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Dressing Symmetries

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Abstract. We study the group of dressing transformations in soliton theories. We show that it is generated by the monodromy matrix. This provides a new proof of their Lie–Poisson property. We treat in detail the examples of the Toda field theories and the Heisenberg model. We show that the group of dressing transformations is the classical precursor of the various manifestations of quantum groups in these models, e.g. algebraic Bethe ansatz, non-local currents, or quantum group symmetries. Finally, we define field multiplets supporting a linear representation of the dressing group and we show that their exchange algebras are encoded in the classical double.

1. Introduction

Are quantum integrable models reducible to quantum group theory? This assertion is best examplified by the algebraic formulation of two-dimensional conformal field theories [1] but remains uncertain for massive integrable models. On one hand, quantum groups [2, 3] and their representation theory are now well understood although some of their relations with the quantum algebraic Bethe Ansatz are still mysterious. On the other hand, a large number of methods for studying integrable models have been developed, cf. e.g. [4, 5], including the most famous algebraic quantum inverse scattering method [6]. More recently, new developments have suggested the use of non-local symmetries for reformulating integrable models as quantum group theories [7, 8, 9]. However, despite suggestive facts, the assertion is still not established by any of these approaches.

The aim of this paper is to try to understand the occurrence of non-local symmetries in classical soliton equations and their relations with semi-classical analogues of quantum group symmetries. We will deal with integrable soliton

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equations having a zero-curvature representation and admitting a Hamiltonian formulation based on the standard transfer matrix formalism. As is well known, this formulation ensures that the models possess integrals of motion in involution. These integrals are computed from the monodromy matrix T as:

$$H_{\text{involution}} = \operatorname{tr}(T) .$$

Natural candidates for the non-local symmetries are the so-called dressing transformations [10, 11, 13]. These are special transformations acting on the solution space of the soliton equations or equivalently on the phase space. They form a usually infinite dimensional symmetry group of the soliton equations. Moreover, this group possesses three remarkable properties:

• *Dressings are Lie–Poisson Actions*
This fundamental property of the group of dressing transformations was proved by Semenov-Tian-Shansky [13]. It is for this reason that the group of dressing transformations is the natural classical precursor of quantum group symmetries. Quite generally, Lie–Poisson action on a symplectic manifold is generated by a non-Abelian Hamiltonian [16]. In the case of dressing transformations we will show that this generator is the monodromy matrix. This means that the generators of the infinitesimal dressing transformations can be thought of as being:

$$H_{\text{dress}} = \operatorname{tr}(X \log T)$$

for any X in the Lie algebra of the dressing group. (A more precise formulation will be given in the following.) This relation indicates that the transfer matrix can be reconstructed from the "Hamiltonians" H_{dress} and $H_{\text{involution}}$.

• *Dressings are Non-Local*
This is true by construction. It implies that the charges which generate the dressing transformations are also non-local. This non-locality is an echo of the Poisson structure defined on the dressing group. Both facts have two related consequences: (i) the dressing transformations are not linearly generated by the charges, and (ii) the Poisson algebra of the charges is not the Lie algebra of the dressing group although it is related: It is a semi-classical deformation of it. However, any algebraic relation in the dressing Lie algebra, e.g. the Serre-like relations, yields a relation in the dressing Poisson algebra.

• *Dressings Induce Field Multiplets*
Because the dressing transformations map solutions of the soliton equations into solutions, they provide a way to gather the fields into non-local field multiplets [14, 15], alternatively named non-local blocks. For example, by dressing local conserved currents one defines a set of non-local conserved currents which form the current multiplets generating the dressing transformations. The field multiplets are closed by dressing and form orbits of the dressing group. The number of multiplets is therefore the number of orbits of the dressing group in the phase space. We do not know under which conditions the quotient of the phase space by the dressing group reduces to a finite number of points. (This notion is the classical analogue of the notion of rationality in massive quantum field theory.) However, a few hints indicate that this is the case in some models invariant under infinite dressing groups. Furthermore, we described how the exchange algebra of the field multiplets is encoded in the classical double algebra.

Section 1 deals with basic facts about Lie–Poisson actions. We generalize the notion of the moment map and explain its relation with the above mentioned non-Abelian Hamiltonians. We give new proofs of the Lie–Poisson properties of the dressing transformations and establish the relation between the non-Abelian Hamiltonians of these transformations and the monodromy matrices.

Section 2 is concerned with the dressing transformations in the Toda field theories. In particular, we show how they provide a classical explanation of the occurrence of quantum group symmetries in two-dimensional conformal field theories. In this case the orbits of the dressing group are in one-to-one correspondence with the fundamental highest weights of the Lie algebra.

In Sect. 3 we study the Heisenberg model. We show that the Poisson algebra of the dressing transformations is a semi-classical Yangian. We describe the relation between the local conserved quantities and the generators of dressing transformations. In this case the orbit space is likely to be reduced to a finite number of points (as local arguments indicate). Moreover, the relation between the algebraic Bethe Ansatz and the representation theory of the dressing group appears clearly.

Although dressing transformations and $\mathscr{T}$au functions are intimately related, e.g. $\mathscr{T}$au functions are orbits of the dressing groups, we reserve their study for a future work.

2. Lie Poisson Actions and Dressing Transformations

In this section we gather a few basic facts about Poisson–Lie groups and their action on symplectic manifolds. This sets up the frame in which dressing transformations fit.

2.1. Poisson–Lie Groups. Let G be a Lie group with Lie algebra $\mathscr{G}$. We will assume that $\mathscr{G}$ is equipped with a non-degenerate bilinear form denoted by tr. We choose an orthonormal basis e_a in $\mathscr{G}$: $\mathrm{tr}(e_a e_b) = \delta_{ab}$. We denote by f_{ab}^{c} the structure constants in $\mathscr{G}$,

$$[e_a, e_b] = f_{ab}^{c} e_c \,.$$

Assume now that G is a Poisson–Lie group [2]. This means that G is a Lie group equipped with a Poisson structure such that the multiplication in G viewed as map $G \times G \to G$ is a Poisson mapping. More explicitly, using the basis (e_a) of $\mathscr{G}$, the Poisson bracket for any functions f_1 and f_2 on G can be written as:

$$\{f_1, f_2\}_G(g) = \sum_{a,b} \eta^{ab}(g)(\nabla_a^R f_1)(g)(\nabla_b^R f_2)(g)\,, \tag{1}$$

where ∇_a^R is the right invariant vector field corresponding to the element $e_a \in \mathscr{G}$. In particular, if we take for f the matrix elements of a representation of G, we get

$$\{g \overset{\otimes}{,} g\}_G = \eta(g)\cdot g \otimes g\,; \quad g \in G\,, \tag{2}$$

where $\eta(g) = \sum_{a,b} \eta^{ab}(g) e_a \otimes e_b$. The Lie Poisson property of the Poisson brackets (2) is equivalent to the fact that $\eta(g)$ is a cocycle [2]: $\eta(hg) = \eta(h) + \mathrm{Ad}\, h \cdot \eta(g)$.

Thanks to this cocycle relation, the bracket $\{,\}_G$ can be used to define a Lie algebra structure on $\mathscr{G}^*$ by the following formula:

$$[d_e \phi_1, d_e \phi_2]_{\mathscr{G}^*} = d_e\{\phi_1, \phi_2\}_G \tag{3}$$

with $d_e\phi \in \mathscr{G}^*$ the differential of the function ϕ on G evaluated at the identity of G. Introducing a basis (δ^a) in $\mathscr{G}^*$, dual to the basis (e_a) in $\mathscr{G}$, the differential at the identity can be written as $d_e\phi = \sum_a \delta^a(\nabla_a^L\phi)(e) \in \mathscr{G}^*$, where ∇_a^L are the left-invariant vector fields on G. In this basis, Eq. (3) gives:

$$[\delta^a, \delta^b]_{\mathscr{G}^*} = C_c^{ab}\delta^c\,, \tag{4}$$

where the structure constants are $C_c^{ab} = (\nabla_c^L \eta^{ab})(e)$. The Lie bracket Eq. (4) satisfies the Jacobi identity thanks to the Jacobi identity for the Poisson bracket in G. We denote by G^* the Lie group with Lie algebra $\mathscr{G}^*$.

In the same way as the Poisson structure on G induces a Lie algebra structure on $\mathscr{G}^*$, the Lie algebra structure on $\mathscr{G}$ induces a Lie–Poisson structure on G^*. We denote by $\{,\}_{G^*}$ these Poisson brackets and by η^* the corresponding cocycle; $\eta^*(\gamma) = \sum_{ab}\eta_{ab}^*(\gamma)\delta^a \otimes \delta^b \in \mathscr{G}^* \otimes \mathscr{G}^*$. It is characterized by the property that $\nabla_L^d \eta_{ab}^*(e) = f_{ab}^d$.

2.2. The Groups G, G and the Double.* We apply the previous results to a more specific situation when η is a coboundary, $\eta(g) = r^\pm - g \otimes g\, r^\pm g^{-1} \otimes g^{-1}$, where $r^\pm = \sum_{ab} r_\pm^{ab} e_a \otimes e_b \in \mathscr{G} \otimes \mathscr{G}$. We assume $r_{12}^+ = -r_{21}^-$, and $r_{12}^+ - r_{12}^- = \mathscr{C}$, where $\mathscr{C} = \sum_a e_a \otimes e_a$ is the tensor Casimir. These conditions ensure the antisymmetry of the Poisson bracket on G which now reads

$$\{g \overset{\otimes}{,} g\}_G = [r^\pm, g \otimes g]; \quad g \in G\,. \tag{5}$$

This bracket is known as the Sklyanin bracket [18]. The Jacobi identity is satisfied if $r^\pm$ are solutions of the classical Yang–Baxter equation:

$$[r_{12}^\pm, r_{13}^\pm] + [r_{12}^\pm, r_{23}^\pm] + [r_{13}^\pm, r_{23}^\pm] = 0\,. \tag{6}$$

Equation (6) is an equation in $\mathscr{G} \otimes \mathscr{G} \otimes \mathscr{G}$, and the indices on $r^\pm$ refer to the copies of $\mathscr{G}$ on which $r^\pm$ is acting. In Eq. (5) we can choose to use either r^+ or r^- as the difference is the tensor Casimir.

Using the bilinear form tr to identify the vector spaces $\mathscr{G}^*$ and $\mathscr{G}$, the elements $r^\pm$ of $\mathscr{G} \otimes \mathscr{G}$ can be mapped into elements $R^\pm \in \mathscr{G} \otimes \mathscr{G}^* \cong \mathrm{End}\,\mathscr{G}$ defined by [12]:

$$R^\pm(X) = \mathrm{tr}_2(r_{12}^\pm(1 \otimes X)); \quad \forall X \in \mathscr{G}\,. \tag{7}$$

Note that we have $R^+ - R^- = \mathrm{Id}$.

