerate on the boundary: the number of linear constraints satisfied by the flows in the fibre over $p \in \mathbb{R}^m$ is equal to the number of boundary hyperplanes containing p. (This follows from Proposition 3.3 above.) Fibres where only this type of degeneration occurs will be tori of dimension lower than m.

In our case, at those points lying in the domains of definition of all the Hamiltonian flows [that is, where all the θ_i lie in $(0, \pi)$] the boundary hyperplanes are given by the equalities corresponding to the inequalities (3.4). They correspond to trinions where the flat connection restricts to an abelian connection.

Because of these degenerations, we do not have a polarization in the strict sense, since the fibres of a genuine polarization should all be smooth manifolds and indeed be tori whose dimension is half the dimension of the symplectic manifold. A similar difficulty was encountered by the authors of [6] in quantizing flag manifolds: to get the correct dimension for the Hilbert space arising from quantization, they found they had to include certain degenerate orbits corresponding to integer values of the action variables. In the present case, we shall also include such orbits in our count of the Bohr-Sommerfeld fibres.

Justification for the inclusion of these extra fibres is provided by the fact that these degenerate fibres of the polarization do admit a global covariant constant section of $\mathscr{L}^k$. Although some Hamiltonian flows degenerate or are not defined, the fundamental group of the fibre degenerates correspondingly. Thus, for fibres of this type where the action variables take integer values, the holonomy representation of the connection on $\mathscr{L}^k$ still sends the entire fundamental group of the fibre to 1.

An important example is the fibre consisting of those points x where $\theta_1(x) = \ldots = \theta_{3g-3}(x) = 0$, which corresponds to those flat connections on Σ^g which extend as flat connections over a handlebody bounding the surface Σ^g. This fibre is "maximally degenerate" in that it is not in the domain of definition of any of our Hamiltonian flows. However, we can explicitly construct a global covariant constant section over this fibre using the Chern-Simons functional, as described below in Sect. 7. This fibre will be of central importance in the companion paper [9].

Our result is

Theorem 6.1. *Let* $x \in B_g$.
(a) *The first homology group (with coefficients in* $\mathbb{Z}$*) of the fibre* $\pi^{-1}(x)$ *is given by*

$$H_1(\pi^{-1}(x)) \simeq \mathbb{Z}^a \oplus \mathbb{Z}_2^b .$$

Here, the free summand may be taken to be generated by the trajectories of the Hamiltonian flows of those functions h_i *for which* $x \in \pi(U_i)$ *and of those* g_γ *for which* $x \in \pi(U_\gamma)$.
(b) *If in addition* $h_i(x)$ *and* $g_\gamma(x)$ *are integral for all* i *and* γ, *with* $h_i(x)$ *an even integer whenever* C_i *is a separating loop (as in Proposition 5.5), the image of the torsion subgroup* $\mathbb{Z}_2^b$ *vanishes in the holonomy representation of* $\pi_1(\pi^{-1}(x))$ *corresponding to the line bundle* $\mathscr{L}^k$.

Remark. Because the holonomy of a line bundle is a homomorphism from the fundamental group into $U(1)$, the holonomy representation of the fundamental group reduces to a representation of the first homology group of the fibre. It will therefore suffice to study the image in this representation of generators of the first homology group in order to verify condition 1′ of Theorem 4.4.

Proof of Theorem 6.1. We shall first prove the result for the case when no boundary circles C_i have the corresponding boundary holonomies θ_i equal to 0 or π; we shall then reduce the general case to this case.

Proposition 6.2. *The conclusion of Theorem 6.1 is true for fibres $\pi^{-1}(x)$, where $x \in B_g^s$.*

Proof. If $x \in \pi(U_i)$ for all i, we obtain the result by Theorem 2.5. Recall that in Sect. 4.2, flat connections A_τ corresponding to points in the fibre above x were formed by gluing together flat connections $A(\theta_{i_1(\gamma)}(x), \theta_{i_2(\gamma)}(x), \theta_{i_3(\gamma)}(x)) = A_\gamma$ on trinions D_γ along the boundary components of the trinions. We must simply extend the construction given in Theorem 2.5 to allow for the possibility that on some trinion D_γ, the flat connection A_γ is reducible: this happens whenever $(\theta_{i_1(\gamma)}(x), \theta_{i_2(\gamma)}(x), \theta_{i_3(\gamma)}(x))$ is not an interior triple, i.e., whenever one of the inequalities (3.4) becomes an equality. Then the stabilizer of A_γ is T. [Since we are assuming all $\theta_i(x) \in (0, \pi)$, A_γ cannot correspond to a representation into $\mathbb{Z}_2$.]

Suppose there are a trinions for which the flat connection A_γ has stabilizer T. By Theorem 2.5, the fibre $\pi^{-1}(x)$ is then $T^{3g-3}/\{T^a \times (\mathbb{Z}_2)^{2g-2-a}\}$, where the action of $T^a \times (\mathbb{Z}_2)^{2g-2-a}$ is by

$$\tau_i \to \Phi_{\gamma'(i)} \tau_i \Phi_{\gamma(i)}^{-1} .$$

Here, $\tau_i \in T$ denotes an element in the i^{th} copy of T in T^{3g-3}, the copy corresponding to the boundary circle C_i. Likewise, $\Phi_{\gamma(i)}, \Phi_{\gamma'(i)}$ denote the elements of T or $\mathbb{Z}_2$ in $T^a \times (\mathbb{Z}_2)^{2g-2-a}$, where $\gamma(i), \gamma'(i)$ designate the two trinions bounding C_i.

The stabilizer group of the action of $T^a \times (\mathbb{Z}_2)^{2g-2-a}$ on T^{3g-3} is thus

$$\mathrm{Stab}[A_\tau] = \{\{\Phi_\gamma\} : \Phi_{\gamma(i)} = \Phi_{\gamma'(i)} \text{ for all } i\} ;$$

in other words, it is T if the flat connections on all trinions are abelian, and $Z(G)$ otherwise. Thus the fundamental group of the fibre is $\mathbb{Z}^{3g-3-a+1}$ if the flat connections on all trinions are abelian, and $\mathbb{Z}^{3g-3-a}$ otherwise.

By Proposition 5.3, the Hamiltonian flow of the function h_i was identified with the action of T on T^{3g-3} given by multiplication on the i^{th} copy of T in T^{3g-3}. The number of linearly independent Hamiltonian vector fields from the torus action is also given by $3g-3-a+1$ (resp. $3g-3-a$) under the above hypotheses. For one begins with $3g-3$ vector fields, which are subject to one linear constraint from each trinion whose associated flat connection is abelian. This follows from Proposition 3.3: indeed, Propositions 3.3 and 3.4 say that the linear equations satisfied by the vector fields are the same as those (3.5) satisfied by the coordinates θ_i. If the number of constraints a is less than $2g-2$, then the constraints are linearly independent: indeed, by considering the trivalent graph one sees that each additional constraint equation introduces a new coordinate θ_i not involved in the previous equations. However, if $a = 2g-2$ (i.e., if the flat connections on *all* trinions are abelian), then since the flat connections A_γ being glued have structure group T, the global connections we obtain by gluing are T connections (see Proposition 5.6). Now the flows of the Hamiltonian functions h_i preserve the subspace $\mathscr{T}_g$ of reducible connections (see Corollary 5.7). In fact, they give g linearly independent Hamiltonian vector fields, so the number of linearly independent vector fields is indeed $g = 3g-3-(a-1)$ in this case, as needed. □

In the following, we shall find a relation between fibres in the polarization of $\bar{\mathscr{S}}_g$ where certain flows degenerate and fibres of moduli spaces corresponding to surfaces (with boundary) of lower genus. To keep track of the corresponding coordinates $\underline{\theta}$, it will be helpful to introduce the following

Definition 6.3. Let Σ^g be a surface (possibly with several boundary components) equipped with a trinion decomposition. A labelling of the trinion decomposition by holonomy angles $\underline{\theta}$ is a set of elements $\theta_j \in [0, \pi]$ associated to each boundary circle C_j in the trinion decomposition (including the components of $\partial \Sigma^g$).

If Σ^g is a closed surface, a point $x \in B_g$ with coordinates $\underline{\theta}(x)$ gives rise to a labelling of a trinion decomposition by holonomy angles $\underline{\theta}(x)$.

We now reduce the proof of Theorem 6.1 to the case proved above, where $\theta_i(x) \in (0, \pi)$. To do so we relate the fibre of our polarization above a point where some of the $\theta_i(x)$ are 0 or π to a product of fibres of polarizations of lower genus surfaces. These surfaces Σ_α are obtained from Σ^g by cutting Σ^g along those boundary circles C_i marked 0 or π. In doing this we will use the following notation:

Let $v: \mathrm{Hom}(\pi_1(\Sigma^g), G) \to \bar{\mathscr{S}}_g$ denote the projection to the quotient under the conjugation action, and let the function $\tilde{\theta}_i: \mathrm{Hom}(\pi_1(\Sigma^g), G) \to \mathbb{R}$ be defined by $\tilde{\theta}_i = \theta_i \circ v$.

