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$\bar{\partial}$-Torsion, Foliations and Holomorphic Vector Fields

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Abstract. In this paper, we investigate the relation between $\bar{\partial}$-torsion and holomorphic vector fields. We consider complex manifolds which fibrate over the torus having a transverse one dimensional holomorphic foliation. The torsion of the total space is then computed in terms of compact leaves. This can be interpreted as a Lefschetz formula for flows of holomorphic vector fields.

1. Introduction

In analogy of recent work of Fried [Fr 3–5], Ruelle [Ru] and older work of Ray and Singer [RS], we will relate topological invariants of complex manifolds to the closed orbits of vector fields or closed leaves of a foliation. We will consider fibrations of compact complex manifolds M

$$\begin{array}{ccc} F & \longrightarrow & M \\ & & \downarrow{\scriptstyle \pi} \\ & & B, \end{array} \tag{1.1}$$

where $B = \mathbb{C}/\Gamma$, $\Gamma = \mathbb{Z} + \tau\mathbb{Z}$ with $\operatorname{Im}\tau > 0$ together with a complex one dimensional holomorphic foliation $\mathscr{F}$ transverse to the fibration. When M is Kähler, Crew and Fried [CF] showed that the existence of a non-vanishing holomorphic vector field on M gives rise to such a foliated fibration. The leaves of $\mathscr{F}$ are then spanned by the flow of the vector field. It should be stressed that in this case, the projection map is not necessarily holomorphic. We will restrict ourselves to the case where π is holomorphically locally trivial.

The invariants one consider are ratios of $\bar{\partial}$-torsion (see [RS]). Given two flat acyclic bundles $\mathscr{E}_1$ and $\mathscr{E}_2$ over B of the same rank, one can construct natural metric invariants of M, namely the ratios

$$\frac{T_p(M, \pi^*\mathscr{E}_1)}{T_p(M, \pi^*\mathscr{E}_2)}, \qquad p = 0, \ldots, \dim_{\mathbb{C}} M, \tag{1.2}$$

where by $T_p(M,E)$ one denotes the p^{th} $\bar{\partial}$-Ray-Singer torsion [RS]. In a previous work [La], using the adiabatic limit techniques and work of Dai [D], Dai, Epstein, and Melrose [DEM], gave an explicit formula for (1.2) in terms of the torsion of B with coefficients in $\mathscr{E}_i \otimes H^{p,q}F$, where $H^{p,q}F$ denotes the cohomology bundle of F over B.

For each closed leaf of the foliation, one can construct a theta-like function which will play the role of the Ruelle Zeta functions (see [Fr 3–5, Ru]). The product of such functions over closed leaves will yield (1.2). The conditions under which this can be done are described in the next paragraph. It should be stressed that the torsion cannot be expressed in terms of local quantities (see [RS], Rosenberg [Ro]). It is therefore an interesting fact that it can be computed in terms of semilocal quantities, namely the closed leaves of a foliation.

In terms of dynamics, this correspondence can be interpreted as a Lefschetz formula for flows (see [Fr 3]). Given such a fibration and a holomorphic vector field spanning the transverse foliation, the ratio of torsions acts for closed orbits in much the same way as the Euler characteristic in the traditional Lefschetz formula.

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2. Main Results

2.1. Preliminaries. Associated to the foliated fibration M, there is a natural holomorphic $\mathbb{Z}^2$ action on the fiber F which will be denoted by

$$\varphi^z_{n,m}: F_z \to F_z, \qquad n, m \in \mathbb{Z}, \tag{2.1}$$

where $z \in B$ and F_z denotes the fiber above z. These maps are naturally induced as the time $n + m\tau$ maps of the holomorphic flow of π^*1. We will consider two cases for this action. In the first case, we will require that for any $z \in B$ and for all $n, m \in \mathbb{Z}$, $n^2 + m^2 \neq 0$,

$$\varphi^z_{n,m} \text{ has isolated fixed points.} \tag{2.2a}$$

In order for several of the quantities that will be defined to exist, one also needs a non-degeneracy condition on the fixed points. This condition is rather technical and arises from the presence of small divisors. We require that there exists a constant $C > 0$ and an positive integer $r > 2$ such that for any $n, m \in \mathbb{Z}$, $n^2 + m^2 > 0$, given a fixed point x of $\varphi^z_{n,m}$, the eigenvalues λ of $\partial\varphi^z_{n,m}(x)$ satisfy

$$|1 - \lambda^k| \geqq \frac{C}{|k|^r}. \tag{2.2b}$$

Where $\lambda = e^{2\pi\theta}$, (2.2b) is just a usual diophantine condition on θ. It is known (see Hardy and Wright [HW]) that for fixed C the set of θ not satisfying this condition has measure of order C. It is thus easy to see that (2.2b) is not satisfied on a set of measure vanishing as C^2. This condition is definitely more restrictive than the requirement of non-degenerate fixed points and will allow us to compute the Atiyah-Bott [AB] indices of periodic points. We will also consider degenerate $\mathbb{Z}^2$

action where the maps

$$\varphi^z_{n,0} = \mathrm{Id}, \qquad n \in \mathbb{Z}. \tag{2.2c}$$

In this case, we require that 2.2a, b) hold for $m \neq 0$.

