Commun. Math. Phys. 150, 209–215 (1992)

Communications in
Mathematical
Physics


# Sharp Estimates for Dirichlet Eigenfunctions in Horn-Shaped Regions

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Received April 28, 1992

**Abstract.** We prove sharp exponential decay for the Dirichlet eigenfunctions in horn-shaped regions in the plane. The estimate is obtained using a method of Carleman which is widely used in the study of harmonic measure. Such estimates can be applied to study the intrinsic ultracontractivity properties of the heat semigroup for such regions.

## 0. Introduction

The purpose of this paper is to prove sharp exponential bounds for the decay of the Dirichlet eigenfunctions in "infinite trumpets" or "horn-shaped" regions. Such estimates, besides being of interest in their own right, can be used to prove *intrinsic ultracontractivity*, IU, for the Dirichlet Laplacian for these regions as in Davies and Simon [4]. Also, our result sharpens Theorem 7.3 in Davies and Simon [4].

Let $\theta:(0,\infty) \to [0,1]$ be continuous and let

$$D_\theta = \{z = (x,y): x > 0, -\theta(x) < y < \theta(x)\}. \tag{0.1}$$

It was proved in Davies and Simon [4] that under the assumptions (i) $\theta'(x)$ bounded and $C^1$, (ii) $\theta(x) \to 0$ as $x \to \infty$, (iii) $\theta'(x) \le 0$ for large $x$ and (iv) $\theta'(x)/\theta(x) \to 0$ as $x \to \infty$, there exist constants $C_1, C_2, D_1, D_2$ such that

$$C_1 \exp\left(-\frac{D_1 x}{\theta(x)}\right) \le \varphi(x,0) \le C_2 \exp\left(-\frac{D_2 x}{\theta(x)}\right), \tag{0.2}$$

where $\varphi$ is the positive eigenfunction of $D_\theta$. Our result, which was motivated by (0.2), is

⋆ Supported in part by NSF

**Theorem 1.** (a) *Suppose* $\theta(x) \downarrow 0$ *as* $x \uparrow \infty$. *Let* $\varphi_\lambda$ *be any* $L^2$*-eigenfunction of the Dirichlet Laplacian in* $D_\theta$ *with eigenvalue* $\lambda$. *For any* $\varepsilon > 0$,

$$\int_{D_\theta} |\varphi_\lambda(x,y)|^2 e^{(1-e)\pi \int_0^x \alpha_\lambda(s)ds}\, dx\, dy \le C_{\varepsilon,\lambda}\,, \tag{0.3}$$

*where*

$$\alpha_\lambda(s) = \sqrt{\max\left(\left(\frac{1}{\theta^2(s)} - \frac{4\lambda}{\pi^2}\right), 0\right)}$$

*and* $C_{\varepsilon,\lambda}$ *depends on* $\lambda$ *and* $\varepsilon$. *If in addition we assume* $\theta \in L^1(dx)$, *we may take* $\varepsilon = 0$ *and obtain* (0.3) *with the constant on the right-hand side depending only on* $\lambda$ *and the area of* $D_\theta$.
(b) *If* $\theta(x) \downarrow 0$ *as* $x \uparrow \infty$ *and* $z = (x,y) \in D_\theta$, *then*

$$|\varphi_\lambda(x,y)| \le C_\lambda \delta(z)^{1/2} e^{-\frac{\pi}{2}\int_0^x a_\lambda(s)ds} \tag{0.4}$$

*where* $\delta(z)$ *is the vertical distance from* $z$ *to* $\partial D$.
(c) *If* $\theta(x) \downarrow 0$ *as* $x \uparrow \infty$, *then*

$$\varphi(x,0) \ge C\theta(x) e^{-\frac{\pi}{2}\int_1^x \frac{ds}{\theta(s)}}$$

*for* $x \ge 1$.

