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Explicit Solution for p–q Duality in Two-Dimensional Quantum Gravity

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Abstract. By using Sato's infinite dimensional Grassmannian theory of the KP hierarchy, we study the global structure of the theory space of 2D quantum gravity coupled to various minimal conformal fields labeled by a pair of integers (p, q). After giving a rigorous proof of the equivalence of Douglas's equation and the Schwinger–Dyson equation (W-constraint on a τ function), we establish the p–q duality of the (p, q) quantum gravity at Green's function level. As an application, we discuss the metamorphosis of operators under unitarity-preserving renormalization group flows.

1. Introduction

Recent developments in two-dimensional (2D) quantum gravity have revealed a great deal of its mathematical and physical structures [1]. However, our understanding of the theory has not yet reached a stage that we can extract enough information for constructing, for example, a general framework for higher dimensional gravities. In order to make further developments in this direction, it is essential to find a universal formulation that describes the operator structures of the whole theory space of 2D gravity. At present, we have two promising formulations; one is based on Douglas's equation [2] and the other on the Schwinger–Dyson (S–D) equation [3, 4] (and an equivalent approach based on the action principle [5, 6]). Either of them, however, is not completely satisfactory, but they are complementary to each other in the following sense. The former is suitable for describing the theory space but not convenient for examining the relations among various operators. On the other hand the latter makes the operator structures manifest but does not give a completely universal description of the whole theory space. In fact, in the latter formulation 2D gravities coupled to (p, q) conformal

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fields with all values of q are unified into a single theory with a set of background sources distinguishing the various values of q. However, one has to consider each value of p separately. Therefore it seems natural to try to construct a universal formulation by generalizing the S–D equation in such a way that both p and q can be varied freely. In this paper, as a first attempt along this line, we first establish a manifest correspondence between Douglas's equation and the S–D equation. We then discuss the relation between the (p, q) and (q, p) theories and show how operators are transformed under the exchange of p and q. Although the results we obtained here are still incomplete, they suggest an existence of a universal formulation of 2D gravity.

This paper is organized as follows. In Sect. 2, in order to make its connection with the S–D equation transparent, we first rewrite Douglas's equation in terms of the Sato theory [7] for the KP hierarchy. Then in Sect. 3, by utilizing the infinite dimensional Grassmannian [8, 9], we show that Douglas's equation rewritten in this form is indeed equivalent to the S–D equation. In Sect. 4, using the results obtained in the preceding sections, we consider the p–q duality and explicitly write down the transformation rules of the operators under the exchange of p and q. Section 5 is devoted to discussion and physical interpretation of our results. There, by using the transformation rules, the operators $\mathcal{O}_k$ ($k = 0 \bmod p$ or q, $k < p + q$) in (p, q) gravity are shown to be redundant in the sense that their sources can be eliminated by a redefinition of the sources of lower dimensional operators. This result is consistent with the BRST cohomological analysis by Lian and Zuckerman [10]. Finally, we apply this analysis to the minimal unitary case, $(p, p + 1)$ gravity, and consider the renormalization group flows from $(p, p + 1)$ gravity to $(p - 1, p)$ gravity.

2. Douglas's Equation in Terms of KP Hierarchy

One systematic formulation of the two-dimensional quantum gravity was given by Douglas [2], which is summarized as follows. First, we consider a pair (P, Q) of differential operators with respect to the cosmological constant t whose commutator is equal to the unity:

$$P = a_p \partial^p + \sum_{n=0}^{p-1} a_n(t) \partial^n, \quad a_p \neq 0 , \tag{2.1a}$$

$$Q = b_q \partial^q + \sum_{n=0}^{q-1} b_n(t) \partial^n, \quad b_q \neq 0 , \tag{2.1b}$$

$$[P, Q] = 1 . \tag{2.1c}$$

Here, ∂ stands for the derivative with respect to t, and P and Q are differential operators of orders p and q, respectively. Furthermore we assume that their highest coefficients, a_p and b_q, are t-independent nonzero constants. We then consider the set $\mathcal{S}_{p,q}$ of all (P, Q) that satisfy the above conditions, and introduce the following equivalence relation in it:

$$(P, Q) \sim \left(c f P f^{-1}, \frac{1}{c} f Q f^{-1} \right) , \tag{2.2}$$

where c is a t-independent arbitrary constant, and $f = f(t)$ is an arbitrary function of t. Obviously, (2.2) defines an equivalence relation in $\mathscr{S}_{p,q}$, and for each equivalence class we can uniquely choose a representative element for which P takes the following form:

$$P = \sum_{n=0}^{p} a_n(t)\partial^n, \quad a_p(t) = 1, \quad a_{p-1}(t) = 0 . \tag{2.3}$$

In this choice of representative element, the free energy $F(t)$ is given by the third coefficient of P as

$$\frac{d^2}{dt^2} F(t) = \frac{2}{p} a_{p-2}(t) . \tag{2.4}$$

Around each point (P, Q) in $\mathscr{S}_{p,q}$ we consider a set of flows which can be expressed in the following form:

$$\begin{aligned} \delta P &= [H, P] , \\ \delta Q &= [H, Q] . \end{aligned} \tag{2.5}$$

