# Non-Commutative Spheres

## III. Irrational Rotations

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*Dedicated to Professor Huzihiro Araki on the occasion of his 60'th birthday*

**Abstract.** Let $A_\theta$ be the irrational rotation algebra, i.e. the $C^*$-algebra generated by two unitaries $U$, $V$ satisfying $VU = e^{2\pi i\theta} UV$, with $\theta$ irrational, and consider the fixed point subalgebra $B_\theta$ under the flip automorphism $U \to U^{-1}$, $V \to V^{-1}$. We prove that $B_\theta$ is an AF-algebra.

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## 1. Introduction

In this paper we continue the study, begun in [BEEK 1] and [BEEK 2], of the fixed point subalgebra of the rotation algebra under the flip. Recall from [Rie] that the rotation algebra $A_\theta$ is the universal $C^*$-algebra generated by two unitaries $U$, $V$ satisfying $VU = \varrho UV$, where $\varrho = e^{2\pi i\theta}$ and $0 \leq \theta < 1$. The flip $\sigma$ is the automorphism of this algebra defined through the requirements

$$\sigma(U) = U^{-1}, \qquad \sigma(V) = V^{-1}. \tag{1.1}$$

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Denote the fixed point algebra under the flip by $B_\theta$, and the crossed product by $C_\theta$. In [BEEK 1] it was established that if $\theta$ is irrational, then $B_\theta$ is the universal $C^*$-algebra generated by two self-adjoint elements $a, b$ satisfying

$$ba^2 + a^2b = 2\lambda aba + 4(1 - \lambda^2)b, \tag{1.2}$$

$$ab^2 + b^2a = 2\lambda bab + 4(1 - \lambda^2)a, \tag{1.3}$$

$$baba = (4\lambda^2 - 1)abab - 2\lambda a^2b^2 + 8\lambda(1 - \lambda^2)(a^2 + b^2 - 1), \tag{1.4}$$

where $\lambda = \cos(2\pi\theta)$. This result was extended to rational $\theta \notin \{0, \frac{1}{2}\}$ in [BEEK 2] while the universal $C^*$-algebra fails to exist if $\theta \in \{0, \frac{1}{2}\}$. The connection between $a$, $b$ and $U, V$ is

$$a = U + U^{-1}, \qquad b = V + V^{-1}. \tag{1.5}$$

When $\theta = p/q$ is rational, it was proved in [BEEK 2] that $B_\theta$ is the subalgebra of the $C^*$-algebra $C(S^2, M_q)$ of continuous functions from the 2-sphere $S^2$ into the algebra of complex $q \times q$ matrices $M_q$ determined up to isomorphism as follows: There are four distinct points $\omega_0, \omega_1, \omega_2,$ and $\omega_3$ in $S^2$ and to each point $\omega_i$ is associated a self-adjoint projection $P_i$ in $M_q$. The dimensions of $P_i$ are all $\dfrac{q-1}{2}$ when $q$ is odd, and

when $q$ is even, $\dim(P_0) = \dfrac{q-2}{2}$ whilst $\dim(P_i) = \dfrac{q}{2}$ for $i = 1, 2, 3$. The algebra $B_\theta$ consists of those functions $f \in C(S^2, M_q)$ such that $f(\omega_i)$ commutes with $P_i$ for $i = 0, 1, 2, 3$.

An analogous result was proved for $C_\theta$, with the difference that $M_q$ is replaced by $M_{2q}$, and $\dim P_i = q$ for $i = 0, 1, 2, 3$, independently of the parity of $q$. (These latter results were extended to other finite subgroups of the canonical action of $SL(2, \mathbf{Z})$ on $A_\theta$ by Farsi and Watling, [FW 1, FW 2, FW 3, FW 4].)

When $\theta$ is irrational, the algebras $B_\theta$ and $C_\theta$ are simple with a unique trace state, [BEEK 1]. Furthermore,

$$K_0(C_\theta) \cong \mathbf{Z}^6, \qquad K_1(C_\theta) \cong 0$$

for all $\theta$, [Kum 2]. A direct argument when $\theta$ is rational is given in [BEEK 2]. In this paper we will prove

**Theorem 1.1.** *The algebras $B_\theta$ and $C_\theta$ are AF-algebras when $\theta$ is irrational.*

Since $B_\theta$ is a corner of $C_\theta$, it suffices to show this for $C_\theta$. In [BEEK 2] we expressed some hope of proving this by approximation by rational $\theta$, but as it is we do not do this directly, but rather use Putnam's tower construction [Put] very much as in [BEK], together with a method of constructing projections in $C_\theta$ which was devised by Kumjian, [Kum 1], modifying Rieffel's method of constructing projections in [Rie].

On the way to proving Theorem 1.1 we will show that $C_\theta$ is an inductive limit of finite direct sums of certain subhomogeneous algebras over the unit interval and some full matrix algebras; see Corollary 7.4 and (7.1)–(7.5). That $C_\theta$ is AF will follow from this by combining with techniques from [BBEK] and [Su]. The strategy is to use unique trace state and simplicity to prove small eigenvalue variation for the inductive limit.

We can also classify the $C_\theta$'s, essentially as the $A_\theta$'s, by computing the range of the trace:

**Theorem 1.2.** *If* $0 < \theta_1, \theta_2 < 1$ *and* $\theta_1, \theta_2$ *are irrational, then* $C_{\theta_1}$ *is isomorphic to* $C_{\theta_2}$ *if and only if* $\theta_1 \in \{\theta_2, 1 - \theta_2\}$.

This contrasts with the rational case where the algebras $C_{p/q}$ and $C_{p'/q'}$ (with $p$, $q$, and also $p'$, $q'$, relatively prime) are isomorphic if and only if $q = q'$, [BEEK 2].

The proof of Theorem 1.2 is independent of the rest of this paper, and is as follows: Since any projection in $B_\theta$ is a projection in $A_\theta$, and the Rieffel projection in $A_\theta$ has a representative which is flip invariant, it follows that the range of the trace on the projections in $B_\theta$ is the same as in $A_\theta$, which is $(\mathbf{Z} + \mathbf{Z}\theta) \cap [0, 1]$. But $B_\theta$ is isomorphic to $eC_\theta e$, where $e$ is a projection in $C_\theta$ with trace $1/2$, and hence the range of the trace on $C_\theta$ is $\frac{1}{2}(\mathbf{Z} + \mathbf{Z}\theta) \cap [0, 1]$. Thus, if $C_{\theta_1}$ and $C_{\theta_2}$ have the same range of the trace, then $\theta_1 = \theta_2$ or $\theta_1 = 1 - \theta_2$, and hence $C_{\theta_1}$ and $C_{\theta_2}$ are non-isomorphic unless $\theta_1$ and $\theta_2$ are related in this way. On the other hand $C_\theta$ and $C_{1-\theta}$ are isomorphic since the isomorphism $u \to v$, $v \to u$ of $A_\theta$ and $A_{1-\theta}$ intertwines the flips of those two algebras. This proves Theorem 1.2.

## 2. Putnam's Tower Construction on T

In this section we will use the identification $\mathbf{T} = \mathbf{R}/\mathbf{Z}$, and by the term interval in $\mathbf{T}$ we will mean closed nonempty intervals where both endpoints (which are supposed to be distinct) lie in the orbit $\mathbf{Z}\theta$ mod 1, where $0 < \theta < 1$ is a fixed irrational number. By a partition of $\mathbf{T}$ will be meant a finite collection of closed intervals with union $\mathbf{T}$ such that the intersection of any pair of the intervals consists of at most one point (which is then an endpoint of both the intervals and thus is contained in $\mathbf{Z}\theta$). Note that the set of intervals are left globally invariant under both $\alpha$ and $\sigma$, where

$$\alpha(t) = t + \theta, \tag{2.1}$$

and

$$\sigma(t) = -t. \tag{2.2}$$

In particular we will consider the partitions of $\mathbf{T}$ determined by the requirements that

$$-(N-1)\theta, \ -(N-2)\theta, \ ..., \ -\theta, 0, \theta, \ ..., (N-2)\theta, (N-1)\theta$$

shall be the set of endpoints, where $N$ is a positive integer. In particular, we will see that these partitions arise from a Putnam tower construction with 3 towers (unless $N$ is very small). For later use, we will choose $N$ in a specific way:

For any positive integer $M$, choose $\delta > 0$ so small that all the translates of $\langle \theta/2 - \delta, \theta/2 + \delta \rangle$ by $m\theta$, with $|m| \leq M + 1$, are pairwise disjoint on $\mathbf{T}$. Then choose $N > 0$ such that

$$N\theta \in \langle \theta/2, \theta/2 + \delta \rangle \tag{2.3}$$

and such that the orbit piece $\{-(N-1)\theta, \ ..., (N-1)\theta\}$ intersects both the intervals $\langle \theta/2, N\theta \rangle$ and $\langle N\theta, \theta/2 + \delta \rangle$. Now let $k\theta$ denote the point in the orbit piece in $\langle N\theta, \theta/2 + \delta \rangle$ which is closest to $N\theta$, and $l\theta$ the point in the orbit piece in $\langle \theta/2, N\theta \rangle$ which is closest to $N\theta$. Thus, $|k| < N$, $|l| < N$ and $[N\theta, k\theta]$, $[l\theta, N\theta]$ are elements in the partition of $\mathbf{T}$ determined by $\{-N\theta, \ ..., N\theta\}$, while $[l\theta, k\theta]$ is an element in the partition of $\mathbf{T}$ determined by $\{-(N-1)\theta, (N-1)\theta\}$.