We briefly recall the definition of the p-Atiyah-Bott index of a point $x \in F$ for a holomorphic map $f: F \to F$ having isolated fixed points. It is defined to be 0, if x is not a fixed point of f and for a fixed point as

$$\mathrm{ind}^p(x, f) = \sum_{q=0}^{\dim_{\mathbb{C}} F} (-1)^q \frac{\mathrm{tr}\, \Lambda^{p,q} \partial f(x)}{\det_{\mathbb{C}}(I - \partial f(x))} \tag{2.3}$$

for $p = 0, \ldots, \dim_{\mathbb{C}} F$. The p^{th} Lefschetz index of f is defined to be

$$\mathscr{I}^p(f) = \sum_{q=0}^{\dim_{\mathbb{C}} F} (-1)^q \mathrm{tr}(H^{p,q}(f)) = \sum \mathrm{ind}^p(x, f), \tag{2.4}$$

where $H^{p,q}(f)$ is the map induced by f on the Dolbeault cohomology of F. We now define the index of a compact leaf.

Definition 2.1. Let $\mathscr{L}$ be a compact leaf of the foliation. When (2.2a, b) hold, we define the index of the leaf as

$$\mathrm{ind}^p_z(\mathscr{L}, \varphi^z_{n,m}) = \sum \mathrm{ind}^p(x, \varphi^z_{n,m}), \tag{2.5}$$

where $p = 0, \ldots, \dim_{\mathbb{C}} F$, $z \in B$ and where the sum runs over all points of $\mathscr{L} \cap F_z$. When the $\mathbb{Z}^2$ action is degenerate, i.e. if (2.2c) holds, we define the index of a compact leaf to be either equal to (2.5) if $m \neq 0$ or equal to the p-Dolbeault characteristic of F when $m = 0$.

As in the product formula for the Ray-Singer torsion (see [La], Dai, Epstein, and Melrose [DEM] or Forman [Fo]), one considers bundles on B arising from unitary representations. Let

$$\varrho: \Gamma \cong \pi_1(B) \to U(n) \tag{2.6}$$

be a representation of $\pi_1(B)$. Let $\mathscr{E}$ be the flat bundle associated to this representation. We will always assume $\mathscr{E}$ to be acyclic. We can now define the dynamical Zeta functions. These are the complex analogues of the dynamical zeta functions introduced by Selberg [Se] and later by Ruelle [Ru] and Fried [Fr 1–5].

Definition 2.2. Let $\mathscr{L}$ be a closed leaf of the foliation. For $z \in B$, $s \in \mathbb{C}$ let

$$\zeta^p_{z,\mathscr{L}}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \omega^p_{z,\mathscr{L}}(t)\, dt, \tag{2.7}$$

where

$$\omega^p_{z,\mathscr{L}}(t) = -\frac{\mathrm{Im}\,\tau}{4\pi t} \sum_{n^2+m^2>0} \mathrm{ind}^p_z(\mathscr{L}, \varphi^z_{n,m})\, \mathrm{tr}(\varrho(n+m\tau)) \exp\left(-\frac{|n+m\tau|^2}{4t}\right). \tag{2.8}$$

2.2. Main Results. We have

Theorem 2.1. *For any closed leaf $\mathscr{L}$ and any $z \in B$, $\zeta^p_{z,\mathscr{L}}$ is an entire function of s.*

Associated to these dynamical Zeta functions, we have dynamical theta functions

Definition 2.3. For a closed leaf $\mathscr{L}$ and for $z \in B$, define

$$\Theta^p_{z,\mathscr{L}}(\mathscr{E}) = \exp\left(-\frac{d}{ds}\bigg|_{s=0} \zeta^p_{z,\mathscr{L}}(s)\right), \tag{2.9a}$$

$$\Theta^p(\mathscr{E}) = \prod_{\mathscr{L}\ \text{compact}} \Theta^p_{z,\mathscr{L}}(\mathscr{E}), \tag{2.9b}$$

$$\Theta_{z,\mathscr{L}}(\mathscr{E}) = \prod_p (\Theta^p_{z,\mathscr{L}}(\mathscr{E}))^{(-1)^p}. \tag{2.9c}$$

We have

Theorem 2.2. *Under the conditions* (2.2), *for any acyclic bundle $\mathscr{E}$ defined by* (2.6), *the dynamical theta functions defined in* (2.9) *exist.*