Here, H is a differential operator with respect to t, and we assume that (P, Q) stays in $\mathscr{S}_{p,q}$ during the flow. In other words, we require that

$$[H, P] = \alpha\partial^p + (\text{terms with lower order derivatives}) , \tag{2.6a}$$

$$[H, Q] = \alpha'\partial^q + (\text{terms with lower order derivatives}) , \tag{2.6b}$$

where α and α' are t-independent constants. As will be discussed shortly below, the general form of such H is given by

$$H = \sum_{n=1}^{p+q} c_n(P^{n/p})_+ + c_0(t) - \alpha PQ , \tag{2.7}$$

where α and c_n $(n = 1, 2, \ldots, p + q)$ are constants and $c_0(t)$ is an arbitrary function. To prove Eq. (2.7), we first note that Eq. (2.6a) implies that $[H + \alpha PQ, P]$ consists of terms with derivatives of order less than p. Therefore as is well known in the theory of pseudo-differential operators, $H + \alpha PQ$ is expressed as

$$H + \alpha PQ = \sum_{n \geqq 1} c_n(P^{n/p})_+ + c_0(t) , \tag{2.8}$$

where c_n's are t-independent constants and $c_0(t)$ is an arbitrary function of t. Taking Eq. (2.6b) into account, we finally find that the summation on the right-hand side of Eq. (2.8) should terminate at $n = p + q$, and Eq. (2.8) is reduced to Eq. (2.7), because of the following equation:

$$\begin{aligned} [(P^{n/p})_+, Q] &= [P^{n/p}, Q] - [(P^{n/p})_-, Q] \\ &= \frac{n}{p} P^{n/p-1} - [(P^{n/p})_-, Q] \\ &= \frac{n}{p} (a_p)^{n/p-1}\partial^{n-p} + (\text{derivatives of order less than } \max\{q, n-p\}) . \end{aligned} \tag{2.9}$$

By a similar argument with P and Q exchanged, we see that the general form of H can also be given in the following form:

$$H = \sum_{n=1}^{p+q} c'_n (Q^{n/q})_+ + c'_0(t) + \alpha' PQ\,. \tag{2.10}$$

So far we have considered flows, (2.5) and (2.7), in the space $\mathscr{S}_{p,q}$. We, however, notice that the flows generated by the last two terms in Eq. (2.7) do not change the equivalence class. This can easily be seen from

$$[c_0(t) + \alpha PQ, P] = c_0(t)P - Pc_0(t) - \alpha P\,,$$
$$[c_0(t) + \alpha PQ, Q] = c_0(t)Q - Qc_0(t) + \alpha Q\,, \tag{2.11}$$

which is nothing but an infinitesimal version of Eq. (2.2). Therefore we find that in the quotient space $\mathscr{S}_{p,q}/\sim$ the general form of flows is represented by the first term of Eq. (2.7), that is,

$$H = \sum_{n=1}^{p+q} c_n (P^{n/p})_+\,. \tag{2.12}$$

Furthermore, it is easy to see that this form of H is compatible with the choice of representative elements (2.3). To be more specific, for such choice of P we can define the Lax operator L by

$$P = L^p, \quad L = \partial + u\partial^{-1} + \dots, \tag{2.13}$$

and rewrite Eq. (2.12) as

$$H = \sum_{n=1}^{p+q} c_n (L^n)_+\,, \tag{2.14}$$

which is nothing but a generator of the KP flows. In the general theory of the KP equation [7], it is known that such flows satisfy the zero-curvature condition. Therefore, the orbit of the flows can be parametrized by $p+q$ variables y_i ($i = 1, 2, \dots, p+q$), and L is given as a function of y_i's that satisfies

$$\frac{\partial}{\partial y_n} L = [(L^n)_+, L]\,. \tag{2.15}$$

Furthermore according to the Sato theory, Eq. (2.15) is equivalent to the following set of equations:

$$L = W\partial W^{-1}\,, \tag{2.16a}$$

$$W = 1 + \sum_{n \geqq 1} w_n(t; y)\partial^{-n}\,, \tag{2.16b}$$

$$\frac{\partial}{\partial y_n} W = B_n W - W\partial^n\,. \tag{2.16c}$$

Here, B_n is a differential operator which is automatically determined by Eq. (2.16c) as

$$B_n = (W\partial^n W^{-1})_+ = (L^n)_+\,. \tag{2.17}$$

Note that the W operator in Eqs. (2.16) has an ambiguity of right multiplication by a constant pseudo-differential operator $C = 1 + c_1\partial^{-1} + c_2\partial^{-2} + \dots$:

$$W \mapsto WC\,. \tag{2.18}$$

Our next step is to express the operators P and Q in terms of the W operator introduced above. As for P, Eqs. (2.13) and (2.16a) immediately give

$$P = W\partial^p W^{-1}\,. \tag{2.19}$$

Then multiplying Eq. (2.1c) by W^{-1} and W from left and from right, respectively, we have

$$[\partial^p, W^{-1}QW] = 1\,. \tag{2.20}$$

From this equation and the fact that Q is of order q we find that $W^{-1}QW$ should have the following form:

$$W^{-1}QW = \frac{1}{p}t\partial^{-p+1} + \sum_{k\leqq q} d_k(y)\partial^k\,. \tag{2.21}$$

Here, the first term on the right-hand side is a special solution of Eq. (2.20), while the second term is the general solution of the homogeneous version of Eq. (2.20) and the $d_k(y)$'s are y-dependent constants. In order to find the y-dependence of $W^{-1}QW$, we consider its y derivatives, $\frac{\partial}{\partial y_n}(W^{-1}QW)$. Using the y-evolution equations given by

$$\frac{\partial}{\partial y_n}W^{-1} = -W^{-1}B_n + \partial^n W^{-1}\,,$$

$$\frac{\partial}{\partial y_n}Q = B_nQ - QB_n\,,$$

$$\frac{\partial}{\partial y_n}W = B_nW - W\partial^n\,, \tag{2.22}$$

we obtain

$$\frac{\partial}{\partial y_n}(W^{-1}QW) = [\partial^n, W^{-1}QW]\,. \tag{2.23}$$

