Separation of Variables in the Classical Integrable $SL(3)$ Magnetic Chain

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Abstract. There are two fundamental problems studied by the theory of hamiltonian integrable systems: integration of equations of motion, and construction of action-angle variables. A third problem, however, should be added to the list: separation of variables. Though much simpler than the two others, it has important relations to quantum integrability. Separation of variables is constructed for the $SL(3)$ magnetic chain – an example of an integrable model associated to a nonhyperelliptic algebraic curve.

1. Introduction

Consider a completely integrable Hamiltonian system with D degrees of freedom. According to the definition of complete integrability due to Liouville-Arnold [1] it means that the system possesses exactly D independent Hamiltonians H_j commuting with respect to the Poisson bracket

$$\{H_j, H_k\} = 0, \quad j, k = 1, \ldots, D. \tag{1}$$

There are three fundamental problems discussed in the theory of integrable systems. They are listed below in the order of decreasing complexity:
- Construction of action-angle variables.
- Integration of equations of motion.
- Separation of variables.

For the wide class of finite-dimensional integrable systems subject to the Inverse Spectral Transform Method an effective integration of equations of motion can be performed using the techniques of algebraic geometry [2]. As for the effective construction of the action-angle variables, it is a more difficult problem [3], especially

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when the reality conditions are carefully taken into account [4]. The case of systems associated to hyperelliptic spectral curves is studied in detail [3,4], there being only some preliminary results for the non-hyperelliptic case [5].

In the present paper the third mentioned problem (separation of variables) is studied. To be precise, the separation of variables is understood here as construction of D pairs of canonical variables x_j, p_j $(j = 1, \ldots, D)$

$$\{x_j, x_k\} = \{p_j, p_k\} = 0, \qquad \{p_j, x_k\} = \delta_{jk}, \tag{2}$$

and D functions Φ_j such that

$$\Phi_j(x_j, p_j, H_1, H_2, \ldots, H_D) = 0, \qquad j = 1, 2, \ldots, D, \tag{3}$$

where H_j are the Hamiltonians (1) in involution.

The above definition is a paraphrase of the usual definition of separability of variables in the Hamilton-Jacobi equation [1]. Note that the canonical transformation from the original variables to (x_j, p_j) may not necessarily be a pure coordinate change, as in textbooks on classical mechanics, but can involve both coordinates and momenta.

The problem in question, being the simplest of the three, is rather neglected in the literature on the subject, though, in our opinion, it deserves attention at least for two reasons. First, the variables (x_j, p_j) serve usually as a raw material for constructing action-angle variables and integrating equations of motion. Second, the problem is interesting for the theory of quantum integrability, since the construction of separated variables usually has direct counterparts in the quantum case [7].

The construction of the variables (x_j, p_j) is well known for the case of the hyperelliptic spectral curve [3, 4, 8, 9] though its relation to the separation of variables is not always stated manifestly. The coordinates x_j are defined as the zeroes of the corresponding Baker-Akhiezer function, and the canonically conjugated momenta p_j, or sometimes $\exp p_j$, usually turn out to be eigenvalues of the corresponding L-operator taken at the values of the spectral parameter equal to x_j. The functions Φ_j are then simply the characteristic polynomials of the L-operator.

In the present paper we study the problem for the nonhyperelliptic case. The $SL(3)$ classical magnetic chain is chosen as a sample toy-model. Having in mind subsequent application to the quantum integrability we make extensive use of the classical r-matrix formalism [10]. We consider a complexified version of the model in order to avoid the additional complications of the real case. Our construction of separated variables is quite elementary and does not involve any sophisticated algebro-geometric techniques.

2. Description of the Model

The model we are going to describe is the nonhomogeneous classical $SL(N)$ magnetic chain. It is in a sense generic for the models related to the $SL(N)$-invariant classical r-matrix [10]. For $N = 2$ the model was introduced in [11, 12]. The continuous version of the model was studied earlier in [13–15]. For a degenerate case (Gaudin model) see Sect. 5. The quantum version of the model is well studied by means of the Bethe ansatz method [16–18].