Our proof of the upper bound, which is very elementary, is based on the well-known method of Carleman which is widely used in the study of harmonic measure; (see Haliste [7]). We have, however, not seen it before in connection to the study of eigenfunctions. After reading a preliminary version of our paper, Professor R. Kauffman informed us that the techniques used in his paper with Evans and Harris [5] to obtain sharp estimates for bounded trumpets would most likely also give our upper bound result. Our lower bound estimate is an immediate consequence of a result of Haliste [7] on the decay of harmonic measure for such regions. Before we prove the theorem we mention some generalizations. First, the trumpet does not have to be symmetric. That is, if $D = \{(x,y)\colon \theta_1(x) < y < \theta_2(x)\}$ with $\theta_2$ positive and decreasing to 0 and $\theta_1$ negative and increasing to 0, then our theorem still holds. In this case the exponential decay is measured with the function $\theta(x) = \theta_2(x) - \theta_1(x)$. It may also be possible, (but we have not checked this), to extend the upper bound part of our theorem to domains in higher dimensions by following the arguments in Friedland and Hayman [6]. We are grateful to T. Wolff and C. Bishop for telling us about some of the literature, and applications of, the Carleman method. Finally we should mention that the the literature on eigenfunction estimates is very exhaustive (and exhausting) and we do not even attempt to discuss it here: (see [1, 3, 4, 5] for some of this literature). We should just mention here that most of the literature we found uses the method of Agmon [1] which as far as we know does not, (at least not directly), apply to the Dirichlet eigenfunctions of regions of $\mathbb{R}^n$ because of the lack of completeness of the metric.

## 1. Proof of Theorem 1

**Lemma 1.1.** *Suppose* $\theta(x) \downarrow 0$ *as* $x \uparrow \infty$. *Define*

$$u_\lambda(x) = \int_{-\theta(x)}^{\theta(x)} |\varphi_\lambda(x,y)|^2 dy\,. \tag{1.1}$$

*There is a real number* $x_\lambda$, *depending only on* $\lambda$, *such that for all* $x > x_\lambda$,

$$|u_\lambda'(x)| \le 2\lambda e^{-\pi \int_{x_\lambda}^{x} \alpha_\lambda(s)ds}\,. \tag{1.2}$$

*Proof.* We first derive a Carleman-type differential inequality following Haliste [7]. Differentiating the function $u_\lambda$ we obtain

$$u_\lambda'(x) = 2\int_{-\theta(x)}^{\theta(x)} \varphi_\lambda(x,y)\frac{\partial \varphi_\lambda}{\partial x}(x,y)dy \tag{1.3}$$

and

$$u_\lambda''(x) = 2\int_{-\theta(x)}^{\theta(x)} \left[\left|\frac{\partial \varphi_\lambda}{\partial x}(x,y)\right|^2 + \frac{\partial^2 \varphi_\lambda}{\partial x^2}(x,y)\varphi_\lambda(x,y)\right]dy\,. \tag{1.4}$$

Substituting $\Delta\varphi_\lambda = -\lambda\varphi$ in (1.4) we find that

$$\begin{aligned} u_\lambda''(x) &= 2\int_{-\theta(x)}^{\theta(x)} \left|\frac{\partial \varphi_\lambda}{\partial x}(x,y)\right|^2 dy - 2\int_{-\theta(x)}^{\theta(x)} \frac{\partial^2 \varphi_\lambda}{\partial y^2}(x,y)\varphi_\lambda(x,y)dy - 2\lambda u_\lambda(x) \\ &= 2\int_{-\theta(x)}^{\theta(x)} \left|\frac{\partial \varphi_\lambda}{\partial x}(x,y)\right|^2 dy + 2\int_{-\theta(x)}^{\theta(x)} \left|\frac{\partial \varphi_\lambda}{\partial y}(x,y)\right|^2 dy - 2\lambda u_\lambda(x)\,, \end{aligned}$$

after an integration by parts in the second term. Applying the Cauchy-Schwartz inequality to (1.3) we get

$$|u_\lambda'(x)|^2 \le 4u_\lambda(x)\int_{-\theta(x)}^{\theta(x)} \left|\frac{\partial \varphi_\lambda}{\partial x}(x,y)\right|^2 dy$$

and thus we obtain

$$u_\lambda''(x) \ge \frac{1}{2}\frac{|u_\lambda(x)|^2}{u_\lambda(x)} + 2\int_{-\theta(x)}^{\theta(x)} \left|\frac{\partial \varphi_\lambda}{\partial y}(x,y)\right|^2 dy - 2\lambda u_\lambda(x)\,. \tag{1.5}$$

