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The Phase Structure of the Two-Dimensional $N = 2$ Wess–Zumino Model*

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Abstract. We construct a convergent cluster expansion for the two-dimensional $N = 2$ Wess–Zumino model, in a region of parameter space where there are multiple phases. As a result of this expansion, we are able to construct the infinite volume field theory and demonstrate exponential decay of correlations. We are also able to investigate the different phases of the model, develop the phase diagram, and show that the free energy of each phase vanishes.

1. Introduction

In the series of papers [10–13] a series of two-dimensional quantum field models were constructed, in finite space-time volume, and some of their properties studied. One of the main results of [10–13] was the existence in these theories, in addition to the usual symmetries of quantum field theory, of an additional symmetry – the supersymmetry. The existence of supersymmetry has important consequences for the behavior of these quantum field models.

The purpose of the present work is of investigate the properties of some of these quantum field models in an infinite space-time volume, and to ascertain to the extent possible the persistence of some of the consequences of supersymmetry in this limit. The main technical tool used in this investigation is the Glimm–Jaffe–Spencer [5, 6] cluster expansion, which allows control of correlation functions in the infinite volume limit. This expansion is applied to our model using methods

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previously applied by Cooper and Rosen [3] and Balaban and Gawedzki [1] to the Yukawa$_2$ model, and using some additional techniques developed in [9] and in the present paper. We restrict our attention to the $N=2$ models for technical reasons explained in [9, 14].

Let us briefly review the properties of the quantum field models in finite volume, as studied in [10–12]. The field content of the $N=2$ Wess–Zumino models studied there is one complex bose field ϕ, and one Dirac spinor field ψ, acting as operators on a Hilbert space $\mathscr{H}=\mathscr{H}_B \otimes \mathscr{H}_F$, the tensor product of the standard free boson representation $\mathscr{H}_B$ and fermion representation $\mathscr{H}_F$. The space $\mathscr{H}$ may be decomposed $\mathscr{H}=\mathscr{H}_+ \otimes \mathscr{H}_-$ into eigenspaces of the unitary generator of chirality γ_5.

In terms of ψ, ϕ, the Hamiltonian of the theory is given formally by

$$H = H_0 + \int_{\Lambda} \left(|V'(\phi)|^2 - |\phi|^2 + \bar{\psi} \begin{bmatrix} V''(\phi)-1 & 0 \\ 0 & V''(\phi)^* - 1 \end{bmatrix} \psi \right) dx, \tag{1.1}$$

where H_0 denotes the sum of the free Hamiltonian for the bose field shifted to a minimum of $|V'|$, and the free Hamiltonian for the Fermi field; here $V(\cdot)$ is a polynomial of degree n. (There will be a choice of representation $H=H_0+H_I$ for each distinct minimum of $|V'|$.) The results of [10–12] imply that the formal operator (1.1) may be regarded as the limit in an appropriate sense of a sequence of regularized selfadjoint unbounded operators on $\mathscr{H}$. Furthermore, it was shown that there exists a selfadjoint Fredholm operator Q on $\mathscr{H}$ such that $Q^2=H$. The operator Q is the supersymmetry generator or "supercharge," and may be written $Q=Q_+ + Q_-$, where $Q_\pm = \frac{1}{4}(1 \mp \gamma_5) Q (1 \pm \gamma_5)$, i.e.

$$\begin{aligned} Q_+ &: \mathscr{H}_+ \to \mathscr{H}_-, \\ Q_- &: \mathscr{H}_- \to \mathscr{H}_+. \end{aligned} \tag{1.2}$$

As Q is Fredholm, it has an index, which was computed in [11]; the result is

$$|\operatorname{ind} Q| = n-1. \tag{1.3}$$

Now by definition of the index,

$$\operatorname{ind} Q = \dim \ker Q_+ - \dim \ker Q_-. \tag{1.4}$$

Since $H=Q^2$, $|\operatorname{ind} Q|>0$ implies the existence of a zero energy state in $\mathscr{H}$, that is a state Ω, such that

$$H\Omega = 0. \tag{1.5}$$

If $\operatorname{ind} Q=0$, there may be no such state; then of necessity the lowest energy state Ψ satisfies $Q\Psi \neq 0$; that is, supersymmetry is spontaneously broken. Of course the vanishing of the index is not sufficient to guarantee broken supersymmetry.

The result (1.3) implies the existence of at least $n-1$ such ground states, and it would be natural to associate these to the zeroes of the polynomial V', and to conjecture that there should be *precisely* $n-1$ such ground states. This result was however beyond the scope of [10–12].