Then by substituting (2.21) in the right-hand side of this equation, we have the following equation:

$$\frac{\partial}{\partial y_n}(W^{-1}QW) = \frac{n}{p}\partial^{n-p} \quad (n = 1, 2, \dots, p+q)\,, \tag{2.24}$$

which together with Eq. (2.21) determines $W^{-1}QW$ as

$$W^{-1}QW = \frac{1}{p}\left(t\partial^{-p+1} + \sum_{k=1}^{p+q} k(y_k - y_k^{(0)})\partial^{k-p} + \lambda\partial^{-p}\right) + \sum_{k\leqq -1} \alpha_k\partial^{k-p}\,, \tag{2.25}$$

where $y_k^{(0)}$, λ and α_k are constants. Since the last term on the right-hand side of this equation can be eliminated by an appropriate redefinition of W given by (2.18),[1] we finally obtain

$$Q = \frac{1}{p} W\left(t\partial^{-p+1} + \lambda\partial^{-p} + \sum_{k=1}^{p+q} kx_k \partial^{k-p} \right) W^{-1} , \tag{2.26}$$

where we have introduced new variables x defined by $x_k = y_k - y_k^{(0)}$. Note that so far we have not used the conditions that P and Q are differential operators. Therefore the general solution of the flow orbit in $\mathscr{S}_{p,q}/\sim$ is expressed as (2.19) and (2.26) provided that the W operator satisfies the Sato equation and the above conditions $((P)_- = (Q)_- = 0)$:[2]

$$\begin{aligned} &\frac{\partial}{\partial x_n} W = B_n W - W\partial^n , \\ &(W\partial^p W^{-1})_- = 0 , \\ &\left(W\left\{ t\partial^{-p+1} + \lambda\partial^{-p} + \sum_{k=1}^{p+q} kx_k \partial^{k-p} \right\} W^{-1} \right)_- = 0 . \end{aligned} \tag{2.27}$$

In the next section, we show that this set of equations is equivalent to the Virasoro constraint on a τ function of the p-reduced KP hierarchy.

3. Equivalence between the S–D Equation and Douglas's Equation

As is shown in refs. [3, 4], analyses based on the S–D equation imply that the square root of the partition function of 2D gravity coupled to $(p, *)$ conformal field is given by a τ function of the KP hierarchy that satisfies the following conditions:

$$\frac{\partial}{\partial x_p} \tau(x) = 0 , \tag{3.1a}$$

$$\mathscr{L}_{-p} \tau(x) = 0 . \tag{3.1b}$$

Here, x_i's $(i = 1, 2, \dots)$ stand for the time variables of the KP hierarchy and $\mathscr{L}_n$'s are the Virasoro generators in the following form:[3]

$$\mathscr{L}_n = \frac{1}{2} \sum_{k+l=-n} klx_k x_l + \sum_{k-l=-n} kx_k \frac{\partial}{\partial x_l} + \frac{1}{2} \sum_{k+l=n} \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_l} .$$

[1] After such redefinition of W, it has no ambiguity any more and hence the corresponding τ function is fixed up to an overall constant

[2] Equation (2.27) were already discussed in ref. [11] from a slightly different point of view. In the next section we will see that the constant λ should be equal to $(-p+1)/2$ in order for Eq. (2.27) to have a nontrivial solution

[3] As was conjectured in refs. [3, 4] and proved in ref. [8], the $W_{1+\infty}$ (or W_p) constraint automatically follows from Eqs. (3.1), which we do not use in the present paper, however. See also ref. [12], where similar analyses were made

Note that the set of equations (3.1) is equivalent to the following weaker conditions which contain three additional constants, a, b and c:

$$\frac{\partial}{\partial x_p}\tau(x) = a\tau(x)\,, \tag{3.2a}$$

$$(\mathscr{L}_{-p} + cx_p)\tau(x) = b\tau(x)\,. \tag{3.2b}$$

In fact, if we require Eq. (3.2), $c = 0$ follows from $\left[\mathscr{L}_{-p} + cx_p, \dfrac{\partial}{\partial x_p}\right]\tau = 0$ and then $a = b = 0$ follows as is shown in ref. [8]. Our aim in this section is to confirm the equivalence between the S–D equation and Douglas's formulation by proving that Eqs. (3.2) are equivalent to Eqs. (2.27). We will see below that the use of the infinite dimensional Grassmannian gives a simple proof of this statement.

Before going into the proof we summarize the relation between pseudo-differential operators and the infinite dimensional Grassmannian. Let H be the linear space of formal Laurent series, that is, $H = \{\sum_{n\in\mathbf{Z}} c_n z^n\}$. Then for any pseudo-differential operator U, we construct a corresponding subspace V_U of H as follows:

$$\begin{aligned} U \mapsto V_U &= [U|_{t=t_0}, (\partial U)|_{t=t_0}, (\partial^2 U)|_{t=t_0}, \ldots] \\ &= \text{linear space spanned by } (\partial^k U)|_{t=t_0}\text{'s } (k = 0, 1, 2, \ldots)\,. \end{aligned} \tag{3.3}$$

Here, the symbol $|_{t=t_0}$ is defined in the following way. Let S be a pseudo-differential operator (e.g. $S = \partial^k U$), and write it in the standard form, $S = \sum_n S_n(t)\partial^n$. Then $S|_{t=t_0} \in H$ is defined by $S|_{t=t_0} = \sum_n S_n(t_0)z^n$. Although in general the mapping (3.3) is not injective, it becomes injective if we restrict the defining domain to the set of pseudo-differential operators having a special form like $U = 1 + \sum_{n=1}^{\infty} u_n(t)\partial^{-n}$. In the latter part of this section the following lemma plays an essential role.

