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Vector Fields on Complex Quantum Groups

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Abstract. Using previous results we construct the q-analogues of the left invariant vector fields of the quantum enveloping algebra corresponding to the complex Lie algebras of type A_{n-1}, B_n, C_n, and D_n. These quantum vector fields are functionals over the complex quantum group $\mathscr{A}$. In the special case A_1 it is shown that this Hopf algebra coincides with $U_q sl(2,\mathbb{C})$.

1. Introduction

We work with the q-deformed function algebras over the complexified groups associated to A_{n-1}, B_n, C_n, and D_n, where $q>0$ is a real parameter. I.e. we consider Hopf algebras which are generated by the matrix functions of the fundamental representation and its hermitian conjugate such that dividing out the unitarity condition yields the quantum groups $SU_q(N)$, $SO_q(N,\mathbb{R})$, $USp_q(N)$. In [DSWZ] a dual Hopf algebra has been constructed thus leading to a q-deformation of the corresponding universal enveloping algebra. In [SWZ, OSWZ] the q-deformed universal enveloping algebra of $sl(2,\mathbb{C})$ was found as an operator algebra on the complex spinor quantum plane. This was also constructed in [CW] by analyzing the differential calculus on the complex quantum groups $Sl_q(n,\mathbb{C})$.

In the real case it is known that the Hopf algebra of regular functionals is generated in some sense by the vector fields which appear in the bicovariant differential calculus on quantum groups [Wor, Jur, Zum, CSWW]. This is proved in [Bur] using the fact that the matrices $L^{+i}{}_j$ and $L^{-i}{}_j$ generating the algebra of regular functionals are upper and lower triangular, respectively.

In the complex case the corresponding matrices $L^{\pm I}{}_J$ introduced in [DSWZ] violate this triangularity. In this paper we prove for the case of A_1 that the $*$-Hopf algebra of regular functionals is generated by the vector fields.

In Sect. 2 we define the vector fields, find some relations between them and construct the Casimir operators of the algebra of regular functionals $U_{\mathscr{R}}$ on the complex quantum group $\mathscr{A}$. In Sect. 3 we concentrate on the case A_1 and show that the vector fields generate a sub-$*$-Hopf algebra of $U_{\mathscr{R}}$. The equivalence of these Hopf algebras is then derived in Sect. 4.

2. Vector Fields on Complexified Quantum Groups

Throughout this paper we are using the notations and conventions of [DSWZ].
Set $(I):=(i,\bar{i})$, $\bar{I}:=(\bar{i},\bar{\bar{i}})=(\bar{i},i)$ $(i,\bar{i}=1,\dots,N)$, where $N=n$ for A_{n-1}, $N=2n+1$ for B_n and $N=2n$ for C_n, D_n. Define then the $2N\times 2N$-matrix

$$T^I{}_J:=\begin{pmatrix} t & 0 \\ 0 & \hat{t} \end{pmatrix}^I{}_J, \tag{2.1}$$

and the $\hat{\mathscr{R}}$-matrix

$$(\hat{\mathscr{R}}_q^{IJ}{}_{KL}):=\begin{pmatrix} \alpha_0\hat{R}_q & 0 & 0 & 0 \\ 0 & 0 & \alpha_1\hat{R}_q & 0 \\ 0 & \alpha_2\hat{R}_q{}^{-1} & 0 & 0 \\ 0 & 0 & 0 & \alpha_3\hat{R}_q{}^{-1} \end{pmatrix} \tag{2.2}$$

with the corresponding $\hat{R}_q$-matrix [FRT] and with $\alpha_i\in\mathbb{C}$ defined through

$$(\alpha_0)^{-n}=(\alpha_1)^{-n}=(\alpha_2)^n=(\alpha_3)^n=q \tag{2.3}$$

for A_{n-1},

$$(\alpha_0)^2=(\alpha_1)^2=(\alpha_2)^2=(\alpha_3)^2=1 \tag{2.4}$$

in the cases of B_n, C_n, D_n and

$$\overline{\alpha_0}\cdot\alpha_3=\overline{\alpha_2}\cdot\alpha_1=1\,. \tag{2.5}$$

We are considering the quantum group

$$\mathscr{A}:=\mathbb{C}\langle T^I{}_J\rangle/(I^{IJ}_{ST},(2.8),(2.9))\,, \tag{2.6}$$

where the ideal is generated by

$$I^{IJ}_{ST}:=\hat{\mathscr{R}}_q^{IJ}{}_{KL}T^K{}_ST^L{}_T-T^I{}_VT^J{}_W\hat{\mathscr{R}}_q^{VW}{}_{ST}\,, \tag{2.7}$$

$$\begin{aligned} &\det(t^i{}_j)-\mathbf{1}=\frac{(-1)^{n-1}}{[n]_q!}q^{-\binom{n}{2}}\varepsilon^{k_1\dots k_n}t^{l_1}{}_{k_1}\cdot\ldots\cdot t^{l_n}{}_{k_n}\varepsilon_{l_1\dots l_n}-\mathbf{1} \quad \text{for } A_{n-1}\,,\\ &t^i{}_s(C^{-1})^{sk}t^l{}_kC_{lj}-\delta^i_j\mathbf{1}\,,\quad (C^{-1})^{ik}t^l{}_kC_{ls}t^s{}_j-\delta^i_j\mathbf{1} \quad \text{for } B_n, C_n, D_n\,,\end{aligned} \tag{2.8}$$

$$\begin{aligned} &\det(\hat{t}^i{}_j)-\mathbf{1}=\frac{(-1)^{n-1}}{[n]_q!}q^{-\binom{n}{2}}\varepsilon^{k_1\dots k_n}\hat{t}^{l_1}{}_{k_1}\cdot\ldots\cdot \hat{t}^{l_n}{}_{k_n}\varepsilon_{l_1\dots l_n}-\mathbf{1} \quad \text{for } A_{n-1}\,,\\ &\hat{t}^i{}_s(C^{-1})^{sk}\hat{t}^l{}_kC_{lj}-\delta^i_j\mathbf{1}\,,\quad (C^{-1})^{ik}\hat{t}^l{}_kC_{ls}\hat{t}^s{}_j-\delta^i_j\mathbf{1} \quad \text{for } B_n, C_n, D_n\,,\end{aligned} \tag{2.9}$$