In the present work we shall extend the construction of the models shared in [10–12] to infinite volume, assuming some technical conditions on the superpotential V. The cluster expansion methods of [1, 3, 5, 6] introduce a space-time lattice to interpolate between a theory defined on a large volume Λ and one defined on the squares of size l contained in Λ, with no interaction between the squares. This enables one to produce estimates uniform in Λ. Unfortunately we are not able to carry out this procedure while preserving the supersymmetry, and hence are unable to establish the existence of supersymmetry. However, using the techniques of [2] we are able to show that a relic of the supersymmetry is preserved in the vanishing of the free energy of the theory; this is the analogue of (1.5). We thus have the following result:

Theorem 1. *Consider the two-dimensional $N=2$ Wess–Zumino model with superpotential*

$$\lambda^{-2}\tilde{W}(\lambda x)+\lambda^{-1}\varpi w(\lambda x),$$

where $\tilde{W}, w$ are polynomials of degree n, $\tilde{W}'$ has $n-1$ distinct zeroes, and $|\tilde{W}''|=1$ at each such zero. Write $H=H_0+H_I$, as above, choosing one minimum of V'. If λ and ϖ are sufficiently small, the Schwinger functions of the model converge to infinite volume Schwinger functions, which are reflection positive, Euclidean invariant, exponentially decaying, and satisfy the axioms of [15]. The $n-1$ choices of the Hamiltonian $H=H_0+H_I$ give rise (by the results of [15]) to $n-1$ single-phase infinite volume limits. The free energy of each such phase is zero.

Remark. A similar result exists for the Schwinger functions of the antiperiodic model. A periodic Schwinger function has an infinite volume limit which is the sum of $n-1$ functions satisfying the axioms of [15].

Proof of Theorem 1, Assuming Theorem 2. The proof of Theorem 1 now follows by standard methods, assuming the results of Theorem 2, by checking each of the OS axioms. The regularity of correlation functions follows by the bound (2.21).

To prove Euclidean invariance, we must prove translation invariance and rotation invariance. We do these separately, introducing periodic boundary conditions for the former and a spatial cutoff in a disc-shaped region for the latter. The cutoff theories then possess the desired invariance properties, and by the cluster expansion each converges to the same infinite volume limit.

It remains to prove Osterwalder–Schrader positivity. To check this, we choose a spatial cutoff in a reflection invariant rectangular region, and note that the infinite volume limit correlation functions in each phase can be taken to be the limits of finite volume correlation functions in the cutoff regions, obtained from a quantum field model with a Hamiltonian equal to the free-field Hamiltonian, with mass term appropriate to the given phase outside the interaction region, and with the full interaction inside this region. This can be proved using methods analogous to those used in [10–12] for the periodic models. These (real time) finite volume correlation functions arise from a Hamiltonian model and are therefore guaranteed to be reflection positive. Thus their infinite volume limits, which again exist and are equal to our infinite volume correlation functions by the cluster expansion, are reflection positive.

The supersymmetric aspect of the theorem, the vanishing of the free energy, in proven (using the methods of [2]) in Sect. 7.3.

2. The Model

We consider the two-dimensional (Euclidean) $N = 2$ Wess–Zumino model, which has the formal Hamiltonian

$$H = H_0 + \int_\Lambda \left(|W'_\lambda(\phi)|^2 - |\phi|^2 + \bar{\psi} \begin{bmatrix} W''_\lambda(\phi) - 1 & 0 \\ 0 & W''_\lambda(\phi)^* - 1 \end{bmatrix} \psi \right) dx, \tag{2.1}$$

where ϕ is a complex scalar field, ψ is a Dirac (complex) fermionic field, H_0 is the free Hamiltonian for a boson and fermion with unit mass. We will require $\tilde{W}'$ to have $n - 1$ distinct zeroes, located at $\lambda\xi_1, \ldots, \lambda\xi_{n-1}$, all of which have a second derivative with absolute value one:

$$|\tilde{W}''(\lambda\xi_i)| = 1. \tag{2.2}$$

Note that such polynomials do exist, e.g.

$$\tilde{W}'(z) = \prod_{k=1}^{n-2} \left(2 \sin \frac{\pi k}{n-1} \right)^{-1} \prod_{k=1}^{n-1} (z - e^{2\pi i k/(n-1)}) \tag{2.3}$$

with zeroes at the $n - 1$ roots of unity, or for the one parameter family of polynomials of degree $2n'$ $(n = 2n' + 1)$:

$$\tilde{W}'_\beta(z) = \prod_{k=1}^{n'-1} \left(2 \sin \frac{\pi k}{n'} \right)^{-1} \prod_{k=1}^{n'} \left[\left(2 \sin \frac{\pi(k+\beta)}{n'} \right)^{-1} (z - e^{2\pi i k/n'})(z - e^{2\pi i(k+\beta)/n'}) \right], \tag{2.4}$$

where $0 < \beta < 1$. Without loss of generality, we can assume that $\xi_1 = 0$ and that $\tilde{W}''(0) = 1$.