1. $\Sigma_\alpha \subset \Sigma^g$ is a surface with several boundary components $C_j^{(\alpha)}$, ε_α is a map from the set of boundary components of Σ_α to $\{0, 1\}$, and the boundary component $C_j^{(\alpha)}$ is labelled by the holonomy angle $\pi\varepsilon_\alpha(j)$. We also define $\mathrm{Hom}^{(\varepsilon_\alpha)}(\pi_1(\Sigma_\alpha), G) = \{\varrho \in \mathrm{Hom}(\pi_1(\Sigma_\alpha), G) \mid \varrho([C_j^{(\alpha)}]) = (-1)^{\varepsilon_\alpha(j)} \text{ for all } j\}$.
2. Let $\tilde{\theta}_{\alpha, l}: \mathrm{Hom}^{(\varepsilon_\alpha)}(\pi_1(\Sigma_\alpha), G) \to [0, \pi]$ be the function given by the holonomy angle around the boundary circle C_l.
3. Let $H_\alpha^{(\varepsilon_\alpha)}$ denote the subset of representations in $\mathrm{Hom}^{(\varepsilon_\alpha)}(\pi_1(\Sigma_\alpha), G)$ for which all of the functions $\tilde{\theta}_{\alpha, l}$ corresponding to boundary circles in the interior of Σ_α take values in $(0, \pi)$.
4. Let $B_\alpha^{(\varepsilon_\alpha)}$ denote the base space corresponding to $H_\alpha^{(\varepsilon_\alpha)}$; that is, $B_\alpha^{(\varepsilon_\alpha)}$ is the image of $H_\alpha^{(\varepsilon_\alpha)}$ under the map $(f_i)_{i \in \mathscr{C}_\alpha}$ corresponding to the subset $\mathscr{C}_\alpha \subseteq \mathscr{I}$ corresponding to those boundary circles C_i which are in Σ_α.
5. Let $v_\alpha^{(\varepsilon_\alpha)}: H_\alpha^{(\varepsilon_\alpha)} \to H_\alpha^{(\varepsilon_\alpha)}/G$ and $\pi_\alpha^{(\varepsilon_\alpha)}: H_\alpha^{(\varepsilon_\alpha)}/G \to B_\alpha^{(\varepsilon_\alpha)}$ be the natural projection maps.
6. Let $x_\alpha \in B_\alpha^{(\varepsilon_\alpha)}$ be a point whose coordinates are the values of the functions $f_j(x_\alpha)$ corresponding to the boundary circles $C_j \in \mathscr{C}_\alpha$.
7. Let $H_\alpha^{(\varepsilon_\alpha)}[x_\alpha]$ denote $(\pi_\alpha^{(\varepsilon_\alpha)} \circ v_\alpha^{(\varepsilon_\alpha)})^{-1}(x_\alpha) \subset H_\alpha^{(\varepsilon_\alpha)}$.

Now all of the results developed in Sects. 2, 3, and 5 about Hamiltonian flows have a straightforward generalization to moduli spaces of representations of fundamental groups of surfaces with boundary, of the type considered above. This can be seen directly as in [10]. Alternatively, we note that the spaces $H_\alpha^{(\varepsilon_\alpha)}[x_\alpha]$, $H_\alpha^{(\varepsilon_\alpha)}$ and their quotients by the action of G are all subspaces of the spaces $\mathrm{Hom}(\pi_1(\Sigma^g), G)$ and its quotient $\bar{\mathscr{S}}_g$, and that these subspaces are preserved by the flows corresponding to the functions h_i and g_γ. Thus these flows give rise to flows on $H_\alpha^{(\varepsilon_\alpha)}[x_\alpha]$, $H_\alpha^{(\varepsilon_\alpha)}$ and their quotients by G, which can in fact be seen to correspond to the Hamiltonian flows of the functions constructed from the $\tilde{\theta}_{\alpha, l}$. We will refer to these flows, by abuse of language, as the flows defined by the functions h_i and g_γ; note that when C_i is in the interior of Σ_α, the flow corresponding to h_i is defined everywhere on $H_\alpha^{(\varepsilon_\alpha)}[x_a]$, $H_\alpha^{(\varepsilon_\alpha)}$ and their quotients by G.

Lemma 6.4. *Let C_i be a nonseparating boundary curve in a surface Σ_α with N boundary components, and consider* $\tilde{\theta}_{\alpha, i}: \mathrm{Hom}^{(\varepsilon_\alpha)}(\pi_1(\Sigma_\alpha), G) \to [0, \pi]$. If $\varepsilon = 0, 1$ *then* $\tilde{\theta}_{\alpha, i}^{-1}(\varepsilon\pi) = G \times \mathrm{Hom}^{(\varepsilon_\beta)}(\pi_1(\Sigma_\beta), G)$, *where Σ_β is the surface formed by cutting Σ_α along C_i, and the map ε_β is obtained from the map ε_α by extending it so it sends the two new boundary circles to ε.*

Proof. Without loss of generality the curve C_i may be assumed to be the generator a_1 in the standard set of generators a_i, b_i, d_j for $\pi_1(\Sigma_\alpha)$, satisfying the relation $\left(\prod_i a_i b_i a_i^{-1} b_i^{-1}\right) d_1 \dots d_N = 1$. Let ϱ be a representation of $\pi_1(\Sigma_\alpha)$ in G. Then if $\varrho(a_1)$ is central, the condition that ϱ be a representation of $\pi_1(\Sigma_\alpha)$ reduces to the condition $\varrho\left(\left(\prod_{i \neq 1} a_i b_i a_i^{-1} b_i^{-1}\right) d_1 \dots d_N\right) = 1$. □

Suppose, on the other hand, that C_i is a boundary circle in a trinion decomposition of Σ_α, which separates Σ_α into surfaces Σ_{β_1} and Σ_{β_2} of genus g_1 and g_2 with N_1+1 (resp. N_2+1) boundary components. Denote by δ the element of $\pi_1(\Sigma_\alpha)$ corresponding to C_i. Then

$$\pi_1(\Sigma_\alpha) = \pi_1(\Sigma_{\beta_1}) *_\delta \pi_1(\Sigma_{\beta_2}),$$

where $\pi_1(\Sigma_{\beta_i})$ have standard generators $a_1^{(i)}, \dots, a_{g_i}^{(i)}, b_1^{(i)}, \dots, b_{g_i}^{(i)}, d_1^{(i)}, \dots, d_{N_i}^{(i)}, \delta$, and the amalgamated product is defined by taking the free group on these generators and imposing the relations[3]

$$\left(\prod_j a_j^{(1)} b_j^{(1)} (a_j^{(1)})^{-1} (b_j^{(1)})^{-1}\right) d_1^{(1)} \dots d_{N_1}^{(1)} = \delta,$$

$$\left(\prod_j a_j^{(2)} b_j^{(2)} (a_j^{(2)})^{-1} (b_j^{(2)})^{-1}\right) d_1^{(2)} \dots d_{N_2}^{(2)} = \delta^{-1}.$$

Thus we have

Lemma 6.5. *Suppose $\varepsilon = 0, 1$ and suppose C_i is a separating boundary loop in Σ_α. Then $\tilde{\theta}_{\alpha,i}^{-1}(\varepsilon\pi) = \{\varrho \in \mathrm{Hom}^{(\varepsilon_\alpha)}(\pi_1(\Sigma_\alpha), G) : \varrho(\delta) = (-1)^\varepsilon\}$ satisfies*

$$\tilde{\theta}_{\alpha,i}^{-1}(\varepsilon\pi) = \mathrm{Hom}^{(\varepsilon_{\beta_1})}(\pi_1(\Sigma_{\beta_1}), G) \times \mathrm{Hom}^{(\varepsilon_{\beta_2})}(\pi_1(\Sigma_{\beta_2}), G),$$

where $\varepsilon_{\beta_1}, \varepsilon_{\beta_2}$ are obtained from the restrictions of the map ε_α to the subsets of boundary components of Σ_α in $\Sigma_{\beta_1}, \Sigma_{\beta_2}$, by extending them to send C_i to ε.

Remark. The identifications given in the previous two lemmas equate the values of the functions $\tilde{\theta}_l$ on elements of $\mathrm{Hom}(\pi_1(\Sigma^g), G)$ with the values of the functions $\tilde{\theta}_{\alpha,l}$ on corresponding elements of $\mathrm{Hom}^{(\varepsilon_\alpha)}(\pi_1(\Sigma_\alpha), G)$: thus the fibre $\bigcap_l \tilde{\theta}_{\alpha,i}^{-1}(y_l) \subset \mathrm{Hom}^{(\varepsilon_\alpha)}(\pi_1(\Sigma_\alpha), G)$ (for any values y_l) is identified with the corresponding fibres in $\mathrm{Hom}^{(\varepsilon_{\beta_1})}(\pi_1(\Sigma_{\beta_1}), G) \times \mathrm{Hom}^{(\varepsilon_{\beta_2})}(\pi_1(\Sigma_{\beta_2}), G)$ (if C_i is separating), or in $G \times \mathrm{Hom}^{(\varepsilon_\beta)}(\pi_1(\Sigma_\beta), G)$ (if C_i is nonseparating).