By Wirtinger's inequality

$$\int_{-\theta(x)}^{\theta(x)} \left| \frac{\partial \varphi_\lambda}{\partial y}(x,y) \right|^2 dy \geq \frac{\pi^2}{4\theta^2(x)} \int_{-\theta(x)}^{\theta(x)} |\varphi(x,y)|^2 dy$$

$$= \frac{\pi^2}{4\theta^2(x)} u_\lambda(x) .$$

This and (1.5) give

$$u''_\lambda(x) \geq \frac{1}{2} \frac{|u'_\lambda(x)|^2}{u_\lambda(x)} + \left( \frac{\pi^2}{2\theta^2(x)} - 2\lambda \right) u_\lambda(x)$$

$$= \frac{1}{2} \frac{|u'_\lambda(x)|^2}{u_\lambda(x)} + \frac{\pi^2}{2} \left( \frac{1}{\theta^2(x)} - \frac{4\lambda}{\pi^2} \right) u_\lambda(x) . \tag{1.6}$$

Since $\theta(x) \downarrow 0$, there is an $x_\lambda$ such that for all $x > x_\lambda$, $\left( \frac{1}{\theta^2(x)} - \frac{4\lambda}{\pi^2} \right) > 1$. Thus for $x > x_\lambda$ we have

$$u''_\lambda(x) \geq |u'_\lambda(x)| \pi \alpha_\lambda(x) . \tag{1.7}$$

Since $u''_\lambda(x) \geq 0$ for all $x > x_\lambda$ and $u_\lambda(x) \to 0$ as $x \to \infty$, we have that $u'_\lambda(x) \leq 0$ for all $x > x_\lambda$. Thus (1.7) gives

$$u''_\lambda(x) \geq -\pi u'_\lambda(x) \alpha_\lambda(x) \tag{1.8}$$

and integrating this inequality gives,

$$|u'_\lambda(x)| \leq |u'_\lambda(x_\lambda)| e^{-\pi \int_{x_\lambda}^{x} \alpha_\lambda(s) ds}$$

for all $x > x_\lambda$.

Applying Green's theorem and using again the fact that $\varphi_\lambda$ vanishes on $\partial D_\theta$ we find that

$$|u'_\lambda(x_\lambda)| = 2 \left| \int_0^{x_\lambda} \int_{-\theta(x)}^{\theta(x)} [|\nabla \varphi_\lambda(x,y)|^2 - \lambda \varphi_\lambda^2(x,y)] dx\, dy \right|$$

$$\leq 2 \max \left( \int_{D_\theta} |\nabla \varphi_\lambda(x,y)|^2 dx\, dy , \int_{D_\theta} |\varphi_\lambda(x,y)|^2 dx\, dy \right) = 2\lambda$$

and this completes the proof of the lemma.

*Proof of* (a). To prove part (a) it is enough to estimate

$$\int_{x_\lambda}^{\infty} \int_{-\theta(x)}^{\theta(x)} |\varphi_\lambda(x,y)|^2 e^{(1-\varepsilon)\pi \int_{x_\lambda}^{x} \alpha_\lambda(s) ds} dx\, dy . \tag{1.9}$$

Substituting

$$u_\lambda(x) = -\int_x^\infty u'_\lambda(t)dt$$

and applying Lemma 5.1 we find that the expression in (1.9) is dominated by

$$2\lambda \int_{x_\lambda}^\infty \left( \int_x^\infty e^{-\pi \int_{x_\lambda}^t \alpha_\lambda(s)ds} dt \right) e^{(1-\varepsilon)\pi \int_{x_\lambda}^x \alpha_\lambda(s)ds} dx$$

$$= 2\lambda \int_{x_\lambda}^\infty \left( \int_x^\infty e^{-\pi \int_x^t \alpha_\lambda(s)ds} dt \right) e^{-\varepsilon\pi \int_{x_\lambda}^x \alpha_\lambda(s)ds} dx\,,$$