The bosonic potential $|W'_\lambda(\phi)|^2$ has minima where $\tilde{W}'$ has zeroes, and the scaling as $\lambda \to 0$ increases the distance between and the depth of the potential wells. For small enough λ this classical behavior will carry over into the quantum theory.

The restriction to mass of absolute value one is for computational convenience and clarity; no new phenomena are expected in the general case, which should be approachable through an extension of our methods (see the appendix to [9]). For small variations about unit mass, our methods work directly. Instead of the previous definition, consider

$$W_\lambda(x) = \lambda^{-2}\tilde{W}(\lambda x) + \lambda^{-1}\varpi w(\lambda x), \tag{2.5}$$

where w is also a polynomial of degree n and ϖ is a small parameter. The perturbation w breaks any artificial symmetry introduced by our mass restriction without significantly altering any of our results.

The partition function corresponding to the above model is

$$Z^\Lambda = \int d\mu(\phi) \exp\left\{ - \int_\Lambda (|W'_\lambda(\phi)|^2 - |\phi|^2) \right\} \det[1 + S\gamma_0\chi_\Lambda(\Upsilon(W''_\lambda(\phi)) - 1], \tag{2.6}$$

where

$$\Upsilon(z) \equiv \begin{bmatrix} z & 0 \\ 0 & z^* \end{bmatrix}, \tag{2.7}$$

and the fermionic propagator is

$$S = \gamma_0(i\partial\!\!\!/ + 1)^{-1} \tag{2.8}$$

where

$$\not{\partial} = \gamma_\mu^E \partial_\mu,$$

with

$$-i\gamma_0 = \gamma_0^E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \gamma_1^E = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix},$$

and $d\mu(\phi)$ is the normalized Gaussian measure with covariance $(-\Delta + 1)^{-1}$. We will also consider the case with periodic covariance.

Reordering our formal expression, we get

$$Z^\Lambda = \int d\mu(\phi) \exp\left\{ -\int_\Lambda (:|W'_\lambda(\phi)|^2: - :|\phi|^2:) dx \right\} \det{}_3[1 + \tilde{K}(\phi)] e^{-\tilde{R}}, \tag{2.9}$$

where

$$\tilde{K}(\phi) = S\gamma_0 \chi_\Lambda (\Upsilon(W''_\lambda(\phi)) - 1) \tag{2.10}$$

and

$$\tilde{R} = \int_\Lambda dx [|W'_\lambda(\phi)|^2 - :|W'_\lambda(\phi)|^2: - |\phi|^2 + :|\phi|^2:] + \tfrac{1}{2} \mathrm{Tr}\, \tilde{K}^2(\phi) - \mathrm{Tr}\, \tilde{K}(\phi). \tag{2.11}$$

It was shown in [10, 12] that the formal expression $\tilde{R}$ is actually finite; i.e. if $\tilde{R}$ is regularized it has a limiting value as the regularization is removed. In finite volume the theory corresponding to (2.9) was constructed in [10, 12]. Our goal here is to prove similar estimates which are uniform in the volume.

In order to make use of the aforementioned finite volume results, we regularize our theory in the same fashion. Thus we have

$$\hat{Z}_\kappa^\Lambda = \int d\mu(\phi) e^{-A^{(\kappa)}} \det[1 + \tilde{\Xi}_\kappa \tilde{K}(\phi_\kappa)], \tag{2.12}$$

where

$$A^{(\kappa)} = \int_\Lambda dx [|W'_\lambda(\phi_\kappa) - \phi_\kappa|^2 + \phi^*(W'_\lambda(\phi_\kappa) - \phi_\kappa) + \phi(W'_\lambda(\phi_\kappa) - \phi_\kappa)^*], \tag{2.13}$$

$\phi_\kappa = \tilde{\Xi}_\kappa \phi$ and $\tilde{\Xi}_\kappa$ is a cutoff whose Fourier transform has compact support. This rather unusual cutoff is necessary to avoid explicitly breaking the supersymmetry.

Equivalently,

$$\hat{Z}_\kappa^\Lambda = \int d\mu(\phi) e^{-\hat{\mathscr{A}}^{(\kappa)}} \det{}_3[1 + \tilde{\Xi}_\kappa \tilde{K}(\phi_\kappa)], \tag{2.14}$$

where

$$\hat{\mathscr{A}}^{(\kappa)} = \int_\Lambda dx [:|W'_\lambda(\phi_\kappa) - \phi_\kappa|^2: + :\phi^*(W'_\lambda(\phi_\kappa) - \phi_\kappa) + \phi(W'_\lambda(\phi_\kappa) - \phi_\kappa)^*:] + \tilde{R}_\kappa \tag{2.15}$$

and

$$\begin{aligned} \tilde{R}_\kappa = {} & \tfrac{1}{2} \mathrm{Tr}\, (\tilde{\Xi}_\kappa \tilde{K}(\phi_\kappa))^2 - \mathrm{Tr}\, \tilde{\Xi}_\kappa \tilde{K}(\phi_\kappa) \\ & + \int_\Lambda dx [|W'_\lambda(\phi_\kappa) - \phi_\kappa|^2 - :|W'_\lambda(\phi_\kappa) - \phi_\kappa|^2: + \phi^*(W'_\lambda(\phi_\kappa) - \phi_\kappa) \\ & + \phi(W'_\lambda(\phi_\kappa) - \phi_\kappa)^* - :\phi^*(W'_\lambda(\phi_\kappa) - \phi_\kappa) + \phi(W'_\lambda(\phi_\kappa) - \phi_\kappa)^*:]. \end{aligned} \tag{2.16}$$