For any loop $C_i \subset \Sigma^g$, we obtain a new (possibly disconnected) surface by cutting Σ^g along C_i. We may cut Σ^g along all boundary circles C_i for which $\theta_i(x) = 0$ or $\theta_i(x) = \pi$, and obtain a collection of surfaces Σ_α, each with several boundary components, and each equipped with a trinion decomposition labelled by holonomy angles. All boundary circles C_j in the interior of Σ_α are labelled by holonomy angles $\theta_j \in (0, \pi)$, while all boundary circles C_j in $\partial\Sigma_\alpha$ are labelled by $\theta_j = 0$ or $\theta_j = \pi$.

Thus, by repeated application of Lemmas 6.4 and 6.5, we therefore relate our fibre $(\pi \circ \nu)^{-1}(x) \subset \mathrm{Hom}(\pi_1(\Sigma^g), G)$ to subspaces of $\mathrm{Hom}(\pi_1(\Sigma_\alpha), G)$, where each Σ_α is a surface with several boundary components. Each Σ_α is equipped with a trinion decomposition labelled by holonomy angles, with all boundary circles C_j in the interior of Σ_α labelled by holonomy angles $\theta_j \in (0, \pi)$, and for which all boundary

[3] Here we have chosen a basepoint in C_i

circles C_j in $\partial\Sigma_\alpha$ are labelled with $\theta_j=0$ or $\theta_j=\pi$. Thus, we relate our fibre $(\pi\circ v)^{-1}(x)\subset\mathrm{Hom}(\pi_1(\Sigma^g),G)$ to the product of a number of fibres $(\pi_\alpha^{(\varepsilon_\alpha)}\circ v_\alpha^{(\varepsilon_\alpha)})^{-1}(x_\alpha)$ $\subset\mathrm{Hom}^{(\varepsilon_\alpha)}(\pi_1(\Sigma_\alpha),G)$, where Σ_α are surfaces with trinion decompositions labelled by holonomy angles.

Summarizing, we have shown

Proposition 6.6. *Let Σ^g be a surface of genus g, and let $\mathscr{I}'$ be an arbitrary subset of $\mathscr{I}$. Let $\underline{\varepsilon}=(\varepsilon_i)_{i\in\mathscr{I}'}:\mathscr{I}'\to\{0,1\}$, and let $\mathrm{Hom}^{\underline{\varepsilon}}(\pi_1(\Sigma^g),G)\subset\mathrm{Hom}(\pi_1(\Sigma^g),G)$ denote the space of representations of $\pi_1(\Sigma^g)$ into G sending the boundary loops C_i, $i\in\mathscr{I}'$ (and only these), to $(-1)^{\varepsilon_i}=\pm1$. This space can be written in terms of representation spaces of surfaces of lower genus, as follows:*

1. *The representation space $\mathrm{Hom}^{\underline{\varepsilon}}(\pi_1(\Sigma^g),G)$ decomposes into a product of representation spaces of lower genus surfaces:*

$$\mathrm{Hom}^{\underline{\varepsilon}}(\pi_1(\Sigma^g),G)=\prod_{\alpha=1}^{N}H_\alpha^{(\varepsilon_\alpha)}\times G^{n_\alpha}\quad \textit{for some}\quad n_\alpha\in\mathbb{Z},\qquad n_\alpha\geqq0\,.$$

2. *The surfaces Σ_α are equipped with trinion decompositions labelled by holonomy angles, and there are functions $\tilde{\theta}_{\alpha,l}:H_\alpha^{(\varepsilon_\alpha)}\to(0,\pi)$ (for all boundary circles C_l in the interior of Σ_α) which generate Hamiltonian flows at all points on $H_\alpha^{(\varepsilon_\alpha)}/G$. Under the identification given in* (1), *there functions $\tilde{\theta}_{\alpha,l}$ agree with the functions $\tilde{\theta}_l$ defined on* $\mathrm{Hom}(\pi_1(\Sigma^g),G)$.
3. *Under the above identification of the coordinates on $\mathrm{Hom}^{\underline{\varepsilon}}(\pi_1(\Sigma^g),G)$ with those on lower genus surfaces, the fibres of the map $\pi\circ v$ decompose into fibres of the corresponding maps on lower genus surfaces, as in part* (1): *i.e., $(\pi\circ v)^{-1}(x)=\prod_\alpha(H_\alpha^{(\varepsilon_\alpha)}[x_\alpha]\times G^{n_\alpha})$.*

The surface Σ_α has a trinion decomposition labelled by holonomy angles, and there are no boundary circles (in the interior of Σ_α) labelled 0 or π: by Proposition 6.2 (and its analog for a surface with several boundary components labelled 0 or π) we thus also have that

Proposition 6.7. *The first homology group (with coefficients in $\mathbb{Z}$) of the fibre $H_\alpha^{(\varepsilon_\alpha)}[x_\alpha]/G$ is generated by the trajectories of the Hamiltonian flows of those functions h_i where C_i is in the interior of Σ_α and those g_γ where D_γ is in the interior of Σ_α.*

In completing the proof of Theorem 6.1, the following observation, which follows from Proposition 5.6, will be useful:

Lemma 6.8. *If $H_\alpha^{(\varepsilon_\alpha)}[x_\alpha]$ contains any abelian representations, then it consists entirely of abelian representations.*

It will be helpful also to have the following explicit characterization of the space $H_\alpha^{(\varepsilon_\alpha)}[x_\alpha]$:

Lemma 6.9. *The space $H_\alpha^{(\varepsilon_\alpha)}[x_\alpha]$ is of one of the following types:*
(a) *If $H_\alpha^{(\varepsilon_\alpha)}[x_\alpha]$ consists of representations into $\mathbb{Z}_2$, then Σ_α is a trinion (with all three boundary circles labelled by holonomy angles 0 or π) and $H_\alpha^{(\varepsilon_\alpha)}[x_\alpha]$ is a point.*
(b) *If $H_\alpha^{(\varepsilon_\alpha)}[x_\alpha]$ consists of reducible representations which are not representations into $\mathbb{Z}_2$, then $H_\alpha^{(\varepsilon_\alpha)}[x_\alpha]$ is of the form $G/T\times T^{m_\alpha}$.*
(c) *If $H_\alpha^{(\varepsilon_\alpha)}[x_\alpha]$ consists of irreducible representations, then $H_\alpha^{(\varepsilon_\alpha)}[x_\alpha]$ is of the form $G/\mathbb{Z}_2\times T^{m_\alpha}$.*

Proof. Part (a) follows immediately from the definition of $H_\alpha^{(\varepsilon_\alpha)}[x_\alpha]$.
(b) In this case, we have the fibration $G/T \to H_\alpha^{(\varepsilon_\alpha)}[x_\alpha] \to H_\alpha^{(\varepsilon_\alpha)}[x_\alpha]/G$. Here $H_\alpha^{(\varepsilon_\alpha)}[x_\alpha]/G$ is a torus (by Theorem 2.5). We may show this fibration is trivial by exhibiting a section of it, i.e., a G equivariant map $H_\alpha^{(\varepsilon_\alpha)}[x_\alpha] \to G/T$. Such a map is given by the restriction map $H_\alpha^{(\varepsilon_\alpha)}[x_\alpha] \to \mathrm{Hom}(\pi_1(D), G)$ to some trinion $D \subset \Sigma_\alpha$ chosen so that the representations in $H_\alpha^{(\varepsilon_\alpha)}[x_\alpha]$ do not restrict on $\pi_1(D)$ to representations into $\mathbb{Z}_2$: the image of this restriction map forms one orbit G/T of the action of G on $\mathrm{Hom}(\pi_1(D), G)$ by conjugation.
(c) The proof is the same as in (b) except that G/T is replaced by $G/\mathbb{Z}_2$, and D is a trinion such that the representations in $H_\alpha^{(\varepsilon_\alpha)}[x_\alpha]$ restrict on $\pi_1(D)$ to irreducible representations. □

We now use the decomposition theorem of Proposition 6.6 to complete the proof of Theorem 6.1. We do this by showing that certain maps induce isomorphisms on the free part of the integer homology of the spaces we consider. There are four such maps. The first map is simply the projection $v[x]:(\pi \circ v)^{-1}(x) \to \pi^{-1}(x)$. The second map is given by the identification of Proposition 6.6: this is a homeomorphism

$$m[x]:(\pi \circ v)^{-1}(x) \to \prod_{\alpha=1}^{N} H_\alpha^{(\varepsilon_\alpha)}[x_\alpha] \times G^{n_\alpha}.$$

The third map is the projection map $P_1[x]: \prod_{\alpha=1}^{N} H_\alpha^{(\varepsilon_\alpha)}[x_\alpha] \times G^{n_\alpha} \to \prod_{\alpha=1}^{N} H_\alpha^{(\varepsilon_\alpha)}[x_\alpha]$. Finally, we consider the projection map $v_\alpha^{(\varepsilon_\alpha)}[x_\alpha]: H_\alpha^{(\varepsilon_\alpha)}[x_\alpha] \to H_\alpha^{(\varepsilon_\alpha)}[x_\alpha]/G$.