The Poisson bracket (5) on G induces a Lie algebra structure on $\mathscr{G}^*$ by the formula (3). Again, identifying the vector spaces $\mathscr{G}$ and $\mathscr{G}^*$ by means of tr, the brackets on $\mathscr{G}^*$ read [12]:

$$[X, Y]_R = [R^\pm(X), Y] + [X, R^\mp(Y)]\,. \tag{8}$$

The Jacobi identity for the commutators (8) follows from the classical Yang–Baxter equation for $r^\pm$. It also implies that $R^\pm$ are Lie algebra homomorphisms from $\mathscr{G}^*$ to $\mathscr{G}$. In particular, $\mathscr{G}_\pm = \mathrm{Im}\,R^\pm$ are subalgebras of $\mathscr{G}$, and $r^\pm \in \mathscr{G}_\pm \otimes \mathscr{G}_\mp$. The group G^* and the algebra $\mathscr{G}^*$ are related to a factorization problem in G specified by the matrices $R^\pm$. First consider $\mathscr{G}^*$; because $R^+ - R^- = \mathrm{Id}$, any $X \in \mathscr{G}$ admits a unique decomposition as:

$$X = X_+ - X_- \quad \text{with} \quad X_\pm = R^\pm(X)\,. \tag{9}$$

In terms of the components X_+ and X_-, the commutator in $\mathcal{G}^*$. becomes:

$$[X, Y]_R = [X_+, Y_+] - [X_-, Y_-] .$$

In particular, the plus and minus components commute in $\mathcal{G}^*$. We denote by G^* the corresponding group. By exponentiation, the group G^* is made of the couples (g_-, g_+) with the composition law,

$$(g_-, g_+)\cdot(h_-, h_+) = (g_-h_-, g_+h_+) . \tag{10}$$

Just as $\mathcal{G} \simeq \mathcal{G}^*$ as vector spaces, we have $G \simeq G^*$ as manifolds: $(g_-, g_+) \in G^* \to g = g_-^{-1}g_+ \in G$. Equivalently, any element $g \in G$ (in a neighbourhood of the identity) admits a unique factorization as:

$$g = g_-^{-1}g_+ \quad \text{with} \quad (g_-, g_+) \in G^* . \tag{11}$$

The group G^* itself becomes a Poisson-Lie group if we introduce on it the Semenov-Tian-Shansky Poisson bracket [13]:

$$\begin{aligned}
\{g_+ \overset{\otimes}{,} g_+\}_{G^*} &= -[r^{\pm}, g_+ \otimes g_+] , \\
\{g_- \overset{\otimes}{,} g_-\}_{G^*} &= -[r^{\mp}, g_- \otimes g_-] , \\
\{g_- \overset{\otimes}{,} g_+\}_{G^*} &= -[r^{-}, g_- \otimes g_+] , \\
\{g_+ \overset{\otimes}{,} g_-\}_{G^*} &= -[r^{+}, g_+ \otimes g_-]
\end{aligned} \tag{12}$$

or, for the factorized element $g = g_-^{-1}g_+$:

$$\begin{aligned}
\{g \overset{\otimes}{,} g\}_{G^*} = &-(g \otimes 1)r^+(1 \otimes g) - (1 \otimes g)r^-(g \otimes 1) \\
&+ (g \otimes g)r^{\pm} + r^{\mp}(g \otimes g) .
\end{aligned} \tag{13}$$

The multiplication in G^* is a Poisson map for the brackets (12). The group G^* is therefore a Poisson–Lie group. Notice that, as it should be, the Lie algebra structure induced by the Poisson brackets (13) on the dual of $\mathcal{G}^*$, i.e. on $\mathcal{G}$, is the original structure on $\mathcal{G}$.

Finally, denoting by (δ^a) the basis of $\mathcal{G}^*$ dual to (e_a) one can calculate the structure constants of $\mathcal{G}^*$,

$$C_c^{ab} = \frac{d}{dt}\eta^{ab}(e^{te_c})\bigg|_{t=0} = -r^{db}f^{\,a}_{\;cd} - r^{ad}f^{\,b}_{\;cd} = r^{da}f^{\,b}_{\;cd} + r^{bd}f^{\,a}_{\;cd} . \tag{14}$$

The Yang–Baxter equation on r implies the Jacobi identity on the constants C_c^{ab}, while from their explicit form Eq. (14), we deduce the following relation valid for all r^{ab}:

$$C_d^{ab}f^{\,d}_{\;cl} - C_l^{bd}f^{\,a}_{\;dc} + C_l^{ad}f^{\,b}_{\;dc} + C_c^{db}f^{\,a}_{\;ld} - C_c^{da}f^{\,b}_{\;ld} = 0 .$$

This is just the cocycle relation. As a consequence the following bracket satisfies the Jacobi identity:

$$\begin{aligned}
[e_a, e_b] &= f^{\,c}_{\;ab}e_c , \\
[\delta^a, e_b] &= f^{\,a}_{\;bc}\delta^c - C_b^{ac}e_c , \\
[\delta^a, \delta^b] &= C_c^{ab}\delta^c .
\end{aligned} \tag{15}$$

These relations will be important later and are the basis of the construction of the classical double [2]. With these relàtions at hand, one can construct a solution of the Yang–Baxter equation

$$r_{12} = \sum_a e_a \otimes \delta^a \in \mathscr{G} \otimes \mathscr{G}^* .$$

This r-matrix will naturally appear in the exchange relations.

2.3. The Action of a Poisson–Lie Group on a Symplectic Manifold. In physics, we are usually used to consider symmetries which are generated by Hamiltonian vector fields. These symmetries preserve the Poisson brackets and are called symplectic transformations. Lie–Poisson actions are generalizations of this notion. The action of a Poisson–Lie group H on a symplectic manifold M is a Lie–Poisson action if the Poisson brackets transform covariantly; i.e. if for any $h \in H$ and any function f_1 and f_2 on M,

$$\{f_1(h\cdot x), f_2(h\cdot x)\}_{H\times M} = \{f_1, f_2\}_M(h\cdot x) . \tag{16}$$

The Poisson structure on $H \times M$ is the product Poisson structure.

Let $X \in \mathscr{H}$, the Lie algebra of H, and denote also by X the vector field on M corresponding to the infinitesimal transformation generated by X.

$$X\cdot f(x) = \frac{d}{dt} f(e^{tX}\cdot x)|_{t=0} . \tag{17}$$

Let $e_a \in \mathscr{H}$ and $e^a \in \mathscr{H}^*$ two dual basis of the Lie algebras $\mathscr{H}$ and $\mathscr{H}^*$, with $\langle e^a, e_b\rangle = \delta^a_b$, where $\langle , \rangle$ denote the pairing between $\mathscr{H}$ and $\mathscr{H}^*$. Then, the infinitesimal form of Eq. (16) becomes [13]:

$$\{e_a\cdot f_1, f_2\}_M + \{f_1, e_a\cdot f_2\}_M + f^{bd}_a(e_b\cdot f_1) e_d\cdot f_2) = e_a\cdot\{f_1, f_2\}_M , \tag{18}$$

where f^{bd}_a are the structure constants of $\mathscr{H}^*$. It follows immediately from Eq. (18) that a Lie–Poisson action cannot be Hamiltonian unless the algebra $\mathscr{H}^*$ is Abelian. However, in general, we have a non-Abelian generalization of Hamiltonian actions [16].

Proposition. *There exists a function Γ, locally defined on M and taking values in the group H^*, such that for any function f on M,*

$$X\cdot f = \langle \Gamma^{-1}\{f, \Gamma\}_M, X\rangle, \quad \forall X \in \mathscr{H} . \tag{19}$$

We will refer to Γ as the non-Abelian Hamiltonian of the Lie–Poisson action.

The proof is the following. Introduce Darboux coordinates (q^i, p^i) on M. Let $\Omega = e^a \Omega_a$ be the $\mathscr{H}^*$-valued one-form defined by $\Omega_a = e^{q^i}_a dp^i - e^{p^i}_a dq^i$, where $e^{q^i}_a$, $e^{p^i}_a$ are the components of the vector field e_a, $e_a = e^{q^i}_a \partial_{q^i} + e^{p^i}_a \partial_{p^i}$. Equation (18) is then equivalent to the following zero-curvature condition for Ω:

$$d\Omega + [\Omega, \Omega]_{\mathscr{H}^*} = 0 .$$

Therefore, locally on M, $\Omega = \Gamma^{-1} d\Gamma$. This proves Eq. (19). The converse is true: an action generated by a non-Abelian Hamiltonian as in Eq. (19) is Lie–Poisson since then we have:

$$\begin{aligned} &X\cdot\{f_1, f_2\}_M - \{X\cdot f_1, f_2\}_M - \{f_1, X\cdot f_2\}_M \\ &= \langle [\Gamma^{-1}\{f_1, \Gamma\}_M, \Gamma^{-1}\{f_2, \Gamma\}_M]_{\mathscr{H}^*}, X\rangle . \end{aligned}$$

The moment map $\mathscr{P}$ for the Lie–Poisson action is defined by:

$$\mathscr{P}: M \to H^*$$
$$x \to \Gamma(x) \quad \text{with} \quad \Gamma(x) = \exp(-Q(x)) \,. \tag{20}$$

We refer to the $\mathscr{H}^*$-valued functions $Q(x)$ as the charges generating the Lie–Poisson action.

As Γ is the pull-back of an element $\gamma \in H^*$ by $\mathscr{P}$, it is natural to require for the Poisson brackets in M of the functions Γ to be the pull-back of the Poisson brackets of the γ's in the group H^*:

$$\{\Gamma \overset{\otimes}{,} \Gamma\}_M = \mathscr{P}^*\{\gamma \overset{\otimes}{,} \gamma\}_{H^*} = \eta^*(\Gamma)\cdot \Gamma \otimes \Gamma \,. \tag{21}$$

Proposition. *Assuming this relation implies that the action defined in Eq.* (19) *is a representation of* $\mathscr{H}$ *on the space of functions* f *on* M:

$$(XY - YX)\cdot f = \langle \Gamma^{-1}\{f, \Gamma\}_M, [X, Y]_{\mathscr{H}}\rangle \,. \tag{22}$$

This is proved by noticing that

$$(XY - YX)\cdot f = \langle (\mathrm{Ad}\,\Gamma^{-1})_{12}\{f, \eta^*_{12}(\Gamma)\}_M, X_1 Y_2\rangle \,,$$

and using that $(\nabla^d_L \eta^*)(\gamma) = \mathrm{Ad}\,\gamma \cdot (\nabla^d_L \eta^*)(e)$ as follows from the cocycle condition. The converse is also true: requiring the relation (22) implies the Poisson brackets (21) for Γ.

2.4. The Dressing Transformations. Dressing transformations are special symmetries of soliton equations which admit a Lax representation [10, 11]. Namely, consider a set of non-linear differential equations for some fields $\phi(x, t)$, which can be written as a zero-curvature condition:

$$[\mathscr{D}_\mu[\phi], \mathscr{D}_\nu[\phi]] = 0 \tag{23}$$

for a connection $\mathscr{D}_\mu[\phi] = \partial_\mu - A_\mu[\phi]$ depending on the fields ϕ. The connection $A_\mu[\phi]$ is usually called the Lax connection; it belongs to some Lie algebra $\mathscr{G}$ and its form depends on the soliton equations. The zero-curvature Eq. (23) is the compatibility condition for an auxiliary linear problem:

$$(\partial_\mu - A_\mu)\Psi(x, t) = 0 \,, \tag{24}$$

where the wave function $\Psi(x, t)$ takes values in the group G. Due to the zero-curvature condition, the Lax connection is a pure gauge: $A_\mu = (\partial_\mu \Psi)\Psi^{-1}$. The wave function $\Psi(x, t)$ is defined up to a right multiplication by a space-time independent group element. This freedom is fixed by imposing a normalization condition on Ψ; e.g. $\Psi(0) = 1$.

Suppose now that the soliton equations (23) admit an Hamiltonian formulation [6, 4, 12, 13] and moreover that the Poisson brackets of the components of the Lax connection (which are deduced from those of the fields ϕ) give rise to the Sklyanin brackets for the function $\Psi(x, t)$:

$$\{\Psi(x) \overset{\otimes}{,} \Psi(x)\} = [r^\pm, \Psi(x) \otimes \Psi(x)] \,, \tag{25}$$

where $r^\pm \in \mathscr{G} \otimes \mathscr{G}$ are solutions of the classical Yang–Baxter equation. As we explained in the previous section, the matrices $r^\pm$ can be used to define two subalgebras $\mathcal{G}_\pm$ of $\mathcal{G}$ and the associated factorization problems, either in the algebra $\mathcal{G}$: $X = X_+ - X_-$ or in the group G: $g = g_-^{-1} g_+$.

