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On the Measure of Gaps and Spectra for Discrete 1D Schrödinger Operators*

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Abstract. We study the Lebesgue measure of gaps and spectra, of ergodic Jacobi matrices. We show that: $|\sigma\backslash A| + |G| \geq v$, where: σ is the spectrum, G is the union of the gaps, A is the set of energies where the Lyaponov exponent vanishes and v is an appropriate seminorm of the potential. We also study in more detail periodic Jacobi matrices, and obtain a lower bound and large coupling asymptotics for the measure of the spectrum. We apply the results of the periodic case, to limit periodic Jacobi matrices, and obtain sufficient conditions for $|G| \geq v$ and for $|\sigma| > 0$.

1. Introduction

In this paper, we study the Lebesgue measure of gaps and spectra, of one dimensional ergodic Jacobi matrices. These are families of operators H_ω on $l^2(Z)$, defined by:

$$H_\omega = H_0 + V_\omega\,, \qquad (H_0 u)(n) = u(n+1) + u(n-1)\,, \qquad (V_\omega u)(n) = V_\omega(n)u(n)\,,$$

where V_ω is a (real) stationary bounded ergodic potential, that is: we consider a probability measure space (Ω, dp), a measure preserving invertible ergodic transformation T, and a bounded measurable real-valued function f, and define: $V_\omega(n) = f(T^n\omega)$.

For such a family $\{V_\omega\}$, it is known [3] that there is a subset σ of R such that $\sigma_\omega \equiv \mathrm{Spec}\,(H_\omega) = \sigma$ for a.e. ω, and that the Lyaponov exponent $\gamma(E)$ exists for every $E \in R$. We denote: $A \equiv \{E \mid \gamma(E) = 0\}$, and $G \equiv [\min \sigma, \max \sigma]\backslash\sigma$, so that G is the union of all the gaps in σ. For each ω we define:

$$v_\omega \equiv \sup_n V_\omega(n) - \inf_n V_\omega(n)\,.$$

Since v_ω is T invariant, we have: $v_\omega = E(v_\omega) \equiv v$ for a.e. ω, where $E(\cdot)$ denotes an integral over ω.

Subclasses of ergodic potentials, that we discuss in more detail, are: periodic potentials and limit periodic potentials. For these classes we consider individual potentials

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$V = \{V(n)\}$ and denote: $H_V \equiv H_0 + V$, $\sigma \equiv \mathrm{Spec}(H_V)$, $v \equiv \sup_n V(n) - \inf_n V(n)$, A and G as before. A potential V is called periodic with period p (p-positive integer), if $V(n+p) = V(n)$ for all n. It is called limit periodic, if it is a uniform limit of periodic potentials, i.e. if there is a sequence $\{V_m\}$ of periodic potentials such that: $\|V - V_m\| \to 0$ as $m \to \infty$. It is a fundamental fact that if V is periodic then: σ is identical to A, and it is also identical to the set of all real E's for which the equation:

$$u(n+1) + u(n-1) + V(n)u(n) = Eu(n) \tag{1.1}$$

has a bounded solution. In this case σ is known to consist of p bands (closed intervals), separated by $p-1$ gaps (some of which may be absent).

The problem of estimating the measure of the spectrum was studied by several authors [1, 8–10] for Harper's equation ($V_\omega(n) = \lambda\cos(2\pi\alpha n + \omega)$). Deift and Simon [4] have obtained the general result (for ergodic potentials): $|A| \leq 4$, and that the equality holds if and only if the potential is a constant.

In this paper, we prove for general ergodic potentials:

Theorem 1.

$$|\sigma \backslash A| + |G| \geq v\,,$$

where $|\cdot|$ denotes Lebesgue measure.

Remark. By Ishii-Pastur-Kotani Theorem [3], the essential closure of A, is the a.c. part of σ, and thus we have: $|\sigma \backslash A| = |\sigma| - |A|$. Since:

$$\max\sigma - \min\sigma \leq 4 + v\,, \tag{1.2}$$

Theorem 1 implies:

$$|A| = |\sigma| - |\sigma\backslash A| = \max\sigma - \min\sigma - |G| - |\sigma\backslash A| \leq 4\,,$$

which is the Deift-Simon result. Therefore, Theorem 1 is related to this result, but stronger.

For periodic potentials, we prove:

Theorem 2. *If V is periodic, with period p, then:*

$$|\sigma| > \frac{4}{(2+v)^{p-1}}\,.$$

Theorem 3. *Let V be a periodic potential, and let η be a positive "coupling constant," then for the potential ηV, in the limit $\eta \to \infty$:*

$$|\sigma| = O(\eta^{1-m})\,,$$

where m is defined by:

$$m = \min\{d(n)\}\,,$$
$$d(n) = \max\{j - k \mid j > k\,,\ V(j) = V(k) = V(n)\,,\ V(l) \neq V(n)\ \forall j > l > k\}$$

(m is the minimum of the distances $\{d(n)\}$, where $d(n)$ is the maximal distance along the potential, between two potential points that have the same value of $V(n)$).

And for limit periodic potentials, we prove:

Theorem 4. *Let V be a (limit periodic) potential, C and α positive numbers, and let $\{V_m\}$ be a sequence of periodic potentials, with periods: $p_m \to \infty$, then:*

(i) *if* $\|V - V_m\| \le C p_m^{-(1+\alpha)}$ *for all* m, *then:* $|G| \ge v$ *and therefore* $|\sigma| \le 4$.
(ii) *if* $\|V - V_m\| \le C m^{-\alpha p_{m+1}}$ *for all* m, *then:* $|\sigma| > 0$.