where $\varepsilon_{i_1\dots i_n}=(-1)^{n-1}\varepsilon^{i_1\dots i_n}=(-q)^{l(\sigma)}$, $l(\sigma)$ is the length (minimal number of transpositions) of the permutation $\sigma=\begin{pmatrix}1 & \dots & n\\ i_1 & \dots & i_n\end{pmatrix}$, $[n]_q=(q^n-q^{-n})/(q-q^{-1})$, $[n]_q!=[1]_q\cdot\ldots\cdot[n]_q$ [CSWW] and C_{ij} is the usual metric [FRT].

With the involution

$$(t^i{}_j)^* := \kappa(\hat{t}^{\hat{j}}{}_{\hat{i}}) \tag{2.10}$$

$\mathscr{A}$ becomes a $*$-Hopf algebra with comultiplication Φ, counit e and antipode κ [DSWZ].

The dual space $\mathscr{A}^*$ of the Hopf algebra $\mathscr{A}$ is an algebra with the convolution product. One can introduce an antimultiplicative involution "$\dagger$" on $\mathscr{A}^*$: For $f \in \mathscr{A}^*$ one sets

$$\forall a \in \mathscr{A}: \quad f^\dagger(a) := \overline{f(\kappa^{-1}(a^*))}. \tag{2.11}$$

In the following we are working mostly with the multiplicative involution "$\bar{\ }$":

$$\bar{f} := f^\dagger \circ \kappa^{-1}. \tag{2.12}$$

We define functionals $L^{\pm I}{}_J \in \mathscr{A}^*$ through their action on the generators of $\mathscr{A}$:

$$\begin{aligned} L^{\pm I}{}_J(\mathbf{1}) &:= \delta^I_J, \\ L^{\pm I}{}_J(T^K{}_L) &:= \hat{\mathscr{R}}_q^{\pm 1 IK}{}_{LJ} \end{aligned} \tag{2.13}$$

and their comultiplication

$$\forall a, b \in \mathscr{A}: \quad L^{\pm I}{}_J(ab) = L^{\pm I}{}_K(a) L^{\pm K}{}_J(b). \tag{2.14}$$

The algebra $U_{\mathscr{R}}$ of regular functionals on $\mathscr{A}$ is the unital algebra generated by $\{L^{\pm I}{}_J\}$ [DSWZ]. It is shown in [DSWZ] that $U_{\mathscr{R}}$ is a $*$-Hopf algebra with comultiplication Δ, counit ε and antipode S.

Now we introduce the matrices

$$\begin{aligned} Y &:= L^+ S(L^-) = \begin{pmatrix} y & 0 \\ 0 & \hat{y} \end{pmatrix}, \\ Y^{-1} &:= L^- S(L^+) = \begin{pmatrix} y^{-1} & 0 \\ 0 & \hat{y}^{-1} \end{pmatrix} \end{aligned} \tag{2.15}$$

with the matrix entries $Y^I{}_J$ and $Y^{-1 I}{}_J \in U_{\mathscr{R}}$.

It follows from the commutation relations of $L^{\pm I}{}_J$ derived in the preceding paper [DSWZ] that

$$\hat{\mathscr{R}}_q(1 \otimes Y)\hat{\mathscr{R}}_q(1 \otimes Y) = (1 \otimes Y)\hat{\mathscr{R}}_q(1 \otimes Y)\hat{\mathscr{R}}_q. \tag{2.16}$$

For convenience we set for any matrix M, $M^I{}_J \in U_{\mathscr{R}}$ the hermitian involution "$*$" with $M^{*\bar{I}}{}_{\bar{J}} := (M^J{}_I)^\dagger$. Using the involution properties of the $L^{\pm I}{}_J$ (see (3.13) of [DSWZ]) one obtains

$$Y^{*J}{}_I = Y^{-1J}{}_I. \tag{2.17}$$

The $Y^I{}_J$ have the comultiplication

$$\Delta(Y^I{}_J) = O^{IK}{}_{LJ} \otimes Y^L{}_K, \tag{2.18}$$

where

$$O^{IK}{}_{LJ} = L^{+I}{}_L S(L^{-K}{}_J). \tag{2.19}$$

A priori the algebra generated by the $Y^I{}_J$ is not a $*$-Hopf subalgebra of $U_{\mathscr{R}}$. However, in Sect. 3 we prove the $*$-Hopf algebra structure in the special case A_1. In Sect. 4 we even show that the $Y^I{}_J$ generate $U_{\mathscr{R}}$.

Similarly, as in [CSWW, Jur, Zum] for the real case we define

$$X = \begin{pmatrix} x & 0 \\ 0 & \hat{x} \end{pmatrix} := \frac{1}{\lambda}(1-Y), \tag{2.20}$$

where $\lambda = (q - q^{-1})$. These elements are the analogues to the linear functionals in [Wor] which correspond to a q-generalization of the left invariant vector fields of the complex Lie group. Now (2.16), (2.17), and (2.20) give

$$\hat{\mathscr{R}}_q(1\otimes X)\hat{\mathscr{R}}_q(1\otimes X) - (1\otimes X)\hat{\mathscr{R}}_q(1\otimes X)\hat{\mathscr{R}}_q = \lambda^{-1}\{\hat{\mathscr{R}}_q^2(1\otimes X) - (1\otimes X)\hat{\mathscr{R}}_q^2\} \tag{2.21}$$