**Lemma 2.1.** *Define* $M$, $\delta$, $N$, $l$, $k$ *as above. Then*

$$k+l<0 \tag{2.4}$$

*and the partition of* $\mathbf{T}$ *defined by the orbit piece* $\{-(N-1)\theta, \ldots, (N-1)\theta\}$ *consists of the intervals*
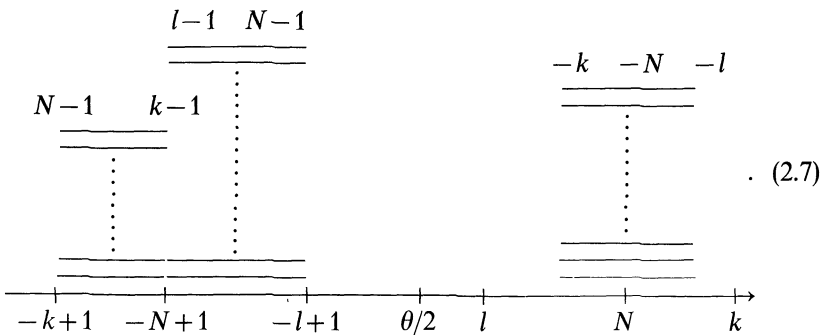
$$[l\theta, k\theta]+m\theta, \quad 0\le m\le -(k+l),$$
$$[(-k+1)\theta, (-N+1)\theta]+m\theta, \quad 0\le m\le N+k-2, \tag{2.5}$$
$$[(-N+1)\theta, (-l+1)\theta]+m\theta, \quad 0\le m\le N+l-2.$$

*Furthermore, this partition consists of the Putnam towers associated to the* $\alpha\sigma$-*invariant set*

$$[(-k+1)\theta, (-l+1)\theta]\cup[l\theta, k\theta]. \tag{2.6}$$

*This set is contained in the interval* $\langle\theta/2-\delta, \theta/2+\delta\rangle$, *and the heights of the three towers are all at least* $2M+2$.

*Proof.* For clarity, let us draw a figure of the whole tower construction (drawn in the case that $k<l$):



$$. \tag{2.7}$$

Here, any integer label $n$ refers to the point $n\theta$. Inspection of the figure above shows that the set of left end points of the intervals occurring runs through the set $\{m; -N+1\le m\le N-1\}$ and each number of this set occurs exactly once. The same is true for the set of right endpoints. Hence all we have to show is that the interiors of the floors of the towers indicated above do not overlap, that is, if $n\theta$ lies in the interior of some floor, then $|n|\ge N$. We check this for the three towers separately.

*Tower 1 from the right.* As for the basement, note that the only $n\theta$ in $\langle l\theta, k\theta\rangle$ with $|n|\le N$ is $N\theta$, by the definition of $k$ and $l$. For the remaining floors $\langle l\theta, k\theta\rangle+m\theta$ we proceed by induction with respect to $m$. If

$$n\theta\in\langle l\theta, k\theta\rangle+m\theta \tag{2.8}$$

with $|n|\le N-1$ and $m\ge 1$, then

$$(n-1)\theta\in\langle\theta, k\theta\rangle+(m-1)\theta, \tag{2.9}$$

and hence, by the induction hypothesis, we must have $n=-N+1$ and $m-1>0$. But as $\sigma$, applied to $l\theta$, $N\theta$, and $k\theta$, gives $-l\theta$, $-N\theta$, $-k\theta$ respectively, and the

whole set $\{-N\theta, (-N+1)\theta, \ldots, (N-1)\theta, N\theta\}$ is $\sigma$-invariant, it follows that

$$\langle l\theta, k\theta \rangle + (m-1)\theta = \langle -k\theta, -l\theta \rangle \qquad (2.10)$$

for this $m$, whence

$$m-1 = -(k+l).$$

This proves simultaneously that

$$k+l<0$$

and that the statement for the first tower holds.

*Towers 2 and 3.* Note that $\alpha$ maps the roof of Tower 1 onto the union of the basements of Towers 2 and 3, and that hence the only point of the form $n\theta$ in $\langle (-k+1)\theta, (-l+1)\theta \rangle$ with $|n| \leq N-1$ is $(-N+1)\theta$. This is seen by subtracting $\theta$ and using that $-N\theta$ is the only point of the form $n\theta$ with $|n| \leq N$ in $\langle -k\theta, -l\theta \rangle$. For the remaining floors of e.g. Tower 2, i.e., $\langle (-N+1)\theta, (-l+1)\theta \rangle + m\theta$, we proceed by induction again: If

$$n\theta \in \langle (-N+1)\theta, (-l+1)\theta \rangle + m\theta,$$

then

$$(n-1)\theta \in \langle (-N+1)\theta, (-l+1)\theta \rangle + (m-1)\theta,$$

and hence $n = -N+1$ by the induction hypothesis. Thus,

$$-N\theta \in \langle (-N+1)\theta, (-l+1)\theta \rangle + (m-1)\theta$$

for this $m$. Since the neighbouring points of $-N\theta$ in $\{-N\theta, \ldots, N\theta\}$ are $-k\theta$ and $-l\theta$ it follows that

$$(-N+1)+(m-1) = -k \quad \text{and} \quad (-l+1)+(m-1) = -l,$$

from which follows

$$m = N-k \quad \text{and} \quad m=0,$$

which is a contradiction. Thus the only restriction on the range of $m$ is that $(-N+1)+m$ and $(-l+1)+m$ should lie in $\{-N+1, \ldots, N-1\}$, i.e. $(-l+1)+m \leq N-1$, i.e. $m \leq N+l-2$. Tower 3 is treated analogously.

Finally, since $\delta$ was chosen such that all translates of $\langle \theta/2-\delta, \theta/2+\delta \rangle$ by $m\theta$, with $|m| \leq M+1$, are pairwise disjoint, and all three basements are contained in this set, it follows that any translate of any basement by $m\theta$, with $|m| \leq M+1$, cannot intersect any other basement. It follows that the height of each of the three towers is at least $2M+2$.

## 3. A Subsidiary Tower Construction

In order to construct finite-dimensional subalgebras of $C_\theta = C(\mathbf{T}) \times_\alpha \mathbf{Z} \times_\sigma \mathbf{Z}_2$, we will have to modify the three-tower construction in Lemma 2.1 and replace it by a six-tower construction. In the case that $k<l$, the new tower construction looks as

follows:



$$(3.1)$$
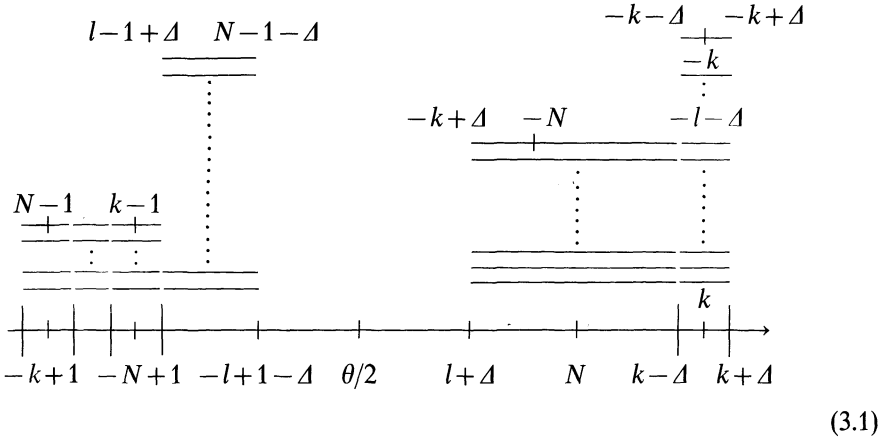
Here, $\varDelta$ is a nonzero integer such that $\varDelta\theta$ is much closer to 0 in $\mathbf{T}$ than any of the points in the orbit $\{-(N-1)\theta, \ldots, (N-1)\theta\}$ are to each other. For definiteness, let us assume that (mod 1)

$$0 < \varDelta\theta \leqq \tfrac{1}{4}\min\{(-l+1)\theta - (-N+1)\theta, (-N+1)\theta - (-k+1)\theta\}. \qquad (3.2)$$

It is then easily verified that the depicted tower construction really is a Putnam tower construction over the basement $[(-k+1-\varDelta)\theta, (-l+1-\varDelta)\theta] \cup [(l+\varDelta)\theta, (k+\varDelta)\theta]$. This basement is still $\alpha\sigma$-invariant ($\alpha\sigma$ interchanges the two pieces). Note also that $\sigma$ maps each of the six towers into themselves except for the first and third tower from the left, which are interchanged, and $\sigma$ reverses the order of the floors, in particular interchanging basements and roofs.