It should be pointed that our proof of Theorem 1 makes use of ideas used by Craig [2], to prove the existence of certain trace formulas. Craig considers continuous Schrödinger operators of the form:

$$H_\omega = -\frac{d^2}{dx^2} + V_\omega(x) \tag{1.3}$$

on $L^2(R)$, and it should not be hard to derive from his work an analog of Theorem 1 for the continuous case. Namely: let H_ω be defined by (1.3), with V_ω continuous bounded stationary ergodic potential, then:

$$|\sigma \backslash A| + |G| \ge \tfrac{1}{2} v\,, \tag{1.4}$$

where σ, A, G, and v are defined analogously to the discrete case (note the factor $\frac{1}{2}$ on the right side, which does not exist in the discrete case). It is also worth pointing out the existence of some asymptotic results for the bandwidth of continuous periodic Schrödinger operators, which are related to Theorem 3 (see [12]).

In Sect. 2 we review some standard results of the theory of periodic Jacobi matrices, and we prove Theorem 1 for the case of periodic potentials. In Sect. 3 we prove Theorem 2, and in Sect. 4 we prove Theorem 3. In Sect. 5 we prove Theorem 4, and finally, in Sect. 6 we prove Theorem 1 for the general ergodic case.

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2. Periodic Jacobi Matrices: A Review

For every periodic potential V, with period p, and for every $k \in R$, $l \in Z$, define:

$$A_l(k) \equiv \begin{pmatrix} V(l+1) & 1 & & & e^{-ikp} \\ 1 & V(l+2) & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ e^{ikp} & & & 1 & V(l+p) \end{pmatrix}$$

$$A_l^- \equiv \begin{pmatrix} V(l+2) & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & V(l+p) \end{pmatrix},$$

then:

Proposition 2.1. (i) $E \in \sigma$ *if and only if Eq.* (1.1) *has a solution* $\{u(n)\}$ *obeying:*

$$u(n+p) = e^{ikp} u(n) \tag{2.1}$$

for all n, *and some real* k.

(ii) *The* p *eigenvalues of* $A_l(k)$ *are independent of* l, *and:*

$$\sigma = \bigcup_k \operatorname{Spec}(A_l(k))\,. \tag{2.2}$$

(iii) *For every l, the p − 1 eigenvalues of* A_l^- *are simple.*

(iv) *The eigenvalues of* $A_l(k)$ *are at most doubly degenerate, and if a given eigenvalue of* $A_l(k)$ *is degenerate then it is also an eigenvalue of* A_l^-.

(v) *For every l and k, the eigenvalues of* A_l^- *separate the eigenvalues of* $A_l(k)$. *That is:*

$$E_1(k) \leq E_{l,1}^- \leq E_2(k) \leq E_{l,2}^- \leq \ldots \leq E_{l,p-1}^- \leq E_p(k) .$$

(vi) *The characteristic polynomial of* $A_l(k)$ *obeys:*

$$\det(z - A_l(k)) = \Delta(z) - 2\cos kp , \tag{2.3}$$

where $\Delta(z)$ *is a polynomial with real coefficients, independent of k and l.*

(vii) σ *is the set of all real E's, for which* $|\Delta(E)| \leq 2$. *It is made of p bands, such that on each band* $\Delta(E)$ *is either increasing or decreasing.*

(viii) *For every l, the eigenvalues of* A_l^- *lie between the bands of* σ. *That is: they lie either in band edges or inside gaps in* σ, *and there is one eigenvalue of* A_l^- *between every two bands.*

Proof. (i) This is the well known Floquet's theorem (also known as Bloch's theorem). For proof see [11].

(ii) If $\{u(n)\}$ is a solution of Eq. (1.1), obeying (2.1), then $(u(l+1), \ldots, u(l+p))^T$ is an eigenvector of $A_l(k)$, and E is the corresponding eigenvalue. Similarly any eigenvector of $A_l(k)$ determines a solution of Eq. (1.1), that obeys (2.1) for the appropriate k. This shows that eigenvalues of $A_l(k)$ are independent of l, and from (i) we obtain (2.2).

(iii) Simplicity of eigenvalues holds [5] for any real tridiagonal matrix $(a_{i,j})$ of order n, obeying $a_{i,i+1}a_{i+1,i} > 0$ for $i = 1, \ldots, n-1$.

(iv) Define:

$$f(z) \equiv \frac{\det(z - A_l^-)}{\det(z - A_l(k))} , \tag{2.4}$$

and let $\{|n\rangle\}_{n=1}^p$ be the standard basis of C^p, $\{E_n\}_{n=1}^p$ the eigenvalues of $A_l(k)$ and $\{|E_n\rangle\}_{n=1}^p$ the corresponding normalized eigenvectors, then we have (using Dirac notation):

$$f(z) = ((z - A_l(k))^{-1})_{11} = \left\langle 1 \left| \sum_{n=1}^{p} \frac{|E_n\rangle\langle E_n|}{z - E_n} \right| 1 \right\rangle = \sum_{n=1}^{p} \frac{|\langle 1 \mid E_n\rangle|^2}{z - E_n} . \tag{2.5}$$

According to (2.5) $f(z)$ has only simple poles. Since the eigenvalues of A_l^- are simple, we obtain statement (iv).