Proposition 6.10. *The maps $v[x]$, $m[x]$, $P_1[x]$, and $v_\alpha^{(\varepsilon_\alpha)}[x_\alpha]$ induce isomorphisms on the free parts of homology groups with coefficients in $\mathbb{Z}$. In particular,*

$$H_1'(\pi^{-1}(x)) \simeq \bigoplus_\alpha H_1'(H_\alpha^{(\varepsilon_\alpha)}[x_\alpha]/G),$$

where the isomorphism carries the trajectories of the flows of the Hamiltonian functions h_i and g_γ on $\pi^{-1}(x)$ (where $x \in \pi(U_i)$ and $x \in \pi(U_\gamma)$, respectively) to their counterparts in $H_\alpha^{(\varepsilon_\alpha)}[x_\alpha]/G$ for appropriate α. (Here, the homology groups H_1 are of the form $\mathbb{Z}^a \oplus \mathbb{Z}_2^b$, as in the statement of Theorem 6.1: we denote by H_1' the $\mathbb{Z}^a$ summand.)

Proposition 6.10, taken along with the identification (Proposition 6.7) of the homology of the fibres $H_\alpha^{(\varepsilon_\alpha)}[x_\alpha]/G$ with the trajectories of the flows h_i and g_γ, allows us to identify the free parts of the first homology of the fibres in Theorem 6.1 as also generated by these trajectories. The torsion in $H_1(\pi^{-1}(x))$ will be dealt with in Proposition 6.12.

Proof of Proposition 6.10. The map $m[x]$ is a homeomorphism by Proposition 6.6; whereas $P_1[x]$ induces an isomorphism on H_1 since G is simply connected. The fact that $v[x]$ and $v_\alpha^{(\varepsilon_\alpha)}[x_\alpha]$ induce isomorphisms on the free parts of H_1 will follow by an argument using path lifting and the exact homotopy sequence of a fibration.[4] We may restrict our attention to the case where none of the spaces $H_\alpha^{(\varepsilon_\alpha)}[x_\alpha]$ that appear are of the type given in Lemma 6.9(a), since these $H_\alpha^{(\varepsilon_\alpha)}[x_\alpha]$ are

[4] The spaces in question all have abelian fundamental groups, as one sees from Lemma 6.9: thus exact homotopy sequences yield the results (6.1) and (6.2) as stated for their first homology groups

just points. We must then show that the maps

$$v_\alpha^{(\varepsilon_\alpha)}[x_\alpha]_* : H'_1(H_\alpha^{(\varepsilon_\alpha)}[x_\alpha]) \to H'_1(H_\alpha^{(\varepsilon_\alpha)}[x_\alpha]/G), \tag{6.1}$$

and

$$v[x]_* : H'_1((\pi \circ v)^{-1}(x)) \to H'_1(\pi^{-1}(x)) \tag{6.2}$$

are isomorphisms.

First we prove statement (6.1). This follows from the fibration

$$G/S \longrightarrow H_\alpha^{(\varepsilon_\alpha)}[x_\alpha] \xrightarrow{v_\alpha^{(\varepsilon_\alpha)}[x_\alpha]} H_\alpha^{(\varepsilon_\alpha)}[x_\alpha]/G, \tag{6.3}$$

where S is the stabilizer of the conjugation action of G on $H_\alpha^{(\varepsilon_\alpha)}[x_\alpha]$. By Lemma 6.8, S is either $\mathbb{Z}_2$ or T globally; thus $\pi_1(G/S) = \mathbb{Z}_2$ or 1, so $H_1(H_\alpha^{(\varepsilon_\alpha)}[x_\alpha], \mathbb{Z}) = H_1(H_\alpha^{(\varepsilon_\alpha)}[x_\alpha]/G, \mathbb{Z})$ up to factors of $\mathbb{Z}_2$.

Now we turn to the proof of (6.2). We introduce the notation [see item (3) of Proposition 6.6]

$$Y = (\pi \circ v)^{-1}(x) = \prod_\alpha H_\alpha^{(\varepsilon_\alpha)}[x_\alpha] \times G^{n_\alpha}. \tag{6.4}$$

The statement (6.2) follows from consideration of the map $Y \xrightarrow{v[x]} Y/G$ given by the quotient of Y by the conjugation action of G on Y. The inverse image $v[x]^{-1}(v[x](p))$ is G/S_p, where S_p is the stabilizer at p of the conjugation action on Y. Unless all $H_\alpha^{(\varepsilon_\alpha)}[x_\alpha]$ consist of reducible connections, S_p is $\mathbb{Z}_2$: in this case $Y \xrightarrow{v[x]} Y/G$ is a fibration and (6.2) follows from the associated exact homotopy sequence.

We thus have reduced the proof of (6.2) to the case where our fibre is such that all $H_\alpha^{(\varepsilon_\alpha)}[x_\alpha]$ consist of reducible connections. In this situation, the group S_p depends on the point $p \in Y$: S_p is still $\mathbb{Z}_2$ unless $p \in Y$ corresponds to a reducible connection on Σ^g, in which case $S_p = T$. Thus the map $v[x]$ is not a fibration in this case. Let us define $Y_{\text{reg}} = \{p \in Y : S_p = \mathbb{Z}_2\}$. Since the map $v[x]$ restricted to Y_{reg} is a fibration, the statement (6.2) will follow from the homotopy exact sequence of $v[x] : Y_{\text{reg}} \to Y_{\text{reg}}/G$ if we can show $\pi_1(Y) = \pi_1(Y_{\text{reg}})$ and $\pi_1(Y/G) = \pi_1(Y_{\text{reg}}/G)$. To check this, we need to prove

Lemma 6.11. *Let $Y = \prod H_\alpha^{(\varepsilon_\alpha)}[x_\alpha] \times G^{n_\alpha}$, where all the $H_\alpha^{(\varepsilon_\alpha)}[x_\alpha]$ consist of reducible connections. Then:*
(a) *Every loop $\sigma : S^1 \to Y$ is homotopic to a loop $\sigma' : S^1 \to Y_{\text{reg}}$ (and likewise with Y and Y_{reg} replaced by Y/G and Y_{reg}/G).*
(b) *If $f_1, f_2 : S^1 \to Y_{\text{reg}}$ are homotopic as maps $S^1 \to Y$ then they are homotopic as maps $S^1 \to Y_{\text{reg}}$ (and likewise with Y and Y_{reg} replaced by Y/G and Y_{reg}/G).*

Now under the assumption that $H_\alpha^{(\varepsilon_\alpha)}[x_\alpha]$ consists of reducible connections, we have [from Lemma 6.9(b)]

$$H_\alpha^{(\varepsilon_\alpha)}[x_\alpha] = T^{m_\alpha} \times G/T. \tag{6.5}$$

Hence Y becomes

$$Y = \prod_\alpha T^{m_\alpha} \times G/T \times G^{n_\alpha}, \tag{6.6}$$

where G acts on T^{m_α} trivially, on G/T by left multiplication, and on G^{n_α} by conjugation. Thus we have

$$\begin{aligned} Y/G &= \prod_{\alpha=1}^N T^{m_\alpha} \times \left(\prod_{\beta=1}^N G/T \times G^{n_\beta} \right) \Big/ G \\ &= \prod_\alpha T^{m_\alpha} \times ((G/T)^{N-1} \times G^A)/T, \end{aligned} \tag{6.7}$$

where $A = \sum_{\alpha} n_\alpha$ and T acts on G by conjugation while it acts on G/T by left multiplication.

To prove Lemma 6.11 for Y, note from (6.6) that Y is a smooth manifold, so the lemma follows provided the codimension of $Y - Y_{\text{reg}}$ is $\geqq 3$ (see [7, Propositions VII.12.4 and VII.12.6]). This is the case except in a few special cases (specifically, $Y = G \times H_\alpha^{(\varepsilon_\alpha)}[x_\alpha]$ or $H_\alpha^{(\varepsilon_\alpha)}[x_\alpha] \times H_\beta^{(\varepsilon_\beta)}[x_\beta]$, where $H_\alpha^{(\varepsilon_\alpha)}[x_\alpha]$, $H_\beta^{(\varepsilon_\beta)}[x_\beta]$ consist entirely of abelian representations, or $Y = G \times G$). For these special cases, (6.2) can be checked directly.

To prove Lemma 6.11 for Y/G, we need to modify the argument of [7, VII.12] so it applies in this case, since Y/G is not a smooth manifold; for this, we shall need a more careful analysis of Y/G near $(Y - Y_{\text{reg}})/G$. Define $Z = (G/T)^{N-1} \times G^A$: then either T acts with stabilizer $\mathbb{Z}_2$ at $z \in Z$, or z is a fixed point of the T action. The space Z/T is smooth wherever T acts with stabilizer $\mathbb{Z}_2$: to find a local model for neighbourhoods in Z/T near the fixed points of the T action, we look at the tangent space to Z at a point z in the manifold of fixed points $W^{N-1} \times T^A \subset Z$ [where $W = N(T)/T$ denotes the Weyl group]. We then have the identification

$$TZ|_z = \mathbb{R}^A \oplus \mathbb{C}^{N-1+A}, \tag{6.8}$$

where $e^{i\theta} \in T$ acts trivially on $\mathbb{R}$, and on $\mathbb{C}$ by multiplication by $e^{2i\theta}$.