For any $g = g_-^{-1} g_+ \in G$, a dressing transformation consists in transforming the variables $\Psi(x, t)$ to the variables $\Psi^g(x, t)$ defined by

$$\Psi^g = \Theta_\pm \Psi g_\pm^{-1} , \tag{26}$$

where the group elements $\Theta_\pm$ are solutions of the factorization problem:

$$\Theta_-^{-1} \Theta_+ = \Psi g \Psi^{-1} . \tag{27}$$

Notice that the two signs give the same result in Eq. (26). Also this transformation preserves the normalization condition $\Psi(0) = 1$. The action on the wave function Eq. (26) induces a gauge transformation on the Lax connection: $A_\mu = (\partial_\mu \Psi) \Psi^{-1}$ is transformed into $A_\mu^g = (\partial_\mu \Psi^g) \Psi^{g-1}$ with:

$$A_\mu^g = (\partial_\mu \Theta_\pm) \Theta_\pm^{-1} + \Theta_\pm A_\mu \Theta_\pm^{-1} . \tag{28}$$

In general [10, 13], and as we will illustrate by examples in the following sections, the factorization problem is constructed in such a way that the form of the Lax connection is preserved by the dressing transformations. This is the main property of these transformations. It implies that they form a (usually called "hidden") symmetry group of the soliton equations.

Proposition. *The commutation relations of the dressing transformations are those of G^* (in particular the plus and minus components commute).*

One can give a direct proof [13, 19]. Consider two elements $g = g_-^{-1} g_+$ and $h = h_-^{-1} h_+$ and transform successively an element Ψ: $\Psi \to \Psi^g \to (\Psi^g)^h$; we have:

$$\Psi^g = \Theta_\pm^g \Psi g_\pm^{-1} \quad \text{with} \quad \Theta_\pm^g = (\Psi g \Psi^{-1})_\pm ,$$
$$(\Psi^g)^h = \Theta_\pm^{hg} \Psi^g h_\pm^{-1} \quad \text{with} \quad \Theta_\pm^{hg} = (\Psi^g h \Psi^{g-1})_\pm . \tag{29}$$

The factorization of $(\Psi^g h \Psi^{g-1})$ can be written as follows:

$$(\Theta_-^{hg})^{-1} \Theta_+^{hg} \equiv \Psi^g h \Psi^{g-1} = \Theta_-^g \Psi (h_- g_-)^{-1} (h_+ g_+) \Psi^{-1} \Theta_+^{g-1}$$

or equivalently,

$$\Theta_\pm^{hg} \Theta_\pm^g = (\Psi (h_- g_-)^{-1} (h_+ g_+) \Psi^{-1})_\pm$$

Inserting this formula into Eq. (29) proves that the multiplication law for the dressing transformations is the same as in G^*.

Proposition. *The action of dressing transformations is a Lie–Poisson action. The non-Abelian generator is the monodromy matrix.*

The fact that the action of dressing transformations is a Lie–Poisson action was first proved in [13] using the classical double. We will give another two line proof of this fact by exhibiting their non-Abelian generator. The infinitesimal form of Eq. (26) is for any $X \in \mathcal{G}$, with $X = X_+ - X_-$:

$$\delta_X \Psi = \Theta_\pm \Psi - \Psi X_\pm \quad \text{with} \quad \Theta_\pm = (\Psi X \Psi^{-1})_\pm . \tag{30}$$

Introducė the monodromy matrix $T(L) = \Psi(L)\Psi^{-1}(0)$, with 0 and L the two boundary values of the coordinate x. Using the ultralocality property, its Poisson bracket with the wave function is:

$$\{\Psi(x) \overset{\otimes}{,} T(L)\} = (1 \otimes T(L)\Psi^{-1}(x))[r^{\pm}, \Psi(x) \otimes \Psi(x)] .$$

Therefore we have:

$$\begin{aligned} \mathrm{tr}_2(1 \otimes XT^{-1}(L)\{\Psi(x) \overset{\otimes}{,} T(L)\}) &= (\Psi(x)X\Psi^{-1}(x))_{\pm}\Psi(x) - \Psi(x)X_{\pm} \\ &= \delta_X \Psi(x) \end{aligned}$$

for any $X \in \mathscr{G}$. This proves that $T(L)$ is the non-Abelian Hamiltonian and shows that the action is Lie–Poisson. See also [17]. We will illustrate this relation in various ways in the next sections.

2.5. Exchange Algebras. Experiences in classical integrable models tell us that a special role is played by collections of field variables whose Poisson brackets are quadratic. These Poisson algebras, which are the classical ancestors of the quantum braiding relations, form the so-called exchange algebras. We are therefore interested in the following

Proposition. *Let $\xi_1(x)$ and $\xi_2(y)$ be two field multiplets. Denote by (δ^a) a basis of the dressing algebra $\mathscr{G}^*$ and (e_a) a dual basis of $\mathscr{G}$. Assume that they satisfy the commutation relations* (15) *of the classical double. Then, the exchange relations:*

$$\begin{aligned} \{\xi_1(x) \overset{\otimes}{,} \xi_2(y)\} &= \sum_a \delta^a(\xi_1(x))e_a(\xi_2(y)); \quad x < y , \\ \{\xi_1(x) \overset{\otimes}{,} \xi_2(y)\} &= -\sum_a e_a(\xi_1(x))\delta^a(\xi_2(y)); \quad x > y \end{aligned} \tag{31}$$

are compatible with the Lie–Poisson property of the dressing transformations.

We have to prove that the following relation (for $x < y$),

$$\delta^a\{\xi_1(x) \overset{\otimes}{,} \xi_2(y)\} = \{\delta^a\xi_1(x) \overset{\otimes}{,} \xi_2(y)\} + \{\xi_1(x) \overset{\otimes}{,} \delta^a\xi_2(y)\} + f^{\,a}_{bd}(\delta^b\xi_1(x))(\delta^d\xi_2(y))$$

is satisfied. By construction, the generators δ^a satisfy the relations $[\delta^a, \delta^b] = C^{ab}_c\delta^c$. Using these commutation relations, the Lie–Poisson property written above turns out to be equivalent to the following condition:

$$\delta^d\xi_1(x)([e_d, \delta^a] + f^{\,a}_{dn}\delta^n + C^{na}_d e_n)(\xi_2(y)) = 0 .$$

It is satisfied if $[\delta^a, e_b] = f^{\,a}_{bc}\delta^c - C^{ac}_b e_c$. This is the second relation of the classical double algebra. The last relation, $[e_a, e_b] = f^{\,c}_{ab}e_c$, comes in by requiring the Jacobi identity for the exchange algebra (31). We will illustrate these exchange algebras in Toda field theories and in the Heisenberg model.

3. Example 1: Dressings in the Toda Field Theories

In this section, we apply the previous results to describe the dressing transformations in Toda field theories. The dressing transformations in the Toda field theories reflect an invariance of these models under a semi-classical version of the quantum algebras $\mathcal{U}_q(\mathcal{G}^*)$. It provides a simple explanation of the occurrence of quantum groups in conformal field theory. Some of these results were announced in [23].

3.1. The Toda Field Theories. Let us first fix the notations. Let $\mathcal{G}$ be a semi-simple Lie Algebra equipped with its Killing form denoted by $(,) = \mathrm{tr}$. We denote by G its Lie group. Let $\mathcal{H}$ be a Cartan subalgebra of $\mathcal{G}$, $\mathcal{G} = \mathcal{N}_- \oplus \mathcal{H} \oplus \mathcal{N}_+$ the Cartan decomposition, $G = N_- H N_+$ the corresponding Gaussian decomposition and $B_\pm = HN_\pm$ the Borel subgroup. We denote by $(e_i^-, \alpha_i^\vee, e_i^+)$, $i = 1, \ldots, \mathrm{rank}\,\mathcal{G}$, the generators associated to the simple roots α_i of $\mathcal{G}$, $\alpha_i \in \mathcal{H}^*$. They satisfy:

$$[\alpha_i^\vee, \alpha_j^\vee] = 0\,,$$

$$[\alpha_i^\vee, e_j^\pm] = \pm\,\alpha_j(\alpha_i^\vee) e_j^\pm\,,$$

$$[e_i^+, e_j^-] = \delta_{ij}\alpha_i^\vee$$

and $(\mathrm{ad}\, e_i^\pm)^{1-a_{ij}} e_j^\pm = 0$ for $i \neq j$, with $a_{ij} = \alpha_j(\alpha_i^\vee)$ the Cartan matrix of $\mathcal{G}$. The algebra $\mathcal{G}$ can be equipped with a natural gradation, sometimes called the principal gradation. It is specified by setting $\deg(\alpha_i^\vee) = 0$; $\deg(e_i^\pm) = \pm 1$.

We denote by (x, t) the space-time coordinates and by $x^\pm = x \pm t$, the light-cone coordinates. We consider the model on a cylinder of radius L (therefore, x goes from 0 to L). Let $\Phi(x, t)$ be a field taking values in the Cartan subalgebra $\mathcal{H}$. The Toda field equations for $\Phi(x, t)$ are:

$$\partial^\mu \partial_\mu \Phi(x, t) = \frac{1}{2} \sum_i \alpha_i^\vee \exp[2\alpha_i(\Phi(x, t))]\,. \tag{32}$$

The sum in Eq. (32) is over the simple roots of $\mathcal{G}$. As is well known, Eq. (32) admits a Lax representation. The Toda equations of motion (32) are equivalent to the zero curvature condition, $[\mathcal{D}_\mu, \mathcal{D}_\nu] = 0$ for the Lax connection, $\mathcal{D}_\mu = \partial_\mu - A_\mu$, defined by,

$$\mathcal{D}_{x^+} = \partial_{x^+} + \partial_{x^+}\Phi + e^{ad\Phi}\mathcal{E}_+\,,$$

$$\mathcal{D}_{x^-} = \partial_{x^-} - \partial_{x^-}\Phi + e^{-ad\Phi}\mathcal{E}_-\,, \tag{33}$$

with $\mathcal{E}_\pm = \sum_i e_i^\pm$. Note that $\deg(\mathcal{E}_\pm) = \pm 1$. Let, as in the previous section, $\Psi(x, t)$ be the normalized wave function; it is specified by $(\partial_\mu - A_\mu)\Psi(x, t) = 0$ and the normalization condition $\Psi(0) = 1$.

The Toda field theories are Hamiltonian systems. The Poisson brackets between the field $\Phi(x, t)$ and its conjugate momentum $\Pi(x, t)$ are introduced in the usual way. As is well known it implies the following Poisson brackets for the G-valued function $\Psi(x, t)$:

$$\{\Psi(x) \overset{\otimes}{,} \Psi(x)\} = [r^\pm, \Psi(x) \otimes \Psi(x)]\,.$$

The matrices $r^\pm$ are given by [20, 21]:

$$r^\pm = \pm\left(\sum_i H_i \otimes H_i + 2\sum_{\alpha > 0} E_{\pm\alpha} \otimes E_{\mp\alpha}\right) \tag{34}$$

with H_i an orthonormalized basis of $\mathcal{H}$ and $E_{\pm\alpha}$ the normalized Chevalley generators. The sum $\sum_{\alpha>0}$ is over all the positive roots of the algebra $\mathcal{G}$. The matrices $r^\pm$ are solutions of the classical Yang–Baxter equation (6).