and

$$x^* + \hat{x} = \lambda x^* \hat{x} = \lambda \hat{x} x^*. \tag{2.22}$$

In the next step we investigate the Casimir operators for $U_{\mathscr{R}}$. We restrict to the A_{n-1}-type. For B_n, C_n, and D_n the results are quite similar. We observe the following:

$$\begin{aligned} L^{\pm i}{}_j y^k{}_l &= \hat{R}_q^{\mp 1\, ki}{}_{vb} \hat{R}_q^{\pm 1\, va}{}_{lc} y^b{}_a L^{\pm c}{}_j, \\ L^{\pm \bar{i}}{}_{\bar{j}} y^k{}_l &= \hat{R}_q^{k\bar{i}}{}_{vb} \hat{R}_q^{-1\, va}{}_{l\bar{c}} y^b{}_a L^{\pm \bar{c}}{}_{\bar{j}}, \\ L^{\pm i}{}_j \hat{y}^{\bar{k}}{}_{\bar{l}} &= \hat{R}_q^{-1\, \bar{k}i}{}_{v\bar{b}} \hat{R}_q^{v\bar{a}}{}_{\bar{l}c} \hat{y}^{\bar{b}}{}_{\bar{a}} L^{\pm c}{}_j, \\ L^{\pm \bar{i}}{}_{\bar{j}} \hat{y}^{\bar{k}}{}_{\bar{l}} &= \hat{R}_q^{\pm 1\, \bar{k}\bar{i}}{}_{v\bar{b}} \hat{R}_q^{\mp 1\, v\bar{a}}{}_{\bar{l}\bar{c}} \hat{y}^{\bar{b}}{}_{\bar{a}} L^{\pm \bar{c}}{}_{\bar{j}}. \end{aligned} \tag{2.23}$$

From (2.23) we derive the Casimir operators in the same way as in [FRT]. We obtain the

Proposition 1. *The elements*

$$\begin{aligned} c_k &:= \mathrm{Tr}(Q y^k), \\ \hat{c}_k &:= \mathrm{Tr}(Q \hat{y}^k) \end{aligned} \tag{2.24}$$

with $k = 1, \ldots, n-1$ *and* $Q = \mathrm{diag}(q^{n-1}, q^{n-3}, \ldots, q^{-(n-1)})$

are the Casimir operators in $U_{\mathscr{R}}$.

3. The Y-Hopf Algebra in $U_q sl(2, \mathbb{C})$

In Sects. 3 and 4 we restrict the above developed formalism to A_1. In the following we are using the definitions

$$y = \begin{pmatrix} y_1 & y_+ \\ y_- & y_2 \end{pmatrix}, \quad \hat{y} = \begin{pmatrix} \hat{y}_1 & \hat{y}_+ \\ \hat{y}_- & \hat{y}_2 \end{pmatrix} \tag{3.1}$$

and analogously for the matrices y^{-1}, $\hat{y}^{-1}$, x, and $\hat{x}$.

For y and $\hat{y}$ we obtain a determinant condition

$$\begin{aligned} y_1 y_2 - q^2 y_+ y_- &= 1, \\ \hat{y}_1 \hat{y}_2 - q^{-2} \hat{y}_- \hat{y}_+ &= 1, \end{aligned} \tag{3.2}$$

which is easily derived from the definition of the $Y^I{}_J$ in terms of the $L^{\pm I}{}_J$. Inserting (2.20) in (3.2) yields

$$\begin{aligned} x_1 + x_2 - \lambda x_1 x_2 + q^2 \lambda x_+ x_- &= 0, \\ \hat{x}_1 + \hat{x}_2 - \lambda \hat{x}_1 \hat{x}_2 + q^{-2} \lambda \hat{x}_- \hat{x}_+ &= 0. \end{aligned} \tag{3.3}$$

The commutation relations (2.21) for the $X^I{}_J$ read explicitly

$$
\begin{aligned}
&[x_1, x_2] = 0, \\
&[x_1, x_+] + \lambda q^{-1} x_+ x_2 = q^{-1} x_+, \\
&[x_1, x_-] - \lambda q^{-1} x_2 x_- = -q^{-1} x_-, \\
&x_2 x_+ - q^2 x_+ x_2 = -q x_+, \\
&x_2 x_- - q^{-2} x_- x_2 = q^{-1} x_-, \\
&[x_+, x_-] - \lambda q^{-1}(x_2 - x_1) x_2 = -q^{-1}(x_2 - x_1), \\
&[\hat{x}_1, \hat{x}_2] = 0, \\
&\hat{x}_1 \hat{x}_2 - q^2 \hat{x}_+ \hat{x}_1 = -q \hat{x}_+, \\
&\hat{x}_1 \hat{x}_- - q^{-2} \hat{x}_- \hat{x}_1 = q^{-1} \hat{x}_-, \\
&[\hat{x}_2, \hat{x}_+] + \lambda q \hat{x}_1 \hat{x}_+ = q \hat{x}_+, \\
&[\hat{x}_2, \hat{x}_-] - \lambda q \hat{x}_- \hat{x}_1 = -q \hat{x}_-, \\
&[\hat{x}_+, \hat{x}_-] + \lambda q \hat{x}_1(\hat{x}_2 - \hat{x}_1) = q(\hat{x}_2 - \hat{x}_1), \\
&[x_1, \hat{x}_1] = \lambda q^{-1} \hat{x}_+ x_-, \\
&[x_1, \hat{x}_+] = 0, \\
&[x_1, \hat{x}_-] = \lambda q^{-1}(\hat{x}_2 - \hat{x}_1) x_-, \\
&[x_1, \hat{x}_2] = -\lambda q \hat{x}_+ x_-, \\
&[x_+, \hat{x}_1] = \lambda q^{-1} \hat{x}_+(x_2 - x_1), \\
&x_+ \hat{x}_+ - q^{-2} \hat{x}_+ x_+ = 0, \\
&x_+ \hat{x}_- - q^2 \hat{x}_- x_+ = \lambda q(\hat{x}_2(x_2 - x_1) - (x_2 - x_1)\hat{x}_1), \\
&[x_+, \hat{x}_2] = -\lambda q \hat{x}_+(x_2 - x_1), \\
&[x_-, \hat{x}_1] = 0, \\
&x_- \hat{x}_+ - q^2 \hat{x}_+ x_- = 0, \\
&x_- \hat{x}_- - q^{-2} \hat{x}_- x_- = 0, \\
&[x_-, \hat{x}_2] = 0, \\
&[x_2, \hat{x}_1] = -\lambda q \hat{x}_+ x_-, \\
&[x_2, \hat{x}_+] = 0, \\
&[x_2, \hat{x}_-] = -\lambda q(\hat{x}_2 - \hat{x}_1) x_-, \\
&[x_2, \hat{x}_2] = \lambda q^3 \hat{x}_+ x_-.
\end{aligned}
\tag{3.4}
$$