In the case that $l < k$, we use the following new tower construction:



$$(3.3)$$
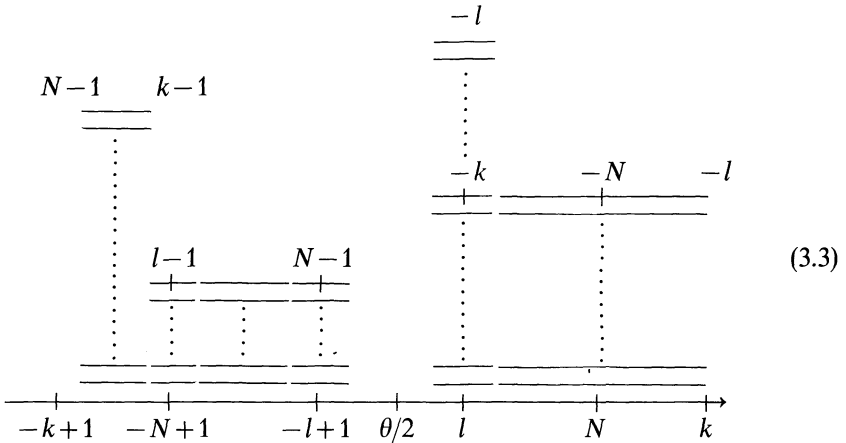
The same remarks, with the obvious modifications, apply to this construction.

In any case, let $Y_1$, $Y_2$, $Y_3$ denote the three ground floors of the wide towers, i.e., towers number 2, 4, and 5 from the left in Fig. (3.1), and let $Y_4$, $Y_5$, $Y_6$ denote the three ground floors of the narrow towers, i.e., towers number 1, 3, and 6 from the left in (3.1). The floors in the towers over $Y_1$, $Y_2$, and $Y_3$ will be called wide floors, and the other floors will be called narrow floors. Let $J_i$ be the number of floors in

the tower over $Y_i$. The numerical value of $J_i$ can be read off from Fig. (3.1) or (3.3). The next lemma follows by inspecting (3.1) and (3.3) in conjunction with Lemma 2.1. It is an analogue of Propositions 1.2 and 1.6 in [BEK].

**Lemma 3.1.** *Adopt the notation and assumptions of Lemma 2.1 as well as the assumptions above. Then the following statements hold:*

$$J_k \geq 2M + 2 \quad for \quad k = 1, \ldots, 6. \tag{3.4}$$

*The sets $\alpha^i(Y_k)$, $i = 0, 1, \ldots, J_k - 1$, $k = 1, \ldots, 6$ form a partition of $\Omega$.* (3.5)

$$\{\sigma(Y_1), \ldots, \sigma(Y_6)\} = \{\alpha^{J_1 - 1}(Y_1), \ldots, \alpha^{J_6 - 1}(Y_6)\} \quad (as\ unordered\ sets). \tag{3.6}$$

*If $I_1, I_2$ are two floors which are adjacent in $\mathbf{T}$,*

*then one is a wide floor and the other a narrow floor.* (3.7)

*The set $Y = Y_1 \cup Y_2 \cup \ldots \cup Y_6$ is invariant under $\alpha\sigma$,*

*and is contained in a $\delta$-neighbourhood of $\theta/2$.* (3.8)

$$\alpha\sigma(Y_k) \cap Y_k = \emptyset \quad for \quad k = 1, \ldots, 6. \tag{3.9}$$

*Remark 3.2.* For (3.8), we assume that $\Delta\theta \mod 1$ has been chosen sufficiently small. As for (3.6), we have $\sigma(Y_k) = \alpha^{J_k - 1}(Y_k)$ for $k = 1, 2, 3$, and for one $k$ in $\{4, 5, 6\}$, say $k = 4$, while $J_5 = J_6$ and $\alpha^{J_5 - 1}(Y_5) = \sigma(Y_6)$ and $\alpha^{J_6 - 1}(Y_6) = \sigma(Y_5)$.

*Remark 3.3.* We will not consider the extent to which the construction of narrow towers and Lemma 3.1 is tied up to our particular choice of partitions. Having any tower construction based on $\mathbf{T}_d$, the Putnam discretization of $\mathbf{T}$ where $\mathbf{T}$ is cut up along the orbit $\mathbf{Z}\theta$, then any floor is a finite union of intervals. Hence, splitting up the towers, we may assume that all the floors are intervals. Cutting off a small, but uniform, piece around each endpoint one obtains a candidate for the floors of the narrow towers of a similar construction. However, it is not clear how one should choose the basements of the new towers in order to ensure the validity of the analogue of Lemma 3.1. As an illustration of the difficulties the reader may wish to verify that if $k < l$ and one tries to build up the narrow towers as in Fig. 3.3 rather as in Fig. 3.1, then the construction works if and only if $l < 0$, and even then one of the narrow towers may have smaller height than $2M + 2$.

## 4. Kumjian's Projections

In this section we will show that if $x_1, \ldots, x_n$ is any finite collection of elements in $C(\mathbf{T}) \subseteq C_\theta = C(\mathbf{T}) \times {}_\alpha \mathbf{Z} \times {}_\sigma \mathbf{Z}_2$ and $\varepsilon > 0$ then there exists a finite-dimensional subalgebra of $C_\theta$ which approximately contains $x_1, \ldots, x_n$ up to $\varepsilon$; see Lemma 4.1.

To this end, equip $\mathbf{T}$ with normalized Haar measure $dt$, and denote the unitary operators implementing $\alpha$, $\sigma$ on $L^2(\mathbf{T})$ by $u(\alpha)$, $u(\sigma)$. The $C^*$-algebra $C(\mathbf{T})$ has a faithful representation on $L^2(\mathbf{T})$ by pointwise multiplication, and as $C(\mathbf{T})$ is abelian and $\mathbf{Z} \times {}_\sigma \mathbf{Z}_2$ is amenable, $C_\theta$ is canonically isomorphic to the $C^*$-algebra on $L^2(\mathbf{T})$ generated by $C(\mathbf{T})$, $u(\alpha)$ and $u(\sigma)$, [Ped]. We thus identify $C_\theta$ with this algebra.