We now briefly recall the definition of the Ray-Singer torsion for compact complex manifolds [RS]. Given a flat acyclic vector bundle E over a compact complex manifold M with metric g_M, let $\Delta_M^{p,q}$ be the $\bar{\partial}$-Laplacian on (p,q) forms over M with coefficients in E. Define the zeta-functions

$$\zeta^p(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_q (-1)^q \operatorname{tr} e^{-t\Delta_M^{p,q}} dt, \tag{2.10}$$

where $p = 0, \ldots, \dim_{\mathbb{C}} M$ and where $s \in \mathbb{C}$. Seeley [Se] showed that ζ^p is analytic for $\operatorname{Re} s$ large enough and that it possesses a meromorphic extension to $\mathbb{C}$ which is regular at $s=0$. Following Ray and Singer [RS], we define

$$T_p(M,E) = \exp \frac{d}{ds}\bigg|_{s=0} \zeta^p(s). \tag{2.11}$$

Note that this torsion is the square of that of [RS]. As noted by Fay [Fa], this definition allows for better extensions of the torsion on bundles arising from non-unitary representations. In the case of fibrated manifolds, one has the product formula

Theorem 2.3. *Let M be a compact complex manifold as in* (1.1). *Consider two bundles* $\mathscr{E}_i$, $i=1,2$ defined by (2.6) of the same rank; assume they are acyclic. Then

$$\frac{T_p(M,\pi^*\mathscr{E}_1)}{T_p(M,\pi^*\mathscr{E}_2)} = \frac{\prod_{p_1+p_2=p} \prod_j T_{p_1}(B, \mathscr{E}_1 \otimes H^{p_2,j}F)^{(-1)^j}}{\prod_{p_1+p_2=p} \prod_j T_{p_1}(B, \mathscr{E}_2 \otimes H^{p_2,j}F)^{(-1)^j}}, \tag{2.12}$$

where $H^{p,q}F$ is the cohomology bundle of F over B and where $p=0,\ldots,\dim_{\mathbb{C}} M$.

A proof of this theorem can be found in [La]. We can now state our main theorem.

Theorem 2.4. *Let M and $\mathscr{E}_i$, $i=1,2$ be as in Theorem 2.2. Assume that* (2.2) *is satisfied. Then, for any $z \in B$,*

$$\frac{T_p(M,\pi^*\mathscr{E}_1)}{T_p(M,\pi^*\mathscr{E}_2)} = \prod_{\mathscr{L}\ \text{closed}} \frac{\Theta^p_{z,\mathscr{L}}(\mathscr{E}_1)\Theta^{p-1}_{z,\mathscr{L}}(\mathscr{E}_1)}{\Theta^p_{z,\mathscr{L}}(\mathscr{E}_2)\Theta^{p-1}_{z,\mathscr{L}}(\mathscr{E}_2)}, \tag{2.13}$$

where $p=0,\ldots,\dim_{\mathbb{C}} M$. By convention we set $\Theta^{-1}_{z,\mathscr{L}} \equiv 1$.

This theorem is reminiscent of the dynamical expressions of the torsion in terms of closed orbits given by Ray and Singer [RS] or Fried [Fr 3–5]. The simplest results of this type express the torsion of hyperbolic manifolds in terms of closed orbits of the geodesic flow (see [RS, Fr 3, 4]). These results were extended by Fried [Fr 5] to general manifolds with strict conditions on the flow. The interesting fact is that (2.13) relates a topological invariant associated to M (see [RS]) to the dynamics on M. As in [Fr 2] this can be interpreted as a Lefschetz formula for flows. In the traditional Lefschetz formula, one relates a global topological invariant to the zeroes of a vector field. Here, the global invariant (the ratio of torsions) is related to semilocal quantities given by the dynamics.

Perhaps the most interesting fact is that by knowing the ratio of torsions for different bundles $\mathscr{E}_i$, $i=1,2$, one can retrieve part of the dynamics on M. This will be illustrated in the next section.

2.3. A Simple Case. In a simple case of a degenerate $\mathbb{Z}^2$ action, that is when (2.2c) is assumed, one can compute the functions given in (2.9) explicitly. These results will partly justify the name of dynamical theta functions. We consider rank 1 bundles and require that the action

$$\varphi_{n,m}=\phi^m$$

is such that ϕ has a finite number of periodic orbits. This implies in particular that the Atiyah-Bott indices are bounded provided (2.2) holds (see [Fr 6]). The simplest examples of such cases are twisted products. Let

$$M=\mathbb{C}\times F/\sim\,, \qquad (x,y)\sim(x+n+m\tau,\phi^m(y)), \tag{2.14}$$

where the map ϕ is as above. The bundles $\mathscr{E}$ we consider are given by (2.6) where

$$\varrho(n+m\tau)=\exp(2\pi i(nv+mu)). \tag{2.15}$$

It is now easy to see that compact leaves correspond to periodic orbits of ϕ. We can thus define the period of a compact leaf $\mathscr{L}$ as the smallest integer α such that

$$\phi^\alpha(x)=x\,, \qquad x\in F_z\cap\mathscr{L}\,. \tag{2.16}$$

It was shown by Fried [Fr 6] that a finite number of periodic orbits implies the periodicity of the Atiyah-Bott indices of ϕ^m. That is,

$$\mathscr{I}^p(\phi^{m+\beta})=\mathscr{I}^p(\phi^m)\,, \qquad m\in\mathbb{Z}\,, \tag{2.17}$$

where $\mathscr{I}^p$ is defined by (2.4). For v, x, and y real, consider the function

$$f(v,x,y)=\exp(2\pi x\,\mathrm{Im}\,\tau(v^2-v+\tfrac{1}{3})+\pi y\,\mathrm{Im}\,\tau(\tfrac{1}{2}+2v)). \tag{2.18}$$