There are more relations among the $Y^I{}_J$ but in the limit $q \to 1$ the commutation relations (3.4) yield the Lie algebra $sl(2, \mathbb{C})$. This can be seen easily with the help of (2.22) and (3.3).

$Y^I{}_J$ and $Y^{-1I}{}_J$ are linearly related. It holds

$$\begin{aligned}
&(y^{-1})_1 = y_2,\\
&(y^{-1})_+ = -q^2 y_+,\\
&(y^{-1})_- = -q^2 y_-,\\
&(y^{-1})_2 = q^2 y_1 + (1-q^2) y_2,\\
&(\hat{y}^{-1})_2 = \hat{y}_1,\\
&(\hat{y}^{-1})_+ = -q^{-2}\hat{y}_+,\\
&(\hat{y}^{-1})_- = -q^{-2}\hat{y}_-,\\
&(\hat{y}^{-1})_1 = q^{-2}\hat{y}_2 + (1-q^{-2})\hat{y}_1.
\end{aligned} \tag{3.5}$$

This fact guarantees that the algebra generated by the $Y^I{}_J$ closes under the action of the "$\dagger$"-involution. Like in the comultiplication [recall (2.18)] the antipode of the $Y^I{}_J$ involves the $O^{IJ}{}_{KL}$. One first observes that the algebra generated by the $O^{IJ}{}_{KL}$ is a sub-$*$-Hopf algebra of $U_{\mathcal{R}}$ and contains the algebra generated by the $Y^I{}_J$. In the next step the $O^{IJ}{}_{KL}$ are expressed in terms of the $Y^I{}_J$, thus showing that the Y-algebra itself is a $*$-Hopf algebra.

The following results are proven by inserting the explicit expansion of the above elements in terms of the $L^{\pm I}{}_J$ and using their properties. From the definition of the $O^{IJ}{}_{KL}$ it follows directly that all the $O^{IJ}{}_{KL}$ can be written as linear combinations of the 16 elements $O^{\bar{2}1}{}_{\bar{1}1}, O^{\bar{2}2}{}_{\bar{1}1}, O^{\bar{2}1}{}_{\bar{1}2}, O^{\bar{2}2}{}_{\bar{1}2}$, and $O^{ij}{}_{kl}\,(i \leqq k)$ which can be rewritten as functions of the $Y^I{}_J$ and $\overline{Y^I{}_J}$ only:

$$\begin{aligned}
&O^{11}{}_{11} = \overline{\hat{y}_1},\\
&O^{12}{}_{21} = (y_1 - \overline{\hat{y}_1}),\\
&O^{11}{}_{12} = q^2\overline{y_-},\\
&O^{12}{}_{22} = y_+ - q^2\overline{\hat{y}_-},\\
&O^{22}{}_{21} = y_-,\\
&O^{22}{}_{22} = y_2,\\
&O^{21}{}_{21} = \hat{y}_1,\\
&O^{21}{}_{22} = \hat{y}_+,\\
&O^{12}{}_{11} = q^{-2}\overline{\hat{y}_+},\\
&O^{\bar{2}1}{}_{\bar{1}1} = \hat{y}_- - q^{-2}\overline{\hat{y}_+},\\
&O^{12}{}_{12} = \overline{y_2},\\
&O^{\bar{2}1}{}_{\bar{1}2} = \hat{y}_2 - \overline{y_2},\\
&O^{\bar{2}2}{}_{\bar{1}2} = \overline{y_+} - q^2 y_-,\\
&O^{11}{}_{21} = \overline{\hat{y}_-} - q^{-2}\hat{y}_+,\\
&O^{11}{}_{22} = q^2(y_+\hat{y}_1 - q^{-2}y_1\hat{y}_+)\overline{y_-},\\
&O^{\bar{2}2}{}_{\bar{1}1} = q^{-2}(\hat{y}_- y_2 - q^2\hat{y}_2 y_-)\overline{\hat{y}_+}.
\end{aligned} \tag{3.6}$$

There is an additional dependence between the $Y^I{}_J$ and $\overline{Y^I{}_J}$:

$$\begin{aligned}\overline{\hat{y}_2}&=y_2+q^2(y_1-\overline{\hat{y}_1}),\\ \overline{y_1}&=\hat{y}_1+q^{-2}(\hat{y}_2-\overline{y_2}).\end{aligned}\tag{3.7}$$

All the $O^{IJ}{}_{KL}$ can now be written in terms of $Y^I{}_J$, $\overline{y_+}$, $\overline{y_-}$, $\overline{y_2}$, $\overline{\hat{y}_1}$, $\overline{\hat{y}_+}$, and $\overline{\hat{y}_-}$.

y_2 and $\hat{y}_1$ are invertible in $U_{\mathscr{R}}$ and

$$\begin{aligned}(y_2)^{-1}&=L^{+1}{}_1(L^{-1}{}_1)^{-1}\\ &=\overline{\hat{y}_1}\left(\mathbf{1}+\sum_{n=1}^{\infty}(-q^2(y_-\overline{y_-}))^n\right),\\ (\hat{y}_1)^{-1}&=L^{+1}{}_1(L^{-2}{}_2)^{-1}\\ &=\overline{y_2}\left(\mathbf{1}+\sum_{n=1}^{\infty}(-q^4(y_-\overline{y_-}))^n\right),\end{aligned}\tag{3.8}$$

where we have used results from [DSWZ] with $q^3(y_-\overline{y_-})=L^{-1}{}_2L^{-2}{}_1=\Delta$. Thus one sees with the help of (3.2) that the Y-algebra is generated by the six elements $y_+, y_-, y_2, \hat{y}_1, \hat{y}_+, \hat{y}_-$ and the inverses of y_2 and $\hat{y}_1$.