Let $\delta$ be a positive number such that

$$\delta < \Delta\theta \mod 1. \tag{4.1}$$

Then all the floors in the new tower construction have length at least $2\delta$. A typical floor has the form $I = [\theta_1, \theta_2]$, where $\theta_1 = n_1\theta \mod 1$, $\theta_2 = n_2\theta \mod 1$ are elements

in the $\theta$-orbit. Following [Kum 1], we will associate a projection $p_I$ to $I$ as follows:
For $t \in [\theta_i - \delta, \theta_i + \delta]$, put

$$\varphi_i(t) = (\theta_i + \delta - t)/2\delta, \quad i = 1, 2, \tag{4.2}$$

and define $f_I \in C(\mathbf{T})$ by

$$f_I(t) = \begin{cases} 1 - \varphi_1(t) & \text{if} \quad \theta_1 - \delta \le t \le \theta_1 + \delta, \\ 1 & \text{if} \quad \theta_1 + \delta \le t \le \theta_2 - \delta, \\ \varphi_2(t) & \text{if} \quad \theta_2 - \delta \le t \le \theta_2 + \delta, \\ 0 & \text{elsewhere}. \end{cases} \tag{4.3}$$

Define $g_{i,I} \in C(\mathbf{T})$ by

$$g_{i,I}(t) = \begin{cases} (\varphi_i(t)(1 - \varphi_i(t)))^{1/2} & \text{if} \quad \theta_i - \delta < t < \theta_i + 2, \\ 0 & \text{elsewhere} \end{cases} \tag{4.4}$$

for $i = 1, 2$, and finally set

$$p_I = f_I + \varepsilon(I)(u(\alpha)^{2n_1}u(\sigma)g_{1,I} + u(\alpha)^{2n_2}u(\sigma)g_{2,I}), \tag{4.5}$$

where $\varepsilon(I) \in \{+1, -1\}$. Using that $n\theta$ is a fixed point for the homeomorphism $\alpha^{2n}\sigma$ of $\mathbf{T}$, one verifies that $p_I$ is indeed a projection, whatever the sign of $\varepsilon(I)$. We now make the following choice for the sign: Put $\varepsilon(I) = +1$ if $I$ is a wide floor, and put $\varepsilon(I) = -1$ if $I$ is a narrow floor. This choice of sign ensures that the boundary terms of the projections belonging to adjacent intervals cancel when the projections are added up, because of (4.4), and as a consequence we have

$$\sum_I p_I = 1, \tag{4.6}$$

where the sum is over all floors in the new tower construction.

For any floor $I$, let $t_I$ denote the middle point of the interval $I \subseteq \mathbf{T}$.

**Lemma 4.1.** *If $x \in C(\mathbf{T})$, then*

$$\lim_{N \to \infty} \left\| x - \sum_I x(t_I)p_I \right\| = 0. \tag{4.7}$$

*Proof.* For given $\varepsilon > 0$ choose $\delta' > 0$ such that $|t - s| < \delta' \Rightarrow |x(t) - x(s)| < \varepsilon$, and choose $N, l, k$ etc. as in Lemma 2.1, with $\delta$ equal to this $\delta'/2$ (or choose $N$ larger). We have

$$x - \sum_I x(t_I)p_I$$

$$= x - \sum_I x(t_I)f_I - \sum_I \varepsilon(I)x(t_I)(u(\alpha)^{2n_1(I)}u(\sigma)g_{1,I} + u(\alpha)^{2n_2(I)}u(\sigma)g_{2,I}). \tag{4.8}$$

The functions $f_I$ form a partition of unity on $\mathbf{T}$, and the support of each $f_I$ has width at most $\delta'$. It follows that

$$\left\| x - \sum_I x(t_I)f_I \right\| < \varepsilon. \tag{4.9}$$

As for the remaining terms, note for example that the operator $u(\alpha)^{2n_1(I)}u(\sigma)g_{1,I}$ lives on $L^2([n_1(I)\theta - \delta, n_1(I)\theta + \delta])$, and as $g_{1,I}$ is symmetric around $n_1(I)\theta$, this subspace of $L^2(\mathbf{T})$ is mapped into itself by $u(\alpha)^{2n_1(I)}u(\sigma)g_1$. Also, there is a unique

floor $J$ such that $n_2(J) = n_1(I)$, i.e., the floor $J$ that intersects $I$ at its left endpoint. Then $\varepsilon(I)$ and $\varepsilon(J)$ have opposite sign, while

$$u(\alpha)^{2n_2(J)} u(\sigma) g_{2,J} = u(\alpha)^{2n(I)} u(\sigma) g_{1,I} \tag{4.10}$$

since $g_{2,J} = g_{1,I}$ by construction. As

$$\|g_{1,I}\| \leqq 1/2 \tag{4.11}$$

and

$$|x(t_I) - x(t_J)| < \varepsilon \tag{4.12}$$

it follows that

$$\|x(t_I)\varepsilon(I)u(\alpha)^{2n_1(I)}u(\sigma)g_{1,I} + x(t_J)\varepsilon(J)U(\alpha)^{2n_2(J)}u(\sigma)g_{2,J}\| \leqq \varepsilon/2. \tag{4.13}$$

Note also that the interval $[n_1(I)\theta - \delta, n_1(I)\theta + \delta]$ is disjoint from all the other intervals around the endpoints of the floors except for the floor $J$ alluded to above. Thus the operator sum

$$\sum_I \varepsilon(I)x(t_I)\left(u(\alpha)^{2n_1(I)}u(\sigma)g_{1,I} + u(\alpha)^{2n_2(I)}u(\sigma)g_{2,I}\right) \tag{4.14}$$

decomposes into a direct sum of operators of the form

$$x(t_I)\varepsilon(I)u(\alpha)^{2n_1(I)}u(\sigma)g_{1,I} + x(t_J)\varepsilon(J)u(\alpha)^{2n_2(J)}u(\sigma)g_{2,J} \tag{4.15}$$

over all adjacent intervals $J, I$ with $J$ to the left. It follows from (3.16) that the norm of the operator sum is also at most $\varepsilon/2$. Combining with (4.8) and (4.9) we obtain

$$\left\|x - \sum_I x(t_I)p_I\right\| < \varepsilon + \varepsilon/2 = 3\varepsilon/2, \tag{4.16}$$

and Lemma 4.1 is proved.

## 5. Finite-Dimensional Subalgebras

We will now define a finite-dimensional subalgebra $A_0$ of $C(\mathbf{T}) \times_\alpha \mathbf{Z} \times_\sigma \mathbf{Z}_2$ which is somewhat analogous to the $A_0$ of [BEK], but in contrast to that case our $A_0$ is not contained in $C(\mathbf{T}) \times_\alpha \mathbf{Z}$. The following lemma is analogous to Lemma 1.5 in [BEK]:

**Lemma 5.1.** *Let $A_0$ be the $C^*$-algebra on $L^2(\mathbf{T})$ generated by $p_{\alpha^i(Y_k)}$, $k = 1, \ldots, 6$, $i = 0, \ldots, J_k - 1$ and $u(\alpha)p_{\mathbf{T}\setminus\sigma(Y)}$, where*

$$p_{\mathbf{T}\setminus\sigma(Y)} = \sum_{k=1}^{6} \sum_{i=0}^{J_k-2} p_{\alpha^i}(Y_k). \tag{5.1}$$

*It follows that $A_0$ is finite dimensional, and the operators*

$$e_{ij}^k = u(\alpha)^i P_{y_k} u(\alpha)^{-j} = u(\alpha)^{i-j} p_\alpha^j(y_k) \tag{5.2}$$

*for $i, j = 0, 1, \ldots, J_k - 1$, $k = 1, 2, \ldots, 6$ constitute a complete set of matrix units for $A_0$. Furthermore, $A_0$ is invariant under $\mathrm{Ad}(u(\sigma))$ and*

$$u(\sigma)e_{ij}^k u(\sigma)^* = e_{J_k-1-i, J_k-1-j}^l, \tag{5.3}$$

*where either $k = l \in \{1, 2, 3, 4\}$, or $\{k, l\} = \{5, 6\}$.*

*Proof.* On comparing with Lemma 3.1 and Remark 3.2, it suffices by [BEK, Lemma 1.5] to show that

$$u(\alpha)p_I u(\alpha)^* = p_{\alpha(I)} \tag{5.4}$$

and

$$u(\sigma)p_I u(\sigma)^* = p_{\sigma(I)} \tag{5.5}$$

whenever $p_I$, $p_{\alpha(I)}$, $p_{\sigma(I)}$ are defined from (4.5) with the same sign on $\varepsilon$, i.e. $\varepsilon(I) = \varepsilon(\alpha(I)) = \varepsilon(\sigma(I))$. Using the notation (4.2)–(4.5) it is clear that

$$u(\alpha)f_1 u(\alpha)^* = f_{\alpha(I)},$$

$$u(\sigma)f_I u(\sigma)^* = f_{\sigma(I)},$$

$$u(\alpha)g_{i,I} u(\alpha)^* = g_{i,\alpha(I)},$$

$$u(\sigma)g_{1,I} u(\sigma) = g_{2,\sigma(I)},$$

$$u(\sigma)g_{2,I} u(\sigma) = g_{1,\sigma(I)},$$

$$u(\alpha)u(\alpha)^{2n_1}u(\sigma)u(\alpha)^* = u(\alpha)^{2(n_1+1)}u(\sigma),$$

$$u(\sigma)u(\alpha)^{2n_1}u(\sigma)u(\sigma) = u(\alpha)^{-2n_1}u(\sigma),$$

and hence (5.4) and (5.5) follow from the definition (4.5).