We denote the usual theta function with characteristic (1/2, 1/2) by θ_1, that is

$$\theta_1(z,\tau)=\sum_{n\in\mathbb{Z}}\exp(\pi i\tau(n+\tfrac{1}{2})^2+2\pi i(n+\tfrac{1}{2})(z+\tfrac{1}{2}))\,. \tag{2.19}$$

We can now state

Theorem 2.5. *Let* $p=0,\ldots,\dim_{\mathbb{C}}F$ *and* $\mathscr{L}$ *be a compact leaf of period* α. *Assume that* (2.14) *holds and let* $\mathscr{E}$ *be defined by* (2.6) *and* (2.15). *Then, if* $0\leqq u\leqq 1$ *and*

$v \not\equiv 0 \pmod 1$ *then*

$$\Theta^p_{\mathscr{L}}(\mathscr{E}) = \mathrm{ep}\left(\frac{\mathrm{Im}\,\tau}{\pi} \sum_{n^2+m^2>0} \frac{e^{2\pi i(nv+m\alpha u)}}{|n+m\alpha\tau|^2} \mathrm{ind}^p(x, \phi^{\alpha m})\right), \tag{2.20a}$$

$$\Theta_{\mathscr{L}}(\mathscr{E}) = f(v, \chi(F), \alpha)|\eta(\alpha\tau)|^{-2}\theta_1(\alpha(u-\tau v), \alpha\tau)\theta_1(\alpha(u-\bar{\tau}v), -\alpha\bar{\tau}), \tag{2.20b}$$

and

$$\Theta^p(\mathscr{E}) = \sum_{j=1}^{\beta} f(v, \chi^p(F), \mathscr{I}^p(\phi^j))(|\eta(\beta\tau)|^{-2}\theta_1(\beta(u-\tau v), \beta\tau)\theta_1(\beta(u-\bar{\tau}v), -\beta\bar{\tau}))^{\mathscr{I}^p(\phi^j)/\beta}, \tag{2.20c}$$

where $\eta(\tau)$ denotes the Dedekind eta function, where $\chi(F)$ and $\chi^p(F)$ denote the Euler characteristic and the Dolbeault characteristic of F, and where β is the period of the p^{th} Lefschetz indices of ϕ.

Since $\theta_1(w, \tau)$ defined by (2.19) is a standard theta function (see Mumford [Mu]), this partly justifies the name of dynamical theta functions given to the $\Theta^p_{z,\mathscr{L}}(\mathscr{E})$ functions.

3. Zeta Functions

3.1. $\bar{\partial}$-Torsion. We will start by computing the torsion terms arising in the product formula (2.12). Note that since B is a torus, for any acyclic bundle $\mathscr{E}$ over B,

$$T_0(B, \mathscr{E}) = T_1(B, \mathscr{E}). \tag{3.1}$$

In order to compute (2.10) it is thus sufficient to compute

$$\prod_q T_0(B, \mathscr{E} \otimes H^{p,q}F)^{(-1)^q}. \tag{3.2}$$

By (3.1), the product over p and $p-1$ of (3.2) yields the numerator or denominator of (2.12). Using the definition of the torsion, it is sufficient to compute

$$\omega^p(t) = -\sum_{q=0}^{\dim_{\mathbb{C}} F} (-1)^q \operatorname{tr} e^{-t\Delta^{p,q}_B}, \tag{3.3}$$

where $\Delta^{p,q}$ denotes the $\bar{\partial}$-Laplacian for the flat metric g_B on B with coefficients $\mathscr{E} \otimes H^{p,q}F$. For $s \in \mathbb{C}$ with $\mathrm{Re}\, s$ large enough define

$$\zeta^p(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \omega^p(t)\, dt. \tag{3.4}$$

By definition, (3.2) is then equal to the derivative at $s=0$ of the analytic extension of $\zeta^p(s)$. Note that even though the bundle we are considering is not unitary, by [Se], the analytic extension is regular at 0 and hence the torsion is well defined. To compute $\omega^p(t)$ we use the techniques of Ray and Singer [RS]. Recall that

$$B = \mathbb{C}/\Gamma, \qquad \Gamma = \mathbb{Z} + \tau\mathbb{Z}, \qquad \mathrm{Im}\,\tau > 0. \tag{3.5}$$