By considering this linear action of T on $TZ|_z$, we see that a local model for a neighbourhood of $[z]$ in Z/T is

$$\mathcal{O} \cong TZ|_z/T = \mathbb{R}^A \times C(\mathbb{C}P^{N-2+A}), \tag{6.9}$$

where $C(\mathbb{C}P^{N-2+A}) = \mathbb{C}P^{N-2+A} \times [0, \infty)/(p, 0) \sim (p', 0)$ is the cone on $\mathbb{C}P^{N-2+A}$.

We are now ready to prove Lemma 6.11(a) for Y/G. We choose the basepoint for $\pi_1(Y/G)$ to lie in the regular locus $\prod_{\alpha} T^{m_\alpha} \times Z_{\text{reg}}/T$, where Z_{reg} is the set of points in Z, where the stabilizer of the T action is $\mathbb{Z}_2$. We must show that any loop $\sigma : S^1 \to Z/T$ can be deformed into Z_{reg}/T keeping the basepoint fixed: then Lemma 6.11(a) will follow from the exact homotopy sequence of the fibration

$$Y_{\text{reg}} \xrightarrow{v[x]} Y_{\text{reg}}/G = \prod_{\alpha} T^{m_\alpha} \times \mathbb{Z}_{\text{reg}}/T .$$

It suffices to prove (cf. [7, Proposition VII.12.4]) that a path σ in Z/T with $\sigma(0) = y_0$, $\sigma(1) = y_1 \in Z_{\text{reg}}/T$ lying entirely in an open neighbourhood of a point in $Z/T - Z_{\text{reg}}/T$ may be deformed (keeping y_0, y_1 fixed) to a path σ' in Z_{reg}/T. But we have seen in (6.9) that such a neighbourhood $\mathcal{O}$ is of the form $\mathcal{O} = \mathbb{R}^4 \times C(\mathbb{C}P^{N-2+A})$, which is in particular contractible. We thus denote our path by $\sigma = (\sigma_1, \sigma_2)$, where $\sigma_1 : [0,1] \to \mathbb{R}^A$ and $\sigma_2 : [0,1] \to C(\mathbb{C}P^{N-2+A})$. Further, there is a path σ' joining y_0 to y_1 and lying entirely within $\mathcal{O} \cap Z_{\text{reg}}/T$. (We obtain this path by projecting y_0 and y_1 on $\mathbb{C}P^{N-2+A}$, and joining their images under the projection by a path ϱ in $\mathbb{C}P^{N-2+A}$. Then, for an appropriate function $\tau : [0,1] \to (0, \infty)$, the desired path is given by $\sigma_2' = (\varrho, \tau) : [0,1] \to \mathbb{C}P^{N-2+A} \times (0, \infty)$. The path $\sigma' = (\sigma_1, \sigma_2')$ is homotopic to σ because $\mathcal{O}$ is contractible.) This proves Lemma 6.11(a).

To prove Lemma 6.11(b) for Y/G, we recall that given an open cover of a topological space X, two homotopic paths in X are homotopic by a sequence of homotopies each of which is the identity outside one of the sets in the open cover. Thus we may assume $f_1, f_2 : [0,1] \to \mathcal{O} \cap Z_{\text{reg}}$. We may project $f_1 \circ f_2^{-1}$ onto

$\mathbb{C}P^{N-2+A}$, and then the existence of the desired homotopy in Z_{reg}/T follows, by an argument similar to that given above, because $\mathbb{C}P^{N-2+A}$ is simply connected. This completes the proof of (6.2) and hence of Proposition 6.10. □

We now show that the $\mathbb{Z}_2$ summands that appeared in Proposition 6.10 vanish in the holonomy representation. More precisely, we have

Proposition 6.12. *Let $x \in B_g$ satisfy* (5.8) *and* (5.9). *Let $\xi \in H_1(\pi^{-1}(x))$ be an element of the torsion subgroup of $H_1(\pi^{-1}(x))$; that is, $2\xi = 0$. Then the holonomy representation of $H_1(\pi^{-1}(x))$ (defined by the connection ∇ on $\mathscr{L}^k$) takes ξ to* 1.

Proof. We must recall how the factors of $\mathbb{Z}_2$ appear. These factors arise for fibres corresponding to labelled trinion decompositions of the surface Σ^g, where the collection of curves C_i labelled with $\theta_i = 0$ or $\theta_i = \pi$ separates Σ^g into two (not necessarily connected) surfaces Σ_1, Σ_2. In our earlier notation, Σ_1 is one of the Σ_α and Σ_2 is $\Sigma^g - \Sigma_\alpha$. An element in this fibre can be represented by a flat a.t.d. connection A on Σ^g, which give rise to flat connections $A_1 = A|_{\Sigma_1}$ and $A_2 = A|_{\Sigma_2}$ on Σ_1 and Σ_2. A loop generating a nonzero torsion element of $H_1(\pi^{-1}(x))$ is then represented by a path A_t of flat a.t.d. connections on Σ^g, where

$$A_t|_{\Sigma_1} = A_1{}^{\zeta_t}, \quad A_t|_{\Sigma_2} = A_2, \tag{6.10}$$

and where $\zeta_t = \mathrm{diag}(e^{i\pi t}, e^{-i\pi t})$ $(0 \leqq t \leqq 1)$ is a path of (constant) gauge transformations from 1 to -1. If $\Sigma^g = \Sigma_1 \cup \Sigma_2$, then the path A_t is a closed loop in $\mathscr{A}_F$. The connection form for the trivial line bundle $\mathscr{A}_F \times \mathbb{C} \to \mathscr{A}_F$ which descends to the line bundle $\mathscr{L} \to \bar{\mathscr{S}}_g$ is given by the one form θ which assigns to an element $a \in T\mathscr{A}|_A$ the number

$$\theta_A(a) = \frac{i}{4\pi} \int_{\Sigma^g} \mathrm{Tr}(A \wedge a) \quad \text{(see [11, p. 412])}.$$

We then must check that the integral of the connection form $\int_0^1 k\theta_{A_t}\left(\frac{dA_t}{dt}\right) dt$ is an integer multiple of $2\pi i$, since the exponential of this integral is the parallel transport in $\mathscr{L}^k$ along the path A. This follows by explicit calculation: we have

$$\begin{aligned}
&\int_{\Sigma^g} \int_{t \in [0,1]} \mathrm{Tr}\left(A_t \wedge \frac{dA_t}{dt}\right) dt \\
&= \int_{\Sigma_1} \int_{t \in [0,1]} \mathrm{Tr}(\zeta_t^{-1} A_1 \zeta_t) \wedge \left(-\zeta_t^{-1} \frac{d\zeta_t}{dt} \zeta_t^{-1} A_1 \zeta_t + \zeta_t^{-1} A_1 \frac{d\zeta_t}{dt}\right) dt \\
&= 2 \int_{\Sigma_1} \int_{t \in [0,1]} \mathrm{Tr}\left(A_1^2 \zeta_t^{-1} \frac{d\zeta_t}{dt}\right) dt \\
&= -2\pi \int_{\Sigma_1} \mathrm{Tr}((\mathrm{diag}(i, -i)) dA_1) \\
&= -2\pi \int_{\partial\Sigma_1} \mathrm{Tr}((\mathrm{diag}(i, -i)) A_1) = 4\pi^2 n,
\end{aligned} \tag{6.11}$$

where n is the number of boundary circles C_j in $\partial\Sigma_1$ for which $\theta_j = \pi$. Thus $\int_0^1 k\theta_{A_t}\left(\frac{dA_t}{dt}\right) dt = i\pi k n$. But it will follow from Lemma 8.2 that n is even when k is odd. This completes the proof. □

This completes the proof of Theorem 6.1. □

We now apply Theorem 6.1 to extend the characterization of the Bohr-Sommerfeld set given in Proposition 5.5 to the singular fibres. Once we have found a point for which there is a covariant constant section over the loop corresponding to the Hamiltonian flow (which will be dealt with in Sect. 7), we shall be able to see (as we did in the case of nonsingular fibres, Proposition 5.5) that the hypotheses of Theorem 4.4 are satisfied, and hence to conclude:

Proposition 6.13. *Let $x \in B_g - B_g^{s,\mathrm{ind}}$. Then x is a Bohr-Sommerfeld point if and only if Eqs. (5.8), (5.9) are satisfied for all boundary circles C_i and trinions D_γ.*

Proof. First, if these conditions are satisfied then x is a Bohr-Sommerfeld point: for the holonomy representation of the fundamental group of $\pi^{-1}(x)$ is generated by the trajectories of the Hamiltonian flows corresponding to a subset of the h_i and g_γ, and by certain $\mathbb{Z}_2$ factors, and by Proposition 6.12, the $\mathbb{Z}_2$ factors vanish in the holonomy representation given by $\mathscr{L}^k$.