## 6. Homogeneous Subalgebras

By adapting the techniques of [BEK] to the present circumstances, we will now prove the following:

**Theorem 6.1.** *Assume that $\theta$ is irrational. Given $\varepsilon > 0$ and elements $x_1, \ldots, x_n \in C(\mathbf{T})$, there exists a $C^*$-subalgebra $B$ of $C_\theta = C(\mathbf{T}) \times_\alpha \mathbf{Z} \times_\sigma \mathbf{Z}_2$ with the same unit as $C_\theta$ such that there exist elements $y_1, \ldots, y_n \in B$ and a unitary $u' \in B$ with*

$$\|y_i - x_i\| < \varepsilon, \qquad i = 1, \ldots, n, \tag{6.1}$$

$$\|u(\alpha) - u'\| < \varepsilon, \tag{6.2}$$

*and $B$ has the form*

$$B \cong M_{J_1} \otimes C(F) \oplus M_{J_2} \oplus \ldots \oplus M_{J_6} \tag{6.3}$$

*with $J_5 = J_6$ and $J_1$ even, where $F$ is a closed subset of $\mathbf{T}$ globally invariant under complex conjugation. Furthermore, $B$ is $\mathrm{Ad}(u(\sigma))$-invariant, and $\sigma$ acts on the canonical unitary $z \to z$ in $1_{J_1} \otimes C(\mathbf{F})$ by sending it into $z \to \bar{z}$. There exist matrix units $e_{ij}^1$ for $M_{J_1} \otimes 1$ and $e_{ij}^k$ for $M_{J_k}$ such that*

$$\sigma(e_{ij}^k) = e_{J_k - i - 1, J_k - j - 1}^k \tag{6.4}$$

*for $k = 1, 2, 3, 4$, and*

$$\sigma(e_{ij}^k) = e_{J_k - i - 1, J_k - j - 1}^l \tag{6.5}$$

*for $\{k, l\} = \{5, 6\}$.*

Before proving the Theorem we state a Corollary.

**Corollary 6.2.** *Assume that $\theta$ is irrational. Given $\varepsilon > 0$ and elements $x_1, \ldots, x_n \in C(\mathbf{T})$ there exists a subalgebra $A$ of $C_\theta$ with the same unit as $C_\theta$ such that*

$$u(\sigma) \in A \tag{6.6}$$

*and there exist elements $y_1, \ldots, y_N$ in $A$ with*

$$\|y_i - x_i\| < \varepsilon \tag{6.7}$$

*and a unitary $u' \in A$ with*

$$\|u(\alpha) - u'\| < \varepsilon, \tag{6.8}$$

*and $A$ has the form*

$$A = B_0 \oplus M_{J_2} \oplus M_{J_2} \oplus M_{J_3} \oplus M_{J_3} \oplus M_{J_4} \oplus M_{J_4} \oplus M_{2J_5}, \tag{6.9}$$

*where*

$$B_0 = \{x \in C(G, M_{2J_1}); \quad x(-1)E = Ex(-1), x(+1)E = Ex(+1)\}. \tag{6.10}$$

*Here, $E$ is a projection in $M_{2J_1}$, of dimension $J_1$, and $G$ is a closed subset of $[-1, 1]$ (when $G \not\ni -1$ (respectively $+1$), the condition $x(-1)E = Ex(-1)$ (respectively $x(+1)E = Ex(+1)$) is vacuous.)*

*Proof of Corollary 6.2.* As $\mathrm{Ad}(u(\sigma))$ acts on the finite-dimensional algebra $B$ as in Theorem 6.1, and $u(\sigma)$ is a self-adjoint unitary, it is clear that the algebra $A$ generated by $u(\sigma)$ and $B$ is isomorphic to a quotient of $B \times_\sigma \mathbf{Z}_2$. Since $\mathrm{Ad}(u(\sigma))$ restricted to the subfactors $M_{J_1} \otimes 1$, $M_{J_2}$, $M_{J_3}$, and $M_{J_4}$ leaves these factors invariant and is inner, it is clear that the corresponding components of the crossed product are $M_{J_1} \otimes 1 \oplus M_{J_1} \otimes 1$, $M_{J_2} \oplus M_{J_2}$, $M_{J_3} \oplus M_{J_3}$, and $M_{J_4} \oplus M_{J_4}$, and hence, by counting dimensions, all we have to show to prove that the corresponding components of $A$ are isomorphic to these is that the corresponding components of $u(\sigma)$ are not contained in the matrix algebra. But it follows from (6.4) that

$$\mathrm{Ad}(u(\sigma))|_{M_{J_k}} = \mathrm{Ad}\begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{pmatrix} \equiv \mathrm{Ad}(u_k). \tag{6.11}$$

But since $\sigma$ reverses the orientation of $\mathbf{T}$, it follows easily from the proof of Theorem 6.1 that if $p$ is a minimal projection in $M_{J_k}$ and $\sigma(p) = q$, then there are projections $p_1$, $p_2$ in $C_\theta$ such that $p_1 p_2 = 0$, $p_1 + p_2 \leq p$ and such that

$$\mathrm{Ad}(u_k)(p_1) = \sigma(p_2), \quad \mathrm{Ad}(u_k)(p_2) = \sigma(p_1).$$

Thus $u(\sigma)$, cut down by the central projection corresponding to $M_{J_k}$, is not a scalar multiple of $u_k$.

Next, as $\sigma$ switches $M_{J_5}$ and $M_{J_6}$, the algebra generated by $M_{J_5} \oplus M_{J_6}$ and the corresponding component of $u(\sigma)$ is equal to the simple crossed product $M_{2J_5}$. The assertion concerning $B_0$ is proved e.g. in [BEEK 3]. The closed set $G$ is the orbit space of $\mathbf{T}$ under the flip $z \rightarrow \bar{z}$; that is, $G$ is the projection of $F$ into the real axis.

*Proof of Theorem 6.1.* The proof closely mimics the proof of Theorem 1.1 in [BEK]. First, we choose one $N$ such that the given elements $x_1, \ldots, x_n$ almost lie in

the algebra $A_0$ of Lemma 5.1. Actually, to ensure that $x_1, \ldots, x_n$ still are approximately contained in the modification $z A_0 z^*$ of $A_0$ introduced later in (6.44), we must choose $N$ so large that $x_1, \ldots, x_n$ have small variation over the sets $\alpha^k(Y)$ and $\alpha^{-k}\sigma(Y)$ for $k = 0, \ldots, M$. Inspection of the proof of Lemma 4.1 shows that $x_1, \ldots, x_n$ can be approximated by linear combinations of the projections $P_{\alpha^k(Y)}$, $P_{\alpha^{-k}\sigma(Y)}$ for $k = 0, \ldots, M$ together with the $P_I$'s corresponding to the remaining floors $I$. Further inspection of the proof of Lemma 4.1 shows that the approximation is uniform in the choice of $\Delta$ in (3.2) and $\delta$ in (4.1); that is, replacing $\delta$ by a smaller $\delta$ we keep the estimate, for the given $N$.

Now, for the moment, consider the sets

$$Y_1 = [l\theta, k\theta],$$
$$Y_2 = [(-k+1)\theta, (-N+1)\theta], \qquad (6.12)$$
$$Y_3 = [(-N+1)\theta, (-l+1)\theta],$$

which are the basements in the original tower construction in Lemma 2.1. By [BEK, Lemmas 1.7 and 1.8], if $Y_i$ is a basement such that one of the $\alpha\sigma$-fixed points $\theta/2$ or $(\theta+1)/2$ lies in the tower over $Y_i$, then the tower over $Y_i$ has an even height $J_i$, and $Y_i$ contains three mutually disjoint intervals $A$, $B$, $C$ such that

$$\alpha^{J_i-1}(A) = \sigma(A), \qquad (6.13)$$

$$\alpha^{J_i-1}(B) = \sigma(C), \qquad (6.14)$$

$$\alpha^{J_i-1}(C) = \sigma(B), \qquad (6.15)$$

and if $k$ is the smallest positive integer such that $\alpha^k\sigma(A) \cap Y_i \neq \emptyset$, then

$$B = \alpha^k\sigma(A), \qquad (6.16)$$

and if $0 \leq j < k$ then

$$A \cap \alpha^j(A) = \emptyset. \qquad (6.17)$$

Now, choose on $N'$ so large that if $k', l' \in \{-(N'-1), \ldots, N'-1\}$ are such that $k'\theta$ is the point in $\{-(N'-1)\theta, \ldots, (N'-1)\theta\}$ which is closest to $N'\theta$ from above and $l'\theta$ the point which is closest to $N'\theta$ from below, then the interval $[l'\theta, k'\theta]$ is contained in the interior of $A$, above. Redefining $A$ as