B is naturally covered by $\mathbb{C}$. It is then possible to relate the heat kernel of $\Delta^{p,q}_B$ denoted by $k^{p,q}_B(t, x, y)$ to the heat kernel of $\Delta^{p,q}_{\mathbb{C}}$ denoted by $k^{p,q}_{\mathbb{C}}(t, x, y)$. One obtains (see [F, RS])

$$k_B^{p,q}(t,x,y)=\sum_{\gamma\in\Gamma} I^{p,q}(\gamma)k_{\mathbb{C}}^{p,q}(t,\tilde{x},\gamma(\tilde{y}))\,, \tag{3.6}$$

where $\tilde{x}$, $\tilde{y}$ have projections x and y, where $\gamma(\tilde{y})=\tilde{y}+n+m\tau$ if $\gamma=n+m\tau$ and where $I^{p,q}(\gamma)$ are the transition functions of the bundles arising in (3.2). One easily finds that

$$I^{p,q}(\gamma)=\varrho(\gamma)H^{p,q}(\varphi_{n,m})\,;\qquad \gamma=n+m\tau\,, \tag{3.7}$$

where ϱ is defined by (2.6) and where the $H^{p,q}(\varphi_{n,m})$ are the functions acting on $H^{p,q}F$ induced by the holonomy $\varphi_{n,m}$ defined earlier. Taking (3.7) at $x=y$ and integrating over B one obtains

$$\operatorname{tr} e^{-t\Delta_B^{p,q}}=-\frac{\operatorname{Im}\tau}{4\pi t}\sum_{n,m\in\mathbb{Z}} I^{p,q}(n+m\tau)e^{-|n+m\tau|^2/4t}\,. \tag{3.8}$$

Hence,

$$\omega^p(t)=-\frac{\operatorname{Im}\tau}{4\pi t}\sum_{n,m\in\mathbb{Z}}\varrho(n+m\tau)\mathscr{I}^p(\varphi_{n,m})e^{-|n+m\tau|^2/4t}\,, \tag{3.9}$$

where $\mathscr{I}^p(\varphi_{n,m})$ are the p-Lefschetz indices (see [AB 1, 2]) of the map $\varphi_{n,m}$ defined by (2.4). We now follow Ray and Singer [RS] and compute the corresponding zeta function. For simplicity, let

$$c_t(n,m)=\varrho(n+m\tau)\mathscr{I}^p(\varphi_{n,m})e^{-|n+m\tau|^2/4t}\,.$$

Then, for $\operatorname{Re}s$ large,

$$\begin{aligned}\zeta^p(s)=&-\frac{1}{\Gamma(s)}\frac{\operatorname{Im}\tau}{4\pi(s-1)}-\frac{\operatorname{Im}\tau}{4\pi\Gamma(s)}\int_0^1 t^{s-2}\sum_{n^2+m^2>0}c_t(n,m)dt\\&+\frac{1}{\Gamma(s)}\int_1^\infty t^{s-1}\operatorname{tr}e^{-t\Delta_B^{p,q}}\,.\end{aligned} \tag{3.10}$$

The right-hand side of (3.10) defines a meromorphic function of the s plane which is the desired continuation. When $\operatorname{Re}s<0$, we can insert (3.9) into the last integral in (3.10) to obtain

$$\zeta^p(s)=-\frac{\operatorname{Im}\tau}{4\pi\Gamma(s)}\int_0^\infty t^{s-2}\sum_{n^2+m^2>0}c_t(n,m)dt\,. \tag{3.11}$$

3.2. Dynamical Zeta Functions. We can now investigate the dynamical zeta functions. We recall that for $z\in B$ and for $p=0,\ldots,\dim_{\mathbb{C}}F$, we defined

$$\omega_{z,\mathscr{L}}^p(t)=-\frac{\operatorname{Im}\tau}{4\pi t}\sum_{n^2+m^2>0}\operatorname{ind}_z^p(\mathscr{L},\varphi_{n,m}^z)\operatorname{trace}(\varrho(n+m\tau))\exp\left(-\frac{|n+m\tau|^2}{4t}\right) \tag{3.12}$$

for any compact leaf $\mathscr{L}$ of the foliation. In order to prove the convergence of (3.12) as well as the properties of the related zeta functions, we need estimates on the index of a compact leaf. Fix $z\in B$ and let $\mathscr{L}$ be a compact leaf. By construction of the $\mathbb{Z}^2$ action, for any two points x_1 and x_2 in $\mathscr{L}\cap F_z$, there exist $(n,m)\in\mathbb{Z}^2$ such that $\varphi_{n,m}^z(x_1)=x_2$. That is, $\mathscr{L}\cap F_z$ is the orbit of a single point which we denote by

$x_{\mathscr{L}}$. Let $\Gamma_{\mathscr{L}}$ be the isotropy subgroup of $x_{\mathscr{L}}$. When non-trivial, this subgroup is generated by two elements ψ_1 and ψ_2 of the action. These two maps fix $x_{\mathscr{L}}$. We can now state