(v) It is enough to show that for each pair of consecutive eigenvalues of $A_l(k)$: $E_j \leq E_{j+1}$, there is an eigenvalue E^- of A_l^-, such that: $E_j \leq E^- \leq E_{j+1}$. We consider 3 cases:

a) Suppose $\langle 1 \mid E_n\rangle = 0$ for $n = j$ or $n = j+1$, then E_n is also an eigenvalue of A_l^- with the eigenvector: $(\langle 2 \mid E_n\rangle, \ldots, \langle p \mid E_n\rangle)^T$.

b) Suppose $E_j = E_{j+1}$, then according to statement (iv), E_j is also an eigenvalue of A_l^-.

c) Suppose $E_j < E_{j+1}$ and $\langle 1 \mid E_n\rangle \neq 0$ for $n = j$ and $n = j+1$, then according to (2.5), $f(z)$ has poles at E_j and E_{j+1}, and a zero at some point on the real axis between E_j and E_{j+1}. This zero is an eigenvalue of A_l^-.

(vi) (2.3) is obtained by "expanding" the determinant: $\det(z - A_l(k))$ in minors, starting with the first line. That $\Delta(z)$ is independent of l, follows from (ii).

(vii) This statement easily follows from (ii), (iv), and (vi).

(viii) Since the band edges are eigenvalues of $A_l(0)$ and $A_l\left(\frac{\pi}{p}\right)$, this statement easily follows from (v), (vi), and (vii). □

Remarks.

a) Proposition 2.1 is a version of "Floquet's theory" for periodic Jacobi matrices. See e.g. [11].
b) Solutions of Eq. (1.1), obeying (2.1), are called: *Bloch wave solutions*, and the corresponding k's are called: *Bloch wave numbers*. From statement (vi), it is enough to consider k in the interval $\left[0, \frac{\pi}{p}\right]$, and this is assumed throughout the rest of this paper.
c) Since eigenvalues of $A_l(k)$ are independent of l, we denote them as $\{E_n(k)\}$ without any reference to l. Throughout the paper, we sometimes refer to $A_l(k)$ without saying anything about l. This should be understood in the sense of "pick some l". As can be seen from statement (vi), each $E_n(k)$ is a strongly monotone C^∞ function of k, from the interval $\left[0, \frac{\pi}{p}\right]$ onto the n^{th} band of σ.
d) $\Delta(E)$ defined by statement (vi), is called the *Discriminant*.
e) Eigenvectors of A_l^-, are solutions of Eq. (1.1) on the "interval" $\{l+1, l+2, \ldots, l+p+1\}$ with Dirichlet boundary conditions, and the corresponding eigenvalues are known in the literature (see e.g. [11]) as *Auxiliary spectra.*

To conclude this section, we show how Proposition 2.1 leads to a simple proof of Theorem 1, for the periodic case. Namely, we prove that if V is periodic then: $|G| \geq v$.

Proof. For every l, denote by $\{E_{l,n}^-\}_{n=1}^{p-1}$ the increasingly ordered eigenvalues of A_l^-, then according to statement (viii) of Proposition 2.1, for every l and m:

$$|G| \geq \sum_{n=1}^{p-1} |E_{l,n}^- - E_{m,n}^-| \geq |\mathrm{Tr}(A_l^-) - \mathrm{Tr}(A_m^-)| = |V(m+1) - V(l+1)| \,. \tag{2.6}$$

Since this holds for every l and m, we obtain:

$$|G| \geq v \,. \quad \square \tag{2.7}$$

3. Proof of Theorem 2

Proof. Let l be some integer. For every real number x, define:

$$B(x) \equiv A_l\left(\frac{\pi}{2p}\right) + x|1\rangle\langle 1| = \begin{pmatrix} V(l+1)+x & 1 & & & e^{-i\frac{\pi}{2}} \\ 1 & V(l+2) & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ e^{i\frac{\pi}{2}} & & & 1 & V(l+p) \end{pmatrix}.$$

For the characteristic polynomial of $B(x)$, we have:

$$\det(z-B(x)) = \det\left(z-A_l\left(\frac{\pi}{2p}\right)\right) - x\det(z-A_l^-) = \Delta(z) - x\det(z-A_l^-)\,. \quad (3.1)$$

Since the value of the discriminant $\Delta(E)$ runs between 2 and -2 on each band of σ, it is seen from (3.1) that if x is such that:

$$|x\det(E-A_l^-)| < 2 \quad (3.2)$$

for every $E \in [\min\sigma, \max\sigma]$ then $\mathrm{Spec}(B(x)) \subset \sigma$ and $B(x)$ has one eigenvalue inside each band of σ. Obviously, the same holds for $B(-x)$ and it follows [similarly to (2.6)] that for such x:

$$|\sigma| > |\mathrm{Tr}(B(x)) - \mathrm{Tr}(B(-x))| = 2|x|\,. \quad (3.3)$$

We have:

$$|\det(E-A_l^-)| = \prod_{n=1}^{p-1} |E - E_{l,n}^-| < \left(\frac{1}{p-1}\sum_{n=1}^{p-1}|E-E_{l,n}^-|\right)^{p-1}, \quad (3.4)$$

and from the fact that $\{E_{l,n}^-\} \subset [\min\sigma, \max\sigma] \subseteq [-2+\min\{V(n)\}, 2+\max\{V(n)\}]$ we have for every $E \in [\min\sigma, \max\sigma]$:

$$\begin{aligned}\sum_{n=1}^{p-1}|E-E_{l,n}^-| &\le \max\left\{\sum_{n=1}^{p-1}|\min\sigma - E_{l,n}^-|, \sum_{n=1}^{p-1}|\max\sigma - E_{l,n}^-|\right\}\\ &= \max\{-(p-1)\min\sigma + \mathrm{Tr}(A_l^-), (p-1)\max\sigma - \mathrm{Tr}(A_l^-)\}\\ &\le (p-1)(2+v)\,. \end{aligned} \quad (3.5)$$