Note that the expression (2.14) corresponds to our formal expression (2.9).

Our analysis will be simplified if our expression for the regularized partition function $\hat{Z}_\kappa^\Lambda$ is replaced with another having the same limit as $\kappa \to \infty$. So we consider

$$Z_\kappa^\Lambda = \int d\mu(\phi) e^{-\tilde{\mathscr{A}}^{(\kappa)}} \det_3 [1 + \tilde{K}(\phi_\kappa)], \tag{2.17}$$

where

$$\begin{aligned}\tilde{\mathscr{A}}^{(\kappa)} &= \int_\Lambda dx [:|W'_\lambda(\phi_\kappa) - \phi_\kappa|^2: + :\phi_\kappa^*(W'_\lambda(\phi_\kappa) - \phi_\kappa) + \phi_\kappa(W'_\lambda(\phi_\kappa) - \phi_\kappa)^*:] + \tilde{R}_\kappa \\ &= \int_\Lambda dx [:|W'_\lambda(\phi_\kappa)|^2: - :|\phi_\kappa|^2:] + \tilde{R}_\kappa . \end{aligned} \tag{2.18}$$

Notice that the counterterms $\tilde{R}_\kappa$ have not been altered (which might change the limiting behavior); we have only changed terms that are expected to behave well in the limit $\kappa \to \infty$. It should be evident that provided both Z_κ^Λ and $\hat{Z}_\kappa^\Lambda$ exist their limits will be the same. The existence of $\hat{Z}_\kappa^\Lambda$ was shown in [10, 12] and we will directly demonstrate the integrability of Z_κ^Λ in later sections.

Similar expressions may be derived for Green's functions using the results of [10–12]. In this case the quantity of interest is

$$\begin{aligned} Z_\kappa^\Lambda(\{f_i, g_j, h_j\}) &= \int d\mu(\phi) e^{-\tilde{\mathscr{A}}^{(\kappa)}} \prod_{i=1}^{\mathscr{I}} \phi^\#(f_i) \mathscr{J}! \operatorname{Tr} \left[\bigwedge_{j=1}^{\mathscr{J}} [1 + \tilde{K}(\phi_\kappa)]^{-1} S h_j(g_j, \cdot) \right] \\ &\quad \cdot \det_3 [1 + \tilde{K}(\phi_\kappa)] e^{-\tilde{R}_\kappa} \\ &= \int d\mu(\phi) e^{-\tilde{\mathscr{A}}^{(\kappa)}} \prod_i \phi^\#(f_i) \det_3 [1 + D^{1/2} \tilde{K}(\phi_\kappa) D^{-1/2}] e^{-\tilde{R}_\kappa} \\ &\quad \cdot \mathscr{J}! \operatorname{Tr} \left[\bigwedge_{j=1}^{\mathscr{J}} [1 + D^{1/2} S \gamma_0 \chi_\Lambda (\Upsilon(W''_\lambda(\phi_\kappa)) - 1) D^{-1/2}]^{-1} \tilde{P}_j \right], \end{aligned} \tag{2.19}$$

where

$$D = (-\Delta + 1)^{1/2}$$

and

$$\tilde{P}_j(\cdot) = D^{1/2} S h_j (D^{-1/2} g, \cdot); \tag{2.20}$$

the test functions $\{f_i\} \subset \mathscr{H}_{-1}$ and $\{g_j, h_j\} \subset \mathscr{H}$ are each localized in a unit square. The insertion of powers of D allows us to compute the determinant of an operator on the space $\mathscr{H}' = L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)$ instead of $\mathscr{H} = \mathscr{H}_{-1/2}(\mathbb{R}^2) \oplus \mathscr{H}_{-1/2}(\mathbb{R}^2)$. The # symbol indicates that some of the test functions f_i may be associated with complex conjugated fields – $\phi^*(f_i)$ – instead of only $\phi(f_i)$. The expression (2.19) will be the starting point for our investigations.