On the other hand, if x is a Bohr-Sommerfeld point then one can verify [as needed for condition (3) of Theorem 4.4] that (5.8) and (5.9) are satisfied, for all i for which $x \in \pi(U_i)$, for all γ for which $x \in \pi(U_\gamma)$, and in fact for *all* i and γ. For $x \in \pi(U_i)$ if $\theta_i(x) \in (0, \pi)$; likewise $x \in \pi(U_\gamma)$ if $\theta_{i_r(\gamma)}(x) \in (0, \pi)$ for all the boundary circles $C_{i_r(\gamma)}$ of the trinion D_γ. Thus if $x \notin \pi(U_i)$ for some i, $h_i(x)$ must be integral, and one of the following possibilities must occur:

$$\begin{aligned}
&\theta_{i_1(\gamma)}(x) = 0, \ \theta_{i_2(\gamma)}(x) = \theta_{i_3(\gamma)}(x) \notin \{0, \pi\}, \\
&\theta_{i_1(\gamma)}(x) = \pi, \ \theta_{i_2(\gamma)}(x) = \pi - \theta_{i_3(\gamma)}(x) \notin \{0, \pi\}, \\
&\theta_{i_1(\gamma)}(x) = \theta_{i_2(\gamma)}(x) = \theta_{i_3(\gamma)}(x) = 0, \\
&\theta_{i_1(\gamma)}(x) = 0, \ \theta_{i_2(\gamma)}(x) = \theta_{i_3(\gamma)}(x) = \pi.
\end{aligned} \tag{6.12}$$

Thus we see that (5.9) is satisfied in all cases. The fact that $h_i(x)$ is an even integer if C_i is a separating loop will then follow from Lemma 8.2. □

7. Normalization of the Action Variables via Branched Covers

The objective of this section is to establish the following result:

Proposition 7.1. *There exist points $x_i \in \pi(U_i) \subset B_g$ (resp. $x_\gamma \in \pi(U_\gamma)$) such that*
1. The function h_i (resp. g_γ) takes an integer value on the fibre $\pi^{-1}(x_i)$ (resp. $\pi^{-1}(x_\gamma)$).
2. There exists a covariant constant section of $\mathscr{L}^k$ along one orbit of the Hamiltonian flow of h_i (resp. g_γ) in $\pi^{-1}(x_i)$ (resp. $\pi^{-1}(x_\gamma)$).

Proposition 7.1 plays an important part in the proof of Propositions 5.5 and 6.13, since the a priori Bohr-Sommerfeld points constructed in Proposition 7.1 allow the moduli space case considered in this paper to be fit into the framework of condition (2) of Theorem 4.4. Thus the construction of these few Bohr-Sommerfeld points allow us to characterize the entire Bohr-Sommerfeld set.

To find the a priori Bohr-Sommerfeld points, we employ constructions from Chern-Simons gauge theory. These constructions will enable us to construct explicit covariant constant sections of the line bundle $\mathscr{L}^k$ restricted to fibres $\pi^{-1}(x_i)$, $\pi^{-1}(x_\gamma)$, for which the action variables h_i and g_γ assume integer values as in Eqs. (5.8), (5.9). Chern-Simons gauge theory enters the picture since the fibres we work with are related to moduli spaces of flat connections on handlebodies. On the fibre corresponding to such a moduli space, the line bundle $\mathscr{L}$ has a global covariant constant section, constructed from the Chern-Simons invariant in [9]. The actual fibres we use correspond to handlebodies which are k-fold branched covers of the handlebody bounding Σ^g; the corresponding covariant constant section descends to a section of $\mathscr{L}^k$ on the fibre. Thus we see an intimate connection between the holonomy of the flat connections in a fibre (related to the branching index) and the power of the line bundle possessing global covariant constant sections on such a fibre.

The present section of the paper is the only one that employs Chern-Simons gauge theory: those readers who are willing to accept on faith that one can find suitable points x_i and x_γ will find that nothing else in the paper depends on the arguments presented here.

7.1. Results on the Chern-Simons Cocycle

We recall from [11] the following construction of the prequantum line bundle $\mathscr{L}$ over $\bar{\mathscr{S}}_g$. As mentioned in Sect. 2, the moduli space is given by $\bar{\mathscr{S}}_g = \mathscr{A}_F/\mathscr{G}$, the space of flat connections on Σ^g modulo gauge transformations. So we begin with the trivial line bundle $\mathscr{A} \times \mathbb{C}$ over the space $\mathscr{A}$ of all connections over Σ^g. This line bundle has a connection ∇ with curvature $2\pi i\omega$, which may be written as a one form θ, defined by

$$\theta_A(a) = \frac{i}{4\pi} \int_{\Sigma^g} \mathrm{Tr}((A - A_0) \wedge a),$$

where $a \in T\mathscr{A}|_A$ and where A_0 denotes the product connection. We lift the action of an element ζ of the gauge group $\mathscr{G}$ to $\mathscr{A} \times \mathbb{C}$ as follows:

$$\zeta : (A, z) \to (A^\zeta, \Theta(A, \zeta)z). \tag{7.1}$$

Here, the *Chern-Simons cocycle* $\Theta(A, \zeta)$ is defined as follows. We choose a path $A(t)$ in $\mathscr{A}$ from the product connection A_0 to A. This path defines a connection $\tilde{A}$ on $\Sigma^g \times I$. We extend the gauge transformation ζ on $\Sigma^g = \Sigma^g \times \{1\}$ to a gauge transformation $\tilde{\zeta}$ on $\Sigma^g \times I$ which is equal to 1 on a neighbourhood of $\Sigma^g \times \{0\}$. The Chern-Simons cocycle is then defined as

$$\Theta(A, \zeta) = \exp 2\pi i[\mathrm{CS}(\tilde{A}^{\tilde{\zeta}}) - \mathrm{CS}(\tilde{A})] \in U(1). \tag{7.2}$$

Here, $\mathrm{CS}(B)$ is the Chern-Simons functional on connections B on $\Sigma^g \times I$,

$$\mathrm{CS}(B) = \frac{1}{8\pi^2} \int_{\Sigma^g \times I} \mathrm{Tr}\left(BdB + \frac{2}{3}B^3\right).$$

The cocycle Θ is independent of the choice of the extension $\tilde{\zeta}$ and of the choice of the path $A(t)$. In [11] it was shown that the lift (7.1) preserved the connection ∇ on $\mathscr{A} \times \mathbb{C}$. Thus (the restriction to $\mathscr{A}_F$ of) the line bundle $\mathscr{A} \times \mathbb{C}$ descends to give a line bundle $\mathscr{L}$ with connection on $\mathscr{A}_F/\mathscr{G}$.

We also note the following facts:

Proposition 7.2. *Let H be a genus g handlebody with boundary Σ^g. Denote by $\mathscr{M}(H)$ the space of conjugacy classes of representations of the handlebody fundamental*

group into G, and by L_H its image in $\bar{\mathscr{S}}_g$ under the restriction map. Then
1. *The map from $\mathscr{M}(H)$ to $\bar{\mathscr{S}}_g$ is injective.*
2. *Given a flat connection A whose gauge equivalence class lies in L_H, denote by $\tilde{A}$ an (arbitrary) extension of A to a flat connection over H. Then the map $\tilde{s}: \mathscr{A}_F \to \mathscr{A}_F \times \mathbb{C}$ given by*

$$\tilde{s}(A) = e^{2\pi i k \, \mathrm{CS}(\tilde{A})}$$

is well-defined (independent of choice of extension) and descends to give a section s of $\mathscr{L}^k$ over L_H. This section is covariant constant.

Proof. To see that this map defines a section of $\mathscr{L}$, we use the construction of the line bundle $\mathscr{L}$ from the trivial bundle $\mathscr{A}_F \times \mathbb{C} \to \mathscr{A}_F$. The definition of $\tilde{s}$ in terms of the Chern-Simons invariant of connections on a three-manifold bounding Σ^g guarantees that it has the correct equivariance property under the action of the gauge group to descend to a section of $\mathscr{L}|_{L_H} \to L_H \subset \bar{\mathscr{S}}_g$. The proof that the section $\tilde{s}$, and therefore the induced section $s: L_H \to \mathscr{L}$, is covariant constant, is given in [9].

7.2. *Covariant Constant Sections over Loops*

The construction of covariant constant sections over the fibre L_H may be generalized to yield covariant constant sections over submanifolds of certain Lagrangian fibres other than L_H: in particular, it will enable us to produce covariant constant sections over certain loops η_i coming from Hamiltonian flows of period 1. The construction uses *branched covers*, in much the same way as these were treated in [8].

We give first, in Proposition 7.3, the proof that if C_i is a *nonseparating* loop, then $\mathscr{L}^k$ has trivial holonomy over an orbit of the Hamiltonian flow of h_i precisely when h_i takes an integer value on that orbit. We will prove this by explicitly constructing a fibre $L_{i,n}$, where there is a covariant constant section of $\mathscr{L}^k$ over each trajectory of the Hamiltonian flow of h_i, and on which h_i takes an integer value. The analogous result when C_i is a separating loop will then be deduced from the nonseparating case in Proposition 7.6.

Suppose then that C_i is a nonseparating boundary circle in the trinion decomposition. There is then a cycle β_i in the trivalent graph representing the trinion decomposition, which contains the edge representing C_i. This cycle corresponds to a simple closed curve ϱ_i in the interior of the handlebody H determined by the trinion decomposition, which has linking number 1 with C_i. For every trinion D_γ, the curve ϱ_i either does not meet D_γ, or else links precisely two of the boundary circles of D_γ with linking number ± 1 and has linking number 0 with the other boundary circle.