From (3.4) and (3.5) we obtain:

$$|\det(E-A_l^-)| < (2+v)^{p-1}\,, \quad (3.6)$$

and thus (3.2) with

$$x = \frac{2}{(2+v)^{p-1}}$$

holds for every $E \in [\min\sigma, \max\sigma]$. From (3.3) the theorem now follows. □

Remark. The above proof is essentially "global" in the sense that it proves Theorem 2 for the total measure of the spectrum, without providing equivalent information on the width of individual bands. It is worth pointing out that a "local" result, which is almost as strong, also holds, and this can be seen as follows:

From Proposition 2.1, we have on the n^{th} band of σ:

$$\Delta(E_n(k)) = 2\cos kp\,, \quad (3.7)$$

from which we obtain by differentiation:

$$\frac{dE_n(k)}{dk} = \frac{-2p\sin kp}{\Delta'(E_n(k))}\,, \quad (3.8)$$

where $\Delta'(E) \equiv \dfrac{d\Delta(E)}{dE}$. Since $\Delta'(E)$ is a polynomial of degree $p-1$, whose zeroes separate the eigenvalues of $A_l(k)$, it can be shown, using similar considerations to (3.4) and (3.5), that for $E \in [\min \sigma, \max \sigma]$:

$$|\Delta'(E)| < p \left[\frac{p}{p-1} (2+v) \right]^{p-1} < pe(2+v)^{p-1} . \tag{3.9}$$

Thus, we obtain from (3.8):

$$\left| \frac{dE_n(k)}{dk} \right| > \frac{2 \sin kp}{e(2+v)^{p-1}} . \tag{3.10}$$

Since the width of the n^{th} band is given by: $\int_0^{\frac{\pi}{p}} \left| \frac{dE_n(k)}{dk} \right| dk$, we obtain for the width |band| of every band in σ:

$$|\text{band}| > \frac{4/e}{p(2+v)^{p-1}} . \tag{3.11}$$

4. Proof of Theorem 3

Lemma 4.1. *Let n be in $\{1, \ldots, p\}$, k in $\left(0, \frac{\pi}{p}\right)$ and let $E_n(k)$ be the n^{th} eigenvalue of $A_l(k)$ and $\{u_n(k,m)\}_{m=-\infty}^{\infty}$ a corresponding normalized Bloch wave solution of Eq. (1.1), then for every integer m:*

$$\frac{dE_n(k)}{dk} = 2p \, \text{Im} \left[u_n(k,m) u_n^*(k,m+1) \right] ,$$

where * denotes complex conjugation.

Remark. The correspondence and normalization of the Bloch wave solution are by the fact that: $|u_n^l(k)\rangle \equiv (u_n(k,l+1), \ldots, u_n(k,l+p))^T$ is a normalized eigenvector of $A_l(k)$, that corresponds to the eigenvalue $E_n(k)$.

Proof. Let δk be a "small" variation of k, and $\delta E_n(k)$ the corresponding variation of $E_n(k)$. To first order in δk, we have: $A_l(k+\delta k) = A_l(k) + \delta A_l(k)$, where:

$$\delta A_l(k) = ip\delta k \begin{pmatrix} & \cdots & 0 & -e^{-ikp} \\ \vdots & & & 0 \\ 0 & & & \vdots \\ e^{ikp} & 0 & \cdots & \end{pmatrix} . \tag{4.1}$$

Since $k \in \left(0, \frac{\pi}{p}\right)$, $E_n(k)$ is nondegenerate (Proposition 2.1), and therefore to first order in δk we have:

$$\begin{aligned} \delta E_n(k) &= \langle u_n^l(k) | \, \delta A_l(k) \, | u_n^l(k) \rangle \\ &= ip\delta k[-e^{-ikp} u_n(k,l+p) u_n^*(k,l+1) \\ &\quad + e^{ikp} u_n(k,l+1) u_n^*(k,l+p)] . \end{aligned} \tag{4.2}$$

Using: $u_n(k, l+p) = e^{ikp} u_n(k, l)$, we obtain from (4.2):

$$\frac{dE_n(k)}{dk} = 2p \, \mathrm{Im} \, [u_n(k,l) u_n^*(k, l+1)] \,,$$

and since l can be chosen at will, the lemma is proven. □

Proof of Theorem 3. Denote: $\lambda = \frac{1}{\eta}$, then the equation:

$$\lambda u(n+1) + \lambda u(n-1) + V(n) u(n) = E u(n) \,, \tag{4.3}$$

has the same Bloch wave solutions of Eq. (1.1) with the potential ηV, and the matrix:

$$A_l(\lambda, k) \equiv \begin{pmatrix} V(l+1) & \lambda & & & \lambda e^{-ikp} \\ \lambda & V(l+2) & \lambda & & \\ & \lambda & \ddots & \ddots & \\ & & \ddots & \ddots & \lambda \\ \lambda e^{ikp} & & & \lambda & V(l+p) \end{pmatrix} , \tag{4.4}$$

has the same eigenvectors of the matrix $A_l(k)$ for the potential ηV. In the limit $\lambda \to 0$ ($\eta \to \infty$) these eigenvectors (and thus the Bloch wave solutions) can be obtained from perturbation theory, by defining:

$$A_0(k) \equiv \begin{pmatrix} 0 & 1 & & & e^{-ikp} \\ 1 & 0 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ e^{ikp} & & & 1 & 0 \end{pmatrix} ,$$
$$B_l \equiv \begin{pmatrix} V(l+1) & & & \\ & V(l+2) & & \\ & & \ddots & \\ & & & \ddots & \\ & & & & V(l+p) \end{pmatrix} , \tag{4.5}$$

and considering $\lambda A_0(k)$ as a perturbation of B_l.

We consider at first, the case of a potential which is nondegenerate along one period, so that: $V(i) \neq V(j)$ for $i \neq j$, $i, j \in \{l+1, \ldots, l+p\}$. In this case, eigenvalues of B_l are nondegenerate, and the corresponding normalized eigenvectors are the standard basis $\{|n\rangle\}_{n=1}^{p}$ of C^p. For each n, an eigenvector of $A_l(\lambda, k)$ corresponding to $|n\rangle$, and to the eigenvalue $V(l+n)$ of B_l, is given by: $P_n(l, k, \lambda) \, |n\rangle$, where $P_n(l, k, \lambda)$ is the spectral projection given by [6]:

$$P_n(l, k, \lambda) = \frac{1}{2\pi i} \oint_{|z - V(l+n)| = \varepsilon} (z - A_l(\lambda, k))^{-1} dz \,. \tag{4.6}$$

This eigenvector is "almost normalized" in the sense that: $\|P_n(l, k, \lambda) \, |n\rangle\| = 1 + O(\lambda)$, and we have:

$$\begin{aligned} (z - A_l(\lambda, k))^{-1} &= (z - B_l - \lambda A_0(k))^{-1} \\ &= (z - B_l)^{-1} + \lambda (z - B_l)^{-1} A_0(k) (z - B_l)^{-1} + \ldots \\ &\quad \ldots + \lambda^n (z - B_l)^{-1} [A_0(k) (z - B_l)^{-1}]^n + \ldots \,. \end{aligned} \tag{4.7}$$

Since $(z - B_l)^{-1}$ is diagonal with exactly one entry that has a singularity inside the circle: $|z - V(l+n)| = \varepsilon$, and since $A_0(k)$ only couple "nearest neighbors" (where p is considered a neighbor of 1), we see from (4.6) and (4.7) that the projection matrix $P_n(l, k, \lambda)$ has exactly one (diagonal) $O(1)$ entry, surrounded by "nearest neighbors" that are $O(\lambda)$, surrounded by next near neighbors that are $O(\lambda^2)$, and so on. The $O(1)$ "center" of $P_n(l, k, \lambda)$ is the n^{th} diagonal entry, and thus the eigenvector $P_n(l, k, \lambda)\,|n\rangle$ has one $O(1)$ term, with two $O(\lambda)$ "nearest neighbors," two $O(\lambda^2)$ next near neighbors and so on. The corresponding normalized Bloch wave solution has a "profile" of the form:

$$\ldots, O(\lambda^2), O(\lambda), O(1), O(\lambda), O(\lambda^2), \ldots, O(\lambda^2), O(\lambda), O(1), O(\lambda), O(\lambda^2), \ldots . \tag{4.8}$$

The terms that are $O(1)$ in such a solution, are a distance p apart, and in the middle between them there are two neighboring terms that are either both $O(\lambda^{\frac{p-1}{2}})$ (if p is odd), or $O(\lambda^{\frac{p}{2}})$ and $O(\lambda^{\frac{p}{2}-1})$ (if p is even).

From Lemma 4.1, we obtain for every n and $k \in \left(0, \frac{\pi}{p}\right)$:

$$\left|\frac{dE_n(k)}{dk}\right| \le \min_m \{2p|u_n(k,m)|\,|u_n(k,m+1)|\}, \tag{4.9}$$

and since all Bloch wave solutions have a "profile" of the form (4.8), we conclude that for every n and k:

$$\left|\frac{dE_n(k)}{dk}\right| = O(\lambda^{p-1}) = O(\eta^{1-p}). \tag{4.10}$$

Since:

$$|\sigma| = \sum_{n=1}^{p} \int_0^{\frac{\pi}{p}} \left|\frac{dE_n(k)}{dk}\right| dk, \tag{4.11}$$

we obtain from (4.10):

$$|\sigma| = O(\eta^{1-p}), \tag{4.12}$$

which proves the theorem for the case of a potential with no degeneracies along its period.