since $\theta(x) \downarrow 0$, $\alpha_\lambda(s) \uparrow$ and we have that the last expression is

$$\le 2\lambda \int_{x_\lambda}^\infty \left( \int_x^\infty e^{-\pi(t-x)\alpha_\lambda(x)} dt \right) e^{\varepsilon\pi(x-x_\lambda)\alpha_\lambda(x_\lambda)} dx$$

$$= 2\lambda e^{\varepsilon\pi x_\lambda \alpha_\lambda(x_\lambda)} \int_{x_\lambda}^\infty e^{\pi x \alpha_\lambda(x)} e^{-\varepsilon\pi x \alpha_\lambda(x_\lambda)} \left( \int_x^\infty e^{-\pi t \alpha_\lambda(x)} dt \right) dx$$

$$= \frac{2\lambda}{\pi} e^{\varepsilon\pi x_\lambda \alpha_\lambda(x_\lambda)} \int_{x_\lambda}^\infty \frac{1}{\alpha_\lambda(x)} e^{-\varepsilon\pi x \alpha_\lambda(x_\lambda)} dx$$

$$\le \frac{2\lambda}{\pi} e^{\varepsilon\pi x_\lambda \alpha_\lambda(x_\lambda)} \int_{x_\lambda}^\infty e^{-\varepsilon\pi x \alpha_\lambda(x_\lambda)} dx$$

which is finite for any $\varepsilon > 0$ and we have proved the first assertion in (a).
If $\theta \in L^1(dx)$ and we take $\varepsilon = 0$ the above calculation shows that

$$\int_{x_\lambda}^\infty \int_{-\theta(x)}^{\theta(x)} |\varphi_\lambda(x,y)|^2 e^{\pi \int_{x_\lambda}^x \alpha_\lambda(s)ds} dx\, dy \le \frac{2\lambda}{\pi} \int_{x_\lambda}^\infty \frac{dx}{\sqrt{\frac{1}{\theta^2(x)} - \frac{4\lambda}{\pi}}}$$

$$\le C_\lambda \int_0^\infty \theta(x) dx\,,$$

and we have the second assertion in part (a).

*Proof of* (b). The same argument used in the proof of (a) shows that if

$$\tilde{u}_\lambda(x) = \int_{-\theta(x)}^{\theta(x)} \left| \frac{\partial \varphi_L}{\partial y}(x,y) \right|^2 dy\,,$$

then $\tilde{u}'_\lambda$ is negative and

$$|\tilde{u}'_\lambda(x)| \le C_\lambda e^{-\pi \int_{x_\lambda}^{x} \alpha_\lambda(s)ds} \tag{1.10}$$

for $x > x_\lambda$, $x_\lambda$ again depending on $\lambda$. By using (1.10) we obtain as above that

$$\tilde{u}_\lambda(x) = \int_x^\infty \tilde{u}'_\lambda(t)dt \le C_\lambda e^{-\pi \int_{x_\lambda}^{x} \alpha_\lambda(s)ds}.$$

Since

$$|\varphi_\lambda(x,y)| = \left| \int_{\theta(x)-\delta(x,y)}^{\theta(x)} \frac{\partial \varphi_\lambda}{\partial y}(x,t)dt \right|,$$

(0.4) follows from the Cauchy-Schwartz inequality.

*Proof of* (c): To prove part (c) we use the following lemma whose proof is exactly as the proof of Theorem 7.1 in Haliste [7]. We use $\omega_z^D(I)$ to denote the harmonic measure of $I$ at $z$ with respect to the domain $D, D$ any domain and $I$ any subset of $\partial D$.