Lemma. *Let U be a pseudo-differential operator and V_U the corresponding subspace of H. Then the following two statements are equivalent*:

(a)
$$\left\{\sum_{m,n\geqq 0} c_{m,n} z^m \left(\frac{d}{dz}\right)^n\right\} V_U \subset V_U\,. \tag{3.4}$$

(b) *There exists a differential operator B such that*[4]

$$U\left\{\sum_{m,n\geqq 0} c_{m,n}(t-t_0)^n\partial^m\right\} = BU\,. \tag{3.5}$$

Proof. The following three statements hold for any pseudo-differential operators X and Y and differential operator B:

(i) $X = Y \Leftrightarrow (\partial^k X)|_{t=t_0} = (\partial^k Y)|_{t=t_0}$ for $\forall k \geqq 0$,

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[4] Strictly speaking, B should be regarded as a formal differential operator in the sense that it is given by a formal Taylor series:

$$B = \sum_{n\geqq 0}\sum_{m\geqq 0}\frac{1}{m!}b_n^{(m)}(t-t_0)^m\partial^n\,.$$

Actually this is sufficient for the proof of the equivalence of Eqs. (3.8) and (3.9)

(ii) $(X(t-t_0)^n\partial^m)|_{t=t_0} = z^m\left(\dfrac{d}{dz}\right)^n (X|_{t=t_0})$,

(iii) $(\partial^k BY)|_{t=t_0} = \sum_{l\geqq 0} \lambda_{k,l}(B)(\partial^l Y)|_{t=t_0} \quad (k\geqq 0)$,

where $\lambda_{k,l}(B) = \sum_{r=\max\{0,k-l\}}^{k} \binom{k}{r} b^{(r)}_{r+l-k}$

for $B = \sum_{n\geqq 0} b_n(t)\partial^n = \sum_{n\geqq 0}\left(\sum_{m\geqq 0}\dfrac{1}{m!}b_n^{(m)}(t-t_0)^m\right)\partial^n$.

Therefore, by applying (i) ~ (iii) to Eq. (3.5), we have

(b) $\Leftrightarrow \exists B$ (differential operator) *such that*

$$\sum_{m,n\geqq 0} c_{m,n}z^m\left(\frac{d}{dz}\right)^n(\partial^k U)|_{t=t_0} = \sum_{l\geqq 0}\lambda_{k,l}(B)(\partial^l U)|_{t=t_0} \quad \text{for } \forall k\geqq 0\,.$$

On the other hand, since V_U is spanned by $(\partial^k U)|_{t=t_0}$ $(k=0,1,2,\dots)$, we have

(a) $\Leftrightarrow \exists \mu_{k,l}$ $(k, l\geqq 0)$ *such that*

$$\sum_{m,n\geqq 0} c_{m,n}z^m\left(\frac{d}{dz}\right)^n(\partial^k U)|_{t=t_0} = \sum_{l\geqq 0}\mu_{k,l}(\partial^l U)|_{t=t_0} \quad \text{for } \forall k\geqq 0\,.$$

The above two statements are equivalent, since for any set of numbers $\mu_{k,l}$ $(k, l\geqq 0)$ we can construct the following formal differential operator B that satisfies $\lambda_{k,l}(B)=\mu_{k,l}$:

$$B = \sum_{n\geqq 0}\sum_{m\geqq 0}\frac{1}{m!}b_n^{(m)}(t-t_0)^m\partial^n,$$

$$b_n^{(m)} = \sum_{k=\max\{0,m-n\}}^{m}(-1)^{m-k}\binom{m}{k}\mu_{k,k+n-m}\,. \quad ∎$$

We now give the proof of the equivalence between Eqs. (3.2) and (2.27). First we note that the Sato equation (for $W = 1+\sum_{k\geqq 1} w_k(t;x)\partial^{-k}$),

$$\frac{\partial}{\partial x_n}W = B_n W - W\partial^n\,, \tag{3.6}$$

is easily solved in terms of the infinite dimensional Grassmannian:

$$V_{W(x)} = e^{-\sum_{k=1}^{\infty} z^k(x_k - x_k^{(0)})}V_{W(x^{(0)})}\,, \tag{3.7}$$

where $V_{W(x)}$ stands for the subspace of H corresponding to $W(x)$. Furthermore as was shown in ref. [8] Eqs. (3.2) are equivalent to

$$z^p V_{W(x)} \subset V_{W(x)},$$

$$\left(z^{-p+1}\frac{d}{dz}+\frac{-p+1}{2}z^{-p}+\sum_{k\geqq 1}k(x_k+\delta_{k,1}t_0)z^{k-p}+cz^{-p}\right)V_{W(x)}\subset V_{W(x)}\,. \tag{3.8}$$

Then by applying Lemma 1 to (3.8), we find that the following equations are equivalent to Eqs. (3.8) and hence to Eqs. (3.2):

$$(W\partial^p W^{-1})_- = 0,$$

$$\left(W\left\{ t\partial^{-p+1} + \left(\frac{-p+1}{2} + c \right)\partial^{-p} + \sum_{k \geqq 1} kx_k \partial^{k-p} \right\} W^{-1} \right)_- = 0\,, \tag{3.9}$$

which are nothing but Eqs. (2.27) if we set x_k's other than $x_1, x_2, \ldots, x_{p+q}$ to zero. Thus we have confirmed the equivalence of the S–D equation and Douglas's equation. In the following sections, we use this equivalence to explore more profound structures of operators in 2D gravity.

4. *p–q* Duality

The system of 2D gravity coupled to $(p, *)$ conformal field is described in a unified way by a τ function of the KP hierarchy that satisfies Eqs. (3.1a) and (3.1b). For example, the 2D gravity coupled to (p, q) conformal field is realized by setting $x_1 = t$, $x_{p+q} =$ const. and $x_i = 0$ $(i \neq 1, p+q)$ [3, 4]. In this sense the S–D equation (3.1) gives a universal description for various values of q, while in order to vary p one has to change the form of the equation itself. In particular, the equivalence between (p, q) theory and (q, p) theory is not manifest in this formalism, nevertheless they should describe the same theory. On the other hand, in Douglas's equation (2.1) p and q appear in a symmetric manner, although the structure of the operators is not transparent. The aim of this section is to give a one-to-one correspondence between the (p,q) and (q, p) theory by using the relation between the two formulations obtained in the preceding sections.