Next, we consider a potential with one double degeneracy along its period, so that: $V(i) \neq V(j)$ for $i \neq j$, $i, j \in \{l+1, \ldots, l+p\}$, except for some $i_0 \neq j_0$ such that $V(i_0) = V(j_0)$. We define:

$$d \equiv \max\{|i_0 - j_0|, p - |i_0 - j_0|\}, \tag{4.13}$$

so that d is the same as $d(i_0)$ [or $d(j_0)$] defined in the theorem. Clearly, for all bands that arise from nondegenerate values of the potential, the proof of the nondegenerate case still holds, and the width of such bands is $O(\eta^{1-p})$. For the two bands that arise from $V(i_0)$ and $V(j_0)$, the effect of the degeneracy is obtained as follows:

Let $n = i_0 - l$ or $n = j_0 - l$, then $P_n(l, k, \lambda)$ defined by (4.6) is now a two dimensional projection. Since $(z - B_l)^{-1}$ now has two entries having singularities inside the circle: $|z - V(l+n)| = \varepsilon$, $P_n(l, k, \lambda)$ has two corresponding "centers" of diagonal $O(1)$ entries at $n = i_0 - l$ and at $n = j_0 - l$. Normalized eigenvectors of $A_l(\lambda, k)$ corresponding to the eigenvalue $V(i_0) = V(j_0)$ of B_l, are given by:

$$|\psi_s\rangle = P_n(l, k, \lambda)\,(a_s|i_0 - l\rangle + b_s|j_0 - l\rangle), \qquad s = 1, 2 \tag{4.14}$$

where a_s and b_s are solutions of:

$$a_s\langle\psi_s \mid i_0 - l\rangle + b_s\langle\psi_s \mid j_0 - l\rangle = 1\,, \qquad a_s\langle\psi_t \mid i_0 - l\rangle + b_s\langle\psi_t \mid j_0 - l\rangle = 0 \quad (4.15)$$

with:

$$t = \begin{cases} 1 & s = 2 \\ 2 & s = 1\,. \end{cases}$$

Since a_s and b_s are both (in general) $O(1)$, the corresponding Bloch wave solutions have "profiles" similar to (4.8), but with two terms that are $O(1)$ along each period. The maximum distance between such $O(1)$ terms is d, and thus (4.9) implies that the width of both bands is $O(\eta^{1-d})$. The total measure of σ has the order of magnitude of the widest bands, and therefore we obtain:

$$|\sigma| = O(\eta^{1-d})\,, \tag{4.16}$$

which proves the theorem for the case of one double degeneracy along the period of the potential.

From here, it should be quite clear how the general case is obtained. For an s-fold degenerate eigenvalue of B_l, the s-dimensional projection $P_n(l,k,\lambda)$ will have s "centers" of diagonal $O(1)$ entries, and the corresponding Bloch wave solutions will have s $O(1)$ terms along each period. Some careful observation shows that the definition of m in the theorem is such that the width of the widest possible bands will be $O(\eta^{1-m})$, from which the theorem follows.

5. Proof of Theorem 4

(i) Denote by G_m the union of the gaps in $\mathrm{Spec}\,(H_0 + V_m) \equiv \sigma_m$, and:

$$v_m \equiv \max_n V_m(n) - \min_n V_m(n)\,.$$

For each gap in σ_m, with measure |gap| larger than $2Cp_m^{-(1+\alpha)}$, there is a corresponding gap in σ with measure larger than $|\mathrm{gap}| - 2Cp_m^{-(1+\alpha)}$. Since there are at most $p_m - 1$ gaps in σ_m, we have:

$$|G| \geq |G_m| - 2C(p_m - 1)p_m^{-(1+\alpha)}\,. \tag{5.1}$$

Since (Theorem 1) $|G_m| \geq v_m$ and $v_m \geq v - 2Cp_m^{-(1+\alpha)}$, we obtain from (5.1):

$$|G| \geq v - 2Cp_m^{-\alpha} \tag{5.2}$$

for all m. In the limit $m \to \infty$, we have: $p_m^{-\alpha} \to 0$, and thus statement (i) follows from (5.2) and (1.2).

(ii) We can obviously assume $p_m < p_{m+1}$ for all m. Denote by $\{(\lambda_{2j-1}, \lambda_{2j})\}_{j=1}^{\infty}$ the gaps in σ, ordered by decreasing measure, that is: $|(\lambda_{2j-1}, \lambda_{2j})| \geq |(\lambda_{2j+1}, \lambda_{2j+2})|$ for all j. For each m, define: $r_m \equiv \sum_{j=p_m}^{\infty} |(\lambda_{2j-1}, \lambda_{2j})|$, then $|G| - r_m$ is the measure of the $p_m - 1$ widest gaps in σ. Using the same argument that lead to (5.1), we obtain for all m:

$$|G| - r_m \geq |G_m| - a_m\,, \tag{5.3}$$

and similarly:

$$|G_m| \geq |G| - r_m - a_m\,, \tag{5.4}$$

where: $a_m \equiv 2C(p_m - 1)m^{-\alpha p_{m+1}}$. Since $a_m, r_m \to 0$ as $m \to \infty$, (5.3) and (5.4) imply: $|G_m| \to |G|$. For every $m < l$, we have:

$$\big||G_m| - |G_l|\big| \le \sum_{j=m+1}^{l} \big||G_j| - |G_{j-1}|\big| \le \sum_{j=m+1}^{l} 4C(p_j - 1)(j-1)^{-\alpha p_j}, \tag{5.5}$$

and by taking $l \to \infty$ in (5.5), we obtain:

$$\big||G_m| - |G|\big| \le \sum_{j=m+1}^{\infty} 4C(p_j - 1)(j-1)^{-\alpha p_j} \equiv \delta_m, \tag{5.6}$$

from which follows:

$$\begin{aligned} |\sigma| &= \max\sigma - \min\sigma - |G| \\ &\ge \max\sigma - \min\sigma - |G_m| - \delta_m \\ &= \max\sigma - \max\sigma_m + \min\sigma_m - \min\sigma + |\sigma_m| - \delta_m \\ &\ge |\sigma_m| - 2Cm^{-\alpha p_{m+1}} - \delta_m . \end{aligned} \tag{5.7}$$

