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On the Classification of $N = 2$ Superconformal Coset Theories*

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Abstract. We show that two dimensional $N = 2$ superconformal field theories cannot be constructed by applying the supersymmetric extension of the GKO construction to the so-called special subalgebras, i.e. subalgebras for which at least one generator associated to a root of the subalgebra does not correspond to a root of the algebra itself. We thus prove the completeness of the classification of $N = 2$ supersymmetric coset models obtained by Kazama and Suzuki. Furthermore we point out that compared to their papers an additional criterion has to be added in the $N = 2$ conditions.

1. Introduction

Coset constructions [4] in conformal field theory have recently undergone an intensive investigation for they allow the construction of many new models within the framework of Kac Moody algebras. In [6] Kazama and Suzuki proposed to use a supersymmetric extension of the GKO construction to obtain new $N = 2$ superconformal field theories.

They considered a reductive subalgebra H of a semi-simple Lie algebra G to perform a supersymmetric coset construction, yielding in all cases an $N = 1$ superconformal field theory. They also gave a necessary and sufficient condition under which this supersymmetry should be enlarged to an $N = 2$ supersymmetry. In a later paper [7] they gave a geometrical interpretation of this criterion which was used in turn to classify all $N = 2$ coset models.

Let us adopt in this note the short-hand convention that the generator in G corresponding to a root is called a root vector. As for reductive subalgebras of reductive Lie-algebras there are two different types (compare e.g. [2,3]): the subalgebra H is called *regular* iff the root vectors of H are also root vectors of G. (The embedding $H \hookrightarrow G$ is always chosen in a way that the Cartan-subalgebra

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Cart(H) of H is mapped into the Cartan-subalgebra Cart(G) of G.) Otherwise it is called a *special* subalgebra. In particular subalgebras satisfying rank H = rank G are always regular for their Cartan-subalgebras are identical.

This note is organized as follows: in Sect. 2 we will shortly review this criterion, showing that an additional criterion has to be added and pointing out that in [7] only the regular subalgebras are dealt with. In Sect. 3 we show that it is not possible to use special subalgebras. In Sect. 4 we state our conclusions.

Throughout this note we shall adopt the following conventions (cf. [6]): let G be a semi-simple Lie-algebra, H a reductive subalgebra. Let

$$\{t^a,\, a = 1 \ldots \dim H\}$$

be a basis of H that can be extended to a basis of G

$$\{t^A,\, A = 1 \ldots \dim G\},$$

in which the Killing-form takes the form $\kappa(t^A, t^B) = \delta^{AB}$. In such a basis the structure constants are known to be totally antisymmetric. We shall denote group indices of generators belonging to G, H and G/H as $A, B, \ldots, a, b, \ldots$ and $\bar{a}, \bar{b}, \ldots$.

2. The $N = 2$ Conditions.

The $N = 2$ conditions spelled out by Kazama and Suzuki in [6] take the following form: enlarged supersymmetry is equivalent to the existence of two totally antisymmetric tensors $h_{\bar{a}\bar{b}}$ and $S_{\bar{a}\bar{b}\bar{c}}$, where the indices take their values in the coset.

They have to satisfy the conditions:

$$h_{\bar{a}\bar{b}} = -h_{\bar{b}\bar{a}} \quad h_{\bar{a}\bar{b}}h_{\bar{b}\bar{c}} = -\delta_{\bar{a}\bar{c}}\,, \tag{1}$$

$$h_{\bar{a}\bar{b}}f_{\bar{b}\bar{c}e} = f_{\bar{a}\bar{b}e}h_{\bar{b}\bar{c}}\,, \tag{2}$$

$$f_{\bar{a}\bar{b}\bar{c}} = h_{\bar{a}\bar{b}}h_{\bar{b}\bar{q}}f_{\bar{p}\bar{q}\bar{c}} + \text{cyclic permutations in } \bar{a},\, \bar{b} \text{ and } \bar{c}\,, \tag{3}$$

$$S_{\bar{a}\bar{b}\bar{c}} = h_{\bar{a}\bar{p}}h_{\bar{b}\bar{q}}h_{\bar{c}\bar{r}}f_{\bar{p}\bar{q}\bar{r}}\,. \tag{4}$$

(1) means that h is a complex structure on G/H, which is H-invariant by (2). (3) is a consistency condition, while (4) can be used to eliminate S in this problem.