We now construct the k-fold branched cover $\tilde{H}$ of H branched over ϱ_i. We have

Proposition 7.3. *The branched cover $\tilde{H}$ is a handlebody.*

Proof. We construct $\tilde{H}$ explicitly as follows. Consider a solid torus $U = D^2 \times S^1 \subset H$ with meridian C_i, which retracts onto the curve $\varrho_i = \{0\} \times S^1$. The solid torus U meets the closure $\overline{H-U}$ of $H-U$ in N disjoint disks $E_1, \ldots, E_N$ (see Fig. 3). The k-fold branched cover $\tilde{H}$ is constructed as the union of a solid torus $\tilde{U} = D^2 \times S^1$ and k copies of $\overline{H-U}$, each of which is joined to $\tilde{U}$ along N disks. The covering map $\tilde{U} \to U$ is the standard map

$$(z,t) \to (z^k, t): \quad D^2 \times S^1 \to D^2 \times S^1 ;$$

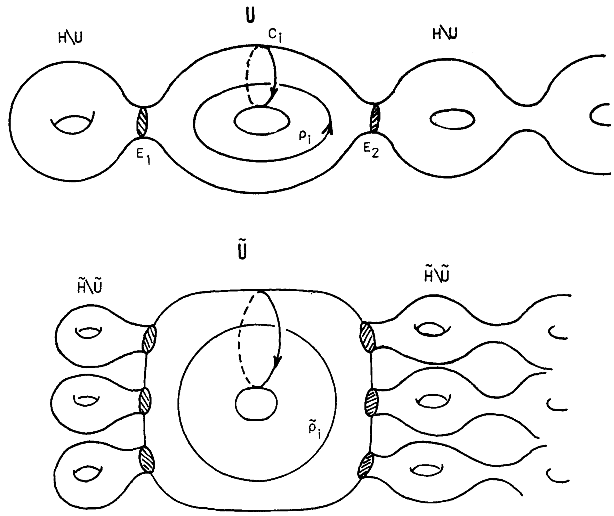

Fig. 3. The handlebody H and its k-fold branched cover $\tilde{H}$, branched over the curve $\varrho_i \subset U \subset H$ (here $k=3$)

the Nk disks in $\partial \tilde{U} \subset \tilde{U}$, along which the k copies of $\overline{(H-U)}$ are attached, are the components of the inverse images of $E_1, \ldots, E_N$ under this map. The branch locus ϱ_i has inverse image $\tilde{\varrho}_i$ under the covering map. The space $\tilde{H}$ is a handlebody; in particular, the boundary of $\tilde{H}$ is a surface $\Sigma^{g'}$ of genus $g'=1+k(g-1)$, which is a k-fold regular cover of Σ^g. $\square$

We consider a particular fibre $L_{i,n}$ of the polarization π of $\bar{\mathscr{S}}_g$, defined by

$$L_{i,n}=\{r^*(\phi)\,|\,\phi \in \mathrm{Hom}(\pi_1(H-\varrho_i), \ \mathrm{Tr}\,\phi([C_i])=2\cos(2\pi n/k)\}/G,$$

where $r:\Sigma^g \to H-\varrho_i$ is the inclusion map. In other words $L_{i,n}$ consists of (restrictions to $\bar{\mathscr{S}}_g$ of) gauge equivalence classes of flat connections on $H-\varrho_i$ for which the holonomy around the meridian C_i of ϱ_i is conjugate to $\mathrm{diag}(\exp 2\pi i n/k, \exp -2\pi i n/k)$. We now have

Proposition 7.4. *Consider the fibre $L_{i,n}$ of the polarization π of $\bar{\mathscr{S}}_g$. Then the natural map $\bar{\mathscr{S}}_g \to \bar{\mathscr{S}}_{g'}$ associated to the covering map $q:\Sigma^{g'} \to \Sigma^g$ takes $L_{i,n}$ into the fibre $L_{\tilde{H}}$ of $\bar{\mathscr{S}}_{g'}$ consisting of gauge equivalence classes of flat connections which extend to flat connections over the handlebody $\tilde{H}$. Moreover, this map lifts to a bundle map $\psi:\mathscr{L}^k \to \tilde{\mathscr{L}}$ from the k^{th} power of the line bundle $\mathscr{L}$ over $\bar{\mathscr{S}}_g$ to the corresponding line bundle $\tilde{\mathscr{L}}$ over $\bar{\mathscr{S}}_{g'}$, which is a map of bundles with connection.*

Proof. Under the covering map, a representation $\phi:\pi_1(H-\varrho_i) \to G$ sending the meridian $[C_i]$ to an element of order k pulls back to a representation $\tilde{\phi}=q^*(\phi):\pi_1(\tilde{H}-\tilde{\varrho}_i) \to G$ which comes from a representation of $\pi_1(\tilde{H})$ into G. In terms of flat connections, a flat connection A of this form on $H-\varrho_i$ pulls back to a connection which extends as a flat connection over all of $\tilde{H}$.

Let $q^*:\mathscr{A}(\Sigma^g) \to \mathscr{A}(\Sigma^{g'})$ denote the map induced on the spaces of connections by the covering map q. Let κ denote the map $\kappa:\mathbb{C} \to \mathbb{C}$ defined by $\kappa(z)=z^k$. Let $\tilde{\psi}=q^* \times \kappa:\mathscr{A}(\Sigma^g) \times \mathbb{C} \to \mathscr{A}(\Sigma^{g'}) \times \mathbb{C}$ denote the map of trivial line bundles induced by q^* and κ. Let $A \in \mathscr{A}(\Sigma^g)$ be a connection on Σ^g, and let $\zeta:\Sigma^g \to G$ be a gauge transformation on Σ^g. The k^{th} power of the Chern-Simons cocycle on Σ^g, $\Theta(A,\zeta)^k$, then lifts to the Chern-Simons cocycle on $\Sigma^{g'}$; that is,

$$\Theta(A,\zeta)^k = \Theta(q^*(A), \zeta \circ q).$$

Thus the map $\tilde{\psi}$ is compatible with the action of the gauge group, and descends to a bundle map $\psi: \mathscr{L}^k \to \tilde{\mathscr{L}}$. It remains to show that the map $\tilde{\psi}$ is in fact a map of bundles with connection. If $a \in T\mathscr{A}(\Sigma^g)|_A$, then $q^*(a)$ can be considered as an element of $T\mathscr{A}(\Sigma^{g'})|_{q^*(A)}$. The connection one form θ' on the trivial bundle $\mathscr{A}(\Sigma^{g'}) \times \mathbb{C}$ given by

$$\theta'_{A'}(a') = \frac{i}{4\pi} \int_{\Sigma^{g'}} \operatorname{Tr}(A' \wedge a')$$

then pulls back to the connection one-form $k\theta$ on $\mathscr{A}(\Sigma^g) \times \mathbb{C}$; that is,

$$k\theta_A(a) = k\frac{i}{4\pi} \int_{\Sigma^g} \operatorname{Tr}(A \wedge a) = \frac{i}{4\pi} \int_{\Sigma^{g'}} \operatorname{Tr}(q^*(A) \wedge q^*(a)) = \theta'_{q^*(A)}(q^*(a)).$$

Thus the bundle map $\psi: \mathscr{L}^k \to \tilde{\mathscr{L}}$ is a map of bundles with connection. □

The flow generated by the Hamiltonian function h_i preserves the fibre $L_{i,n}$. Further, the map of $L_{i,n}$ to $L_{\tilde{H}}$ is injective. Thus we may pull back the covariant constant section over $L_{\tilde{H}}$ to get a covariant constant section over $L_{i,n}$. Since such a section exists, the holonomy of ∇ around loops in $L_{i,n}$ generated by the Hamiltonian flow of h_i is 1.

Since h_i takes the integer value $2n$ on the fibre $L_{i,n}$, this suffices to establish:

Proposition 7.5. *If C_i is a nonseparating loop in Σ^g, then $\mathscr{L}^k$ has trivial holonomy over an orbit of the Hamiltonian flow of h_i precisely when h_i takes an integer value on that orbit.*

We now turn our attention to

Proposition 7.6. *The result of Proposition 7.5 holds also when C_i is a separating loop in Σ^g.*

Proof. By adding another handle, we may construct a surface Σ^{g+1} in which C_i is a nonseparating loop. There is then a map $\Sigma^{g+1} \to \Sigma^g$ which collapses the extra handle to a point. Thus we get a map $\bar{\mathscr{S}}_g \to \bar{\mathscr{S}}_{g+1}$ which is compatible with the Hamiltonian flow of h_i, and with $(\mathscr{L}, \nabla)$. We wish to show that a covariant constant section of $\mathscr{L}^k$ exists over the orbit of the Hamiltonian flow of h_i through $y \in \bar{\mathscr{S}}_g$ whenever $h_i(y) \in \mathbb{Z}$: but this may now be inferred using Proposition 7.5 from the corresponding fact for $\bar{\mathscr{S}}_{g+1}$. □

We now turn our attention to normalizing the Hamiltonian function g_γ.