The model in question is described in terms of the variables $S_{\alpha\beta}^{(m)}$, $\Big(\alpha, \beta = 1, \ldots, N;\ m = 1, \ldots, M;\ \sum_{\alpha=1}^{N} S_{\alpha\alpha}^{(m)} = 0\Big)$ subject to the Poisson brackets

$$\{S_{\alpha_1\beta_1}^{(m)}, S_{\alpha_2\beta_2}^{(n)}\} = (S_{\alpha_1\beta_2}^{(m)}\delta_{\alpha_2\beta_1} - S_{\alpha_2\beta_1}^{(m)}\delta_{\alpha_1\beta_2})\delta_{mn} \tag{4}$$

which define the Kirillov-Kostant Poissonian structure on the direct product of M orbits of the coadjoint action of the Lie group $SL(N)$ on $sl(N)^*$, see e.g. [10]. It is well known that the center of the Poisson algebra is generated by the eigenvalues $l_\alpha^{(m)}$ of the matrices $S^{(m)}$

$$\det(u + S^{(m)}) = \prod_{\alpha=1}^{N}(u + l_\alpha^{(m)}), \qquad \sum_{\alpha=1}^{N} l_\alpha^{(m)} = 0. \tag{5}$$

We shall assume that $l_\alpha^{(m)}$ are fixed numbers. The Poisson bracket (4) is thus nondegenerate on the manifold (5) having dimension $2D = MN(N-1)$ for the case of generic orbit (all eigenvalues of $S^{(m)}$ are distinct). In what follows we always assume that the orbit is generic.

Let Z be an invertible $N \times N$ number matrix having N distinct eigenvalues, let δ_m ($m = 1, \ldots, M$) be some fixed numbers, and u be a complex parameter (spectral parameter). Consider the product (monodromy matrix)

$$T(u) = Z(u - \delta_M + S^{(M)}) \ldots (u - \delta_2 + S^{(2)})(u - \delta_1 + S^{(1)}). \tag{6}$$

Proposition 1. *Matrix elements of $T(u)$ have the following quadratic Poisson brackets:*

$$\{T_{\alpha_1\beta_1}(u), T_{\alpha_2\beta_2}(v)\} = \frac{1}{u - v}(T_{\alpha_2\beta_1}(u)T_{\alpha_1\beta_2}(v) - T_{\alpha_1\beta_2}(u)T_{\alpha_2\beta_1}(v)). \tag{7}$$

The proof (see [10]) is based on the fact that the factors $(u - \delta_m + S^{(m)})$ have the same Poisson brackets (7) which reproduce themselves for the product $T(u)$ (Lie-Poisson group structure).

Using the notation [10] $\overset{1}{T} \equiv T \otimes \mathrm{id}$, $\overset{2}{T} \equiv \mathrm{id} \otimes T$ one can put the formula (7) into a compact form

$$\{\overset{1}{T}(u), \overset{2}{T}(v)\} = \frac{1}{u - v}[\mathscr{P}, \overset{1}{T}(u)\overset{2}{T}(v)], \tag{8}$$

where $\mathscr{P}$ is the permutation operator in $\mathbf{C}^N \otimes \mathbf{C}^N$.

Let the spectral invariants $t_\nu(u)$ of the matrix $T(u)$ be defined as the elementary symmetric polynomials of its eigenvalues

$$t_\nu(u) \equiv \mathrm{tr} \overset{\nu}{\bigwedge} T(u), \qquad \nu = 1, \ldots, N.$$

For example,

$$t_1(u) = \mathrm{tr}\, T(u), \qquad t_2(u) = \tfrac{1}{2}(\mathrm{tr}^2\, T(u) - \mathrm{tr}\, T^2(u)), \ldots$$
$$t_N(u) = \det T(u) \equiv d(u).$$

Note that the central functions $l_\alpha^{(m)}$ are contained in the determinant $d(u) \equiv \det T(u)$, see (5).

Proposition 2. *The non-leading coefficients at powers of u of the polynomials $t_\nu(u)$, $\nu = 1, \ldots, (N - 1)$, form a commutative, with respect to the Poisson bracket (8), family of $MN(N-1)/2$ independent Hamiltonians.*

Proof. The polynomial $t_\nu(u)$ having power νM in u contributes νM Hamiltonians (its leading coefficient is a number), the total number of Hamiltonians is $M(1 + 2 + \ldots + (N - 1)) = MN(N-1)/2$. The commutativity of $t_\nu(u)$

$$\{t_\mu(u), t_\nu(v)\} = 0 \quad \forall u, v$$

is a direct consequence of the fundamental Poisson bracket (8), see [10]. The independence of the integrals of motion is proven in [19, 20] for a different model (Gaudin model, see Sect. 5) but the proof is valid also for our case. Note that the assumption made concerning nondegeneracy of the spectrum of the matrix Z is essential for the independence of $t_\nu(u)$.