Now we are able to express all $\overline{Y^I{}_J}$ in terms of $Y^I{}_J$ and the inverses of y_2 and $\hat{y}_1$: For $\overline{y_2}$ and $\overline{\hat{y}_1}$ we obtain by a simple calculation

$$\overline{y_2}=(\hat{y}_1)^{-1}(\mathbf{1}+q^4y_-\overline{y_-})\tag{3.9}$$

and

$$\overline{\hat{y}_1}=(y_2)^{-1}(\mathbf{1}+q^2y_-\overline{y_-}).\tag{3.10}$$

With the help of (3.9) and (3.10) the relations

$$y_2\overline{y_-}=q^{-4}\hat{y}_+\overline{y_2}\tag{3.11}$$

and

$$[\overline{y_-},y_-]=0\tag{3.12}$$

obtained from the commutation relations of the $L^{\pm I}{}_J$ lead to

$$y_A\overline{y_-}=q^{-4}(y_2)^{-1}\hat{y}_+(\hat{y}_1)^{-1}\tag{3.13}$$

with the definition

$$y_A:=\mathbf{1}-(y_2)^{-1}\hat{y}_+(\hat{y}_1)^{-1}y_-.\tag{3.14}$$

The element y_A is invertible since

$$y_A(\mathbf{1}+q^4y_-\overline{y_-})=(\mathbf{1}+q^4y_-\overline{y_-})y_A=\mathbf{1}.\tag{3.15}$$

y_A^{-1} can be expanded as a power series in $(y_2)^{-1}\hat{y}_+(\hat{y}_1)^{-1}y_-$ which converges (compare the discussion in [DSWZ] for the element Δ). Thus

$$\overline{y_-}=q^{-4}(y_A)^{-1}(y_2)^{-1}\hat{y}_+(\hat{y}_1)^{-1}.\tag{3.16}$$

Using y_A we can rewrite (3.9) and (3.10)

$$\overline{y_2}=(\hat{y}_1)^{-1}(y_A)^{-1}\tag{3.17}$$

and

$$\overline{\hat{y}_1}=\hat{y}_1(y_2)^{-1}(\hat{y}_1)^{-1}(y_A)^{-1}.\tag{3.18}$$

In the next step $(L^{+1}{}_1)^2$ is calculated.

$$(L^{+1}{}_1)^2 = L^{+1}{}_1 L^{-2}{}_2 (L^{-2}{}_2)^{-1} L^{+1}{}_1 = (y_2)^{-1}(\hat{y}_1)^{-1}(y_A)^{-1}, \tag{3.19}$$

and thus we obtain

$$\overline{\hat{y}_+} = q^2 (L^{+1}{}_1)^2 y_- = q^2 (y_2)^{-1}(\hat{y}_1)^{-1}(y_A)^{-1} y_- . \tag{3.20}$$

Expanding $(\overline{y_+}\hat{y}_1)$ in $L^{\pm I}{}_J$ and using their commutation relations one arrives at

$$\overline{y_+} = (y_2\hat{y}_- + y_-\hat{y}_1 - (\hat{y}_1)^{-1}(y_A)^{-1}y_-)(\hat{y}_1)^{-1} . \tag{3.21}$$

In a similar manner we obtain

$$\overline{\hat{y}_-} = (\hat{y}_1 y_+ + \hat{y}_+ y_2 - \hat{y}_1 (y_2)^{-1}(\hat{y}_1)^{-1}(y_A)^{-1}\hat{y}_+)(y_2)^{-1} \tag{3.22}$$

and therefore, the O-algebra can be expressed by the $Y^I{}_J$ only.

We have now shown that the Y-algebra[1] is a sub-$*$-Hopf algebra in $U_q sl(2,\mathbb{C})$. In this approach we mainly used the algebraic properties of $U_{\mathscr{R}}$.

A second approach uses the convolutive action of the $Y^I{}_J$ as differential operators on $\mathscr{A}$. This is presented in the following. From [DSWZ] one obtains the fundamental commutation relations between the generators of $\mathscr{A}$ and the Y-algebra

$$Y^I{}_J T^V{}_W = \hat{\mathscr{R}}_q^{IA}{}_{BS} \hat{\mathscr{R}}_q^{BR}{}_{JW} T^V{}_A Y^S{}_R . \tag{3.23}$$

For convenience we introduce a new index notation:

$$(\Omega) = (\omega, \bar{\omega}) = (11, 21, 12, 22, \overline{1}\overline{1}, \overline{2}\overline{1}, \overline{1}\overline{2}, \overline{2}\overline{2}) = (1, -, +, 2, \bar{1}, \bar{-}, \bar{+}, \bar{2}) .$$