We claim that these conditions are equivalent to the subsequent one:

Theorem. *Let t be the orthogonal complement of H with respect to the Killing-form κ of G. (G is semi-simple, so κ is non-degenerate.) The model $[G/H]$ is $N = 2$ supersymmetric if and only if there exists a direct sum decomposition of vector spaces:*

$$t = t_+ \oplus t_- \tag{5}$$

(direct sum of vector spaces), where $\dim t_+ = \dim t_-$*, such that t_+ and t_- separately form closed Lie algebras and*

$$\kappa|_{t_\pm} \equiv 0\,, \tag{6}$$

when restricted to $t_\pm$ respectively.

Remark. Note that compared to [7] we have added the condition (6).

Proof. We shall use the ideas outlined in [7], but will dwell on (6).

Suppose first that $[G/H]$ is $N = 2$ supersymmetric. Define $t_\pm$ to be the eigenspaces corresponding to the eigenvalues $\pm i$ of the complex structure h. Then $t = t_+ \oplus t_-$, $\dim t_+ = \dim t_-$ is immediate. Using (1)–(4) it is easy to show that

$$[t_\pm^{\bar{a}}, t_\pm^{\bar{b}}] = i/2(f_{\bar{a}\bar{b}\bar{c}} \pm 1/iS_{\bar{a}\bar{b}\bar{c}})t_\pm^{\bar{c}} ,$$

where $t_\pm^{\bar{a}}$ denotes the component of $t^{\bar{a}}$ in $t_\pm$. $t_\pm$ thus close under the Lie-bracket. Let $h_\pm^{(1)}$, $h_\pm^{(2)} \in t_\pm$ be arbitrary elements. Calculating by use of the antisymmetry (1) of h yields:

$$\begin{aligned}\kappa(h_\pm^{(1)}, h_\pm^{(2)}) &= (\pm 1/i)\kappa(hh_\pm^{(1)}, h_\pm^{(2)}) = -(\pm 1/i)\kappa(h_\pm^{(1)}, hh_\pm^{(2)}) \\ &= -\kappa(h_\pm^{(1)}, h_\pm^{(2)}) = 0 .\end{aligned}$$

On the other hand, given a decomposition like (5), define h by requiring $t_\pm$ to be the eigenspaces of h corresponding to the eigenvalues $\pm i$, assuring that the second equation of (1) is fulfilled. Then (2), (3) can be shown to follow from the fact that $t_\pm$ are subalgebras, while (6) implies the first part of (1): let $r, s \in t$ be arbitrary elements, then $r = r_+ + r_-$, $s = s_+ + s_-$, where $s_\pm, r_\pm \in t_\pm$. Then

$$\begin{aligned}\kappa(hr, s) &= \kappa(ir_+ - ir_-, s_+ + s_-) = i\kappa(r_+, s_-) - i\kappa(r_-, s_+) \\ &= -\kappa(r_+ + r_-, is_+ - is_-) = -\kappa(r, hs).\end{aligned}$$ □

In [7] a sequential method is used to reduce the case rank $H <$ rank G to the equal rank case. This method necessitates the existence of an intermediate subalgebra, satisfying:

$$H \subseteq H \oplus U(1)^{\text{rank}\, G - \text{rank}\, H} \subseteq G \tag{7}$$

(direct sum of Lie-algebras).

Proposition. *Such an algebra exists if and only if H is a regular subalgebra.*

Proof. First, let H be a regular subalgebra. Then the root space H^* of H can be canonically embedded into the root space G^* of G. Let

$$\{\beta^{(i)},\ i = 1, \ldots \text{rank}\, G - \text{rank}\, H\}$$

be a basis for the orthogonal complement of H^* in G^* relative to the Killing-form. The generators of the $U(1)$-factors can then be shown in a Cartan-Weyl basis to be:

$$\left\{ (\beta^{(i)}, H) = \sum_{j=1}^{\text{rank}\, G} \beta_j^{(i)} H^j , i = 1, \ldots \text{rank}\, G - \text{rank}\, H \right\}.$$