In Sect. 2, we found that the flow orbit in the space $\mathscr{S}_{p,q}/\sim$ of a pair of differential operators (P, Q) satisfying Eq. (2.1) is represented as follows by a pseudo-differential operator W that obeys Eq. (2.27):

$$P = W\partial^p W^{-1}$$

$$Q = \frac{1}{p} W\left(t\partial^{-p+1} + \frac{-p+1}{2}\partial^{-p} + \sum_{k=1}^{p+q} kx_k \partial^{k-p} \right) W^{-1}\,, \tag{4.1}$$

where W has the form $W = 1 + \sum_{k=1}^{\infty} w_k(t; x)\partial^{-k}$. In this expression we have chosen such representative elements that P has the standard form (2.3). Obviously we can change the representative elements in such a way that Q becomes the standard form. In fact if we introduce $\bar{P}$ and $\bar{Q}$ defined by

$$\bar{P} = \frac{p}{(p+q)x_{p+q}} e^{at} Q e^{-at}\,,$$

$$\bar{Q} = -\frac{(p+q)x_{p+q}}{p} e^{at} P e^{-at},$$

$$a = \frac{(p+q-1)x_{p+q-1}}{q(p+q)\,x_{p+q}}\,, \tag{4.2}$$

then obviously we have $(-\bar{Q}, \bar{P}) \sim (P, Q)$ in the sense of (2.2) and $\bar{P}$ is in the standard form (2.3). Furthermore $(\bar{P}, \bar{Q})$ satisfies Eq. (2.1) with p and q exchanged. Therefore, the general argument given in Sect. 2 implies that there uniquely exist a pseudo-differential operator $\bar{W}$ and a set of constants $\bar{x}_k$ $(k = 1, 2, \ldots, p+q)$ such that

$$\bar{P} = \bar{W}\partial^q \bar{W}^{-1},$$
$$\bar{Q} = \frac{1}{q}\bar{W}\left(t\partial^{-q+1} + \frac{-q+1}{2}\partial^{-q} + \sum_{k=1}^{p+q} k\bar{x}_k\partial^{k-q}\right)\bar{W}^{-1}, \tag{4.3}$$

where $\bar{W}$ has the form $\bar{W} = 1 + \sum_{k=1}^{\infty} \bar{w}_k(t, \bar{x})\partial^{-k}$ and satisfies the Sato equation. By combining Eqs. (4.1), (4.2) and (4.3), we obtain

$$U\partial^q U^{-1} = \frac{1}{(p+q)x_{p+q}}\left(t\partial^{-p+1} + \frac{-p+1}{2}\partial^{-p} + \sum_{k=1}^{p+q} kx_k\partial^{k-p}\right), \tag{4.4a}$$

$$U\frac{-p}{q(p+q)x_{p+q}}\left(t\partial^{-q+1} + \frac{-q+1}{2}\partial^{-q} + \sum_{k=1}^{p+q} k\bar{x}_k\partial^{k-q}\right)U^{-1} = \partial^p, \tag{4.4b}$$

where U is defined by

$$U = W^{-1}e^{-at}\bar{W}. \tag{4.5}$$

From Eq. (4.4b) we immediately see that

$$\bar{x}_{p+q} = -\frac{q}{p}x_{p+q}, \tag{4.6}$$

by comparing the coefficient of ∂^p. In order to make the equations transparent, we introduce the following notations:

$$a_{p+q-k} = \frac{kx_k}{(p+q)x_{p+q}}, \quad \bar{a}_{p+q-k} = \frac{k\bar{x}_k}{(p+q)\bar{x}_{p+q}}, \tag{4.7}$$

$$\bar{\partial} = U\partial U^{-1}, \quad \bar{t} = UtU^{-1}. \tag{4.8}$$

Obviously $\bar{\partial}$ and $\bar{t}$ satisfy

$$[\bar{\partial}, \bar{t}] = 1 \tag{4.9}$$

and have the following forms:

$$\bar{\partial} = \partial + \ldots,$$
$$\bar{t} = t\partial^0 + \ldots. \tag{4.10}$$

Then Eqs. (4.4) can be rewritten in a manifestly p–q symmetric form:

$$\bar{\partial}^q = \partial^q\left(1 + \sum_{k=1}^{p+q-1} a_k\partial^{-k}\right) + \frac{1}{(p+q)x_{p+q}}\left(t\partial^{-p+1} + \frac{-p+1}{2}\partial^{-p}\right),$$
$$\partial^p = \bar{\partial}^p\left(1 + \sum_{k=1}^{p+q-1} \bar{a}_k\bar{\partial}^{-k}\right) + \frac{1}{(p+q)\bar{x}_{p+q}}\left(\bar{t}\bar{\partial}^{-q+1} + \frac{-q+1}{2}\bar{\partial}^{-q}\right). \tag{4.11}$$

Although to obtain a complete solution of (4.11) is not an easy task, a simple power counting shows that for the first $p + q$ terms we can write

$$\bar{\partial} = \partial\left(1 + \sum_{k=1}^{p+q-1} a_k \partial^{-k}\right)^{1/q} + \frac{t}{q(p+q)x_{p+q}} \partial^{2-p-q} + O(\partial^{1-p-q}),$$

$$\partial = \bar{\partial}\left(1 + \sum_{k=1}^{p+q-1} \bar{a}_k \bar{\partial}^{-k}\right)^{1/p} + \frac{\bar{t}}{p(p+q)\bar{x}_{p+q}} \bar{\partial}^{2-p-q} + O(\bar{\partial}^{1-p-q}). \quad (4.12)$$