By virtue of the proposition and since the number of Hamiltonians constructed $D = MN(N-1)/2$ equals exactly half dimension of the phase space the system is completely integrable. Now we can turn to the problem of constructing the separated variables.

Conjecture 1. *There exist functions $\mathscr{A}$ and $\mathscr{B}$ on $GL(N)$ such that the following two assertions are true. First, $\mathscr{A}(T)$ is an algebraic function and $\mathscr{B}(T)$, respectively, is a polynomial of degree $D = MN(N-1)/2$ of the matrix elements $T_{\alpha\beta}$. Second, the variables x_j, P_j $(j = 1, \ldots, D)$ defined from the equations*

$$\mathscr{B}(T(x_j)) = 0, \qquad P_j = \mathscr{A}(T(x_j)) \tag{9}$$

have the Poisson brackets

$$\{x_j, x_k\} = \{P_j, P_k\} = 0, \qquad \{P_j, x_k\} = P_j \delta_{jk} \tag{10}$$

and, besides, are bound to the Hamiltonians $t_\nu(u)$ by the relations

$$\det(P_j - T(x_j)) = 0. \tag{11}$$

The last relation means simply that P_j is an eigenvalue of the matrix $T(u)$ when $u = x_j$. Putting $P_j = \exp p_j$ we observe that (11) fits the form (3) since the spectral invariants of $T(u)$ contain only the integrals of motion.

In the present paper we prove Conjecture 1 for the cases $N = 2$ and $N = 3$.

3. $SL(2)$ Case

Though the construction of the separation variables for $N = 2$ is described in [8] we reproduce it here in order to fix notation and to prepare the discussion of the more difficult $N = 3$ case.

The system has M degrees of freedom. The spectral invariants of $T(u)$ are

$$t(u) \equiv t_1(u) \equiv \operatorname{tr} T(u), \qquad d(u) \equiv t_2(u) \equiv \det T(u),$$

the trace $t(u)$ containing M integrals of motion.

Define $\mathscr{A}$ and $\mathscr{B}$ as [8]

$$\mathscr{A}(T) \equiv T_{11}, \qquad \mathscr{B}(T) \equiv T_{12} \tag{12}$$

and x_j, P_j respectively by the formulas (9). For the polynomial $B(u) \equiv \mathscr{B}(T(u))$ to have M zeroes it is necessary that its leading coefficient Z_{12} be nonzero. It can always be done by a similarity transform $QT(u)Q^{-1}$ which affects neither basic Poisson brackets (8), nor Hamiltonians $t(u)$, since the matrix Z, by assumption, has nondegenerate spectrum.

Since the matrix $T(u)$ becomes triangular at $u = x_j$ the quantity P_j is an eigenvalue of $T(x_j)$ and satisfies therefore the secular equation (11) which in the two-dimensional case takes the form

$$P_j^2 - t(x_j)P_j + d(x_j) = 0, \qquad j = 1, \ldots, M.$$

Note that the secular equation defines a *hyperelliptic* algebraic curve relating P_j and x_j.

To prove Conjecture 1 it remains to calculate the Poisson brackets of P's and x's.

Theorem 1. *The Poisson brackets for P_j and x_j are given by* (10).

Proof. Let $A(u) = \mathscr{A}(T(u))$ and $B(u) = \mathscr{B}(T(u))$. Taking particular values of indices in (7) one obtains the identities

$$\{A(u), A(v)\} = 0, \tag{13}$$

$$\{B(u), B(v)\} = 0, \tag{14}$$

$$\{A(u), B(v)\} = \frac{A(u)B(v) - B(u)A(v)}{u - v}. \tag{15}$$

The commutativity of B's (14) entrains obviously the commutativity of x_j [zeroes of $B(u)$]. The Poisson brackets including P_j can be calculated using the implicit definition of x_j. From $B(x_j) = 0$ it follows that