Although our cluster expansion and the expression (2.19) assume a particular choice of the minimum of the potential we expand around (we expand around ξ_1), we could have just as easily chosen one of the other $n-2$ minima or taken periodic boundary conditions. For any of these choices, we have the following:

Theorem 2. *For λ and ϖ sufficiently small,*
(a) *The Green's functions $Z_\kappa^\Lambda(\{f_i, g_j, h_j\})$ have a limit as $\kappa \to \infty$, $Z^\Lambda(\{f_i, g_j, h_j\})$.*
(b) *The infinite volume limit of $Z^\Lambda(\{f_i, g_j, h_j\})/Z^\Lambda \equiv S^\Lambda(\{f_i, g_j, h_j\})$ exists, and*

has a convergent cluster expansion. Thus the resulting connected Schwinger functions S_C exhibit exponential decay in the distance between the supports of the functions $\{f_i, g_j, h_j\}$ in the form appropriate to [15]:

$$S_C(\{f_i, g_j, h_j\}) \leq (\mathscr{I}!)^{1/2}(c_B)^{\mathscr{I}} \prod_{i=1}^{\mathscr{I}} \|f_i\|_{L^2}(c_F)^{\mathscr{I}} \prod_{j=1}^{\mathscr{I}} (\|g_j\|_{\mathscr{H}} \|h_j\|_{\mathscr{H}}) \sup_{\Gamma \supset \operatorname{supp}\{f_i, g_j, h_j\}} \exp -c_D|\Gamma| \tag{2.21}$$

for appropriate (positive) values of the constants c_B, c_F and c_D.

This theorem follows via standard methods from the cluster decay bound of Proposition 7.1 and the lower bound of Proposition 7.3.

3. Cluster Expansion

We proceed to analyze our starting expression (2.19) as in [1, 5–8]. We first define the block-spin configurations, by restricting the field ϕ to be near one of the $n-1$ minima ξ_i, within blocks of size d.

Let Σ be a function from d-blocks to $\{1, \ldots, n-1\}$, where $\Sigma(\Delta)$ represents which phase the field ϕ is in inside Δ. Let

$$h(x) = \xi_{\Sigma(\Delta)} \quad \text{for} \quad x \in \Delta. \tag{3.1}$$

Since ϕ is complex, we cannot immediately use the characteristic function of [1] or [8] to restrict ϕ to be near one of the ξ_i. However, we can associate with each of the minima ξ_i a neighborhood Ξ_i so that $\xi_i \in \Xi_i$ (see Fig. 1), where the choice of region is unimportant, so long as they scale with λ in the same way as the underlying ξ_i. Then define

$$\chi_q(\xi) = \frac{1}{\pi} \int_{\Xi_q} e^{-|\xi - z|^2} d^2 z. \tag{3.2}$$

Note that $\sum_{q=1}^{n-1} \chi_q(x) = 1$.

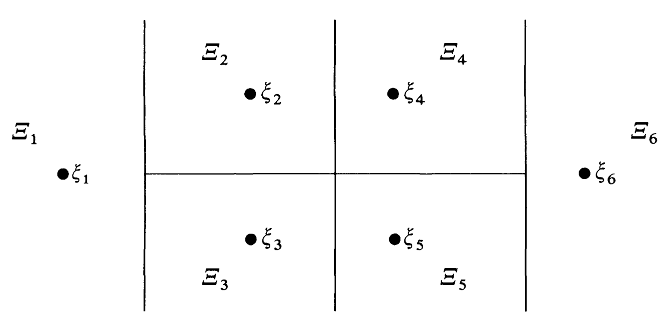

Fig. 1. Example of six minima and their associated regions

Defining the mean-field $(\phi)_\Delta = d^{-2}\int_\Delta \phi(x)d^2x$, we have the characteristic function that restricts ϕ to a particular block-spin configuration:

$$\prod_\Delta \chi_{\Sigma(\Delta)}((\phi)_\Delta). \tag{3.3}$$

Then $\sum_\Sigma \prod_\Delta \chi_{\Sigma(\Delta)}((\phi)_\Delta)$ is a partition of unity, and $Z_\kappa^\Lambda(\{f_i, g_j, h_j\}) = \sum_\Sigma Z_\Sigma$, where

$$Z_\Sigma = \int d\mu(\phi) \prod_\Delta \chi_{\Sigma(\Delta)}((\phi)_\Delta) e^{-\mathscr{A}(\kappa)} \prod_i \phi^\#(f_i) \det_3[1 + D^{1/2}\tilde{K}(\phi_\kappa)D^{-1/2}] e^{-\tilde{R}_\kappa}$$
$$\cdot \mathscr{J}! \operatorname{Tr}\left[\bigwedge_{j=1}^{\mathscr{J}} [1 + D^{1/2} S\gamma_0 \chi_\Lambda(\Upsilon(W_\lambda''(\phi_\kappa)) - 1)D^{-1/2}]^{-1}\tilde{P}_j\right]. \tag{3.4}$$

We expect Z_Σ to have decay in the size of the phase boundary $|\partial\Sigma|$.