Actually this form is sufficient for solving a_k in terms of $\bar{a}_k$ because we need just $p + q - 1$ relations among them. Therefore, the problem is reduced to finding the condition for the following two Laurent series to be inverse functions of each other:

$$w = z\left(1 + \sum_{n=1}^{\infty} a_n z^{-n}\right)^{1/q},$$

$$z = w\left(1 + \sum_{n=1}^{\infty} \bar{a}_n w^{-n}\right)^{1/p}. \quad (4.13)$$

This problem is solved by the following series of manipulations:

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \oint dz\, z^{n-1} \left(\frac{w}{z}\right)^q \\ &= \frac{1}{2\pi i} \oint dw \frac{dz}{dw} z^{n-1} \left(\frac{z}{w}\right)^{-q} \\ &= \frac{1}{2\pi i} \oint dw\, w^{n-1} \left(1 + \sum_{m=1}^{\infty} \bar{a}_m w^{-m}\right)^{(n-p-q)/p} \left(1 + \sum_{m=1}^{\infty} (1 - m/p)\bar{a}_m w^{-m}\right). \end{aligned} \quad (4.14)$$

The last expression is easily evaluated and we finally obtain

$$a_n = -\frac{q}{p} \sum_{l \geqq 1} \frac{1}{l} \binom{(n-p-q)/p}{l-1} \sum_{\substack{m_1, \ldots, m_l \geqq 1 \\ m_1 + \ldots + m_l = n}} \bar{a}_{m_1} \ldots \bar{a}_{m_l}, \quad (4.15)$$

where a_k and $\bar{a}_k$ were defined in (4.7).[5]

As for the partition function, we can in principle write down the relation between the (p, q) and (q, p) theories by first solving Eq. (4.4) for U and then translating Eq. (4.5) in terms of τ functions. However, in order to solve U from Eq. (4.4) we have to keep the higher-order terms in (4.12) which we did not need in the

[5] When a_k and $\bar{a}_k$ are related by (4.15), we can prove the following identities which generalize Eq. (4.15):

$$\frac{1}{r} \sum_{l=1}^{\infty} \binom{r/q}{l} A_{s+r}^{(l)} + \frac{1}{s} \sum_{l=1}^{\infty} \binom{s/p}{l} \bar{A}_{s+r}^{(l)} = 0,$$

where r and s are arbitrary integers satisfying $s + r \geqq 1$, and

$$A_n^{(l)} = \sum_{\substack{m_1, \ldots, m_l \geqq 1 \\ m_1 + \ldots + m_l = n}} a_{m_1} \ldots a_{m_l}, \quad \bar{A}_n^{(l)} = \sum_{\substack{m_1, \ldots, m_l \geqq 1 \\ m_1 + \ldots + m_l = n}} \bar{a}_{m_1} \ldots \bar{a}_{m_l}$$

derivation of Eq. (4.15). On the other hand, if we are satisfied with the second derivative of the free energy with respect to the cosmological constant t, the p–q duality relation can easily be obtained. In fact, we have

$$P = \partial^p + pu\partial^{p-2} + \dots, \quad 2u = \frac{\partial^2}{\partial t^2} \ln Z ,$$

$$\bar{P} = \partial^q + q\bar{u}\partial^{q-2} + \dots, \quad 2\bar{u} = \frac{\partial^2}{\partial t^2} \ln \bar{Z} , \tag{4.16}$$

where Z and $\bar{Z}$ are the partition functions of the (p, q) and (q, p) theories, respectively. Then Eqs. (4.1) and (4.2) give the following relation between u and $\bar{u}$:

$$\bar{u} = u + \frac{(p+q-2)x_{p+q-2}}{q(p+q)x_{p+q}} - \frac{q-1}{2}\left(\frac{(p+q-1)x_{p+q-1}}{q(p+q)x_{p+q}}\right)^2 . \tag{4.17}$$

In order to see physical consequences of the p–q duality discussed above, we consider the case $(p, q) = (2, 3)$ as an example. As was stated in Sect. 3, the partition function $Z_{(2,*)}$ of $(2, *)$ theory is given by a τ function of the 2-reduced KP hierarchy, τ_2, which satisfies Eq. (3.1b) for $p = 2$:

$$\sqrt{Z_{(2,*)}} = \tau_2(x_1, x_3, x_5, x_7, \dots) . \tag{4.18}$$

Here τ_2 depends only on x_k ($k \neq 0 \bmod 2$). By setting $x_k = 0$ for $k \geqq 7$, we have the partition function of (2, 3) theory in the $(2, *)$ description and $u(x_1, x_3, x_5)$ is expressed as

$$u(x_1, x_3, x_5) = \frac{\partial^2}{\partial x_1^2} \ln \tau_2(x_1, x_3, x_5, \text{othere } x_k\text{'s} = 0) . \tag{4.19}$$

Similarly if we regard the (2, 3) theory as a special case of $(3, *)$ theory, we have

$$\bar{u}(x_1, x_2, x_4, x_5) = \frac{\partial^2}{\partial x_1^2} \ln \tau_3(x_1, x_2, x_4, x_5, \text{other } x_k\text{'s} = 0) , \tag{4.20}$$

where τ_3 is a τ function of the 3-reduced KP hierarchy which satisfies Eq. (3.1b) for $p = 3$ and depends only on x_k ($k \neq 0$ mod 3). Although u and $\bar{u}$ describe the same theory, they appear to have different structures; u depends on three variables while $\bar{u}$ involves four variables. However, the p–q duality gives a complete one-to-one correspondence between them. In fact, by applying Eqs. (4.6), (4.15) and (4.17) to this case, $(p, q) = (2, 3)$, we have

$$\bar{u}(\bar{x}_1, \bar{x}_2, \bar{x}_4, \bar{x}_5) = u(x_1, x_3, x_5) - \frac{3}{10}\frac{\bar{x}_3}{\bar{x}_5} + \frac{2}{25}\frac{\bar{x}_4^2}{\bar{x}_5^2} , \tag{4.21}$$