3.2. Dressing Transformations in the Toda Models. The dressing transformations are associated to a factorization problem in the group G. For the Toda field theories, this factorization is specified by the classical $r^{\pm}$-matrices given in Eq. (34). It can be described as follows. Any element $g \in G$ admits a unique decomposition as $g = g_-^{-1} g_+$ with $g_\pm \in B_\pm = HN_\pm$ and such that g_- and g_+ have inverse components on the Cartan torus. In practice it is given by half splitting the Gaussian decomposition of g. The infinitesimal version of this factorization problem consists in decomposing any element $X \in \mathscr{G}$ as $X = X_+ - X_-$ with $X_\pm \in (\mathscr{H} \oplus \mathscr{N}_\pm)$ such that X_+ and X_- have opposite components on $\mathscr{H}$.

The dressing transformations are defined as in Eqs. (26) and (27): $\Psi(x) \to \Psi^g(x) = \Theta_\pm \Psi(x) g_\pm^{-1}$ with $\Theta_-^{-1}\Theta_+ = \Psi(x) g \Psi^{-1}(x)$. They induce gauge transformations on the Lax connection: $A_\mu = (\partial_\mu \Psi)\Psi^{-1}$ is transformed into $A_\mu^g = (\partial_\mu \Psi^g)\Psi^{g-1}$. The factorization problem described above is cooked up such that the form of the Lax connection is preserved by these transformations. The proof of this statement essentially relies on the fact that the gauge transformations (28) can be implemented using either Θ_- or Θ_+. One first shows that the degrees of the components of the Toda Lax connection are preserved by the dressing and then one verifies that the connection can be written as in Eq. (33).

The (non-local) gauge transformations Eq. (28) of the Lax connection induce transformations of the Toda fields $\Phi(x, t)$. Because the form of the Lax connection is preserved by these transformations, the induced actions map a solution of the Toda equations $\Phi(x, t)$ into another solution $\Phi^g(x, t)$ (which, in general, possess non-trivial topological numbers). The transformations of the Toda fields can be described as follows. Factorize $\Theta_\pm$ as:

$$\Theta_\pm = (\Psi g \Psi^{-1})_\pm = K_\pm^g M_\pm^g$$

with $M_\pm^g \in N_\pm$ and $K_\pm^g \in H$. According to the factorization problem, the components of Θ_- and Θ_+ on the Cartan torus are inverse: $K_-^g K_+^g = 1$. Put $K_\pm^g = \exp \Delta_\pm^g$ with $\Delta_\pm^g \in \mathscr{H}$: $\Delta_-^g + \Delta_+^g = 0$. Then, by looking at the exact expression of the transformed Lax connection A_μ^g, one deduces that:

$$\Phi^g = \Phi - \Delta_+^g = \Phi + \Delta_-^g \,. \tag{35}$$

By construction, if Φ is a solution of the Toda equations, so is the dressed field Φ^g. The relation between Φ^g and Φ is non-local because $\Delta_\pm^g$ are expressed in a non-local way in terms of Φ. We have: $\exp(\pm 2\lambda_{\max}(\Delta_\pm^g)) = \langle \lambda_{\max} | \Psi g \Psi^{-1} | \lambda_{\max} \rangle$, for any highest weight $\lambda_{\max}$.

We explicitly checked in [23] that the action defined in Eq. (26) is a Lie Poisson action provided that the group G^* is equipped with the Semenov-Tian-Shansky Poisson brackets (12).

3.3. Dressings of the Chiral Fields. If $\mathscr{G}$ possesses highest weight representations, the Toda field equations admit a remarkable chiral splitting. Requiring that the Kac–Moody algebra possesses integrable highest weight representations selects the finite dimensional Lie algebras or the affine algebras but excludes the loop algebras. Let us denote by ρ a highest weight vector representation of $\mathscr{G}$ acting on the linear space V, and by $|\lambda_{\max}\rangle$ its highest weight vector. To any representation ρ we associate two fields, $\xi(x, t)$ and $\bar{\xi}(x, t)$, taking values in V and in its dual V^*. They are defined by:

$$\xi(x) = \langle \lambda_{\max} | (e^{-\Phi(x)} \Psi(x)) \,, \tag{36}$$

$$\bar{\xi}(x) = (\Psi^{-1}(x) e^{-\Phi(x)}) | \lambda_{\max} \rangle \,, \tag{37}$$

$\xi(x)$ is a line vector, $\bar{\xi}(x)$ is a column vector. The Toda field equations imply that these fields are chiral: $\partial_{x^-}\xi = \partial_{x^+}\bar{\xi} = 0$. The Toda fields $\Phi(x)$ can be reconstructed from the fields $\xi(x)$ and $\bar{\xi}(x)$;

$$\exp(-2\lambda_{\max}(\Phi)(x)) = \xi(x)\cdot\bar{\xi}(x)\,.$$

The Poisson brackets between the fields $\xi(x)$ and $\bar{\xi}(x)$ are derived from the Sklyanin brackets (25) as in [22]. They are the following semi-classical exchange relations:

$$\begin{aligned}
\{\xi(x) \overset{\otimes}{,} \xi(y)\} &= -\,\xi(x)\otimes\xi(y)r^{\pm}; && \text{for } x \gtrless y\,,\\
\{\bar{\xi}(x) \overset{\otimes}{,} \bar{\xi}(y)\} &= -\,r^{\mp}\,\bar{\xi}(x)\otimes\bar{\xi}(y); && \text{for } x \gtrless y\,,\\
\{\xi(x) \overset{\otimes}{,} \bar{\xi}(y)\} &= (\xi(x)\otimes 1)r^{-}(1\otimes\bar{\xi}(y)); && \forall x, y\,,\\
\{\bar{\xi}(x) \overset{\otimes}{,} \xi(y)\} &= (1\otimes\xi(y))r^{+}(\bar{\xi}(x)\otimes 1)); && \forall x, y\,.
\end{aligned} \tag{38}$$

Proposition. *The action on the chiral fields $\xi(x,t)$ and $\bar{\xi}(x,t)$ induced by the dressing transformation is remarkably simple*:

$$\begin{aligned}
\xi^{g}(x,t) &= \xi(x,t)\cdot g_{-}^{-1}\,,\\
\bar{\xi}^{g}(x,t) &= g_{+}\cdot\bar{\xi}(x,t)\,.
\end{aligned} \tag{39}$$

It gives another formula for the dressed Toda fields:

$$\exp(-2\lambda_{\max}(\Phi^{g})(x)) = \xi(x)\cdot g\cdot\bar{\xi}(x)\,.$$

The transformation properties of the chiral fields $\xi(x,t)$ and $\bar{\xi}(x,t)$ shows clearly that the plus and minus components in G^* commute because the $g_{\pm}$ components of g act separately on the two chiral sectors. The proof of Eq. (39) is very simple:

$$\begin{aligned}
\xi^{g} &= \langle\lambda_{\max}|(e^{-\Phi^{g}}\Psi^{g})\\
&= e^{-\lambda_{\max}(\Phi^{g})}\langle\lambda_{\max}|(\Theta_{-}\Psi)g_{-}^{-1}\\
&= e^{-\lambda_{\max}(\Phi^{g}-\Phi-\Delta_{-}^{g})}\langle\lambda_{\max}|(e^{-\Phi}\Psi)g_{-}^{-1}\\
&= \xi\cdot g_{-}^{-1}\,.
\end{aligned} \tag{40}$$

In Eq. (40) we implemented the transformation using Θ_{-} and g_{-}, and we used the property that $|\lambda_{\max}\rangle$ is a highest weight vector and the relation (35). The transformation law Eq. (39) of the chiral fields probably provides the simplest way to derive the Poisson structure Eq. (12) of the dressing group by demanding that the form of Eqs. (38) is preserved under the action Eq. (39).

3.4. Non-Local Charges and the Monodromy Matrix. Let us first check that the monodromy matrix $T = \Psi(L)$ is the non-Abelian Hamiltonian generating the dressing transformations. The Poisson brackets between the chiral fields and the monodromy matrix are:

$$\begin{aligned}
\{\xi(x) \overset{\otimes}{,} T\} &= -\,\xi(x)\otimes Tr^{-}\,,\\
\{\bar{\xi}(x) \overset{\otimes}{,} T\} &= (1\otimes T)r^{+}(\bar{\xi}(x)\otimes 1)\,.
\end{aligned}$$

Therefore, by a direct computation, we find, for any $X \in \mathscr{G}$:

$$\mathrm{tr}_2(1 \otimes X T^{-1}\{\xi(x) \overset{\otimes}{,} T\}) = -\,\xi(x) X_- = \delta_X \xi(x)\,,$$
$$\mathrm{tr}_2(1 \otimes X T^{-1}\{\bar{\xi}(x) \overset{\otimes}{,} T\}) = X_+ \bar{\xi}(x) = \delta_X \bar{\xi}(x)\,. \tag{41}$$

Equations (41) are the infinitesimal actions of the dressings (39) by $g \simeq 1 + X_+ - X_-$. As explained in Sect. 1, the Poisson algebra of the charges which generate the dressing follow from the Poisson brackets of the monodromy matrix:

$$\{T \overset{\otimes}{,} T\} = [r^\varepsilon, T \otimes T]\,.$$

To give a more precise description of this algebra, we introduce a more adequate parametrisation of the monodromy matrix T. This parametrization is obtained by factorizing $T = T_-^{-1} T_+$ with:

$$T_\pm = D^{\pm 1} M_\pm \quad \text{with } D \in \exp \mathscr{H},\, M_\pm \in \exp \mathscr{N}_\pm\,.$$

The Poisson brackets for the matrix T then become

$$\{D \overset{\otimes}{,} D\} = 0\,,$$
$$\{M_\pm \overset{\otimes}{,} D\}(M_\pm^{-1} \otimes D^{-1}) = \pm \frac{1}{2} \sum_i (M_\pm H_i M_\pm^{-1} \otimes H_i - H_i \otimes H_i)\,,$$
$$\{M_+ \overset{\otimes}{,} M_-\} = 0\,,$$
$$\{M_\pm \overset{\otimes}{,} M_\pm\}(M_\pm^{-1} \otimes M_\pm^{-1}) = (r^\varepsilon - M_\pm \otimes M_\pm r^\varepsilon M_\pm^{-1} \otimes M_\pm^{-1})|_{\mathscr{N}_\pm \otimes \mathscr{N}_\pm}\,.$$

Notice that the Poisson brackets between M_+ and M_- are zero and that the matrix D is a generating function for charges in involution. The matrices $T_\pm$ are related to the charges Q_i and $Q_{\pm\alpha}$ by:

$$D^2 = \exp\left(\sum_i Q_i H_i\right),$$
$$M_\pm = \exp\left(2 \sum_{\alpha > 0} Q_{\pm\alpha} E_{\pm\alpha}\right).$$

Proposition. *The dressing Poisson algebra is generated by the charges $Q_i^\vee = (\alpha_i^\vee, Q) = \sum_j (\alpha_i^\vee, H_j) Q_j$ and $Q_{\pm\alpha_i}$, where α_i are the simple roots. The only relations among these generators are:*

$$\{Q_i^\vee, Q_j^\vee\} = 0\,,$$
$$\{Q_i^\vee, Q_{\pm\alpha_j}\} = \pm\, a_{ij} Q_{\pm\alpha_j}\,,$$
$$\{Q_{\pm\alpha_i}, Q_{\mp\alpha_j}\} = 0\,, \tag{42}$$

plus the deformed Serre-like relations:

$$\sum_{0 \leq 2k \leq 1 - a_{ij}} B_{ij}^{2k} Q_{\pm\alpha_i}^{2k} \underbrace{\{Q_{\pm\alpha_i}, \{\cdots, \{Q_{\pm\alpha_i}, Q_{\pm\alpha_j}\} \cdots\}}_{1 - a_{ij} - 2k \ \ \mathit{factors}} = 0\,, \tag{43}$$

where the B_{ij}-coefficients are given by the generating function $B_{ij}(x)$:

$$B_{ij}(x) = \sum_{0 \leqq 2k \leqq 1 - a_{ij}} B_{ij}^{2k} x^{2k} = \prod_{0 \leqq 2p \leqq -a_{ij}} [1 - (\alpha_i, \alpha_j + p\alpha_i)^2 x^2] .$$

For the proofs see [24]. Equations (43) are exactly the semi-classical limits of the quantum Serre relations. As it should be this Poisson algebra is the semi-classical limit of $\mathscr{U}_q(\mathscr{G}^*) \simeq \mathscr{U}_q(\mathscr{B}_+) \times \mathscr{U}_q(\mathscr{B}_-)$, which is not quite the same as the semi-classical limit of $\mathscr{U}_q(\mathscr{G})$.