According to Theorem 2, we have:

$$|\sigma_m| > \frac{4}{(2+v_m)^{p_m - 1}}, \tag{5.8}$$

and since: $v_m \le v + 2Cm^{-\alpha p_m} \le v + 2C$, (5.7) and (5.8) imply:

$$|\sigma| > \frac{4}{(2+v+2C)^{p_m-1}} - \sum_{j=m+1}^{\infty} 4Cp_j(j-1)^{-\alpha p_j} \tag{5.9}$$

for all m. Since we assume: $p_m < p_{m+1}$ for all m, the negative term on the right in (5.9), converges to zero as $m \to \infty$ "faster" than the positive term. Thus, for some m, the right-hand side of (5.9) is positive, and we obtain: $|\sigma| > 0$. $\square$

6. Proof of Theorem 1

For each ω, define:

$$g^{\omega}_{nm}(z) \equiv ((z - H_\omega)^{-1})_{nm}, \tag{6.1}$$

then:

Lemma 6.1. *For every pair ω, n, and a closed curve Γ around σ_ω:*

$$-V_\omega(n) = \frac{1}{2\pi i} \oint_\Gamma z \frac{d}{dz} (\ln g^{\omega}_{nn}(z)) dz .$$

Proof. Since:

$$\sum_k (z - H_\omega)_{nk} g^{\omega}_{km}(z) = \delta_{nm}, \tag{6.2}$$

we have for each n:

$$-g^{\omega}_{n-1,n}(z) + (z - V_\omega(n))g^{\omega}_{nn}(z) - g^{\omega}_{n+1,n}(z) = 1, \tag{6.3}$$

$$-g^{\omega}_{nn}(z) + (z - V_\omega(n+1))g^{\omega}_{n+1,n}(z) - g^{\omega}_{n+2,n}(z) = 0, \tag{6.4}$$

$$-g^{\omega}_{n-2,n}(z) + (z - V_\omega(n-1))g^{\omega}_{n-1,n}(z) - g^{\omega}_{nn}(z) = 0. \tag{6.5}$$

By isolating $g^\omega_{n+1,n}(z)$ from (6.4), and $g^\omega_{n-1,n}(z)$ from (6.5), substituting in (6.3) and then isolating $g^\omega_{nn}(z)$, we obtain:

$$g^\omega_{nn}(z) = \left[\frac{1}{z - V_\omega(n) - \frac{1}{z - V_\omega(n+1)} - \frac{1}{z - V_\omega(n-1)}}\right] \times \left[1 + \frac{1}{z - V_\omega(n-1)} g^\omega_{n-2,n}(z) + \frac{1}{z - V_\omega(n+1)} g^\omega_{n+2,n}(z)\right]. \quad (6.6)$$

Since, for large $|z|$, $g^\omega_{nm}(z) = O(|z|^{-1})$, (6.6) can be written as:

$$\begin{aligned} g^\omega_{nn}(z) &= \left[\frac{1}{z - V_\omega(n)} + O(|z|^{-3})\right] \times [1 + O(|z|^{-2})] \\ &= \frac{1}{z - V_\omega(n)} + O(|z|^{-3}), \end{aligned} \quad (6.7)$$

and by differentiating the logarithm of $g^\omega_{nn}(z)$, we obtain in the limit $|z| \to \infty$:

$$z \frac{d}{dz} \ln g^\omega_{nn}(z) = \frac{-z}{z - V_\omega(n)} + O(|z|^{-2}). \quad (6.8)$$

By considering a circular curve Γ around the origin ($\Gamma \equiv \{\mathrm{Re}^{i\theta} \mid 0 \le \theta < 2\pi\}$), with radius $R \to \infty$, the contribution of the $O(|z|^{-2})$ term to the integral is seen to vanish, and thus:

$$\frac{1}{2\pi i} \oint_\Gamma z \frac{d}{dz} (\ln g^\omega_{nn}(z)) dz = \frac{1}{2\pi i} \oint_\Gamma \frac{-z}{z - V_\omega(n)} dz = -V_\omega(n). \quad \square \quad (6.9)$$

Proof of Theorem 1. Let Γ be a rectangular curve, that surrounds σ_ω at a distance ε from each side of the real axis, and a distance δ from each edge of σ_ω, then from Lemma 6.1 we have:

$$-V_\omega(n) = \frac{1}{2\pi i} \oint_\Gamma z \frac{d}{dz} \left(\ln g^\omega_{nn}(z) - i\, \frac{\pi}{2}\right) dz, \quad (6.10)$$

where:

$$\oint_\Gamma = \int_{\min \sigma_\omega - \delta + i\varepsilon}^{\min \sigma_\omega - \delta - i\varepsilon} + \int_{\min \sigma_\omega - \delta - i\varepsilon}^{\max \sigma_\omega + \delta - i\varepsilon} + \int_{\max \sigma_\omega + \delta - i\varepsilon}^{\max \sigma_\omega + \delta + i\varepsilon} + \int_{\max \sigma_\omega + \delta + i\varepsilon}^{\min \sigma_\omega - \delta + i\varepsilon}. \quad (6.11)$$