$$A := [l'\theta, k'\theta] \qquad (6.18)$$

and

$$B := \alpha^k\sigma(A), \qquad C := \sigma\alpha^{J_i-1}(B), \qquad (6.19)$$

we see that $A$, $B$, $C$ still has the properties (6.14)–(6.17) above; the only problem is property (6.13). To ensure this property, we must examine the proof of Lemma 1.8 in [BEK] more closely. We see that $Y_1$ has a $\sigma\alpha^{J_1-1}$-fixed point $\omega$, which in our concrete setting has to be $\dfrac{1-J_i}{2}\theta$ or $\dfrac{1-J_i}{2}\theta + \dfrac{1}{2}$, and $A$ is taken to be a small $\sigma\alpha^{J_i-1}$-invariant neighborhood of $\omega$ in $Y$. Hence, in order that $[l'\theta, k'\theta]$ shall be $\sigma\alpha^{J_1-1}$-invariant, we must choose $N'$ so that $N'\theta$ is very close to the fixed point $\omega$. For this, let us show the following elementary lemma:

**Lemma 6.3.** *For* $n = 1, 2, 3, \ldots$ *let* $N_n$ *be the* $n^{\text{th}}$ *nonzero integer with the property that* $N_n\theta$ *is strictly closer to* $\omega$ *than any* $k\theta$ *with* $|k| < |N_n|$. *It follows that there exists an*

$n_0 > 0$ *such that if* $n \geq n_0$, *then* $N_n > 0$ *and if* $k', l' \in \{-(N_n-1), \ldots, (N_n-1)\}$ *are such that* $k'\theta$ *is the point in* $\{-(N_n-1)\theta, \ldots, (N_n-1)\theta\}$ *which is closest to* $N_n\theta$ *from above and* $l'\theta$ *the point which is closest to* $N_n\theta$ *from below, then*

$$\{k', l'\} = \{N_{n-1}, -N_{n-1} - J_i + 1\}, \tag{6.20}$$

*as sets. As a consequence,*

$$\sigma\alpha^{J_i-1}[l'\theta, k'\theta] = [l'\theta, k'\theta]. \tag{6.21}$$

*Proof.* Note that as $\sigma\alpha^{J_i-1}\omega = \omega$, the two points

$$k\theta, (-k - J_i + 1)\theta,$$

which are conjugate under $\sigma\alpha^{J_i-1}$, have the same distance to $\omega$. Thus, if $k$ is an integer with $|k| > J_i - 1$ and $k$ is negative, then $(-k - J_i + 1)$ is a positive integer with smaller absolute value than $k$ such that $(-k - J_i + 1)\theta$ has the same distance to $\omega$ as $k\theta$. Thus, $N_n > 0$ when $|N_n| > J_i - 1$.

Let $\varepsilon > 0$ be such that if $I$ is any interval of length $\varepsilon$, then the translates $\alpha^k I$, with $|k| \leq J_i$, are all disjoint. Choose $n_0$ so large that

$$N_{n_0-1} > J_i - 1 \tag{6.22}$$

and

$$|N_{n_0-1}\theta - \omega| < \varepsilon/4. \tag{6.23}$$

Then, if $n \geq n_0 - 1$, the translates

$$\alpha^k[N_n\theta - \varepsilon/2, N_n\theta + \varepsilon/2]$$

for $|k| \leq J_i$ are all disjoint, and it follows that

$$N_{n+1} > N_n + J_i.$$

Thus, if $n \geq n_0$, then both the points

$$N_{n-1}\theta, (-N_{n-1} - J_i + 1)\theta$$

lie in the set

$$\{-(N_n-1)\theta, \ldots, (N_n-1)\theta\}$$

and also these two points are conjugate under $\sigma\alpha^{J_i-1}$. It is then clear from the definition of $N_n$, $k'$, $l'$ that

$$\{k', l'\} = \{N_{n-1}, -N_{n-1} - J_i + 1\}. \tag{6.20}$$

This ends the proof of Lemma 6.3.

By Lemma 6.3, we may redefine $A$ as in (6.18), and still retain all the properties (6.13)–(6.17). Following [BEK, (1.33)] we now define

$$X = A \cup \alpha\sigma(A). \tag{6.24}$$

The Putnam tower construction over $X$ is then exactly like the construction over $Y$ described in Lemma 2.1; we have just replaced $N$, $l$, $k$ by $N'$, $l'$, $k'$. Also, as $N' > N$, the partition of **T** defined by the new tower construction is finer than the previous one. Now choose the $\Delta$ in (3.2) so that

$$0 < \Delta\theta \leq \tfrac{1}{4}\min\{(-l'-1)\theta - (-N'+1)\theta, (-N'+1)\theta - (-k'+l)\theta\} \tag{6.25}$$

and the $\delta$ in (4.1) so that

$$\delta < \varDelta\theta \bmod 1, \tag{6.26}$$

with the new $\varDelta$. We will, furthermore, assume that $\varDelta\theta$ and the interval $A$ are chosen so small that when $X$ is modified from (2.7) to (3.1), then the resulting new $A$ is still contained in one of the three wide basements of the new $Y$, and the properties (6.13)–(6.17) still hold for the modified $A$ inside the modified $Y$-towers. Actually, when referring to $X$, $Y$ from now on, we shall refer to the modified basements in the tower construction (3.1) rather than the original basements in (2.7).

Use the new values of $\delta$, $\varDelta$ when defining $A_0$ from the towers over $Y$, and define another finite-dimensional subalgebra $A_1$ of $C_\theta$ by using the tower construction over $X$ and the same values of $\varDelta$, $\delta$. Since the partition of **T** defined by the towers over $X$ is a refinement of the partition defined by the towers over $Y$, it is clear from Definition 4.5 that the $p_I$'s for the intervals in the $Y$-partition are sums of $p_I$'s for the intervals in the $X$-partition, and using Lemma 5.1 it is then clear that

$$A_0 \subseteq A_1. \tag{6.27}$$

From now on, we follow [BEK, Sect. 1] closely, just replacing $\chi_I$ by $p_I$ for all intervals $I$. So, define

$$
\begin{aligned}
v_0 &= \sum_{k=1}^{6} \sum_{i=0}^{J_i-2} e_{i+1\,i}^k + \sum_{k=1}^{6} e_{0\,J_k-1}^k \\
&= u(\alpha) P_{\Omega\setminus\sigma(Y)} + \sum_{k=1}^{6} u(\alpha)^{1-J_k} P_{\alpha^{J_k-1}(Y_k)} \\
&= u(\alpha) P_{\Omega\setminus\sigma(Y)} + \sum_{k=1}^{6} P_{Y_k} u(\alpha)^{1-J_k}
\end{aligned}
\tag{6.28}
$$

and

$$u_0 = u(\alpha) v_0^* = P_{\Omega\setminus Y} + \sum_{k=1}^{6} u(\alpha)^{J_k} P_{Y_k}. \tag{6.29}$$

Then, define $v_1$, $u_1$ correspondingly from the towers over $X$, and verify

$$\mathrm{Ad}(v_0 u(\sigma))(v_1 v_0^*) = (v_1 v_0^*)^* \tag{6.30}$$

as in [BEK, Lemma 1.9]. If $X = X_1 \cup X_2 \cup X_3 \cup X_4 \cup X_5 \cup X_6$ is the partition of $X$ defined by the new tower construction, then for any $k$ such that $\sigma$ maps the tower over $X_k$ into itself, i.e. for $k = 1, 2, 3, 4$, the number of floors in this tower contained in $Y$ is odd, and hence the restriction of $v_1 v_0^*$ to the corresponding central projection in $A_1$ has odd order, see [BEK, Lemma 1.10]. Consequently there exists a unitary operator $w \in A_1$ such that