The action of these charges on the chiral fields is described as follows:

$$\{\xi(x), Q_i\} = \xi(x) \cdot H_i ,$$

$$\{\xi(x), Q_{\alpha_i}\} + (\xi(x) \cdot H_{\alpha_i}) Q_{\alpha_i} = \xi(x) \cdot E_{-\alpha_i} ,$$

$$\{\xi(x), Q_{-\alpha_i}\} = 0 ,$$

Similarly with the other chirality. The extra piece in the bracket between the charges Q_{α_i} and the chiral fields arises from the non-locality of the charges: it reflects the semi-classical exchange relations between the chiral fields and the non-local currents.

The fields $\xi(x, t)$ and $\bar{\xi}(x, t)$ are the classical analogues of the field multiplets which transform covariantly under the action of the quantum group symmetry in the $W_{\mathscr{G}}$-invariant conformal field theories, (theories which include the minimal conformal models for $\mathscr{G} = su(2)$) [26, 27].

4. Example 2: The Heisenberg Model

As a second example, we consider the Heisenberg chain. We restrict ourselves to the isotropic case, although the same considerations also apply to the non-isotropic cases. Besides being one of the simplest and most studied models, our present interest lies in the fact that the group of dressing transformations is infinite dimensional. We will show that the Poisson algebra of these transformations form a Yangian. Recall that according to Drinfeld [2], the Yangian $Y(\mathscr{G})$ is an associative algebra with unity, generated by the elements Q_i^0 and Q_i^1 with the defining relations

$$[Q_i^0, Q_j^0] = f_{ijk} Q_k^0 ,$$

$$[Q_i^0, Q_j^1] = f_{ijk} Q_k^1 ,$$

$$[Q_i^1, [Q_j^1, Q_k^0]] - [Q_i^0, [Q_j^1, Q_k^1]] = A_{lmn}^{ijk} \{Q_l^0, Q_m^0, Q_n^0\}_{\mathrm{sym}} ,$$

$$[[Q_i^1, Q_j^1], [Q_k^0, Q_l^1]] + [[Q_k^1, Q_l^1], [Q_i^0, Q_j^1]]$$
$$= (A_{stu}^{ijr} f_{klr} - A_{stu}^{klr} f_{ijr}) \{Q_s^1, Q_t^0, Q_u^0\}_{\mathrm{sym}} , \quad (44)$$

where f_{ijk} are the structure constants of $\mathscr{G}$, $A_{rst}^{ijk} = \frac{1}{24} f_{ira} f_{jsb} f_{ktc} f^{abc}$ and $\{x_1, x_2, x_3\}_{\mathrm{sym}} = \sum_{i \neq j \neq k} x_i x_j x_k$. It is a Hopf algebra with comultiplication

$$\Delta Q_i^0 = Q_i^0 \otimes 1 + 1 \otimes Q_i^0 ,$$

$$\Delta Q_i^1 = Q_i^1 \otimes 1 + 1 \otimes Q_i^1 - \frac{1}{2} f_{ijk} Q_j^0 \otimes Q_k^0 . \quad (45)$$

The antipode and the counit can be deduced from this comultiplication. The Yangian $Y(\mathscr{G})$ acts on the fields Φ through the adjoint action, which for the generators Q_i^0 and Q_i^1 is given by:

$$(\mathrm{Ad}\, Q_i^0)\cdot\Phi = [Q_i^0, \Phi]\,,$$
$$(\mathrm{Ad}\, Q_i^1)\cdot\Phi = [Q_i^1, \Phi] + \frac{1}{2} f_{ijk}[Q_j^0, \Phi]Q_k^0\,. \tag{46}$$

Our purpose is to obtain the semiclassical version of the above formulae by investigating the group of dressing transformations in the Heisenberg model.

4.1. Definition of the Model. We start with the definition of the model. Consider a spin variable $S(x)$:

$$S(x) = \sum_{i=1}^{3} S^i(x)\sigma_i \quad \text{with} \quad \sum_{i=1}^{3} S^i(x)^2 = s^2\,,$$

where the σ_i are the Pauli matrices with $[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$ and $\mathrm{tr}(\sigma_j\sigma_k) = 2\delta_{jk}$. Introduce the Poisson bracket

$$\{S^i(x), S^j(y)\} = \varepsilon^{ijk}S^k(x)\delta(x-y)\,.$$

The Hamiltonian is

$$H_1 = -\frac{1}{4}\int_0^L dx\, \mathrm{tr}(\partial_x S \partial_x S)\,.$$

The equations of motion deduced from this Hamiltonian read

$$\partial_t S = -\frac{i}{2}[S, \partial_x^2 S] = \frac{i}{2}\partial_x[S_x, S]\,. \tag{47}$$

Notice that these equations are the conservation laws for a $su(2)$-valued current. The integrability of the Heisenberg model relies on the fact that Eqs. (47) can be written as a zero curvature condition

$$(\partial_x + A_x)\Psi(x,\lambda) = 0\,,$$
$$(\partial_t + A_t)\Psi(x,\lambda) = 0$$

with Lax connection:

$$A_x = \frac{i}{\lambda}S(x)\,,$$
$$A_t = -\frac{2is^2}{\lambda^2}S(x) + \frac{1}{2\lambda}[S(x), \partial_x S(x)]\,. \tag{48}$$

The Lax connection (48) is an element of the $su(2)$ loop algebra $\widetilde{su(2)} = su(2) \otimes C[\lambda, \lambda^{-1}]$. An important ingredient is the transport matrix $T(x,\lambda)$; it is defined as

$$\Psi(x,\lambda) = T(x,\lambda)\Psi(0,\lambda)\,,$$
$$T(x,\lambda) = P\exp\left[-\int_0^x A_x(y,\lambda)\,dy\right]. \tag{49}$$

The monodromy matrix $T(\lambda)$ is simply $T(L, \lambda)$. As is well known, the importance of the monodromy matrix lies in the fact that one can calculate the Poisson bracket of its matrix elements. One finds [18]:

$$\{T(\lambda) \overset{\otimes}{,} T(\mu)\} = \frac{1}{2}[r(\lambda, \mu), T(\lambda) \otimes T(\mu)] , \tag{50}$$

where

$$r(\lambda, \mu) = \frac{\sum_i \sigma_i \otimes \sigma_i}{\lambda - \mu} . \tag{51}$$

From this result, it follows that $\mathrm{tr}(T(\lambda))$ generates quantities in involution.

4.2. The Transfer Matrix and the Local Conserved Charges. From its definition, $T(x, \lambda)$ is analytic in λ with an essential singularity at $\lambda = 0$. From Eq. (49), we can easily find an expansion around $\lambda = \infty$:

$$T(x, \lambda) = 1 - \frac{i}{\lambda}\int_0^x dy\, S(y) - \frac{1}{\lambda^2}\int_0^x dy\, S(y)\int_0^y dz\, S(z) \cdots .$$

This development in $1/\lambda$ has an infinite radius of convergence. We will study it later in more detail in relation with the dressing transformations.

To find the structure of $T(x, \lambda)$ around $\lambda = 0$ is more delicate but important as it provides the local conserved charges in involution. The main point [6] is to notice that there exists a *local* gauge transformation, *regular* at $\lambda = 0$, such that

$$T(x, \lambda) = g(x) D(x) g^{-1}(0) , \tag{52}$$

where $D(x)$ is a diagonal matrix: $D(x) = \exp(id(x)\sigma_3)$. We can choose g to be unitary, and, since g is defined up to a diagonal matrix, we can require that it has a real diagonal:

$$g = \frac{1}{(1 + v\bar{v})^{\frac{1}{2}}}\begin{pmatrix} 1 & v \\ -\bar{v} & 1 \end{pmatrix} .$$

The differential equation for T becomes a differential equation for g and d:

$$g^{-1}\partial_x g + i(\partial_x d)\sigma_3 + \frac{i}{\lambda} g^{-1} S g = 0 . \tag{53}$$

Projecting Eq. (53) on the Pauli matrices σ_i's gives differential equations for v and d:

$$\partial_x v = -\frac{i}{\lambda}(S_- + 2vS_3 - S_+ v^2) ,$$

$$\partial_x d = \frac{1}{2\lambda}(-2S_3 + vS_+ + \bar{v}S_-) .$$

The first of these equations is a Ricatti equation for $v(x)$. Expanding in λ the functions $v(x)$ and $d(x)$ as:

$$\partial_x d(x) = -\frac{s}{\lambda} + \sum_{n=0}^{\infty} \rho_n(x)\lambda^n ,$$

$$v(x) = \sum_{n=0}^{\infty} v_n(x)\lambda^n ; \quad v_0 = \frac{S_3 - s}{S_+} .$$

The Ricatti equation (53) becomes:

$$2is\,v_{n+1} = -v_n' + iS_+ \sum_{m=1}^{n} v_{n+1-m}v_m ,$$

$$\rho_n = \frac{1}{2}(v_{n+1}S_+ + \bar{v}_{n+1}S_-) . \tag{54}$$

Note that $v(x)$ is regular at $\lambda = 0$. Equations (54) recursively determine the functions $v_n(x)$ and $\rho_n(x)$ as local functions of the dynamical variables $S^i(x)$. This describes the asymptotic behaviour of $T(\lambda)$ at $\lambda = 0$. The asymptotic series become convergent if we regularize the model by discretizing the space interval.

Concerning the monodromy matrix $T(\lambda)$, since $g(x)$ is local and if we assume periodic boundary conditions, we can write

$$T(\lambda) = \cos P_0(\lambda)\,\mathrm{Id} + i\sin P_0(\lambda) M(\lambda) ,$$

where $M(\lambda) = M(L; \lambda)$ with $M(x, \lambda) = g(x)\sigma_3 g^{-1}(x)$ and

$$P_0(\lambda) = \int_0^L dx(\partial_x d) . \tag{55}$$

The trace of the transfer matrix $\mathrm{tr}(T(\lambda))$ is:

$$\mathrm{tr}(T(\lambda)) = 2\cos P_0(\lambda) . \tag{56}$$

Using the previous result at $\lambda = 0$, we can use $P_0(\lambda)$ as a generating function for the commuting local conserved quantities:

$$I_n = \int_0^L dx\,\rho_n(x) .$$

The first ones are

$$I_0 = \frac{i}{4s}\int_0^L dx \log\left(\frac{S_+}{S_-}\right)\cdot \partial_x S_3 ,$$

$$I_1 = -\frac{1}{16s^3}\int_0^L dx\,\mathrm{tr}(\partial_x S \partial_x S) ,$$

$$I_2 = \frac{i}{64s^5}\int_0^L dx\,\mathrm{tr}(S[\partial_x S, \partial_x^2 S]) .$$

I_0 and I_1 correspond to momentum and energy respectively.