**Lemma 1.2.** *Let* $\theta(x) \downarrow 0$ *and let* $D_1 = \{(x,y) : x > 1, -\theta(x) < y < \theta(x)\}$ *and* $I_1 = \{x = 1\} \cap D_\theta$. *Then for* $x > 1$,

$$\omega_{(x,0)}^{D_1}(I_1) \ge C\theta(x) e^{-\frac{\pi}{2}\int_1^x \frac{ds}{\theta(s)}}.$$

We now prove part (c). Since $\varphi$ is superharmonic, and since $\varphi$ vanishes on $\partial D$, we have

$$\begin{aligned} \varphi(x,0) &\ge \int_{I_1} \varphi(\xi) d\omega_{(x,0)}^{D_1}(\xi) \\ &\ge \int_{\widehat{I_1}} \varphi(\xi) d\omega_{(x,0)}^{D_1}(\xi), \end{aligned} \tag{1.11}$$

where $\hat{I}_1 = \{z \in I_1 : \text{dist}(z, \partial D) > C_1 l(I_1)\}$, $l(I_1)$ is the length of $I_1$. By Lemma 2.3 in Bañuelos and Davis [2], there exists a constant $C_1$ such that

$$\omega_z^{D_1}(\hat{I}_1) \ge C_2 \omega_z^{D_1}(I_1) \tag{1.12}$$

for all $z \in \{x = 2\} \cap D_\theta$. By (1.11), (1.12), and Lemma 1.2, we have

$$\begin{aligned} \varphi(x,0) &\ge C\left(\inf_{z \in \hat{I}_1} \varphi(z)\right)\theta(x) e^{-\frac{\pi}{2}\int_1^x \frac{dt}{\theta(s)}} \\ &= C\theta(x) e^{-\frac{\pi}{2}\int_1^x \frac{dt}{\theta(s)}} \end{aligned}$$

which proves part (c).

If $z = (x, y) \in D_\theta$ it follows as in Bañuelos and Davis [2], [Inequality (1.1)], that

$$\varphi(z) \geq \theta(x) e^{-\varrho_{D_\theta}(z,x)} \varphi(x, 0)\,, \tag{1.13}$$

where $\varrho_{D_\theta}(z, x)$ is the quasi-hyperbolic distance in $D_\theta$. It is not difficult to show that for any of our $D_\theta$, $\varrho_{D_\theta}(z, x) \leq C/d_{D_\theta}(z)$, where $d_{D_\theta}(z)$ is the distance from $z$ to $\partial D_\theta$. Thus (1.13) and part (c) of our theorem give

$$\log \frac{1}{\varphi(z)} \leq C/d_{D_\theta}(z) + \frac{\pi}{2} \int_1^x \frac{dt}{\theta(t)} + \log \frac{1}{\theta(x)}\,. \tag{1.14}$$

Such estimates and the logarithmic Sobolev inequality can be used to show IU for $D_\theta$ as in Davies and Simon [4]. Finally we mention that if $\theta$ satisfies the smoothness assumptions in Davies and Simon [4], then

$$\omega_z^{D_1}(I_1) \geq C d_{D_\theta}(z) \omega_{(x,0)}^{D_1}(I_1)$$

and since $\theta(x)$ is decreasing, the argument above gives

$$\varphi(z) \geq C_1 d_D(z) \exp\left(\frac{-C_2 x}{\theta(x)}\right). \tag{1.15}$$

That (1.15) should be true was remarked by Davies and Simon [4], p. 366.

## References

1. Agmon, S.: Lectures on exponential decay of solutions of second elliptic equation: Bounds on eigenfunctions of $N$-body Schrödinger operators. Math. Notes, Princeton **29** (1982)
2. Bañuelos, R., Davis, B.: A geometrical characterization of intrinsic ultracontractivity for planar domains with boundaries given by the graphs of functions. Preprint
3. Davies, B.: Heat kernels and spectral theory. Cambridge: Cambridge University Press 1989
4. Davies, B., Simon, B.: Ultracontracitvity and heat kernels for Schrödinger operators and Dirichlet Laplacians. J. Funct. Anal. **59**, 335–395 (1984)
5. Evans, W.D., Haris, D.J., Kaufman, R.M.: Boundary behaviour of Dirichlet eigenfunctions of second order elliptic operators. Math. Z. **204**, 85–115 (1990)
6. Friedland, S., Hayman, W.K.: Eigenvalue inequalities for the Dirichlet problem on spheres and the growth of subharmonic functions. Comment. Math. Helvetici **51**, 133–161 (1976)
7. Haliste, M.: Estimates of harmonic measure. Arkiv för Math. **6**, 1–31 (1965)

Communicated by B. Simon