$$0 = \{F, B(x_j)\} = \{F, B(u)\}_{u=x_j} + B'(x_j)\{F, x_j\}$$

or

$$\{F, x_j\} = -\frac{\{F, B(u)\}_{u=x_j}}{B'(x_j)}$$

for any function F. In the same way we have

$$\{P_j, F\} = \{A(x_j), F\} = \{A(u), F\}_{u=x_j} + A'(x_j)\{x_j, F\}.$$

Now it is easy to prove that $\{P_j, x_k\} = P_j \delta_{jk}$. Starting with

$$\{P_j, x_k\} = \{A(u), x_k\}_{u=x_j} + A'(x_j)\{x_j, x_k\},$$

expanding the first term further, noting that the second term is already shown to vanish (14), and using (15) we arrive at

$$\{P_j, x_k\} = -\frac{\{A(u), B(v)\}_{\substack{u=x_j \\ v=x_k}}}{B'(x_k)} = \frac{1}{x_j - x_k} \frac{B(x_j)A(x_k) - A(x_j)B(x_k)}{B'(x_k)}.$$

The last expression vanishes for $x_j \neq x_k$ due to $B(x_j) = B(x_k) = 0$ and is evaluated via L'Hôpital rule for $x_j = x_k$ to produce the proclaimed result. The commutativity of P's can be shown in the same way starting from (13).

4. $SL(3)$ Case

Let now $N = 3$. The polynomial $T(u)$ takes values in 3×3 matrices.

$$T(u) = \begin{pmatrix} T_{11}(u) & T_{12}(u) & T_{13}(u) \\ T_{21}(u) & T_{22}(u) & T_{23}(u) \\ T_{31}(u) & T_{32}(u) & T_{33}(u) \end{pmatrix}.$$

The system has $D = 3M$ degrees of freedom. The spectral invariants of the matrix $T(u)$

$$\begin{aligned} t_1(u) &\equiv \operatorname{tr} T(u) &&= \lambda_1 + \lambda_2 + \lambda_3 \\ t_2(u) &\equiv \tfrac{1}{2}(\operatorname{tr}^2 T(u) - \operatorname{tr} T^2(u)) &&= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 \\ d(u) &\equiv \det T(u) &&= \lambda_1\lambda_2\lambda_3 \end{aligned}$$

are the coefficients of the characteristic polynomial for $T(u)$

$$\det(\lambda - T(u)) = \lambda^3 - t_1(u)\lambda^2 + t_2(u)\lambda - d(u)$$

which defines a *nonhyperelliptic* algebraic curve.

It is convenient to introduce the matrix $\mathscr{U}(T)$ for any $T \in GL(3)$,

$$\begin{aligned} \mathscr{U}(T) &\equiv T \wedge T \equiv (T^{-1})^t \det T \\ &= \begin{pmatrix} T_{22}T_{33} - T_{23}T_{32} & T_{23}T_{31} - T_{21}T_{33} & T_{21}T_{32} - T_{22}T_{31} \\ T_{13}T_{32} - T_{12}T_{33} & T_{11}T_{33} - T_{13}T_{31} & T_{12}T_{31} - T_{11}T_{32} \\ T_{12}T_{23} - T_{13}T_{22} & T_{13}T_{21} - T_{11}T_{23} & T_{11}T_{22} - T_{12}T_{21} \end{pmatrix} \end{aligned}$$

whose elements $\mathscr{U}_{\alpha\beta}$ are algebraic adjuncts of $T_{\alpha\beta}$.

Let $U(u) = \mathscr{U}(T(u))$. The Poisson brackets for T and U are calculated easily from (8):

$$\{\overset{1}{T}(u), \overset{2}{U}(v)\} = -\frac{1}{u-v}[\mathscr{P}^{t_2}, \overset{1}{T}(u)\overset{2}{U}(v)], \tag{16}$$

$$\{T_{\alpha_1\beta_1}(u), U_{\alpha_2\beta_2}(v)\} = \frac{1}{u-v}\sum_{\gamma=1}^{3}(-\delta_{\alpha_1\alpha_2}T_{\gamma\beta_1}(u)U_{\gamma\beta_2}(v)$$
$$+ T_{\alpha_1\gamma}(u)U_{\alpha_2\gamma}(v)\delta_{\beta_1\beta_2}), \tag{17}$$

$$\{\overset{1}{U}(u), \overset{2}{U}(v)\} = \frac{1}{u-v}[\mathscr{P}, \overset{1}{U}(u)\overset{2}{U}(v)], \tag{18}$$

$$\{U_{\alpha_1\beta_1}(u), U_{\alpha_2\beta_2}(v)\} = \frac{1}{u-v}(U_{\alpha_2\beta_1}(u)U_{\alpha_1\beta_2}(v) - U_{\alpha_1\beta_2}(u)U_{\alpha_2\beta_1}(v)) \tag{19}$$

[the superscript t_2 in (16) denotes the transposition with respect to the second space in $\mathbf{C}^3 \otimes \mathbf{C}^3$].