4.3. Dressing Transformations. We now describe the group of dressing transformations for the Heisenberg model. Our first task is to specify the Lie algebras $\mathscr{G}_\pm$ entering the factorization problem. The matrices $r^\pm$ correspond to expanding the matrix $r(\lambda, \mu)$ either in powers of (λ/μ) or in powers of (μ/λ); e.g.:

$$r^+(\lambda, \mu) = -\sum_{n=0}^{\infty} \lambda^n \sigma_i \otimes \left(\frac{\sigma_i}{\mu^{n+1}}\right),$$

$$r^-(\lambda, \mu) = \sum_{n=0}^{\infty} \left(\frac{\sigma_i}{\lambda^{n+1}}\right) \otimes (\mu^n \sigma_i).$$

Introducing on the loop algebra $\widetilde{su(2)}$ the following bilinear form

$$\mathrm{Tr}(X(\lambda)\, Y(\lambda)) = -\oint \frac{d\lambda}{2i\pi} \mathrm{tr}(X(\lambda)\, Y(\lambda)),$$

we can rewrite $r^\pm$ as projection operators. For any $X(\lambda) = \sum_i \lambda^i X_i \in \widetilde{su(2)}$, we define as in Eq. (9),

$$X_\pm(\lambda) = R_\pm(X(\lambda)) = -\oint \frac{d\mu}{2i\pi} \mathrm{tr}_2\left(r_{12}^\pm(\lambda, \mu) 1 \otimes X(\mu)\right). \tag{57}$$

We have

$$X_+(\lambda) = \sum_{n \geqq 0} \lambda^n X_n, \tag{58}$$

$$X_-(\lambda) = -\sum_{n<0} \lambda^n X_n. \tag{59}$$

Therefore, $\mathscr{G}_\pm = \mathrm{Im}\, R_\pm$ are the subalgebras of $\widetilde{su(2)}$ with elements of the form Eq. (58) and Eq. (59) respectively. Notice that any $X(\lambda) \in \widetilde{su(2)}$ has a unique decomposition,

$$X(\lambda) = X_+(\lambda) - X_-(\lambda).$$

This defines our factorization (or Riemann–Hilbert) problem. Remark that the transfer matrix $T(x, \lambda)$ belongs to the subgroup $G_- = \exp(\mathscr{G}_-)$. This is an important particularity of the Heisenberg model.

From the factorization problem, one can define the dressing transformations. Let $X(\lambda) \in \widetilde{su(2)}$ and set

$$\Theta_X(x, \lambda) = (TXT^{-1})(x, \lambda) = \Theta_+(x, \lambda) - \Theta_-(x, \lambda).$$

A dressing transformation is a gauge transformation with either Θ_+ or Θ_-,

$$\delta_X A_x = -[A_x, \Theta_+] - \partial_x \Theta_+.$$

The gauge transformation with Θ_- gives the same result. If $X \in \mathscr{G}_-$, then $\delta_X S(x) = 0$ since $\Theta = -\Theta_-$. For $X \in \mathscr{G}_+$, one has:

$$\begin{aligned}\delta_X S(x) &= -[S(x), \Theta_+(x, \lambda)|_{\lambda=0}] \\ &= -\oint \frac{d\lambda}{2i\pi\lambda} [S(x), \Theta_X(x, \lambda)].\end{aligned} \tag{60}$$

This is proved by noticing that [28]:

$$\begin{aligned}\delta_X S &= -i\mathrm{Res}_{\lambda=0}(\delta_X A_x)\\ &= i\partial_x \mathrm{Res}_{\lambda=0}(\Theta_+) + i\mathrm{Res}_{\lambda=0}([A_x, \Theta_+]) .\end{aligned}$$

One can give a more explicit description of these transformations.

Proposition. *Let* $X = i\lambda^n v$, *with* $v^+ = v$, *and denote* δ_X *by* δ_v^n. *Then*:

$$\delta_v^n S(x) = i[Z_v^n(x), S(x)] , \tag{61}$$

where the functions $Z_v^k(x)$ *can be computed recursively by*

$$\partial_x Z_v^k(x) + i[S(x), Z_v^{k-1}(x)] = 0; \quad Z_v^0 = v . \tag{62}$$

This can be proved as follows. Let us define the functions $Z_v^k(x)$ by:

$$(TvT^{-1})(x, \lambda) = \sum_{k=0}^{\infty} \lambda^{-k} Z_v^k(x) . \tag{63}$$

The differential equations satisfied by these functions are consequences of those satisfied by the transport matrix $T(x, \lambda)$. For $X = i\lambda^n v$, we can expand $\Theta_X(x, \lambda)$ as:

$$\Theta_X(x, \lambda) = i\sum_{k=0}^{\infty} \lambda^{n-k} Z_v^k(x) .$$

With these notations, the factorization problem has a simple solution

$$\Theta_+ = i\sum_{k=0}^{n} \lambda^{n-k} Z_v^k, \quad \Theta_- = -i\sum_{k=n+1}^{\infty} \lambda^{n-k} Z_v^k .$$

It is then also a simple exercise to check that the form of A_x is preserved. In fact

$$\begin{aligned}\delta_X A_x &= -[A_x, \Theta_+] - \partial_x \Theta_+\\ &= \lambda^{-1}[S, Z_v^n] - i\sum_{k=0}^{n-1} \{\partial_x Z_v^{k+1} + i[S, Z_v^k]\}\lambda^{n-k-1}\\ &= \lambda^{-1}[S, Z_v^n] ,\end{aligned}$$

the last sum vanishes by virtue of Eq. (62) and we are left with $\delta_v^n S(x) = i[Z_v^n(x), S(x)]$. This proves Eq. (61). One can check similarly that the form of A_t is unchanged, and its variation is compatible with Eq. (61). This in particular implies that the equations of motion are invariant. In other words, the transformations (61) are symmetries of the equations of motion. This can also be checked directly.

It is proved in the appendix that this action is Lie–Poisson, i.e.

$$\{{}^gS_i(x), {}^gS_j(y)\}_{G\times M} = \varepsilon_{ijk}\,{}^gS_k(x)\delta(x-y) ,$$

where for an infinitesimal transformation ${}^gS = S + \delta_X S$. Taking $X = i\sum_{n\geqq 0}\sum_{i=1}^{3} \xi_n^i \lambda^n \sigma_i$, the Poisson bracket on G reads

$$\{\xi_n^i, \xi_m^j\}_G = \varepsilon^{ijk}\xi_{n+m+1}^k . \tag{64}$$

The shift in the grading (i.e. $n + m + 1$ instead of $n + m$) is due to the choice of the bilinear form on $\widetilde{su(2)}$ made in Eq. (57).

4.4. The Non-Local Charges and the Monodromy Matrix. We already remarked that the equations of motion have the form of a conservation law

$$\partial_t J_t - \partial_x J_x = 0$$

with $J_t = S$ and $J_x = \frac{i}{2}[S_x, S]$. Since the dressing transformations are symmetries of the equations of motion, by dressing this local current we produce new currents which form an infinite multiplet of non-local conserved currents,

$$\begin{aligned} J_t^{n,v} &= \delta_v^n S = i[Z_v^n, S]\,, \\ J_x^{n,v} &= -\frac{1}{2}[\partial_x[Z_v^n, S], S] - \frac{1}{2}[S_x, [Z_v^n, S]]\,, \end{aligned} \tag{65}$$

for any $n \geqq 0$ and $v \in su(2)$. The charges are defined by:

$$Q_v^n = \int_0^L J_t^{n,v}(x)\, dx = Z_v^{n+1}(L)\,.$$

Since the currents are non-local, the charges are not conserved. We have

$$\frac{d}{dt} Q_v^n = J_x^{n,v}(L) - J_x^{n,v}(0)$$

or

$$\frac{d}{dt} Q_v^n = \frac{1}{2}[[S_x(L), S(L)], Q_v^{n-1}] - 2is^2[Q_v^{n-2}, S(L)]\,, \quad n \geqq 2\,.$$

Nevertheless, these charges are important because, as we will see, they are the generators of the dressing transformations.

We now show that they are equivalent to the knowledge of the monodromy matrix T. Let

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}; \quad AD - BC = 1\,.$$

Let us define

$$Q_{ij}(\lambda) = \frac{1}{2}\mathrm{tr}(T\sigma_i T^{-1}\sigma_j)(\lambda) = \delta_{ij} + \sum_{p=0}^{\infty} \lambda^{-p-1} Q_{ij}^p\,,$$

where $Q_{ij}^p = \frac{1}{2}\mathrm{tr}(Z_{\sigma_i}^{p+1}\sigma_j)$. The $Q_{ij}(\lambda)$ are the generating functions of the non-local charges. Introduce the antisymmetric part of the matrix elements $Q_{ij}(\lambda)$,

$$Q_i(\lambda) = \sum_{j,k=0}^{3} \varepsilon_{ijk} Q_{jk}(\lambda)\,.$$

The quantities $Q_i(\lambda)$ are quadratic functions of the matrix elements of $T(\lambda)$. But, one can invert these relations and express $T(\lambda)$ in terms of the $Q_i(\lambda)$'s;

Proposition. *The relation between the transfer matrix and the non-local charges $Q_i(\lambda)$ is:*

$$T(\lambda) = \frac{1}{2} W(\lambda)\mathrm{Id} - \frac{i}{2} W^{-1}(\lambda) \sum_i Q_i(\lambda)\sigma_i$$

with

$$W(\lambda) = \sqrt{2 + 2\sqrt{1 - \frac{1}{4}\vec{Q}^2(\lambda)}} .$$

Using this result, we can express the matrix Q_{ij} in terms of the charges Q_i. Setting

$$\mathscr{A}(\lambda) = \frac{1}{2}\begin{pmatrix} 0 & Q_3 & -Q_2 \\ -Q_3 & 0 & Q_1 \\ Q_2 & -Q_1 & 0 \end{pmatrix}$$

we find

$$Q(\lambda) = \mathrm{Id} + \mathscr{A} + 2W^{-2}\mathscr{A}^2 .$$

So, the charges $Q_i(\lambda)$ contains the same amount of information as the transfer matrix.

We will show in the next section that the charges $Q_i(\lambda)$ as well as $T(\lambda)$ generate the dressing transformations, and that under Poisson bracket the $Q_i(\lambda)$'s generate a Yangian. It is interesting in this context to examine more closely the relation between the $Q_i(\lambda)$ and the matrix elements of $T(\lambda)$. We have

$$Q_+(\lambda) = Q_1(\lambda) + iQ_2(\lambda) = 2iW[\vec{Q}^2(\lambda)]C(\lambda) , \tag{66}$$

$$Q_-(\lambda) = Q_1(\lambda) - iQ_2(\lambda) = 2iW[\vec{Q}^2(\lambda)]B(\lambda) . \tag{67}$$

In view of Eq. (66, 67), one can speculate that, in this case, the algebraic Bethe Ansatz [6] is nothing else but the construction of highest weight representations of $Y(su(2))$, the symmetry group of our model.

Moreover

$$\mathrm{tr}\, T(\lambda) = W[\vec{Q}^2(\lambda)] .$$

Therefore, $\vec{Q}^2(\lambda)$ is also a generating function for commuting quantities. Its relation with the generating function $P_0(\lambda)$ is:

$$\vec{Q}^2(\lambda) = 4\sin^2(2P_0(\lambda)) .$$

However, expanding $\vec{Q}^2(\lambda)$ around $\lambda = \infty$ gives non-local commuting quantities while expanding $P_0(\lambda)$ around $\lambda = 0$ gives the local commuting quantities. The links between them are hidden in the subtleties of the analytic properties of the monodromy matrix. These analytic properties should be encoded into the representation theory of the Yangian.