where x and $\bar{x}$ are related by

$$x_1 = \bar{x}_1 - \frac{4\bar{x}_2\bar{x}_4}{5\bar{x}_5} - \frac{9\bar{x}_3^2}{20\bar{x}_5} + \frac{18\bar{x}_3\bar{x}_4^2}{25\bar{x}_5^2} - \frac{4\bar{x}_4^4}{25\bar{x}_5^3} ,$$

$$x_2 = \bar{x}_2 - \frac{6\bar{x}_3\bar{x}_4}{5\bar{x}_5} + \frac{32\bar{x}_4^3}{75\bar{x}_5^2} ,$$

$$
\begin{aligned}
x_3 &= \bar{x}_3 - \frac{4\bar{x}_4^2}{5\bar{x}_5}\,, \\
x_4 &= \bar{x}_4\,, \\
x_5 &= -\frac{2}{3}\bar{x}_5\,.
\end{aligned}
\tag{4.22}
$$

Since the left-hand side of Eq. (4.21) does not depend on $\bar{x}_3$, we can restrict the variables so that $\bar{x}_3 = 4\bar{x}_4^2/15\bar{x}_5$ without losing any information and we have

$$
\bar{u}(\bar{x}_1, \bar{x}_2, \bar{x}_4, \bar{x}_5) = u(x_1, x_3, x_5)\,,
\tag{4.23}
$$

for the variables satisfying

$$
\begin{aligned}
x_1 &= \bar{x}_1 - \frac{4\bar{x}_2\bar{x}_4}{5\bar{x}_5}\,, \\
x_3 &= -\frac{8\bar{x}_4^2}{15\bar{x}_5}\,, \\
x_5 &= -\frac{2}{3}\bar{x}_5\,.
\end{aligned}
\tag{4.24}
$$

To obtain the relation between Z and $\bar{Z}$ we must integrate Eq. (4.21) twice and the integration constants are determined by KP flows. A straightforward but long calculation leads to the following relation between the two partition functions Z and $\bar{Z}$:

$$
\begin{aligned}
&\ln\sqrt{Z(x_1, x_3, x_5)} - \frac{x_1 x_3^3}{25x_5^2} + \frac{3x_3^5}{625x_5^3} + \frac{x_1^2 x_3}{10x_5} + \frac{1}{40}\ln x_5 \\
&= \ln\sqrt{\bar{Z}(\bar{x}_1, \bar{x}_2, \bar{x}_4, \bar{x}_5)} + \frac{2\bar{x}_1^2\bar{x}_4^2}{25\bar{x}_5^2} + \frac{\bar{x}_1\bar{x}_2^2}{5\bar{x}_5} - \frac{16\bar{x}_1\bar{x}_2\bar{x}_4^3}{125\bar{x}_5^3} \\
&\qquad + \frac{128\bar{x}_1\bar{x}_4^6}{9375\bar{x}_5^5} - \frac{4\bar{x}_2^3\bar{x}_4}{75\bar{x}_5^2} + \frac{32\bar{x}_2^2\bar{x}_4^4}{625\bar{x}_5^4} - \frac{512\bar{x}_2\bar{x}_4^7}{46875\bar{x}_5^6} \\
&\qquad + \frac{4096\bar{x}_4^{10}}{5859375\bar{x}_5^8} + \frac{1}{15}\ln\bar{x}_5\,.
\end{aligned}
\tag{4.25}
$$

Here the functions added to $\ln\sqrt{Z}$ and $\ln\sqrt{\bar{Z}}$ can be regarded as local counter-terms to make (2, 3) and (3, 2) gravity theories equivalent.

Equations (4.23) and (4.24) have a rather interesting implication. As is well known, 2D gravity coupled to the Ising model is described by τ_3 if one keeps $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_7$ and sets other $\bar{x}_k$'s to zero. Then $\bar{x}_2$ and $\bar{x}_5$ are identified with the sources for the spin operator and the mass operator, respectively, and $\bar{x}_1$ and $\bar{x}_4$ with those for the cosmological term and a higher gravitational operator, respectively. With this interpretation, the (3, 2) theory can be regarded as the limit of $\bar{x}_7 \to 0$, or equivalently as the limit of infinite mass, $m \to \infty$ [3]. Therefore Eqs. (4.23) and (4.24) indicate that the four operators corresponding to $\bar{x}_1, \bar{x}_2, \bar{x}_4$ and $\bar{x}_5$, which were independent in the case of the Ising model, are recombined into three independent ones as in Eq. (4.24). In other words, $\mathbf{Z}_2$-odd operators, which correspond to $\bar{x}_2$ and $\bar{x}_4$, form $\mathbf{Z}_2$-invariant bound state operators in the limit of $m \to \infty$. In general, if we consider a renormalization group flow in the direction of unitary deformation, operators will be recombined rather wildly, and this seems to be the greatest obstacle for constructing a completely universal description of 2D gravity.

5. Discussions

One of the most important problems of 2D gravity at present is to construct a formalism which describes the whole theory space in a unified manner and gives manifest relations among the operators. In this paper, as a first step in this direction, we have considered the relation between the two formulations, Douglas's equation and the S–D equation. Actually we have proved the equivalence of them using the Sato theory of the KP hierarchy, and then we have explicitly given the duality relation between the (p, q) and (q, p) theories. We expect that the results obtained through this analysis give a clue for constructing a universal description of the 2D quantum gravity. In fact, Eq. (4.11) is completely symmetric under the exchange of p and q and still compatible with the infinite dimensional Grassmannian structure. Therefore, if we succeed in introducing not only $p + q$ sources but also arbitrary sources to Eq. (4.11), it would give a universal description of the whole theory space. In the following, however, we restrict our consideration to the first $p + q$ sources and present applications of the p–q duality.