Lemma 3.1. *Let $z \in B$ and $\mathscr{L}$ be a compact leaf. Then there are positive constants $c(p, \mathscr{L})$ and α such that for any $(n,m) \in \mathbb{Z}^2$,*

$$|\mathrm{ind}_z^p(\mathscr{L}, \varphi_{n,m}^z)| \leqq c(|n|+|m|)^\alpha . \tag{3.13}$$

Proof. Since we consider compact leaves, by the definition of the index, it is sufficient to prove (3.13) for a single point x in $\mathscr{L} \cap F_z$. The isotropy subgroup of x is $\Gamma_{\mathscr{L}}$. Hence, if

$$x = \varphi_{n,m}^z(x), \qquad \varphi_{n,m}^z = \psi_1^k \psi_2^l \tag{3.14}$$

for some $(k,l) \in \mathbb{Z}^2$. Note that $|k|+|l| \leqq |n|+|m|$. $\partial\psi_1(x)$ and $\partial\psi_2(x)$ are two commuting complex matrices. We can therefore arrange their eigenvalues $\lambda_1, \ldots, \lambda_N$ and $\mu_1, \ldots, \mu_N$ in such a way that the eigenvalues of $\partial\psi_1(x)^k \partial\psi_2(x)^l$ are $\lambda_i^k \mu_i^l$, $i = 1, \ldots, N$. They will be denoted by $\beta_i(k,l)$. From the definition of the Atiyah-Bott indices, we have to compute

$$\frac{\mathrm{tr}\, \Lambda^p \partial\psi_1^k \partial\psi_2^l}{\det_{\mathbb{C}}(I - \partial\psi_1^k \partial\psi_2^l)} . \tag{3.15}$$

The eigenvalues of $\Lambda^p \partial\psi_1^k \partial\psi_2^l$ are of the given by $\beta_{i_1} \ldots \beta_{i_p}$, where $i_1 < \ldots < i_p$. We will denote the one with the largest modulus by $\beta_{j_1} \ldots \beta_{j_p}$. Thus, there is a constant $C(p)$ such that we can estimate (3.15) by

$$C(p) \left| \frac{\beta_{j_1} \cdots \beta_{j_p}}{\prod\limits_{k=1,\ldots,p} (1-\beta_{j_k})} \right| \left| \frac{1}{\prod\limits_{i \neq j_1,\ldots,j_p} (1-\beta_i)} \right|$$

which is equal to

$$C(p) \frac{1}{\left| \prod\limits_{k=1,\ldots,p} (\beta_{j_k}^{-1} - 1) \right|} \frac{1}{\left| \prod\limits_{i \neq j_1,\ldots,j_p} (1-\beta_i) \right|} . \tag{3.16}$$

We now make use of (2.2b). For $(k,l) \neq (0,0)$, we have

$$|1 - \lambda^k \mu^l| > \frac{C}{(|k|+|l|)^r} . \tag{3.17}$$

To see this, assume that it is false. Then, we can fin k and l such that (3.17) is false. Let $d = \gcd(k,l)$. Then

$$\frac{C}{(|k|+|l|)^r} \geqq |1 - \lambda^k \mu^l| > \frac{C}{|d|^r} ,$$

where the last inequality is obtained by applying (2.2b) to $(\lambda^\alpha \mu^\beta)^d$ with $\alpha d = k$, $\beta d = l$. Since $|k|+|l| = |d|(|\alpha|+|\beta|)$, one immediately has a contradiction. Using (3.17), we estimate (3.16) by

$$C^{-N}(|k|+|l|)^{Nr} ,$$

and hence, one obtains (3.13). □

It is now possible to show that $\omega^p_{z,\mathscr{L}}(t)$ is well defined for $t>0$. We define a function

$$\phi(t)=\frac{1}{4\pi t}\sum_{n,m} e^{-(n^2+m^2)/4t}. \tag{3.18}$$

By the Poisson summation formula, we can rewrite $\phi(t)$ as

$$\sum_{n,m} e^{-4\pi t(n^2+m^2)}. \tag{3.19}$$

Hence, for any positive integers k and l we have

$$t^k\left(\frac{d}{dt}\right)^l\left(\phi(t)-\frac{1}{4\pi t}\right)\to 0, \quad \text{as} \quad t\to\infty, \tag{3.20a}$$

and

$$t^{-k}\left(\frac{d}{dt}\right)^l\left(\phi(t)-\frac{1}{4\pi t}\right)\to 0, \quad \text{as} \quad t\to 0. \tag{3.20b}$$

Now using Lemma 3.1, we have

$$|\omega^p_{z,\mathscr{L}}|\leqq c\frac{\operatorname{Im}\tau}{4\pi t}\sum_{n^2+m^2>0}(|n|+|m|)^\alpha\,|\operatorname{tr}\varrho(n+m\tau)|\exp\left(-\frac{|n+m\tau|^2}{4t}\right).$$

Hence, using the definition of $\phi(t)$, there are positive integers γ and δ such that

$$|\omega_{z,\mathscr{L}}(t)|\leqq c\left|t^\gamma\left(\frac{d}{dt}\right)^\delta\phi(t)\right|. \tag{3.21}$$