The first and third terms of (6.11), are easily seen to vanish in the limit $\varepsilon \to 0$. Since $g^\omega_{nn}(z)$ obeys:

$$g^\omega_{nn}(z) = \int_R \frac{1}{z - E} d\mu_n(E), \quad (6.12)$$

where μ_n is a positive measure supported on σ_ω, we have for every real x, and $\varepsilon > 0$:

$$g^\omega_{nn}(x + i\varepsilon) = (g^\omega_{nn}(x - i\varepsilon))^*. \quad (6.13)$$

Thus, the second and fourth terms in (6.11) have the same imaginary parts (the real parts of those two terms cancel each other), and we obtain:

$$-V_\omega(n) = \lim_{\varepsilon\to 0^+} \frac{1}{\pi} \int\limits_{\min\sigma_\omega-\delta-i\varepsilon}^{\max\sigma_\omega+\delta-i\varepsilon} \mathrm{Im}\left[z\,\frac{d}{dz}\left(\ln g^\omega_{nn}(z) - i\,\frac{\pi}{2}\right)\right] dz\,. \tag{6.14}$$

Since $g^\omega_{nn}(z)$ is Herglotz, it is known (see [7] and references therein), that the limit:

$$\lim_{\varepsilon\to 0^+} g^\omega_{nn}(x - i\varepsilon) \equiv g^\omega_{nn}(x - i0)\,, \tag{6.15}$$

exists (and is finite and non-zero) for a.e. $x \in R$. Therefore, integrating (6.14) by parts leads to:

$$\begin{aligned}-V_\omega(n) &= \frac{1}{\pi}\, x\,\mathrm{Im}\left[\ln g^\omega_{nn}(x-i0) - i\,\frac{\pi}{2}\right]\Bigg|_{\min\sigma_\omega-\delta}^{\max\sigma_\omega+\delta} \\ &\quad - \frac{1}{\pi}\int\limits_{\min\sigma_\omega-\delta}^{\max\sigma_\omega+\delta} \mathrm{Im}\left[\ln g^\omega_{nn}(x-i0) - i\,\frac{\pi}{2}\right] dx\,.\end{aligned} \tag{6.16}$$

From (6.12), it is clear that $g^\omega_{nn}(x - i0)$ is negative for $x < \min\sigma_\omega$ and positive for $x > \max\sigma_\omega$. Therefore, by letting $\delta \to 0$ in (6.16), we obtain:

$$V_\omega(n) = \tfrac{1}{2}\,(\max\sigma_\omega + \min\sigma_\omega) + \tfrac{1}{2}\,S_G(n,\omega) + \tfrac{1}{2}\,S_\sigma(n,\omega)\,, \tag{6.17}$$

where:

$$\begin{aligned}\frac{1}{2}\,S_G(n,\omega) &= \frac{1}{\pi}\int\limits_{G_\omega} \mathrm{Im}\left[\ln g^\omega_{nn}(x-i0) - i\,\frac{\pi}{2}\right] dx\,,\\ \frac{1}{2}\,S_\sigma(n,\omega) &= \frac{1}{\pi}\int\limits_{\sigma_\omega} \mathrm{Im}\left[\ln g^\omega_{nn}(x-i0) - i\,\frac{\pi}{2}\right] dx\,.\end{aligned} \tag{6.18}$$

From (6.17), we have for every m and n:

$$V_\omega(n) - V_\omega(m) = \tfrac{1}{2}\,(S_G(n,\omega) - S_G(m,\omega)) + \tfrac{1}{2}\,(S_\sigma(n,\omega) - S_\sigma(m,\omega))\,, \tag{6.19}$$

and therefore:

$$|V_\omega(n) - V_\omega(m)| \le \tfrac{1}{2}\,|S_G(n,\omega) - S_G(m,\omega)| + \tfrac{1}{2}\,|S_\sigma(n,\omega) - S_\sigma(m,\omega)|\,. \tag{6.20}$$

Simon [7], had shown that for every n, for a.e. ω: $\mathrm{Re}\, g^\omega_{nn}(x - i0) = 0$ for a.e. $x \in A$. Since $g^\omega_{nn}(z)$ is Herglotz, we also have: $g^\omega_{nn}(x - i0) \neq 0$ for a.e. $x \in R$. Thus, for every n, for a.e. ω, we have: $\mathrm{Im}\,[\ln g^\omega_{nn}(x - i0) - i\,\frac{\pi}{2}] = 0$ for a.e. $x \in A$. Since also: $|\mathrm{Im}\,[\ln g^\omega_{nn}(x - i0) - i\,\frac{\pi}{2}]| \le \frac{\pi}{2}$, we obtain for every n, for a.e. ω:

$$|S_\sigma(n,\omega)| \le |\sigma_\omega \backslash A|\,, \tag{6.21}$$

and similarly:

$$|S_G(n,\omega) \le |G_\omega|\,. \tag{6.22}$$

Equations (6.20), (6.21), and (6.22), imply that for a.e. ω, for every m and n:

$$|\sigma_\omega \backslash A| + |G_\omega| \ge |V_\omega(n) - V_\omega(m)|\,, \tag{6.23}$$

and thus, for a.e. ω:

$$|\sigma_\omega \backslash A| + |G_\omega| \ge v_\omega\,, \tag{6.24}$$

from which the theorem follows. □

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