First, we consider the problem of redundant operators. In (p, q) theory, the operators $\mathcal{O}_k$ ($k = 0 \bmod q$ or $k = 0 \bmod p$) are known to lack physical interpretation. For example, if we describe $(p, p + 1)$ theory in terms of the S–D equation of $(p, *)$ type, we have $2p - 1$ sources x_k ($k = 1, \ldots, 2p + 1$; $k \neq p$, $2p$) and they correspond to the ϕ^l operators of the ϕ^{2p-2} theory as

$$\begin{array}{cccccccc} x_1 & x_2 & \cdots & x_{p-1} & x_{p+1} & x_{p+2} & \cdots & x_{2p-1} & x_{2p+1} \\ \updownarrow & \updownarrow & \cdots & \updownarrow & & \updownarrow & \cdots & \updownarrow & \\ 1 & \phi & \cdots & \phi^{p-2} & & \phi^{p-1} & \cdots & \phi^{2p-4} & \end{array} \tag{5.1}$$

and x_{2p+1} should be regarded as a source for the Lagrangian itself. However, $\mathcal{O}_{q=p+1}$ is a rather mysterious operator in the sense that it has no corresponding field in the above list, and indeed it does not appear in the BRST cohomology [10]. In the following, we show that the operators $\mathcal{O}_k$ ($k = 0 \bmod q$, $k < p + q$) in the $(p, *)$ formalism of (p, q) gravity are redundant in the sense that their sources can be absorbed into the sources of lower dimensional operators through their analytic redefinition.[6]

We consider (p, q) gravity in the $(p, *)$ formalism and apply the p–q duality equation (4.17). Since its left-hand side does not depend on $\bar{x}_q$, and Eq. (4.15) reads as

$$x_k = \bar{x}_k + \bar{x}_{p+q} \times \left(\text{a polynomial of } \frac{\bar{x}_j}{\bar{x}_{p+q}} \text{ for } j > k \right), \tag{5.2}$$

[6] The same problem has also been considered in ref. [13] and the operator $\mathcal{O}_q$ in the case of $p < q$ is interpreted there as a boundary operator

the KP flow of the parameter $\bar{x}_q$ causes the change only of the parameters x_j with j less than or equal to q. Thus, by setting $\bar{x}_q$ to a special value, x_q can be set to zero accompanied with an analytic redefinition of $x_1, \ldots, x_{q-1}$:[7]

$$\begin{aligned} u(x_1, &\ldots, x_{p-1}, x_{p+1}, \ldots, x_{q-1}, x_q, x_{q+1}, \ldots, x_{p+q}) \\ &= u(x'_1, \ldots, x'_{p-1}, x'_{p+1}, \ldots, x'_{q-1}, 0, x_{q+1}, \ldots, x_{p+q}) \\ &\qquad + (\text{an additional term appearing only for the case } p = 2)\,, \end{aligned} \tag{5.3}$$

where $x'_i (i = 1, \ldots, q-1)$ have a form such as

$$x'_i = x_i + x_{p+q} \times \left(\text{a polynomial of } \frac{x_j}{x_{p+q}} \quad \text{for } j > i \right). \tag{5.4}$$

Similarly, the sources x_k with $k = q, 2q, \ldots, [(p+q)/q]q$ can also be set to zero by an analytic redefinition of lower dimensional sources.[8] Thus, starting from the highest dimensional source, $x_{[(p+q)/q]q}$, we can eliminate all the sources x_k with $k = q, 2q, \ldots, [(p+q)/q]q$. In this sense the operators corresponding to these sources are redundant and it is natural to set them to zero from the beginning. In particular, in the minimal unitary case ($(p, p+1)$ theory), this procedure ends with the operators which can be interpreted in terms of the Ginzburg–Landau potential. For example, for the case of $(p, q) = (2, 3)$ we can easily follow this prescription for Eqs. (4.21) and (4.22), and obtain[9]

$$u(x_1, x_3, x_5) = u\left(x_1 - \frac{3x_3^2}{10x_5}, 0, x_5 \right) - \frac{x_3}{5x_5}\,. \tag{5.5}$$

The above result agrees with a BRST cohomological analysis by Lian and Zuckerman [10] which excludes these operators $\mathcal{O}_k$ ($k = 0 \bmod q$) as well as $\mathcal{O}_k$ ($k = 0 \bmod p$) as BRST trivial operators, although our analysis is restricted to the first $(p+q)$ operators.

It is now possible to consider the renormalization group flow from $(p, p+1)$ to $(p-1, p)$ as follows. Starting from $(p, p+1)$ theory with x_{p+1} eliminated as above, we obtain $(p, p-1)$ theory by setting x_{2p+1} to zero. Then the p–q duality maps this theory to $(p-1, p)$ theory and the above prescription eliminates the redundant variable to give a new Ginzburg–Landau potential. Consideration along this line

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[7] Equation (5.3) can also be obtained by considering the characteristic curve of the S–D equation: $\mathscr{L}_{-p}\tau = 0$ or equivalently,

$$\frac{1}{2} \sum_{k+l=p} k l x_k x_l + \sum_{k=p+1}^{p+q} k x_k \partial_{k-p} \ln \tau = 0\,.$$

[8] $[a]$ denotes a maximal integer $\leqq a$

[9] At the partition function level, we have the following relation between the new partition functions:

$$\ln \sqrt{Z\left(x_1 - \frac{3x_3^2}{10x_5}, 0, x_5 \right)} + \frac{1}{40} \ln x_5 = \ln \sqrt{\bar{Z}\left(\bar{x}_1 - \frac{4\bar{x}_2\bar{x}_4}{5\bar{x}_5} + \frac{16\bar{x}_4^4}{125\bar{x}_5^3}, 0, 0, \bar{x}_5 \right)} + \frac{1}{15} \ln \bar{x}_5$$

suggests the possibility of constructing a universal description of the minimal unitary series coupled to 2D gravity.

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