Hence by (3.20) and (3.21), for every integer $k>0$, we obtain

$$|t^{-k}\omega^p_{z,\mathscr{L}}(t)|\to 0, \quad \text{as} \quad t\to 0, \tag{3.22a}$$

and

$$|t^k\omega^p_{z,\mathscr{L}}(t)|\to 0, \quad \text{as} \quad t\to\infty. \tag{3.22b}$$

Theorem 2.1 can now be proved. Recall that the p^{th} dynamical zeta function associated to a compact leaf $\mathscr{L}$ is defined by

$$\zeta^p_{z,\mathscr{L}}(s)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\omega^p_{z,\mathscr{L}}(t)dt, \tag{3.23}$$

where $s\in\mathbb{C}$. Using (3.22), one sees that

$$\int_0^1 t^{s-1}\omega^p_{z,\mathscr{L}}(t)dt \quad \text{and} \quad \int_1^\infty t^{s-1}\omega^p_{z,\mathscr{L}}(t)dt$$

define entire functions of s, which proves Theorem 2.1.

4. Proof of the Main Theorems

We can now prove the main theorems. We first begin by a summation formula.

Proposition 4.1. *Assume that* (2.2) *holds. Then, for any* $z \in B$ *and any* $t > 0$,

$$\omega^p(t) = -\frac{\operatorname{Im}\tau}{4\pi t} + \sum_{\mathscr{L}\text{ compact}} \omega^p_{z,\mathscr{L}}(t), \tag{4.1}$$

where $\omega^p(t)$ *is defined by* (3.9).

Proof. For simplicity, let

$$c_t(n,m) = \operatorname{tr}(\varrho(n+m\tau))\exp\left(-\frac{|n+m\tau|^2}{4t}\right). \tag{4.2}$$

Let M be a positive integer. By (2.2b), for $|n|+|m| \leqq M$, there are finitely many compact leaves for which $\operatorname{ind}^p_z(\mathscr{L}, \varphi_{n,m})$ is non-zero. Hence, by the definition of the index of a leaf, we have

$$\sum_{0<|n|+|m|\leqq M} \mathscr{L}^p(\varphi_{n,m})c_t(n,m) = \sum_{\mathscr{L}\text{ compact}} \sum_{0<|n|+|m|\leqq M} c_t(n,m)\operatorname{ind}^p_z(\mathscr{L}, \varphi_{n,m}). \tag{4.3}$$

We note that the left-hand side of (4.3) converges to

$$\omega^p(t) + \frac{\operatorname{Im}\tau}{4t} \tag{4.4}$$

as M goes to infinity. Hence, letting M tend to infinity and using the definition of $\omega^p_{z,\mathscr{L}}(t)$, we obtain (4.1). □

The proof of Theorems 2.2 and 2.4 is a triviality. Theorem 2.1 guarantees that the functions $\Theta^p_{z,\mathscr{L}}(\mathscr{E})$ are well defined for compact leaves $\mathscr{L}$. Hence, the functions $\Theta_{z,\mathscr{L}}(\mathscr{E})$ exist for any $z \in B$ and for any acyclic bundle $\mathscr{E}$. For $\Theta^p(\mathscr{E})$, we rewrite (4.1) in terms of zeta functions. For this, we use (3.11). Note that the integrand in (3.11) is precisely the sum over compact leaves $\mathscr{L}$ of $\omega^p_{z,\mathscr{L}}(t)$. Hence, for $\operatorname{Re} s < 0$, one has

$$\zeta^p(s) = \sum_{\mathscr{L}\text{ compact}} \zeta^p_{z,\mathscr{L}}(s), \tag{4.5}$$

where ζ^p is defined by (3.4) and where $\zeta^p_{z,\mathscr{L}}$ is defined by (3.20). Both sides define functions which are analytic for s near 0. Hence, (4.5) holds at $s=0$ for the functions and their derivatives. Taking the derivatives of both sides of (4.6) at $s=0$ and taking the exponential yields (2.9b) and shows that $\Theta^p(\mathscr{E})$ is well defined. Now by the definition of $\omega^p(t)$, the derivative of $\zeta^p(s)$ at $s=0$ yields the logarithm of (3.2). Hence, using (2.9b) and the product formula (2.12) one easily obtains Theorem 2.4.