Standard analysis of determinants of the type appearing in (3.4) is essentially an application of the formula

$$\det(1+x) = \exp \operatorname{Tr} \log(1+x) \tag{3.5}$$

and is only effective when $\Upsilon(W_\lambda''(\phi_k) - 1)$ is small. However, the characteristic function $\chi_{\Sigma(\Delta)}$ can force $W_\lambda''(\phi_\kappa)$ away from 1 when the different phases of the theory have different masses. Fortunately this is exactly the case we considered in [9, 14], where we saw that a local unitary transformation can "rotate" the determinant from one minimum to another.

Let

$$\zeta = \gamma_0 \chi_\Lambda [\Upsilon(W_\lambda''(\phi_\kappa)) e^{2i\alpha\gamma_5} - 1 - \partial\!\!\!/ \alpha\gamma_5] \tag{3.6}$$

with $\alpha \in C_0^\infty(\mathbb{R}^2)$ chosen such that $|\partial\alpha| < 1$ and $e^{-2i\alpha} = \tilde{W}_\lambda''(\xi_{\Sigma(\Delta)})$ if Σ is constant in a neighborhood of Δ. Then we have

Lemma 3.1. *Let $\Xi_{\kappa'}$ be the cutoff operator on $\mathscr{H}'$ given by*

$$(\Xi_{\kappa'} f)(x) = \int e^{-p^2/\kappa'^2} \hat{f}(p) \frac{d^2p}{(2\pi)^2} e^{ip\cdot x}. \tag{3.7}$$

Then

$$\det_3[1 + \tilde{K}(\phi_\kappa)] = \det_3[1 + S\zeta] \exp R'(\phi_\kappa), \tag{3.8}$$

where

$$R'(\phi_\kappa) = \lim_{\kappa' \to \infty} \operatorname{Tr} \Xi_{\kappa'} \left[S\zeta - \tilde{K}(\phi_\kappa) - \frac{1}{2}(S\zeta)^2 + \frac{1}{2}\tilde{K}^2(\phi_\kappa)\right] + \frac{1}{4\pi}\|\partial\alpha\|_2^2. \tag{3.9}$$

The preceding lemma combined with our choice of α will allow us to use standard perturbative techniques to analyze the determinant, since ζ is small when ϕ is near a minimum: $\zeta(\xi_{\Sigma(\Delta)}) = O(\varpi)$ if $d(\Delta, \partial\Sigma)$ is sufficiently large (it would be zero except for our perturbation w).

As a result of the lemma,

$$Z_\Sigma = \int d\mu(\phi) \prod_\Delta \chi_{\Sigma(\Delta)}((\phi)_\Delta) e^{-\mathscr{A}(\kappa)} \prod_i \phi^\#(f_i) \det_3[1 + D^{1/2} S\zeta D^{-1/2}] e^{R'(\phi_\kappa) - \tilde{R}_\kappa}$$
$$\cdot \mathscr{J}! \operatorname{Tr}\left[\bigwedge_{j=1}^{\mathscr{J}} [1 + D^{1/2} S\zeta D^{-1/2}]^{-1} P_j\right], \tag{3.10}$$

where (as in [9, 14])

$$P_j(\cdot)=(D^{-1/2}\tilde{g}_j,\cdot)D^{1/2}S\tilde{h}_j \tag{3.11}$$

and

$$\tilde{h}_j=Uh_j,\quad \tilde{g}_j=U^{-1}g_j, \tag{3.12}$$

with

$$U=e^{i\alpha\gamma_5}. \tag{3.13}$$

Notice that there is a formal cancellation in the counterterms so that

$$\tilde{R}-R'\approx \lim_{\text{regularization}\to 0}\left(\text{bosonic terms}+\operatorname{Tr}[(S\zeta)^2/2-S\zeta]+\frac{1}{4\pi}\|\partial\alpha\|_2^2\right), \tag{3.14}$$

again indicating that perturbative methods should be effective. These terms will be dealt with more carefully later on.

We wish to shift the field ϕ so that we are near the minimum of the potential in each block. As in [5–8], let

$$g_c(x)=(\eta(-\Delta+\eta)^{-1}h)(x), \tag{3.15}$$

where η is a constant to be specified later. The function g_c is a smooth version of h, but we would prefer to shift by a function almost as smooth but more localized – the function g specified by (2.3.3)–(2.3.5) of [8]. Then we have

$$\begin{aligned}Z_\Sigma=\mathscr{I}!\int d\mu(\phi)\prod_\Delta\chi_{\Sigma(\Delta)}((\phi+g)_\Delta)\exp\left\{-\int_\Delta(:|W'_\lambda((\phi+g)_\kappa)|^2:-:|(\phi+g)_\kappa|^2:)dx\right\}\\ \cdot\prod_i(\phi+g)^\#(f_i)\operatorname{Tr}\left[\bigwedge_{j=1}^{\mathscr{I}}[1+K((\phi+g)_\kappa)]^{-1}P_j\right]\det{}_3[1+K((\phi+g)_\kappa)]\\ \cdot\exp-[R_{(\kappa)}+\textstyle\int[g^*(-\Delta+1)g+2\operatorname{Re}\phi^*(-\Delta+1)g]dx]\end{aligned} \tag{3.16}$$

where

$$K=D^{1/2}S\zeta D^{-1/2} \tag{3.17}$$

and

$$R_{(\kappa)}=\tilde{R}_\kappa((\phi+g)_\kappa)-R'((\phi+g)_\kappa). \tag{3.18}$$