As another application of the relation between Q_i and T, we give a formula for the coproduct of the charges Q_i. Recall that following an idea of [25], and as is well known in the formalism of integrable systems [6], the classical counterpart of the coproduct on T is given by

$$\Delta T_{ij} = (T'T)_{ij} ,$$

where T and T' are the transport matrices on two adjacent intervals. Then

$$\Delta Q_{ij} = \frac{1}{2}\mathrm{tr}(T'T\sigma_i T^{-1}T'^{-1}\sigma_j)$$

and using $T\sigma_i T^{-1} = \sum_l Q_{il}\sigma_l$, we obtain

$$\Delta Q_{ij} = \sum_j Q_{il} Q'_{lj} .$$

Inserting the precise form of Q in terms of $\mathscr{A}$, we find

$$\Delta \mathscr{A} = \mathscr{A} + \mathscr{A}' + \frac{1}{2}[\mathscr{A}, \mathscr{A}'] + W^{-2}(\mathscr{A}^2\mathscr{A}' + \mathscr{A}'\mathscr{A}^2)$$
$$+ W'^{-2}(\mathscr{A}\mathscr{A}'^2 + \mathscr{A}'^2\mathscr{A}) + 2W^{-2}W'^{-2}[\mathscr{A}^2, \mathscr{A}'^2] .$$

If we expand in λ, $Q_i(\lambda) = \sum_{n=0}^{\infty} \lambda^{-n-1} Q_i^n$, we get

Proposition. *For* $n = 0$ *and* 1, *we have*:

$$\Delta Q_i^0 = Q_i^0 + Q_i'^0 ,$$
$$\Delta Q_i^1 = Q_i^1 + Q_i'^1 - \frac{1}{4}\varepsilon_{ikl} Q_k^0 Q_l'^0 .$$

These are exactly the comultiplications in the Yangian, cf. *Eq.* (44).

4.5. Charges as Generators of Dressing Transformations. We now calculate the Poisson bracket of the charges Q_i with the dynamical variable S. We will first show that the non-Abelian Hamiltonian of the transformations (60) is the monodromy matrix and then, we show that the charges Q_i also generate these transformations. We start form

$$T_1^{-1}(\lambda, x)\{T_1(\lambda, x), S_2(y)\}$$
$$= \frac{1}{2\lambda}\theta(x-y) \sum_{i=1}^{3} \sigma_i \otimes [T(\lambda, y)\sigma_i T^{-1}(\lambda, y), S(y)] . \tag{68}$$

This is easily established from the linear system or from the definition of $T(\lambda, x)$ as a path ordered exponential. With this formula, we can check immediately that the generator of the dressing transformations is $T(\lambda)$ itself. Indeed, setting $x = L$, multiplying Eq. (68) by X on the first space and taking the trace, we get

$$\mathrm{tr}_1(X_1 T_1^{-1}(\lambda)\{T_1(\lambda), S_2(y)\}) = \frac{1}{\lambda}[T(\lambda, y) X T^{-1}(\lambda, y), S(y)] .$$

Choosing $X = i\lambda^n v$ and using Eqs. (63, 61), we find

$$\delta_v^n S(y) = \oint \frac{d\lambda}{2i\pi} \mathrm{tr}_1(\lambda^n i v_1 T_1^{-1}(\lambda)\{T_1(\lambda), S_2(y)\}) = \mathrm{Tr}(X T_2^{-1}\{S_1(y), T_2\})$$

which is exactly what was to be expected: $T(\lambda)$ is the generator of dressing transformations. To express this formula in terms of the charges Q_i, one can also calculate the Poisson bracket of $\Theta_v = TvT^{-1}$ and S. We find

$$\{Q_{ij}, S(y)\} = \varepsilon_{ijk}\delta_k^n S(y) - \sum_{p=0}^{n-1} \varepsilon_{ikl} Q_{lj}^{n-p-1} \delta_k^p S(y)$$

or else

$$\delta_i^n S(y) = \frac{1}{2}\{Q_i^n, S(y)\} + \frac{1}{2}\sum_{p=0}^{n-1} [Q^{n-p-1} - \mathrm{tr}(Q^{n-p-1})\mathrm{Id}]_{ik} \delta_k^p S(y) .$$

This equation can be interpreted in two different ways: (i) this is a system of equations allowing to express recursively $\delta_v^n S(y)$ in terms of Poisson brackets; or (ii) it allows to express the dressing $\delta_X S$ in a non-linear way in terms of the charges Q_i.

Proposition. *For $n = 0$ or 1, we have:*

$$\delta_i^0 S(y) = \frac{1}{2}\{Q_i^0, S(y)\},$$

$$\delta_i^1 S(y) = \frac{1}{2}\{Q_i^1, S(y)\} - \frac{1}{8}\varepsilon^{ijk} Q_j^0 \{Q_k^0, S(y)\}. \tag{69}$$

These equations are the semi-classical analogue of the adjoint action of the Yangian, Eq. (46).

4.6. The Poisson Algebra of the Charges. It remains to calculate the Poisson brackets of the charges. For this purpose, it is enough to calculate the Poisson brackets of Θ. We find

$$\begin{aligned}\{\Theta_v(\lambda, x) \overset{\otimes}{,} \Theta_w(\mu, x)\} &= \frac{1}{2}\frac{1}{\lambda - \mu}\sum_i [\sigma^i, \Theta_v(\lambda, x)] \otimes [\sigma^i, \Theta_w(\mu, x)] \\ &\quad - \frac{1}{2}\frac{1}{\lambda - \mu}\sum_i \Theta_{[\sigma^i, v]}(\lambda, x) \otimes \Theta_{[\sigma^i, w]}(\mu, x).\end{aligned}$$

Expanding the factors $\frac{1}{\lambda - \mu}$ $\left(\text{it does not matter whether we expand in } \frac{\mu}{\lambda} \text{ or in } \frac{\lambda}{\mu}\right)$, we find

$$\begin{aligned}\{Q_{ij}^n, Q_{pq}^m\} &= 2(\delta_{iq} Q_{pj}^{n+m} - \delta_{jq} Q_{pi}^{n+m} + \delta_{ip} Q_{jq}^{n+m} - \delta_{jp} Q_{iq}^{n+m}) \\ &\quad - 2\sum_{a=0}^{n-1} (\delta_{jq} Q_{ir}^{n-1-a} Q_{pr}^{m+a} - Q_{iq}^{n-1-a} Q_{pj}^{m+a} \\ &\quad - \delta_{ip} Q_{rj}^{n-1-a} Q_{rq}^{m+a} + Q_{pj}^{n-1-a} Q_{iq}^{m+a}).\end{aligned}$$

Multiplying by $\varepsilon_{ijk}\varepsilon_{pql}$, we get

$$\begin{aligned}\{Q_k^n, Q_l^m\} &= 8\mathscr{A}_{kl}^{n+m} - 4\sum_{a=0}^{n-1} \big[(\Sigma^{n-1-a}\mathscr{A}^{m+a})_{kl} - (\mathscr{A}^{n-1-a}\Sigma^{m+a})_{kl} \\ &\quad + \mathrm{tr}(\Sigma^{m+a})\mathscr{A}_{kl}^{n-1-a} - \mathrm{tr}(\Sigma^{n-1-a})\mathscr{A}_{kl}^{m+a}\big],\end{aligned}$$

where Σ denotes the symmetric part of Q: $\Sigma(\lambda) = 2W^{-2}\mathscr{A}^2 = \sum_{n=0}^{\infty}\lambda^{-n-1}\Sigma^n$. For $n = 0$ or 1, we get

$$\{Q_k^0, Q_l^m\} = 4\varepsilon_{klr} Q_r^m,$$

$$\{Q_k^1, Q_l^m\} = 8\mathscr{A}_{kl}^{m+1} - 4[\mathrm{tr}(\Sigma^m)\mathscr{A}_{kl}^0 - (\mathscr{A}^0\Sigma^m)_{kl}].$$

The first equation shows that Q_k^0 generate an $su(2)$ subalgebra and that Q_l^m transform in the adjoint representation. Setting $m = 1$ or 2 gives

$$\{Q_k^1, Q_l^1\} = 4\varepsilon_{klr} Q_r^2 + \frac{1}{2}(\vec{Q}^0 \cdot \vec{Q}^0)\varepsilon_{klr} Q_r^0,$$

$$\{Q_k^1, Q_l^2\} = 4\varepsilon_{klr} Q_r^3 + (\vec{Q}^0 \cdot \vec{Q}^1)\varepsilon_{klr} Q_r^0 + \frac{1}{2}\varepsilon_{krs} Q_s^0 Q_r^1 Q_l^0.$$

With the help of these results it is easy to prove the following:

Proposition. *The dressing Poisson algebra is generated by* Q_k^0 *and* Q_l^1. *These charges satisfy the semi-classical analogues of the defining relations of the Yangians*:

$$\{Q_k^0, Q_l^0\} = 4\varepsilon_{klr} Q_r^0 \,,$$
$$\{Q_k^0, Q_l^1\} = 4\varepsilon_{klr} Q_r^1 \,,$$
$$\{Q_\lambda^1, \{Q_\mu^1, Q_\nu^0\}\} - \{Q_\lambda^0 \{Q_\mu^1, Q_\nu^1\}\} = \alpha_{\lambda\mu\nu\alpha\beta\gamma} Q_\alpha^0 Q_\beta^0 Q_\gamma^0 \,,$$
$$\{\{Q_\lambda^1, Q_\mu^1\}, \{Q_\rho^0, Q_\sigma^1\}\} + \{\{Q_\rho^1, Q_\sigma^1\}, \{Q_\lambda^0, Q_\mu^1\}\}$$
$$= 8(a_{\lambda\mu\nu\alpha\beta\gamma}\varepsilon_{\rho\sigma\nu} + a_{\rho\sigma\nu\alpha\beta\gamma}\varepsilon_{\lambda\mu\nu}) Q_\alpha^0 Q_\beta^0 Q_\gamma^1 \,. \tag{70}$$

The last two equations are the semi-classical Serre relations since in the semi-classical limit we have

$$\{Q_\alpha^0, Q_\beta^0, Q_\gamma^1\}_{\mathrm{sym}} = Q_\alpha^0 Q_\beta^0 Q_\gamma^1 \,.$$

Above, $a_{\lambda\mu\nu\alpha\beta\gamma} = \frac{2}{3}\varepsilon_{\lambda\alpha i}\varepsilon_{\mu\beta j}\varepsilon_{\nu\gamma k}\varepsilon_{ijk}$.

These relations are proved directly using all the relations we gave. There is no extra relation between the charges Q_n^0 and Q_k^1 since there is no extra relation between the generators δ_n^0 and δ_k^1 in the $su(2)$ loop algebra.

Notice that the quantization of the algebra of the dressing transformations is canonical in the variables Q_n^0 and Q_k^1 (i.e. one goes from the semi-classical relations to the quantum relations just by transforming the Poisson brackets into commutators), but it is not canonical in terms of the transfer matrix.

4.7. Exchange Algebra. We now exhibit the exchange algebra in the Heisenberg model. The simplest multiplets are built from the variables

$$\Theta_v(x, \lambda) = T(x, \lambda) v T^{-1}(x, \lambda) = \sum_{k=0}^{\infty} \lambda^{-k} Z_v^k(x) \,.$$

So, we consider the vector $\xi^v(x, \lambda)$ with components

$$\xi^v(x, \lambda) = \{\Theta_v(x, \lambda), \delta_{\sigma_{i_1}}^{n_1} \Theta_v(x, \lambda), \delta_{\sigma_{i_2}}^{n_2} \delta_{\sigma_{i_1}}^{n_1} \Theta_v(x, \lambda), \cdots\} \,.$$

From Eq. (61) we have $Z_v^n(x) = -\delta_v^{n-1} \int_0^x S(y)dy$, so that this multiplet is really built from the multiple dressing of $\int_0^x S(y)dy$. We now show that this set of fields satisfy an exchange relation as in Eq. (31). This will be proved by induction on the degree, defined to be the number of times we act with δ.