$$wP_{\Omega\setminus Y} = P_{\Omega\setminus Y}, \tag{6.31}$$

$$w^{2M} = v_1 v_0^*, \tag{6.32}$$

$$\mathrm{Ad}(v_0 u(\sigma))(w) = w^*, \tag{6.33}$$

$$\|1 - w\| \leq \pi/2M; \tag{6.34}$$

see [BEK, Lemma 1.11], and a unitary operator $u \in A_1$ such that

$$uP_{\Omega \setminus Y} = P_{\Omega \setminus Y}, \tag{6.35}$$

$$u^M P_{Y_i} u^{-M} \geqq w^{-M} P_X w^M, \tag{6.36}$$

$$\mathrm{Ad}(v_0 u(\sigma))(u) = u, \tag{6.37}$$

$$\|1 - u\| \leqq \pi/M; \tag{6.38}$$

see [BEK, Lemma 1.12]. Now, defining a unitary operator $z$ in $A_1$ by

$$z = \sum_{k=0}^{M} v_0^k w^{M-k} u^{M-k} v_0^{-k} P_{\alpha^k(Y)} + \sum_{k=0}^{M} u(\sigma) v_0^k w^{M-k} u^{M-k} v_0^{-k} u(\sigma) P_{\alpha^{-k}\sigma(Y)}$$

$$+ P_{\Omega \setminus \left\{ \bigcup_{k=0}^{M} \alpha^k(Y) \cup \left( \bigcup_{k=0}^{M} \alpha^{-k}\sigma(Y) \right) \right\}} \tag{6.39}$$

one verifies that

$$z P_{Y_i} z^* \geqq P_X, \tag{6.40}$$

$$z u(\sigma) = u(\sigma) z, \tag{6.41}$$

$$z v_0 z^* v_0^* P_Y = v_1 v_0^* P_Y, \tag{6.42}$$

$$\|z v_0 z^* - v_1\| \leqq 3\pi/2M; \tag{6.43}$$

see [BEK, Lemma 1.13].

Now, define

$$B = C^*(z A_0 z^*, u_1), \tag{6.44}$$

where we recall that

$$u_1 = u(\alpha) v_1^*. \tag{6.45}$$

We will verify that $B$ has the properties in Theorem 6.1. First, note that as $z$ commutes with the projections

$$P_{\alpha^k(Y)}, P_{\alpha^{-k}\sigma(Y)}, \quad k = 0, \ldots, M,$$

as well as with the subprojections in $A_0$ of

$$P_{\Omega \setminus \left\{ \bigcup_{k=0}^{M} \alpha^k(Y) \cup \left( \bigcup_{k=0}^{M} \alpha^{-k}\sigma(Y) \right) \right\}},$$

it follows that all of these projections belong to $z A_0 z^*$. Since the diameter of the set $Y$ can be chosen arbitrarily small at the outset, it follows from Lemma 4.1 and its proof that for given $\varepsilon > 0$ and elements $x_1, \ldots, x_n \in C(\mathbf{T})$, for $N$ large enough there exists elements $y_1, \ldots, y_n \in z A_0 z^*$ with

$$\|y_i - x_i\| < \varepsilon, \quad i = 1, \ldots, n.$$

This is (6.1).

Next, as $u_1 \in B$ and $v_0 \in A_0$, we have

$$u' = u_1 z v_0 z^* \in B, \tag{6.46}$$

and as $u(\alpha) = u_1 v_1$ we have

$$\|u' - u(\alpha)\| = \|z v_0 z^* - v_1\| \leqq 3\pi/2M \tag{6.47}$$

by (6.43). Thus, if $M$ is chosen large enough, $u(\alpha)$ is approximately contained in $B$, which is (6.2). The proof of the remaining statements of Theorem 6.1 is almost identical to the end of the proof of Theorem 1.1 in [BEK]. In particular, the partial unitary

$$V = \sum_{k=0}^{J_i-1} (zv_0^k P_{Y_i} z^*) u_1 (zP_{Y_i} v_0^{-k} z^*)$$

$$= \sum_{k=0}^{J_i-1} (ze_{k0}^i z^*) u_1 (ze_{0k}^i z^*)$$

with support $z \left( \sum_{k=0}^{J_i-1} e_{kk}^i \right) z^*$ is the canonical generator of the $C(F)$-part of $B$ in (6.3); that is, $F$ is the spectrum of this partial unitary. As

$$ze_{00}^i z^* = zP_{Y_i} z^* \geqq P_X$$

by (6.40), and $u_1$ acts as the identity on $P_{\Omega\backslash X}$ by (6.29), it follows that

$$F \equiv \mathrm{Sp}\, V = \mathrm{Sp}(u_1).$$

In [Put] and [BEK], one now used the fact that $u_1$ was contained in the same $K_1$-class as $u(\alpha)$, which is non-trivial in $C(\mathbf{T}) \times_\alpha \mathbf{Z}$, to conclude that $F = \mathbf{T}$. However, in the present case the definition of the projections $P_{X_k}$ and thus of $u_1$ involves the operator $u(\sigma)$, and so $u_1 \notin C(\mathbf{T}) \times_\alpha \mathbf{Z}$. Therefore we cannot conclude from this argument that $F \doteq \mathbf{T}$ in our case. In the previous case one could also conclude that $\mathrm{Sp}(u_1) = \mathbf{T}$ by observing that $u_1$ is the unitary on $L^2(X)$ which is defined by the return map on $X$, which is minimal as a map on the discretization of $X$ obtained by cutting at all points on the orbit $\mathbf{Z}\theta$. We have not been able to turn this into an argument that the present $u_1$ has full spectrum.

## 7. Basic Building Blocks

In order to prove from Corollary 6.2 that $C_\theta$ is an AF algebra, we will replace $B_0$ with a "large" subalgebra which is easier to describe in terms of a certain number of subalgebras which are defined as follows:

$$C_{n,k} = M_n \otimes C_k, \quad k = -1, 0, 1, 2, \quad n = 1, 2, \ldots, \tag{7.1}$$

where

$$C_{-1} = \mathbf{C}, \tag{7.2}$$

$$C_0 = C([-1,1]) \tag{7.3}$$

= the universal $C^*$-algebra generated by an $x = x^*$ with $-1 \leqq x \leqq 1$,

$$C_1 = \{f \in C([0,1], M_2); f(0) \in \mathbf{C} \oplus \mathbf{C}\} \tag{7.4}$$

= the universal $C^*$-algebra generated by $x, v$ satisfying $x = x^*$, $-1 \leqq x \leqq 1$, $v = v^*$, $v^2 = 1$, $vxv = -x$, and

$$C_2 = \{f \in C([-1,1], M_2); f(-1) \in \mathbf{C} \oplus \mathbf{C} \text{ and } f(1) \in \mathbf{C} \oplus \mathbf{C}\} \tag{7.5}$$

= the universal algebra generated by $u, v$ satisfying $v = v^*$, $v^2 = 1$, $uu^* = u^*u = 1$, $vuv = u^*$.

The statements about $C_1$ and $C_2$ follow from the fact that the crossed product of $C([-1,1])$ by the flip $(\sigma f)(x) = f(-x)$ is just $C_1$ and the crossed product of $C(\mathbf{T})$ by the flip $(\sigma f)(z) = f(\bar{z})$ is just $C_2$. The embedding of $x$, $v$ into $C_1$ is given by

$$x : t \in [0,1] \rightarrow \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}, \tag{7.6}$$

$$v : t \in [0,1] \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{7.7}$$

$\left(\text{and then } \mathbf{C} \oplus \mathbf{C} \text{ is skewly embedded into } M_2 \text{ as the two eigensubspaces of } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$, and the embedding of $u, v$ into $C_2$ is given by

$$u : t \in [-1,1] \rightarrow \begin{pmatrix} t + i\sqrt{1-t^2} & 0 \\ 0 & t - i\sqrt{1-t^2} \end{pmatrix}, \tag{7.8}$$

$$v : t \in [-1,1] \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{7.9}$$

Recall from [Kum 2] that $C_2$ can also be characterized as the universal $C^*$-algebra generated by two self-adjoint unitaries $v_1$ and $v_2$. The connection with the other characterization is $v = v_1$, $u = v_1 v_2$.

We call the following elements the canonical generators for $C_k$:

$$C_{-1} : 1, \tag{7.10}$$

$$C_0 : x, 1, \tag{7.11}$$

$$C_1 : x, v, 1, \tag{7.12}$$

$$C_2 : u, v, 1. \tag{7.13}$$

Thus, $C_{n,k}$ is the universal $C^*$-algebra generated by elements $e_{ij}$, $i, j = 1, \ldots, n$ satisfying

$$e_{ij}^* = e_{ji}, \quad e_{ij} e_{kl} = \delta_{jk} e_{il}, \quad \sum_{i=1}^{n} e_{ii} = 1, \tag{7.14}$$

together with the canonical generators of $C_k$, and the latter are assumed to commute with the $e_{ij}$'s. We will call $e_{ij}$, together with the canonical generators of $C_k$, the canonical generators of $C_{n,k}$.