Since g is smooth, we can make g_κ arbitrary close to g by taking κ large. Since we are ultimately interested in $\kappa\to\infty$, we replace (3.16) by the simpler expression

$$\begin{aligned}Z_\Sigma=\mathscr{I}!\int d\mu(\phi)\prod_\Delta\chi_{\Sigma(\Delta)}((\phi+g)_\Delta)\exp\left\{-\int_\Delta(:|W'_\lambda(\phi_\kappa+g)|^2:-:|\phi_\kappa+g|^2:)dx\right\}\\ \cdot\prod_i(\phi+g)^\#(f_i)\operatorname{Tr}\left[\bigwedge_{j=1}^{\mathscr{I}}[1+K(\phi_\kappa+g)]^{-1}P_j\right]\det{}_3[1+K(\phi_\kappa+g)]\\ \cdot\exp-[R_\kappa+\textstyle\int[g^*(-\Delta+1)g+2\operatorname{Re}\phi^*(-\Delta+1)g]dx],\end{aligned} \tag{3.19}$$

where

$$R_\kappa=\tilde{R}_\kappa(\phi_\kappa+g)-R'(\phi_\kappa+g). \tag{3.20}$$

We leave our notation unchanged although (3.19) is not equal to (3.16) as they have the same $\kappa \to \infty$ limit. Differences between g and g_κ will be $O(\kappa^{-N})$ for some large N; this shows that even the counterterm divergences behave acceptably.

We also wish to divide the action into terms that give decay along the boundary of Σ and those that do not. Thus we write

$$\mathscr{A}_\kappa = R_\kappa + \int_\Lambda [:|W'_\lambda(\phi_\kappa + g)|^2: - \eta:|\phi_\kappa + g - h|^2: - (1-\eta):|\phi_\kappa|^2:]dx \tag{3.21}$$

and $F = \sum_{i=1}^{4} F_i$ with

$$F_1 = \eta \int_\Lambda |h-g|^2 dx + \int_\Lambda |\nabla g|^2 dx, \tag{3.22}$$

$$F_2 = (1-\eta) \int_{\mathbb{R}^2 \setminus \Lambda} |h-g|^2 dx, \tag{3.23}$$

$$F_3 = 2\,\mathrm{Re}\left(\eta \int_\Lambda \phi^*(g-h)dx + \int_\Lambda \phi^*(-\Delta) g\, dx \right), \tag{3.24}$$

$$F_4 = 2(1-\eta)\,\mathrm{Re} \int_{\mathbb{R}^2 \setminus \Lambda} \phi^*(g-h)dx. \tag{3.25}$$

Here F_2 and F_4 are boundary terms, F_1 is the term that gives decay on $\partial\Sigma$, and F_3 is a correction to that. We neglect the difference between ϕ and ϕ_κ in F_3 and F_4; as with the difference in regularizations of Z_κ and $\hat{Z}_\kappa$ discussed previously, this makes no difference in the $\kappa \to \infty$ limit. Thus we write

$$Z_\Sigma = \mathscr{J}! \int d\mu(\phi) \prod_\Delta \chi_{\Sigma(\Delta)}((\phi+g)_\Delta) e^{-\mathscr{A}-F} \prod_i (\phi+g)^\#(f_i)$$
$$\cdot \mathrm{Tr}\left[\bigwedge_{j=1}^{\mathscr{J}} [1 + K(\phi_\kappa + g)]^{-1} P_j \right] \det{}_3 [1 + K(\phi_\kappa + g)]. \tag{3.26}$$

The Glimm–Jaffe–Spencer [5, 6] cluster expansion is essentially an application of the fundamental theorem of calculus. For a partially decoupled function $F(s)$ depending on parameters $\{s_b\}$ we have

$$F(1) = \sum_{\Gamma \in \mathscr{B}(\Sigma)} \int_0^1 ds_\Gamma \partial_s^\Gamma F(s), \tag{3.27}$$

where

$$\partial_s^\Gamma = \prod_{b \in \Gamma} \partial_{s_b}, \quad ds_\Gamma = \prod_{b \in \Gamma} ds_b, \tag{3.28}$$

and we set $s_b = 0$ for all $b \notin \Gamma$.