The experience of the Inverse Spectral Transform Method and, in particular, the $SL(2)$ case suggests that in the $SL(3)$ case the separated coordinates x_j, $j = 1, \ldots, 3M$ should be defined as zeroes of some polynomial $B(u)$ of degree $3M$ and the corresponding momenta p_j should be bound to x_j by the secular equation

$$P_j^3 - t_1(x_j)P_j^2 + t_2(x_j)P_j - d(x_j) = 0, \qquad P_j = \exp p_j.$$

It means that P_j should be an eigenvalue of the matrix $T(x_j)$. Therefore, there must exist such a similarity transformation

$$T(x_j) \to \tilde{T}(x_j) = K_j T(x_j) K_j^{-1}$$

for each j that the matrix $\tilde{T}(x_j)$ is block-triangular

$$\tilde{T}_{12}(x_j) = \tilde{T}_{13}(x_j) = 0 \tag{20}$$

and P_j is the eigenvalue of $T(x_j)$ splitted from the upper block

$$P_j = \tilde{T}_{11}(x_j). \tag{21}$$

The problem is reduced thus to determining the polynomial $B(u)$ and the matrices K_j. Let us take the simplest possible triangular, one-parametric matrix $K(k)$

$$K(k) = \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that the matrix

$\tilde{T}(u, k)$
$\equiv K(k) T(u) K^{-1}(k)$

$$= \begin{pmatrix} T_{11}(u) + k T_{21}(u) & T_{12}(u) + k T_{22}(u) - k T_{11}(u) - k^2 T_{21}(u) & T_{13}(u) + k T_{23}(u) \\ T_{21}(u) & T_{22}(u) - k T_{21}(u) & T_{23}(u) \\ T_{31}(u) & T_{32}(u) - k T_{31}(u) & T_{33}(u) \end{pmatrix}$$

depends on *two* parameters: u and k. Therefore, we can consider the condition (20) as the set of two algebraic equations

$$\begin{aligned} \tilde{T}_{12}(x, k) &\equiv T_{12}(x) + k T_{22}(x) - k T_{11}(x) - k^2 T_{21}(x) = 0, \\ \tilde{T}_{13}(x, k) &\equiv T_{13}(x) + k T_{23}(x) = 0, \end{aligned} \tag{22}$$

for two variables x and k. Eliminating k from (22), one obtains the polynomial equation for x,

$$T_{23}(T_{12} T_{23} - T_{13} T_{22}) - T_{13}(T_{13} T_{21} - T_{11} T_{23}) = 0 \tag{23}$$

or

$$T_{23}(x) U_{31}(x) - T_{13}(x) U_{32}(x) = 0. \tag{24}$$

Since the matrix Z is assumed to have simple spectrum, the leading coefficient $\mathcal{B}(Z)$ of the polynomial $\mathcal{B}(T(u))$ can always be made nonzero by a similarity transformation $Q T(u) Q^{-1}$, Eq. (24) being thus of degree $3M$.

Expressing k from $\tilde{T}_{13} = 0$ as $k = -T_{13}(x)/T_{23}(x)$ and substituting it into the definition (21) of P one obtains

$$P = T_{11}(x) + kT_{21}(x) = -\frac{U_{32}(x)}{T_{23}(x)}. \tag{25}$$

So, we have constructed $3M$ pairs of variables x_j, P_j. To prove Conjecture 1 it remains to show that they have good Poisson brackets.

Theorem 2. *The Poisson brackets for x_j and P_j are given by* (10).