We are now ready for the reformulation of Corollary 6.2.

**Corollary 7.1.** *Assume that $\theta$ is irrational. Given $\varepsilon > 0$ and elements $x_1, \ldots, x_n \in C_\theta$ there exists a subalgebra $A$ of $C_\theta$ with the same unit as $C_\theta$ such that $A$ is a finite direct sum of basic building blocks $C_{n,k}$, and elements $y_1, \ldots, y_n \in A$ such that*

$$\|x_i - y_i\| < \varepsilon, \quad i = 1, \ldots, m. \tag{7.15}$$

*Furthermore, if one of the basic building blocks $C_{n,0}$ or $C_{n,1}$ occurs in $A$, then $C_{n,2}$ does not occur, and in that case there is an positive integer $J$ such that the $C_{n,0}$'s occurring are all $C_{2J,0}$ and the $C_{n,1}$'s occurring are all $C_{J,1}$. In any case $C_{n,1}$'s occur at most twice and $C_{n,2}$'s at most once.*

*Proof.* Referring to Corollary 6.2, it is clear that any finite subset of the algebra $B_0$ occurring there can be approximated by elements in a subalgebra of $B_0$ of the form described in the present corollary, by dividing $G$ into sufficiently small clopen subsets.

Our next aim is to show that any separable $C^*$-algebra with the approximation property of Corollary 7.1 is in fact an inductive limit of finite direct sums of basic building blocks.

**Theorem 7.2.** *Let $A$ be a unital separable $C^*$-algebra, and assume that for any $\varepsilon > 0$, and any finite number $x_1, ..., x_n$ of elements in $A$ there exists a $C^*$-subalgebra $B$ of $A$ with the same unit as $A$, such that $B$ is isomorphic to a finite direct sum of basic building blocks $C_{n,k}$, and there exist elements $y_1, ..., y_n \in B$ with $\|y_i - x_i\| < \varepsilon$ for $i = 1, ..., n$. Then $A$ is an inductive limit of a sequence*

$$A_1 \to A_2 \to A_3 \to ...,$$

*where each $A_k$ is a finite direct sum of basic building blocks.*

The proof of Theorem 7.2 is patterned on the proof of Theorem 2.1 in [BEK], and thus on the proofs in [Bra, Gli]. First we establish the following lemma.

**Lemma 7.3.** *Let $A$ be a unital $C^*$-algebra and $B$ a $C^*$-subalgebra of $A$ with the same unit as $A$ such that $B$ is a direct sum of basic building blocks, and let $x_1, ..., x_m \in B$.*

*It follows that for any $\varepsilon > 0$ there exists a $\delta > 0$ (depending on $B$ and $x_1, ..., x_m$) such that for any $C^*$-subalgebra $C$ of $A$ with the property that the distance of each of the generators of each of the basic building blocks of $B$ from $C$ is less that $\delta$, there exists a morphism $\phi : B \to C$ with*

$$\|\phi(x_i) - x_i\| \leqq \varepsilon \|x_i\| \tag{7.16}$$

*for $i = 1, ..., m$.*

*Proof.* The proof of this lemma is almost identical to the proof of Lemma 2.3 in [BEK] or to Lemma 4.2 in [Ell]. In either case the idea is that the relations of the generators defining $B$ is stable in the sense that if one has a set of elements in $C$ which approximately satisfy the relations, then they can be perturbed by a small amount to exactly satisfy the relations. We give an outline of the argument:

The first step is to approximate $x_1, ..., x_n$ by polynomials in the generators of the basic building blocks for $B$. This done, it is clear that if we have estimates like (7.16) for the canonical generators, with a smaller $\varepsilon$, we have the estimates (7.16) themselves. So assume that $\delta$ has been chosen small. If

$$B = \sum_{i=1}^{I} {}^{\oplus} C_{n_i, k_i} = \sum_{i=1}^{I} {}^{\oplus} M_{n_i} \otimes C_{k_i},$$

where the sum is finite, consider the finite dimensional subalgebra

$$B_0 = \sum_{i=1}^{I} {}^{\oplus} M_{n_i} \otimes 1,$$

and let $e_{jl}^{n_i}$ be a complete set of matrix units for $B_0$. By [Gli, Lemma 1.10] or [Bra, Lemma 2.1] there exists a set of matrix units $f_{jl}^{n_i}$ in $C$ such that $e_{jl}^{n_i}$ is close to $f_{jl}^{n_i}$ for each $n_i, j, l$, and these matrix units span a subalgebra $C_0$ of $C$ which is isomorphic to $B_0$. By integrating $\mathrm{Ad}(u)$ over $u$ in the unitary group of $C_0$, it is clear that we can approximate the $x, u, v$-generators by elements in the relative commutant $C_0' \cap C$ of

$C_0$ in $C$, and by cutting these down by the central projections $f^{n_i} = \sum_j f_{jj}^{n_i}$ in $C_0$, we may also assume that the approximants sit inside the appropriate central projection. Hence, by universality of the algebras $C_0$, $C_1$, $C_2$ the problem of defining $\phi$ boils down to showing that if the relations defining these algebras are approximately verified by some elements, a small perturbation of these elements will exactly verify the relations. For $C_0$ this is trivial, for $C_2$ the argument is essentially given in the proof of Lemma 2.3 in [BEK], so let's do $C_1$: Assume that we have the approximate relations

$$x \cong x^*, \ \|x\| \stackrel{<}{\cong} 1, \ v \cong v^*, \ v^2 \cong 1 \quad \text{and} \quad vxv \cong -x.$$

First take the self-adjoint part of $v$ and modify it by spectral theory so that $v = v^*$ and $v^2 = 1$. Then take the self-adjoint part of $x$ and modify $x$ by spectral theory so that $x = x^*$ and $\|x\| \leq 1$. Then, as the new $v$, $x$ are close to the old ones, $vxv \cong -x$ even after modification. Hence the element $\frac{1}{2}(x - vxv)$ is close to $x$, and replacing $x$ by this latter element we exactly obtain $vxv = -x$.

This ends the proof of Lemma 7.3.

*Proof of Theorem 7.2.* The proof of Theorem 7.2 from Lemma 7.3 is now almost a word-for-word rendering of the proof of Theorem 2.1 in [BEK] from Lemma 2.3 there, with the difference that the morphisms in the inductive system are no longer necessarily injective. Apart from Lemma 7.3, the only input in the proof is separability. A similar proof is the proof of Theorem 4.3 from Lemma 4.2 in [Ell].

**Corollary 7.4.** *Assume that $\theta$ is irrational. Then the algebra $C_\theta$ is the inductive limit of a sequence of algebras which are finite direct sums of basic building blocks $C_{n,k}$. Furthermore, there are the same restrictions on the basic building blocks actually occurring in one of the algebras in the sequence as in the concluding remarks of Corollary 7.1.*

*Proof.* This is clear from Corollary 7.1 and Theorem 7.2, and the proof of Theorem 7.2.

## 8. Small Eigenvalue Variation

In this section we will prove Theorem 1.1 by combining techniques from [BBEK] and [Su]. Actually, Theorem 1.1 follows from the following theorem in conjunction with Corollary 7.4.

**Theorem 8.1.** *Let $C$ be a simple unital $C^*$-algebra with a unique trace state, and assume that $C$ is the inductive limit of a sequence of algebras which are finite direct sums of basic building blocks $C_{n,k}$. It follows that $C$ is an AF-algebra.*

*Proof.* Our basic building blocks are a subclass of the basic building blocks considered in [Su], which are $C^*$-subalgebras of $C(\Omega, M_n)$, where $\Omega$ is a finite connected graph such that the subalgebra has diagonal block form at some vertices in $\Omega$. It is proved in [Su], Theorem 1 that if $C$ has real rank zero, then $K_*(C)$ with the graded dimension range is a complete invariant for $C$. For our special basic building blocks, $K_1 = 0$, and hence it follows from Su's classification that our algebras are AF if they have real rank zero. To prove that $C$ has real rank zero, we just copy the proof of $1 \Rightarrow 5$ in Theorem 1.3 of [BBEK], where the same thing is proved in the case that the basic building blocks are full homogeneous

algebras over spaces of dimension at most 2; that is, one first establishes small eigenvalue variation and then proves that $C$ has real rank zero. We omit the details, but would also like to remark that one could prove directly that $C$ is an AF-algebra from small eigenvalue variation by essentially the same argument as in [BEK].

This argument also occurs in [Ell 2].

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