Having introduced the operators $O^\Pi{}_\Omega$ through

$$\Delta(Y^\Omega) = O^\Omega{}_\Pi \otimes Y^\Pi \tag{3.24}$$

[compare (2.18)] one tries to express them in terms of the Y^Ω. In the first step we restrict only to the action on the subalgebras of $\mathscr{A}$ generated by either $(t^i{}_j)$ or $(\hat{t}^{\bar{i}}{}_{\bar{j}})$ because there $L^{-i}{}_j = 0$ for $i<j$ and $i>j$, respectively, and therefore Δ vanishes on both of these sectors [DSWZ]. In order to construct the $O^\Pi{}_\Omega$ from its restricted operators it is sufficient to find a decomposition of the operators $O^\Pi{}_\Omega$ into

$$O^\Pi{}_\Omega = \tilde{O}^\Pi{}_R \tilde{\tilde{O}}^R{}_\Omega \tag{3.25}$$

such that

$$\begin{aligned} \forall \alpha \in \langle (t^i{}_j) \rangle, \hat{\beta} \in \langle (\hat{t}^{\bar{i}}{}_{\bar{j}}) \rangle : \quad & \tilde{O}^\Pi{}_\Omega(\alpha\hat{\beta}) = \tilde{O}^\Pi{}_\Omega(\alpha) e(\hat{\beta}) , \\ & \tilde{\tilde{O}}^\Pi{}_\Omega(\alpha\hat{\beta}) = e(\alpha) \tilde{\tilde{O}}^\Pi{}_\Omega(\hat{\beta}) , \end{aligned} \tag{3.26}$$

i.e. into factors which act nontrivially only on one of these subalgebras. To make the following more transparent we restrict the action of $Y^I{}_J$ to monomials of the form $(t^1{}_1)^k (t^1{}_2)^l$ and $(\hat{t}^{\bar{1}}{}_{\bar{1}})^m (\hat{t}^{\bar{1}}{}_{\bar{2}})^n$. For $(t^1{}_1)^k (t^1{}_2)^l$ we use the abbreviation (k, l) and for

[1] I.e. the algebra generated by the $Y^I{}_J$ and by the convergent power series in $Y^I{}_J$ introduced above

$(\hat{t}^1{}_1)^m(\hat{t}^1{}_2)^n$ we use (m, n). From (3.23) we obtain

$$\begin{aligned}
y_1(k,l) &= q^{k-l}(k,l)y_1 + \lambda[k]_q(k-1,l+1)y_- + \lambda q^{k-l+1}[l]_q(k+1,l-1)y_+ \\
&\quad + \lambda^2 q^{-1}[k+1]_q[l]_q(k,l)y_2\,, \\
y_-(k,l) &= (k,l)y_- + \lambda q^{-1}[l]_q(k+1,l-1)y_2\,, \\
y_+(k,l) &= (k,l)y_+ + \lambda q^{l-k}[k]_q(k-1,l+1)y_2\,, \\
y_2(k,l) &= q^{l-k}(k,l)y_2\,, \\
y_1(m,n) &= (m,n)y_1 + \lambda q^{n-m}[m]_q(m-1,n+1)y_-\,, \\
y_-(m,n) &= q^{n-m}(m,n)y_-\,, \\
y_+(m,n) &= q^{m-n}(m,n)y_+ - \lambda[m]_q(m-1,n+1)(y_1-y_2) \\
&\quad - \lambda^2 q^{n-m+2}[m]_q[m-1]_q(m-2,n+2)y_-\,, \\
y_2(m,n) &= (m,n)y_2 - \lambda q^{n-m+2}[m]_q(m-1,n+1)y_-\,, \\
\hat{y}_1(k,l) &= (k,l)\hat{y}_1 + \lambda q^{-1}[l]_q(k+1,l-1)\hat{y}_+\,, \\
\hat{y}_-(k,l) &= q^{k-l}(k,l)\hat{y}_- - \lambda q^{k-l+1}[l]_q(k+1,l-1)(\hat{y}_1-\hat{y}_2) \\
&\quad - \lambda^2 q^{k-l+2}[l]_q[l-1]_q(k+2,l-2)\hat{y}_+\,, \\
\hat{y}_+(k,l) &= q^{l-k}(k,l)\hat{y}_+\,, \\
\hat{y}_2(k,l) &= (k,l)\hat{y}_2 - \lambda q[l]_q(k+1,l-1)\hat{y}_+\,, \\
\hat{y}_1(m,n) &= q^{n-m}(m,n)\hat{y}_1\,, \\
\hat{y}_-(m,n) &= (m,n)\hat{y}_- - \lambda q[n]_q(m+1,n-1)\hat{y}_1\,, \\
\hat{y}_+(m,n) &= (m,n)\hat{y}_+ - \lambda q^{n-m+2}[m]_q(m-1,n+1)\hat{y}_1\,, \\
\hat{y}_2(m,n) &= q^{m-n}(m,n)\hat{y}_2 - \lambda q^{m-n+1}[n]_q(m+1,n-1)\hat{y}_+ \\
&\quad - \lambda[m]_q(m-1,n+1)\hat{y}_- + \lambda^2 q[m]_q[n+1]_q(m,n)\hat{y}_1\,.
\end{aligned} \tag{3.27}$$

It is now possible to express the above introduced convolutive action by eight operators A, B, C, D, K, L, M, N, where A, B, K, L only operate on the $\langle(t^i{}_j)\rangle$-sector and C, D, M, N only operate on $\langle\hat{t}^{\bar{i}}{}_{\bar{j}}\rangle$. They are defined on ordered monomials as follows [set $(k,l,k',l',m,n,m',n') := (t^1{}_1)^k(t^1{}_2)^l(t^2{}_1)^{k'}(t^2{}_2)^{l'}(\hat{t}^1{}_1)^m(\hat{t}^1{}_2)^n(\hat{t}^2{}_1)^{m'}(\hat{t}^2{}_2)^{n'}$]:

$$\begin{aligned}
A(k,l,k',l',m,n,m',n') &= q^{l+l'-1}[l']_q(k,l,k'+1,l'-1,m,n,m',n') \\
&\quad + q^{2l'+l-k'-1}[l]_q(k+1,l-1,k',l',m,n,m',n')\,, \\
B(k,l,k',l',m,n,m',n') &= q^{2k+k'-l-1}[k']_q(k,l,k'-1,l'+1,m,n,m',n') \\
&\quad + q^{k+k'-1}[k]_q(k-1,l+1,k',l',m,n,m',n')\,, \\
K(k,l,k',l',m,n,m',n') &= (k+k')(k,l,k',l',m,n,m',n')\,, \\
L(k,l,k',l',m,n,m',n') &= (l+l')(k,l,k',l',m,n,m',n')\,, \\
C(k,l,k',l',m,n,m',n') &= q^{n+n'-1}[n']_q(k,l,k',l',m,n,m'+1,n'-1) \\
&\quad + q^{2n'+n-m'-1}[n]_q(k,l,k',l',m+1,n-1,m',n')\,, \\
D(k,l,k',l',m,n,m',n') &= q^{2m+m'-n-1}[m']_q(k,l,k',l',m,n,m'-1,n'+1) \\
&\quad + q^{m+m'-1}[m]_q(k,l,k',l',m-1,n+1,m',n')\,,
\end{aligned} \tag{3.28}$$

$$M(k,l,k',l',m,n,m',n')=(m+m')(k,l,k',l',m,n,m',n'),$$
$$N(k,l,k',l',m,n,m',n')=(n+n')(k,l,k',l',m,n,m',n'),$$

where $[n]_q=(q^n-q^{-n})/(q-q^{-1})$ as defined in Sect. 2. In (3.28) only the action of the operators on monomials is presented.

Using these definitions we rewrite the Y^Ω as

$$\begin{aligned}
y_1&=q^{K+L}+q^{-(K+L+2)}-q^{L-K-2},\\
y_-&=\lambda A q^{-L}=\lambda q^{-L-1}A,\\
y_+&=\lambda B q^{-2K+L+1}=\lambda q^{-2K+L-2}B,\\
y_2&=q^{L-K},\\
\hat{y}_1&=q^{N-M}-\lambda^2 AD q^{-L-2M+N+3},\\
\hat{y}_-&=-\lambda C q^{K-L-N+2}+\lambda^3 A^2 D q^{-3L+K-2M+N+8}\\
&\quad+\lambda A q^{K-2L+2}(q^{M+N+2}+q^{-M-N}-q^{N-M}(1+q^2)),\\
\hat{y}_+&=-\lambda D q^{L-K-2M+N+3},\\
\hat{y}_2&=q^{M+N+2}+q^{-M-N}-q^2\hat{y}_1\,.
\end{aligned}\tag{3.29}$$

The restrictions of $O^\Pi{}_\Omega$ on the separate sectors [see (3.27) where the coefficients correspond to the action of these restricted operators] can be expressed in terms of either A, B, K, L or C, D, M, N, respectively. We have thus found the decomposition of $O^\Pi{}_\Omega$ into $\tilde{O}^\Pi{}_\Omega$ and $\tilde{\tilde{O}}^\Pi{}_\Omega$ [compare (3.25) and (3.26)]. After reexpressing A, B, C, D, K, L, M, N in Y^Ω by inverting (3.29) we arrive at

$$\begin{aligned}
(\tilde{O}^\pi{}_\omega)&=\begin{pmatrix}(y_2)^{-1} & y_+(y_2)^{-1} & q^2y_-(y_2)^{-1} & y_1-(y_2)^{-1}\\ 0 & 1 & 0 & y_-\\ 0 & 0 & 1 & y_+\\ 0 & 0 & 0 & y_2\end{pmatrix},\\
(\tilde{O}^{\bar\pi}{}_{\bar\omega})&=\begin{pmatrix}1 & 0 & y_- & 0\\ -q^2y_-(y_2)^{-1} & (y_2)^{-1} & -q^4(y_-)^2(y_2)^{-1} & q^2y_-(y_2)^{-1}\\ 0 & 0 & y_2 & 0\\ 0 & 0 & -q^2y_- & 1\end{pmatrix},\\
(\tilde{\tilde{O}}^\pi{}_\omega)&=\begin{pmatrix}1 & -q^{-2}\hat{y}_+(y_2)^{-1} & 0 & 0\\ 0 & y_A\hat{y}_1 & 0 & 0\\ y_C & -q^{-2}\hat{y}_+(y_2)^{-1}y_C & (y_A\hat{y}_1)^{-1} & -y_C\\ 0 & \hat{y}_+(y_2)^{-1} & 0 & 1\end{pmatrix},\\
(\tilde{\tilde{O}}^{\bar\pi}{}_{\bar\omega})&=\begin{pmatrix}y_A\hat{y}_1 & 0 & 0 & 0\\ y_B & 1 & 0 & 0\\ \hat{y}_+(y_2)^{-1} & 0 & 1 & 0\\ \hat{y}_2+q^2\hat{y}_1-q^2y_A\hat{y}_1-(y_A\hat{y}_1)^{-1} & y_C & y_B(y_A\hat{y}_1)^{-1} & (y_A\hat{y}_1)^{-1}\end{pmatrix},\\
(\tilde{O}^{\bar\pi}{}_\omega)&=(\tilde{O}^\pi{}_{\bar\omega})=(\tilde{\tilde{O}}^{\bar\pi}{}_\omega)=(\tilde{\tilde{O}}^\pi{}_{\bar\omega})=0,
\end{aligned}\tag{3.30}$$

where $y_A^{\dagger}$ is given by (3.14) and

$$\begin{aligned} y_B &= -y_-\hat{y}_2 + y_-\hat{y}_1 + y_2\hat{y}_- - q^2(y_-)^2\hat{y}_+(y_2)^{-1}, \\ y_C &= q^{-2}\hat{y}_+(y_2)^{-1}(y_A\hat{y}_1)^{-1}. \end{aligned} \tag{3.31}$$

The $O^{\Pi}{}_{\Omega}$ constructed with the help of (3.25) and (3.30) coincide with the results from the algebraic approach.