Proposition. *The exchange relations for the* ξ*'s are*

$$\{\xi_1^v(x, \lambda), \xi_2^w(y, \mu)\} = \sum_{n,i} \delta_{\sigma_i}^n \xi_1^v(x, \lambda)\, e_n^{\sigma_i} \xi_2^w(y, \mu); \quad x < y \,,$$

where $\delta_{\sigma_\imath}^n$ *and* $e_n^{\sigma_i}$, *whose definitions are given below, satisfy the commutation relations*

$$[\delta_{\sigma_i}^n, \delta_{\sigma_j}^m] = -2\varepsilon_{ijk}\delta_{\sigma_k}^{n+m} \,,$$
$$[\delta_{\sigma_\imath}^n, e_m^{\sigma_j}] = -\varepsilon_{ijk}\delta_{\sigma_k}^{n-m-1} - 2\varepsilon_{ijk} e_{m-n}^{\sigma_k} \,,$$
$$[e_n^{\sigma_\imath}, e_m^{\sigma_j}] = -\varepsilon_{ijk} e_{n+m+1}^{\sigma_k} \,.$$

Above, we implicitly assume $\delta_{\sigma_\imath}^n, e_n^{\sigma_i} = 0$ *if* $n < 0$.

Proof. We first notice that for any functional $F[S](x)$ depending only on the field $S(z)$, $0 \leqq z < x$, we have from the ultralocality property when $x < y < L$,

$$T^{-1}(L)\{F[S](x), T(L)\} = T^{-1}(y)\{F[S](x), T(y)\} = T^{-1}(x)\{F[S](x), T(x)\} .$$

Using the fact that $T(\lambda)$ is the generator of dressing transformations, for such a function we have

$$\{F[S](x), \Theta_w(y, \mu)\} = \sum_{i = 1,3,n \geqq 0} \delta^n_{\sigma_i} F[S](x) e^{\sigma_i}_n \Theta_w(y, \mu) \quad x < y ,$$

where

$$e^{\sigma_i}_n \Theta_w(y, \mu) = \frac{i}{2}\mu^{-n-1}\Theta_{[\sigma_i, w]}(y, \mu) . \tag{71}$$

Notice that when acting on $\Theta_w(y, \mu)$, we have

$$[e^{\sigma_i}_n, e^{\sigma_j}_m] = -\varepsilon_{ijk} e^{\sigma_k}_{n+m+1} .$$

Setting $F[S](x) = \xi^v_1(x, \lambda)$, this proves the exchange relation for $\xi^w_2(y, \mu) = \Theta_w(y, \mu)$. We now use a multi-index notation $\delta^{a_p} \equiv \delta^{n_p}_{\sigma_{i_p}}$ and similarly for e_a. We denote by C^{ab}_c the structure constants of the Lie algebra $\mathscr{H}$ of the group of dressing transformations in the basis (δ^a), and by f^c_{ab} those of $\mathscr{H}^*$. Notice that the structure constants appearing in Eq. (71) are precisely the f^c_{ab}. We proceed now by induction on the degree of $\xi^w_2(y, \mu)$. Assuming that the relation is true for $\{\xi^v_1(x, \lambda), \xi^w_2(y, \mu)\}$ up to some degree, we want to show that it is true for $\{\xi^v_1(x, \lambda), \delta^a \xi^w_2(y, \mu)\}$. From the Lie–Poisson property we find

$$\{\xi^v_1(x, \lambda), \delta^a \xi^w_2(y, \mu)\} = \sum_b \delta^b \xi^v_1(x, \lambda)\, e_b(\delta^a \xi^w_2(y, \mu)) ,$$

where

$$e_b(\delta^a \xi^w_2(y, \mu)) = \delta^a(e_b \xi^w_2(y, \mu)) - f^a_{bc}\delta^c \xi^w_2(y, \mu) + C^{ac}_b e_c \xi^w_2(y, \mu) .$$

This shows that the exchange relation takes the proper form at any level. Moreover the recursion relation defining the e_a's implies by construction the commutation relation

$$[\delta^a, e_b] = f^a_{bc}\delta^c - C^{ac}_b e_c . \tag{72}$$

The last step consists in proving by induction the commutation relation between the e_a's, using their recursive definition. Assume that we have for some vector ξ,

$$[e_a, e_b](\xi) = f^c_{ab} e_c(\xi),$$

we want to show that the same relation is true for $\delta^c\xi$. An explicit calculation using Eq. (72) shows that we have

$$\begin{aligned}[e_a, e_b](\delta^c \xi) = \{&\delta^c[e_a, e_b] + C^{dc}_a[e_b, e_d] - C^{dc}_b[e_a, e_d] \\ &+ (C^{dl}_b f^c_{ad} - C^{dl}_a f^c_{bd})e_l + (f^c_{bd} f^d_{al} - f^c_{ad} f^d_{bl})\delta^l\}(\xi) .\end{aligned}$$

Using the induction hypothesis and Eq. (72) again, we get

$$\begin{aligned}[e_a, e_b]\delta^c(\xi) = {}& f^d_{ab} e_d(\delta^c \xi) \\ &+ (C^{dl}_b f^c_{ad} - C^{dl}_a f^c_{bd} + C^{dc}_a f^l_{bd} - C^{dc}_b f^l_{ad} - C^{cl}_d f^d_{ab})e_l(\xi) \\ &+ (f^d_{ab} f^c_{dl} + f^c_{bd} f^d_{al} - f^c_{ad} f^d_{bl})\delta^l(\xi) .\end{aligned}$$

The coefficient of $\delta^l(\xi)$ vanishes due to the Jacobi identity on the Lie algebra $\mathscr{H}^*$, and the coefficient of $e_l(\xi)$ vanishes due to the cocycle condition which is easily seen to be satisfied.

5. Appendix

We prove that the action of the dressing transformations in the Heisenberg model is Lie–Poisson. Let ${}^gS$ be the dressed spin variable. We want to show

$$\{{}^gS^i(x), {}^gS^j(y)\} = \varepsilon^{ijk}\, {}^gS^k(x)\delta(x-y)\,,$$

or in tensor notation

$$\{{}^gS_1(x), {}^gS_2(y)\} = \frac{i}{2}[C_{12}, {}^gS_2(x)]\delta(x-y)\,,$$

where $C_{12} = \sum_{i=1}^3 \sigma^i \otimes \sigma^i$ is the Casimir element. Infinitesimally, we have

$${}^gS(x) = S(x) + \delta_X S(x)$$

with

$$\delta_X S(x) = -i\partial_x \oint \frac{dz}{2i\pi}\Theta_X(z,x)\,. \tag{73}$$

We have to check

$$\{\delta_X S_1(x), S_2(y)\}_M + \{S_1(x), \delta_X S_2(y)\}_M + \{\delta_X S_1(x), \delta_X S_2(y)\}_G$$
$$= \frac{i}{2}[C_{12}, \delta_X S_2(x)]\delta(x-y)\,, \tag{74}$$

where $\{\ \}_M$ is the Poisson bracket on phase space, and $\{\ \}_G$ is the Poisson bracket on the group of dressing transformations. Using Eq. (68), we get

$$\{\Theta_1(x), S_2(y)\}_M = -\frac{1}{2\lambda}\theta(x-y)T_1(x)[[T_1^{-1}(y)C_{12}T_1(y), X_1], S_2(y)]T_1^{-1}(x)\,. \tag{75}$$

Consider first the term proportional to $\delta(x-y)$ in the left-hand side of Eq. (74). It comes from the derivatives in Eq. (73) acting on $\theta(x-y)$ in the first two terms of this expression. Explicitly:

$$-\frac{i}{2}\delta(x-y)\left\{\oint \frac{dz}{2i\pi z}T_1(x)[[T_1^{-1}(x)C_{12}T_1(x), X_1], S_2(x)]T_1^{-1}(x)\right.$$
$$\left. - \oint \frac{dz}{2i\pi z}T_2(x)[[T_2^{-1}(x)C_{12}T_2(x), X_2], S_1(x)]T_2^{-1}(x)\right\}$$

or else

$$-\frac{i}{2}\delta(x-y)\oint \frac{dz}{2i\pi z}\{[[C_{12}, \Theta_1(z,x)], S_2(x)] - [[C_{12}, \Theta_2(z,x)], S_1(x)]\}\,.$$

Using the fact that $[C_{12}, S_1(x)] = -[C_{12}, S_2(x)]$ and $[C_{12}, \Theta_1] = -[C_{12}, \Theta_2]$ we can rewrite this as

$$-\frac{i}{2}\delta(x-y)\oint\frac{dz}{2i\pi z}\{-[[C_{12}, \Theta_2(z,x)], S_2(x)] + [[C_{12}, S_2(x)], \Theta_2(z,x)] .$$

Finally, using the Jacobi identity, we get

$$-\frac{i}{2}\delta(x-y)\oint\frac{dz}{2i\pi z}[C_{12}, [S_2(x), \Theta_2(z,x)]] = \frac{i}{2}\delta(x-y)[C_{12}, \delta_X S_2(x)] .$$

Next we consider the terms obtained when the derivatives act on the other factors. Assume $x > y$. These terms read

$$-\frac{i}{2}\partial_x\oint\frac{dz}{2i\pi z}T_1(x)[[T_1^{-1}(y)C_{12}T_1(y), X_1], S_2(y)]T_1^{-1}(x) . \tag{76}$$

We must compare with

$$\begin{aligned}\{\delta_X S_1(x), \delta_X S_2(y)\}_G &= \partial_x\partial_y\oint\frac{dz}{2i\pi}\oint\frac{dz'}{2i\pi}T_1(z,x)T_2(z',y)\\ &\quad\times\{X_1(z), X_2(z')\}_G T_1^{-1}(z,x)T_2^{-1}(z',y)\\ &= -i\partial_x\oint\frac{dz}{2i\pi}\oint\frac{dz'}{2i\pi z'}[S_2(y), T_1(z,x)T_2(z',y)\\ &\quad\times\{X_1(z), X_2(z')\}_G T_1^{-1}(z,x)T_2^{-1}(z',y) .\end{aligned}$$

Suppose

$$\{X_1(z), X_2(z')\}_G = \frac{1}{2}\left[\frac{C_{12}}{z-z'}, X_1(z)\right] \quad |z'| < |z| . \tag{77}$$

Deforming the integration contour $|z'| < |z|$ to $|z'| > |z|$, we get a contribution of the residue at $z' = z$,

$$\begin{aligned}-\frac{i}{2}\partial_x\oint\frac{dz}{2i\pi z}\{&[S_2(y), T_1(x)T_1^{-1}(y)C_{12}T_1(y)X_1T_1^{-1}(x)]\\ &-[S_2(y), T_1(x)X_1T_1^{-1}(y)C_{12}T_1(y)T_1^{-1}(x)]\}\end{aligned}$$

matching exactly Eq. (76). The remaining integral $|z'| > |z|$ gives no contribution since we must expand

$$\frac{1}{z-z'} = -\frac{1}{z'}\sum_{n=0}^{\infty}\left(\frac{z}{z'}\right)^n$$

and since there is already a factor $\frac{1}{z'}$, we see that the order of the pole at $z' = 0$ is $\geqq 2$.

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