Proof. Let

$$\mathscr{A}(T) \equiv \frac{\mathscr{U}_{32}(T)}{T_{23}}, \qquad \mathscr{B}(T) \equiv T_{23}\mathscr{U}_{31}(T) - T_{13}\mathscr{U}_{32}(T). \tag{26}$$

Putting $A(u) = \mathscr{A}(T(u))$, $B(u) = \mathscr{B}(T(u))$ and using (7), (17), (19) one easily calculate the following Poisson brackets:

$$\{A(u), A(v)\} = \{B(u), B(v)\} = 0, \tag{27}$$

$$\{A(u), B(v)\} = \frac{1}{u-v}\left(A(u)B(v) - B(u)A(v)\frac{T_{23}^2(v)}{T_{23}^2(u)}\right) \tag{28}$$

from which the wanted Poisson brackets for x_j and P_j are derived immediately in the same manner as in the $SL(2)$ case.

Remark. As N. Reshetikhin pointed out to us, the polynomial $\mathscr{B}(T)$, see (26), is invariant under the similarity transform QTQ^{-1} acting on the first and second row/column, $Q \in SL(2) \subset SL(3)$. The $SL(2)$-invariance of $\mathscr{B}(T)$ entrains invariance of x_j and P_j. The meaning of this fact is still unclear.

5. Gaudin Model

The above construction of separated variables can be applied also to another integrable system – Gaudin model – which was introduced first in the quantum variant [21], see also [22, 23]. Its classical version turned out to be a useful example for developing a general group-theoretic approach to integrable systems [19, 20].

The model is formulated in terms of the same $SL(N)$ variables $S_{\alpha\beta}^{(m)}$ as in Sect. 2, see (4). Consider the matrix function

$$\mathscr{T}(u) \equiv \mathscr{Z} + \sum_{m=1}^{M} \frac{S^{(m)}}{u - \delta_m}, $$

where $\{\delta_m\}_{m=1}^{M}$ are some fixed parameters and $\mathscr{Z}$ is a traceless number matrix having N distinct eigenvalues. In contrast with $T(u)$, see (8), the matrix $\mathscr{T}(u)$ has linear Poisson brackets.

Proposition 3.

$$\{\overset{1}{\mathscr{T}}(u), \overset{2}{\mathscr{T}}(v)\} = \frac{1}{u-v}[\mathscr{P}, \overset{1}{\mathscr{T}}(u) + \overset{2}{\mathscr{T}}(v)]. \tag{29}$$

The proof is a matter of direct computation [10].

Consider now the spectral invariants $\tau_\nu(u)$ of the matrix $\mathscr{T}(u)$

$$\tau_\nu(u) \equiv \mathrm{tr}\,\mathscr{T}(u)^\nu, \qquad \nu = 1, \ldots, N.$$

Note that $\tau_\nu(u)$ is a meromorphic function of u

$$\tau_\nu(u) = \zeta_\nu + \sum_{m=1}^{M} \sum_{\alpha=1}^{\nu} \frac{\tau_{m,\nu}^\alpha}{(u - \delta_m)^\alpha}.$$

Note that $\zeta_\nu = \mathrm{tr}\,\mathscr{Z}^\nu$ and $\tau_{m,\nu}^\nu = \mathrm{tr}[S^{(m)}]^\nu = \sum_{\alpha=1}^{N} [l_\alpha^{(m)}]^\nu$ are numbers, see (5). The following proposition, analogous to Proposition 2, states the complete integrability of the system.

Proposition 4. *The quantities* $\tau_{m,\nu}^\alpha$, $(m = 1, \ldots, M; \nu = 2, \ldots, N; \alpha = 1, \ldots,$ $(\nu - 1))$, *form a commutative, with respect to the Poisson bracket* (29), *family of* $MN(N - 1)/2$ *independent Hamiltonians.*

Proof. It is easy to compute the total number of Hamiltonians $M(1 + 2 + \ldots + (N - 1)) = MN(N - 1)/2$ which equals exactly half dimension of the phase space. The commutativity of the spectral invariants of $\mathscr{T}(u)$ follows directly from (29), see [19, 20]. The independence of the Hamiltonians is also proven there.

The analog of Conjecture 1 for the Gaudin model is presented below.