Having found the $O^{\Pi}{}_{\Omega}$ in terms of the Y^{Σ} it is now possible to construct the antipode $S(Y^{\Sigma})$ as function of the Y^{Π}. There are two possibilities to derive that. The first derivation starts from the Hopf relation

$$m \circ (\mathrm{id} \otimes S) \circ \Delta = \eta \circ \varepsilon, \tag{3.32}$$

where m is the multiplication and η is the unit map of the algebra and uses invertible elements of the algebra to solve (3.32) for the $S(Y^{\Pi})$. For the second derivation one expands $S(Y^{\Pi})$ into products $L^{-I}{}_J L^{+K}{}_L$ and uses the commutation relations of $L^{\pm I}{}_J$ [DSWZ] to express the $L^{-I}{}_J L^{+K}{}_L$ in terms of the $O^{\Pi}{}_{\Omega}$. Both derivations yield

$$\begin{aligned} S(y_1) &= y_2 + y_- y_D(y_A\hat{y}_1 y_2)^{-1}, \\ S(y_-) &= -y_-(y_A\hat{y}_1 y_2)^{-1}, \\ S(y_+) &= -\hat{y}_+ - \hat{y}_1 y_D(y_A\hat{y}_1 y_2)^{-1}, \\ S(y_2) &= \hat{y}_1(y_A\hat{y}_1 y_2)^{-1}, \\ S(\hat{y}_1) &= y_2(y_A\hat{y}_1 y_2)^{-1}, \\ S(\hat{y}_-) &= -y_- - y_2 y_E(y_A\hat{y}_1 y_2)^{-1}, \\ S(\hat{y}_+) &= -\hat{y}_+(y_A\hat{y}_1 y_2)^{-1}, \\ S(\hat{y}_2) &= \hat{y}_1 + \hat{y}_+ y_E(y_A\hat{y}_1 y_2)^{-1} \end{aligned} \tag{3.33}$$

with

$$\begin{aligned} y_D &= y_+\hat{y}_1 - q^{-2}y_1\hat{y}_+, \\ y_E &= \hat{y}_- y_2 - q^2\hat{y}_2 y_-. \end{aligned} \tag{3.34}$$

4. $U_q sl(2, \mathbb{C})$ in the Y-Hopf Algebra

In this section we demonstrate the equivalence of the Y-algebra and $U_{\mathscr{R}}$. For that purpose we consider (3.19),

$$(L^{+1}{}_1)^2 = (y_A\hat{y}_1 y_2)^{-1}.$$

We define the functional $\sqrt{(L^{+1}{}_1)^2}$ as follows:

$$\begin{aligned} &\sqrt{(L^{+1}{}_1)^2}(\mathbf{1}) := 1, \\ &\sqrt{(L^{+1}{}_1)^2}(T^I{}_J) := \sqrt{(L^{+1}{}_1)^2(T^I{}_J)}, \\ \forall a, b \in \mathscr{A}: \quad &\sqrt{(L^{+1}{}_1)^2}(ab) := \sqrt{(L^{+1}{}_1)^2}(a)\sqrt{(L^{+1}{}_1)^2}(b), \end{aligned} \tag{4.1}$$

where the root is taken such that $\sqrt{(L^{+1}{}_1)^2}(T^I{}_J)=L^{+1}{}_1(T^I{}_J)$. Then we have the following

Proposition 2. *$\sqrt{(L^{+1}{}_1)^2}$ defined above as a function of the $Y^I{}_J$ is a well defined algebra homomorphism on $\mathscr{A}$ and equals $L^{+1}{}_1$.*

From the definition of $Y^I{}_J$ one obtains

$$\begin{aligned}
&L^{+2}{}_2L^{+1}{}_2=qy_+\hat{y}_1-q^{-1}y_1\hat{y}_+\,,\\
&L^{+2}{}_2L^{+\bar{2}}{}_{\bar{1}}=q^{-1}\hat{y}_-y_2-q\hat{y}_2y_-\,,\\
&L^{+2}{}_2L^{-1}{}_1=y_2\,,\\
&L^{+2}{}_2L^{-1}{}_2=-q^{-1}\hat{y}_+\,,\\
&L^{+2}{}_2L^{-2}{}_1=-qy_-\,,\\
&L^{+2}{}_2L^{-2}{}_2=\hat{y}_1\,,\\
&L^{+2}{}_2L^{+2}{}_2=y_A\hat{y}_1y_2=\hat{y}_1y_2-y_-\hat{y}_+\,.
\end{aligned}\tag{4.2}$$

Applying $L^{+1}{}_1$ to (4.2) from the left yields all functionals generating $U_{\mathscr{R}}$ as functions of the $Y^I{}_J$ thus proving the equivalence of $U_{\mathscr{R}}$ and the Y-algebra. In particular, we found again the q-deformed Lorentz algebra.

Throughout the paper we considered the algebra generated by the $Y^I{}_J$ as a subset of $U_{\mathscr{R}}$. Certainly, there are more relations in the Y-algebra than (2.16), (2.17), (3.2), (3.5) – we used such additional relations in the case of A_1 in order to show the equivalence to $U_{\mathscr{R}}$. We did not investigate whether the Y-algebra can be abstracted from $U_{\mathscr{R}}$ such that $\mathbb{C}\langle Y'^I{}_J\rangle/((2.16),(2.17),(3.2),(3.5))$ becomes a $*$-Hopf algebra if the comultiplication for the generators $Y'^I{}_J$ is given through (2.18) with the $O^{IJ}{}_{KL}$ as functions of the $Y'^I{}_J$. Then the above presented Y-algebra is a $*$-Hopf algebra representation of the Y'-algebra. It is interesting whether one can find a general scheme to obtain the results of Sects. 3 and 4 to show the Hopf structure of the Y-algebra introduced in Sect. 2 and its equivalence to $U_{\mathscr{R}}$ in general for the cases A_{n-1}, B_n, C_n, and D_n.

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