5. Degenerate Actions

We now proceed to compute the dynamical theta functions in the case of degenerate $\mathbb{Z}^2$ actions as defined in (2.13). We will follow closely Ray and Singer [RS]. In the following, let $\mathscr{L}$ be a compact leaf of period α and denote the period of the p^{th} Lefschetz index by β. We have to compute the zeta functions associated to ω^p, $\omega^p_{z,\mathscr{L}}(t)$ and to $\sum(-1)^p\omega^p_{z,\mathscr{L}}(t)$. First note that

$$\operatorname{ind}^p_z(\mathscr{L}, \phi^m) = 0 \quad \text{if} \quad m \neq k\alpha.$$

Using the definitions of these functions and the method yielding (3.13), we obtain that

$$\zeta^p(s) = -\sum_{j=1}^{\beta} \frac{\Gamma(1-s)}{\Gamma(s)} \frac{\operatorname{Im}\tau}{4\pi} \sum_{n^2+m^2>0} \varrho(n+m\beta\tau)\mathscr{I}^p(\phi^j)\left(\frac{4}{|n+m\beta\tau|^2}\right)^{1-s}, \tag{5.1a}$$

$$\zeta^p_{z,\mathscr{L}}(s) = -\frac{\Gamma(1-s)}{\Gamma(s)} \frac{\operatorname{Im}\tau}{4\pi} \sum_{n^2+m^2>0} \varrho(n+m\alpha\tau)\operatorname{ind}^p_z(\mathscr{L},\phi^{m\alpha})\left(\frac{4}{|n+m\alpha\tau|^2}\right)^{1-s}, \tag{5.1b}$$

and

$$\begin{aligned}\zeta_{z,\mathscr{L}}(s) = &-\alpha\frac{\Gamma(1-s)}{\Gamma(s)} \frac{\operatorname{Im}\tau}{4\pi} \sum_{n^2+m^2>0} \varrho(n+m\alpha\tau)\left(\frac{4}{|n+m\alpha\tau|^2}\right)^{1-s} \\ &-\chi(F)\frac{\Gamma(1-s)}{\Gamma(s)} \frac{\operatorname{Im}\tau}{4\pi} \sum_{n\neq 0} \varrho(n)\left(\frac{4}{n^2}\right)^{1-s}.\end{aligned} \tag{5.1c}$$

For (5.1a), we used the periodicity of the p-Lefschetz indices. For (5.1c), we used that

$$\sum_p (-1)^p \operatorname{ind}^p(x,\phi) = 1$$

for a holomorphic function ϕ (the alternate sum of Atiyah-Bott indices is the classical Lefschetz index), that $\mathscr{L}\cap F_z$ contains α points and that for $m=0$ the alternate sum of the p^{th} Lefschetz indices yields the Euler characteristic. Note that these zeta functions all vanish at $s=0$. Their derivatives at $s=0$ are therefore equal to the sums on the right-hand side of (5.1) taken at $s=0$. These sums are not absolutely convergent at $s=0$. To check their convergence, we follow [RS]. Let $w\in\mathbb{C}$, with $\operatorname{Im} w>0$ and consider

$$\begin{aligned}\frac{\operatorname{Im}\tau}{\pi} \sum_{n^2+m^2>0} \varrho(n+mw)\frac{1}{|n+mw\tau|^{2-2s}} = &\frac{\operatorname{Im}\tau}{\pi}\sum_{n\neq 0}\varrho(n)\frac{1}{n^{2-2s}} \\ &+\frac{\operatorname{Im}\tau}{\pi}\sum_{m\neq 0}\varrho(mw)\sum_n \varrho(n)\frac{1}{|n+mw|^{2-2s}}.\end{aligned} \tag{5.2}$$

By our definition of the representation, the first sum on the right-hand side of (5.2) converges at $s=0$ and is equal to

$$2\pi\operatorname{Im}\tau(v-v^2-\tfrac{1}{3}). \tag{5.3}$$

For the second sum let

$$A_n = e^{2\pi i n v}\frac{\sin\pi(n+1)v}{\sin\pi v}$$

and

$$b_n = |mw+n|^{2s-2}.$$

Hence

$$\left|\sum_n e^{2\pi i n v}|m\tau+n|^{2s-2}\right| = \left|\sum_n (A_n - A_{n-1})b_n\right| = \left|\sum_n A_n(b_n - b_{n-1})\right|.$$

The last sum is easily bounded by

$$\frac{1}{\sin\pi v}\frac{2}{|m|(\operatorname{Im} w)^{2-2s}}. \tag{5.4}$$

Using Kronecker second limit, one can explicitly compute this second sum at $s=0$. (See [RS] and Siegel [Si].) It is equal to

$$-\frac{\operatorname{Im}\tau}{\operatorname{Im}w}\log\left(\left|\prod_{k=-\infty}^{+\infty}(1-e^{2\pi i(|k|w-\varepsilon_k z)})\right|^2\right) \tag{5.5}$$

with $\varepsilon_k=\operatorname{sign}(k+\frac{1}{2})$ and $z=(\operatorname{Im}w/\operatorname{Im}\tau)u-wv$. We can now prove Theorem 2.5. The boundedness of the Atiyah-Bott indices and (5.3), (5.4) guarantee the convergence of the sum in (5.1b) at $s=0$. Hence, taking the exponential of the derivative of (5.1b) yields (2.20a). For (2.20b) and (2.20c), we use the standard product formulas of theta functions (see Mumford [Mu] or Siegel [Si]) to compute the exponential of the derivatives of (5.1a, c). With (5.3) and (5.5), it is easy to see that one obtains the formulas of Theorem 2.5.

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