Conjecture 2. *Let* $\mathscr{A}$ *and* $\mathscr{B}$ *be the same functions on* $GL(N)$ *as that in Conjecture 1. Then the variables* x_j *and* p_j *defined by the equations*

$$\mathscr{B}(\mathscr{T}(x_j)) = 0, \qquad p_j = \mathscr{A}(\mathscr{T}(x_j)) \tag{30}$$

have the canonical Poisson brackets (2) *and, besides, are bound to the Hamiltonians* $\tau_{m,\nu}^\alpha$ *by the relation* $\det(p_j - \mathscr{T}(x_j)) = 0$.

The separation of variables for the $N = 2$ and $N = 3$ cases is performed now in the same manner as, respectively, in Sects. 3 and 4.

Theorem 3. *The Conjecture 2 is true for* $N = 2$.

Proof. In the $SL(2)$ case, the functions $\mathscr{A}$ and $\mathscr{B}$ on $GL(2)$ are defined by the formulas (12). Like in Sect. 3, we can always suppose that $\mathscr{B}(\mathscr{Z}) \neq 0$. The variables $x_j, p_j, j = 1, \ldots, N$ are then determined by Eq. (30).

Let $A_G(u) = \mathscr{A}(\mathscr{T}(u))$ and $B_G(u) = \mathscr{B}(\mathscr{T}(u))$. Taking particular matrix elements of (29) one obtains

$$\{A_G(u), A_G(v)\} = \{B_G(u), B_G(v)\} = 0, \tag{31}$$

$$\{A_G(u), B_G(v)\} = -\frac{B_G(u) - B_G(v)}{u - v}. \tag{32}$$

The rest of the proof follows that of Theorem 1.

Theorem 4. *Conjecture 2 is true for* $N = 3$.

Proof. In the $SL(3)$ case define $\mathscr{A}, \mathscr{B}$ by the formulas (26) and, like in Sect. 4, suppose that $\mathscr{B}(\mathscr{Z}) \neq 0$.

Let again $A_G(u) = \mathscr{A}(\mathscr{T}(u))$ and $B_G(u) = \mathscr{B}(\mathscr{T}(u))$. It suffices to establish the Poisson brackets

$$\{A_G(u), A_G(v)\} = \{B_G(u), B_G(v)\} = 0, \tag{33}$$

$$\{A_G(u), B_G(v)\} = \frac{1}{u-v}\left(B_G(v) - B_G(u)\frac{T_{23}^2(v)}{T_{23}^2(u)}\right), \tag{34}$$

since the remaining calculation is standard.

It is possible to verify the above Poisson brackets directly, using (29). It is simpler, however, to avoid long computations and to use the fact [21, 22] that the Gaudin model is in fact a degenerate case of the magnetic chain. To be precise, let us replace $S^{(m)}$ in (6) by $\varepsilon S^{(m)}$, Z by $1 + \varepsilon\mathscr{Z}$, and divide $T(u)$ by $\prod_{m=1}^{M}(u - \delta_m)$. Then, in the first order in ε, we have $T(u)\Big/\prod_{m=1}^{M}(u - \delta_m) = 1 + \varepsilon\mathscr{T}(u) + O(\varepsilon^2)$. The Poisson brackets (29) are obtained, respectively, as the linearization of the quadratic Poisson brackets (8).

To conclude the proof, it remains to notice that

$$A(u) = 1 + \varepsilon A_G(u) + O(\varepsilon^2), \qquad B(u) = \varepsilon^3 B_G(u) + O(\varepsilon^4)$$

and that the Poisson brackets (33), (34) are obtained in the leading order in ε from (27), (28).

6. Unsolved Problems

The natural question arises whether the construction of separated variables presented here for the $SL(2)$ and $SL(3)$ cases can be generalized to the $SL(N)$ case and further, to the integrable systems associated with classical r-matrices corresponding to other simple Lie algebras. Hopefully, the generalization will elucidate the geometric and algebraic meaning of the construction.

The $SL(N)$ case is presently under study. The problem consists in finding a multiparametric family of matrices $K(k_1, \ldots, k_Q)$ such that after eliminating k's from the system $T_{12}(x) = \ldots = T_{1N}(x) = 0$ the resulting equation for x provide the necessary number of commuting zeroes. Another challenging object of study is the Kowalewski top which can be considered as a Gaudin model for $Sp(4) \simeq SO(5)$ group [24].

Since the construction of separated variables for the $SL(2)$ case has the direct quantum counterpart [6, 7, 8], it seems reasonable to conjecture that the same is true for the $SL(N)$ case.

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