Proposition 7.7. *The function g_γ takes integer values on orbits of the Hamiltonian flow over which there is a covariant constant section of $\mathscr{L}^k$.*

Proof. We let $\mathscr{T}'_g \subset \mathscr{T}_g \subset \bar{\mathscr{S}}_g$ denote the subspace of conjugacy classes y of *reducible* representations [i.e., representations $\pi_1(\Sigma^g) \to T \subset G$] for which $\theta_{i_1(\gamma)}(y) + \theta_{i_2(\gamma)}(y) + \theta_{i_3(\gamma)}(y) = 2\pi$ (cf. Lemma 3.5). If we examine Hamiltonian flows beginning at a point $y \in \mathscr{T}'_g$, then since the flows of the h_i preserve $\mathscr{T}'_g$ (see Corollary 5.7), so does the flow corresponding to g_γ, i.e., the Hamiltonian vector field X_{g_γ} lies in the tangent space to $\mathscr{T}'_g$. (This flow is defined on the open subset $\mathscr{T}'_g \cap U_\gamma$ of $\mathscr{T}'_g$.)

However, the flow corresponding to g_γ is *trivial* restricted to $\mathscr{T}_g'$, since ω restricts to give a (nondegenerate) symplectic form on $\mathscr{T}_g'$ and $g_\gamma - k$ (and hence also dg_γ) is identically zero on $\mathscr{T}_g'$, while X_{g_γ} is tangent to $\mathscr{T}_g'$. Thus there is a global covariant constant section of $\mathscr{L}$ over the (trivial) loop arising from the Hamiltonian flow of g_γ through $y \in \mathscr{T}_g'$. This, combined with the observation that $g_\gamma = k$ on $\mathscr{T}_g'$, suffices to establish the Hamiltonian function g_γ is normalized so as to take integer values on orbits over which there is a covariant constant section of $\mathscr{L}^k$. □

This completes the proof of Proposition 7.1.

8. Counting the Bohr-Sommerfeld Orbits

We now conclude by counting the Bohr-Sommerfeld fibres of our polarization, to determine the dimension of the putative quantization. We shall see that the number of fibres obtained agrees with the Verlinde formula for the dimension of the quantization in a Kähler polarization.

We have seen in Theorem 4.4 that to find the Bohr-Sommerfeld orbits, it suffices to verify that there are points $x_i, x_\gamma \in B_g$, for which we can construct covariant constant sections of $\mathscr{L}^k$ on the orbits of the Hamiltonian flow of h_i (resp. g_γ) on $\pi^{-1}(x_i)$ [resp. $\pi^{-1}(x_\gamma)$] and for which the Hamiltonian functions h_i (resp. g_γ) take integer values. The existence of these points was proved in Sect. 7 (see Proposition 7.1). The resulting characterization of the Bohr-Sommerfeld set was given in Propositions 5.5 and 6.13. We may restate the situation as follows:

Theorem 8.1. *The set P^{bs} of Bohr-Sommerfeld points in B_g is given by*

$$P^{\mathrm{bs}} = \{x \in B_g \mid h_i, g_\gamma \text{ take integer values on } \pi^{-1}(x)\ \forall i, \gamma\},$$

where the value of h_i must be an even integer if C_i is a separating curve.

In terms of the coordinates $\underline{\theta} = \underline{\theta}(x)$ of the point $x \in B_g$, this condition reduces to

$$P^{\mathrm{bs}} = \left\{ x \in B_g \;\middle|\; \begin{array}{lll} \text{(a)} & \theta_i(x) = \pi l_i/k & \text{with } l_i = 0, 1, \ldots, k \text{ for all } i, \\ & & \text{and } l_i \in 2\mathbb{Z} \text{ if } C_i \text{ is separating;} \\ \text{(b)} & l_{i_1(\gamma)} + l_{i_2(\gamma)} + l_{i_3(\gamma)} \in 2\mathbb{Z} & \text{for all } \gamma \end{array} \right\} \tag{8.1}$$

The condition (8.1)(a) arises from the integrality condition for h_i, while (b) arises from the integrality condition for g_γ.

To obtain our final result, we recall that a set of values $\underline{\theta} = (\theta_1, \ldots, \theta_{3g-3})$ arises as the image $(\theta_1(x), \ldots, \theta_{3g-3}(x))$ of a point $x \in B_g$ if and only if the conditions (3.4) are satisfied for the triples $(\theta_{i_1(\gamma)}, \theta_{i_2(\gamma)}, \theta_{i_3(\gamma)})$ corresponding to each trinion D_γ. As discussed in Sect. 2, trinion decompositions correspond to trivalent graphs, where

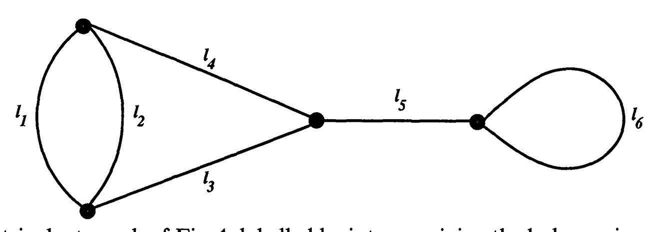

Fig. 4. The trivalent graph of Fig. 1, labelled by integers giving the holonomies of connections in the corresponding Bohr-Sommerfeld leaf

one associated a vertex to each trinion and an edge to each boundary circle joining two trinions, as in Fig. 1. Therefore, the Bohr-Sommerfeld points may be seen to correspond to labelled trivalent graphs, where one assigns the integer label $l_i = 0, 1, \ldots, k$ to each edge i (see Fig. 4).

In terms of the labels $l_i \in \mathbb{Z}$ such that $\theta_i = \pi l_i / k$ is the holonomy angle around a boundary circle, $\underline{\theta}$ actually is equal to $\underline{\theta}(x)$ for a point $x \in B_g$ if and only if

$$|l_{i_1(\gamma)} - l_{i_2(\gamma)}| \leqq l_{i_3(\gamma)} \leqq \min\{(l_{i_1(\gamma)} + l_{i_2(\gamma)}), 2k - (l_{i_1(\gamma)} + l_{i_2(\gamma)})\}.$$

Thus we see that the Bohr-Sommerfeld set is in $1-1$ correspondence with labellings of a trivalent graph by integers in $[0, k]$ such that for each vertex with labels l_1, l_2, l_3, the following conditions are satisfied:

$$\begin{aligned} &\text{(a)} \qquad |l_1 - l_2| \leqq l_3 \leqq l_1 + l_2, \\ &\text{(b)} \qquad l_1 + l_2 + l_3 \leqq 2k, \qquad\qquad (8.2) \\ &\text{(c)} \qquad l_1 + l_2 + l_3 \in 2\mathbb{Z}, \end{aligned}$$

and such that edges corresponding to separating boundary circles are labelled by even integers. Equations (8.2a, b) arise from the condition (3.4) that the point $\underline{\theta}$ be in B_g, while Eq. (8.2c) results from the integrality condition for g_γ.

An elementary combinatorial argument establishes that

Lemma 8.2. *Suppose $\Sigma^g = \Sigma_1 \cup \Sigma_2$, where $\Sigma_1 \cap \Sigma_2 = \bigcup_{j \in \mathcal{I}'} C_j$. Suppose that the boundary circles C_j of Σ^g are labelled by integers l_j such that $l_{i_1(\gamma)} + l_{i_2(\gamma)} + l_{i_3(\gamma)} \in 2\mathbb{Z}$ for each trinion γ. Then $\sum_{j \in \mathcal{I}'} l_j$ is even. (In particular, separating boundary circles are always labelled by even integers.)*

Proof. The corresponding trivalent graph is separated in two components S_1, S_2 corresponding to Σ_1 and Σ_2. We denote by $V(S_1)$ and $E(S_1)$ the sets of vertices and edges in S_1, and consider the sum

$$I = \sum_{\gamma \in V(S_1)} l_{i_1(\gamma)} + l_{i_2(\gamma)} + l_{i_3(\gamma)} \in \to \mathbb{Z} \quad [\text{by (8.2c)}].$$

This sum is equal to $\sum_{j \in \mathcal{I}'} l_j + 2 \sum_{i \in E(S_1)} l_i$, and hence $\sum_{j \in \mathcal{I}'} l_j$ must be even. □

Thus the condition that separating boundary circles be labelled by even integers follows automatically from (8.2): in other words, the solutions of (8.2) are in bijective correspondence with the points in the Bohr-Sommerfeld set.

Equations (8.2a–c) are the *quantum Clebsch-Gordan conditions*: this system of equations arises from the "fusion rules" in conformal field theory, specifically in the $SU(2)$ Wess-Zumino-Witten model [14]. The number of labellings of a trivalent graph satisfying (8.2) is also known to give the dimension of the quantization $\mathcal{H} = H^0(\Sigma^g, \mathcal{L}^k)$ associated to a Kähler polarization (i.e., the space of holomorphic sections of the line bundle $\mathcal{L}^k$).

Our final result is then

Theorem 8.3. *Consider a fixed trinion decomposition of a two-manifold Σ^g; it gives rise to a trivalent graph and to a real polarization of $\bar{\mathcal{S}}_g$. There is a one-to-one correspondence between the set of Bohr-Sommerfeld fibres of the real polarization and the set of labellings (by integers in $[0, k]$) of the edges of the trivalent graph satisfying the quantum Clebsch-Gordan conditions* (8.2a–c) *at each vertex.*

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