Conversely, suppose there is a chain of subalgebras like (7). Let E be a root vector of H. An arbitrary element $h \in \text{Cart}(G)$ can be decomposed according to (7) like

$$h = h' + \sum_{i=1}^{\text{rank}\, G - \text{rank}\, H} u_i .$$

$h' \in \text{Cart}(H)$, u_i multiples of the generators of the $U(1)$ factors. Due to the direct sum structure in (7) one finds:

$$\text{ad}_h(E) = \left[h' + \sum u_i, E\right] = [h', E] \propto E .$$

Thus every root vector of H is also a root vector of G. □

3. Special Subalgebras

One may now ask whether it is possible to use special subalgebras in order to construct new $N = 2$ supersymmetric coset models. We give a negative answer by the following

Theorem. *Let $H \hookrightarrow G$ be a special subalgebra, H reductive, G semisimple. Then the model $[G/H]$ cannot be $N = 2$ supersymmetric.*

Proof. Indirect proof
Let $\Phi_{\pm}^{G/H}$ denote the set of root vectors corresponding to the positive respectively negative roots of G respectively H, $\langle \Phi_{\pm}^{G/H} \rangle$ the vector spaces generated by the corresponding set. Recall that we have chosen the embedding such that Cart(H) $\hookrightarrow$ Cart(G).

Lemma 1. *Without loss of generality we can assume that*

$$t_{\pm} \subseteq \text{Cart}(G) \oplus \langle \Phi_{\pm}^{G} \rangle .$$

Proof. Equation (6) implies (see e.g. [3, p. 20]) that $t_{\pm}$ is solvable and thus contained in a maximal solvable subalgebra, a *Borel-subalgebra.* As is shown in [5, p. 84] any two Borel subalgebras are conjugated under inner automorphisms, which are known to let the Cartan-subalgebra fix (compare e.g. [3, p. 106]). The result now follows from the fact that

$$b_{\pm} := \text{Cart}(G) \oplus \langle \Phi_{\pm}^{G} \rangle$$

are Borel subalgebras. □

Lemma 2. *The positive roots of H can be chosen in a way to guarantee*

$$\langle \Phi_{\pm}^{H} \rangle \subseteq \langle \Phi_{\pm}^{G} \rangle .$$

Proof. We can argue like in the proof of Lemma 1, but have to use automorphisms and Borel subalgebras of H this time, to deduce

$$\langle \Phi_{\pm}^{H} \rangle \subseteq \text{Cart}(G) \oplus \langle \Phi_{\pm}^{G} \rangle .$$

Let $E_+ = \eta_+ + h_0$ be a root vector of H, $h_0 \in \text{Cart}(G)$, $\eta_+ \in \langle \Phi_{+}^{G} \rangle$. There is a $h_H \in \text{Cart}(H)$ with

$$[h_H, E_+] = \lambda(h_H) E_+ , \tag{8}$$

where $\lambda(h_H) \neq 0$. (The roots are non-zero regarded as functionals on Cart(H).)

Suppose $h_0 \neq 0$. G is semisimple, so κ is not degenerate, i.e. there is $h' \in \text{Cart}(G)$ with $\kappa(h', h_0) \neq 0$. Now

$$\kappa(h', [h_H, E_+]) = \kappa(h', \lambda(h_H) E_+) = \lambda(h_H) \kappa(h', h_0) \neq 0 .$$

But using the invariance of κ we find a contradiction:

$$\kappa(h', [h_H, E_+]) = \kappa([h', h_H], E_+) = 0 .$$ □

Proof of the theorem. H being a special subalgebra there is a root vector E^+ of H (let E^- denote the root vector corresponding to the negative root) and $h_0 \in \text{Cart}(G)$ such that $[h_0, E^\pm]$ is not proportional to $E^\pm$. We may indeed assume that $h_0 \in t \cap \text{Cart}(G)$. Thus

$$Y_\pm := [h_0, E^\pm] \neq 0, \quad Y_\pm \in [H, t] \subseteq t\,,$$

where the last inclusion can be deduced from the antisymmetry of the structure constants of G in our basis and the fact that H closes under the Lie-bracket. Lemma 2 implies

$$E^+ = \sum_i \lambda_i E^{\alpha_i}, \text{ where all coefficients are non-zero.}$$

In the generic case the coefficients are complex numbers, but we will assume that they are real by absorbing the complex phase in the definition of E^{α_i}.
Thus

$$E^- = \sum_i \lambda_i E^{-\alpha_i}$$

and

$$Y_\pm = [h_0, E^\pm] = \pm \sum_i \lambda_i \alpha_i(h_0) E^{\pm\alpha_i} \in \langle \Phi^G_\pm \rangle$$

yielding

$$Y_\pm \in \langle \Phi^G_\pm \rangle \cap t \subseteq t_\pm$$

by Lemma 1.

We now claim that restricted to $V_\pm := \langle E^{\pm\alpha_i} \rangle_R \cap t$ the functional

$$f_\pm(\cdot) := \kappa(\cdot, Y_\mp)$$

does not vanish. The subscript R indicates that we consider real linear combinations of the $E^{\pm\alpha_i}$ only.

Using the well-known properties of the Killing-form in a Cartan-Weyl-basis it is clear that $f_\pm$ vanishes on the orthogonal complement of the complexification of $V_\pm$. The Killing-form is not degenerate and $Y_\pm \neq 0$, so $f_\pm$ cannot vanish on the complexification of $V_\pm$ and hence on $V_\pm$.

We have thus proven the existence of

$$s_+ = \sum \mu_i E^{\alpha_i} \in t_+, \;\; \mu_i \text{ real}$$

satisfying

$$\kappa(s_+, Y_-) \neq 0\,. \tag{9}$$

The hermitian conjugate $s_- = \sum \mu_i E^{-\alpha_i} \in t_-$ obeys $\kappa(s_-, Y_+) \neq 0$.

Let P denote the projector on Cart(G). Using (9) and the invariance of κ we see

$$0 \neq \kappa(s_\pm, Y_\mp) = \kappa([s_\pm, E^\mp], h_0)\,. \tag{10}$$

On the other hand

$$P[s_\pm, E^\mp] = \pm \sum \mu_i \lambda_i (\alpha_i H) \tag{11}$$

shows that the two vectors on the left-hand side corresponding to the upper respectively lower choice of the signs are equal up to sign.

Now suppose there is an $N = 2$ supersymmetric model. Equation (5) yields a direct sum decomposition

$$\mathrm{Cart}(G) \cap t = (\mathrm{Cart}(G) \cap t_+) \oplus (\mathrm{Cart}(G) \cap t_-)\,, \tag{12}$$

in particular we can decompose

$$h_0 = h_0^+ + h_0^-\,, \quad h_\pm \in t_\pm \cap \mathrm{Cart}(G)\,.$$

$t_\pm$ have to be subalgebras, so for all $h_\pm \in t_\pm \cap \mathrm{Cart}(G)$ we need

$$[h_\pm, s_\pm] \in t_\pm\,.$$

We thus find due to the orthogonality of t and H,

$$\begin{aligned} &\kappa([h_\pm, s_\pm], E^\mp) = 0 \\ \Leftrightarrow\ &\kappa(h_\pm, [s_\pm, E^\mp]) = 0\,, \end{aligned} \tag{13}$$

and using (11),

$$\kappa(h_\mp, [s_\pm, E^\mp]) = 0\,. \tag{14}$$

Calculating and inserting the results (13) and (14) we find

$$\kappa(h_0, [s_\pm, E^\mp]) = \kappa(h_0^+ + h_0^-, [s_\pm, E^\mp]) = 0$$

in contradiction to Eq. (10). □

4. Conclusion

The theorem proven in Sect. 3 assures indeed the completeness of the classification of $N = 2$ superconformal coset models obtained in [7]. All these models are thus given for simple G by regular subalgebras for which both $\mathrm{rank}\, G - \mathrm{rank}\, H = 2n$, $n = 0, 1, \ldots$ and $G/H \times U(1)^{2n}$ is kählerian.

One may wonder why special subalgebras have not been dealt with in [7]. One reason may have been that, since special subalgebras necessarily have

$$\mathrm{rank}\, H < \mathrm{rank}\, G\,, \tag{15}$$

they are a priori not so tempting for superstring theories, which have inspired much of the work in this field. Indeed spacetime supersymmetry requires not only $N = 2$ world-sheet supersymmetry, but also integral charges for the $U(1)$ of the $N = 2$ superconformal algebra [1]. Now subalgebras satisfying (15) are known not to fulfill this condition unless they are twisted [8]. This explains the focussing of interest on equal rank models and thus on regular subalgebras.

We finally point out that our additional condition (6) in the $N = 2$ criterion does not affect the results in [7]. This is because the criterion is applied there in the equal rank case only, where

$$t_\pm \subseteq \langle \Phi_\pm^G \rangle\,,$$

i.e. (6) is automatically fulfilled.

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