We introduce a decomposed version of Z_Σ – we choose (as in [9, 14]) to use the decoupling scheme of [1]. We denote the subset of bounds on which we decouple by $\mathscr{B}(\Sigma)$, to be chosen later. Let Δ, Δ' be l-lattice squares, and take $s \in [0,1]^{\mathscr{B}(\Sigma)}$. Then define

$$H(s, \Delta, \Delta') = \sum_{\text{finite } \gamma \subset \mathscr{B}(\Sigma)} \prod_{b \in \gamma} s_b \prod_{b \notin \gamma} (1 - s_b) \frac{\bar{C}^{\mathscr{B}(\Sigma) \setminus \gamma}(\Delta, \Delta')}{\bar{C}(\Delta, \Delta')}, \tag{3.29}$$

where

$$\bar{C}^{\Gamma}(\Delta,\Delta')=\int_{\Delta} dx \int_{\Delta'} dy C^{\Gamma}(x,y), \quad C^{\Gamma}=(-\Delta_{\Gamma}^{D}+m_c^2)^{-1}; \tag{3.30}$$

Δ_{Γ}^{D} is the Laplacian with Dirichlet boundary conditions on Γ and $m_c>0$ is a sufficiently small constant to be chosen later.

Lemma 3.2. *$H(s,\Delta,\Delta')$ has the following properties [1]:*
(a) $0 \leqq H(s,\Delta,\Delta') \leqq 1$
(b) $|\partial_s^{\gamma} H(s,\Delta,\Delta')| \leqq e^{O(l)} e^{2m_c d(\Delta,\Delta') - \delta m_c d(\gamma,\Delta,\Delta')} G_1(\gamma,\delta_1) e^{(c-\delta_2 l)|\gamma|}$,
where G_1 gives decay in the size of γ and is defined in Eq. (5.12).

Definition 3.3. *Let $\mathcal{O}$ be an operator on $\mathcal{H}'$. Then we define*

$$\mathcal{O}_s \equiv \sum_{\Delta} \chi_{\Delta} \mathcal{O} \chi_{\Delta} + \sum_{\Delta \neq \Delta'} H(s,\Delta,\Delta') \chi_{\Delta} \mathcal{O} \chi_{\Delta'}.$$

We use this definition to produce decoupled versions of the operators $K, P, \tilde{K}, \mathcal{A}$ and R.

Let $\mathcal{U}=D^{1/2}SD^{1/2}$; note that $\mathcal{U}$ is unitary. Define

$$K(s) \equiv \mathcal{U}_s (D^{-1/2})_s \zeta (D^{-1/2})_s, \tag{3.31}$$

$$P_j(s) \equiv ((D^{-1/2})_s \tilde{g}_j, \cdot)(D^{1/2}S)_s \tilde{h}_j, \tag{3.32}$$

with $\tilde{K}(s)$, $\mathcal{A}(s)$, $R(s)$ defined similarly.

Finally, we also have the partially-decoupled measure $d\mu_s$, defined as the normalized Gaussian measure with covariance

$$C_s = \sum_{\Gamma \subset \mathcal{B}(\Sigma)} \left[\prod_{b \in \Gamma} s_b \prod_{b \in \mathcal{B}(\Sigma) \setminus \Gamma} (1-s_b) \right] C^{\mathcal{B}(\Sigma)\setminus\Gamma}, \tag{3.33}$$

as in [5].

Then the decoupled version of Z_{Σ} is

$$Z_{\Sigma,s} = \mathcal{J}! \int d\mu_s(\phi) \prod_{\Delta} \chi_{\Sigma(\Delta)}((\phi+g)_{\Delta}) e^{-\mathcal{A}(s)-F} \prod_i (\phi+g)^{\#}(f_i)$$

$$\cdot \mathrm{Tr}\left[\bigwedge_{j=1}^{\mathcal{J}} [1+K(s)(\phi_{\kappa}+g)]^{-1} P_j(s) \right] \det{}_3 [1+K(s)(\phi_{\kappa}+g)]. \tag{3.34}$$

Note that $Z_{\Sigma,s}$ factors on connected regions whose boundary consists of bounds with $s_b=0$.

Applying the G-J-S expansion to $Z_{\Sigma,s}$ gives

Proposition 3.4.

$$Z_{\kappa}(\{f_i,g_j,h_j\}) = \sum_{\Sigma} Z_{\Sigma,s=1} = \sum_{\Sigma} \sum_{\Gamma \in \mathcal{B}(\Sigma)} \int ds_{\Gamma} \partial_s^{\Gamma} Z_{\Sigma,s}. \tag{3.35}$$

The s-derivatives in (3.35) can act either on the explicit s-dependence present in the integrand, or upon the measure $d\mu_s$:

$$Z_{\Sigma,s=1} = \sum_{\Gamma \in \mathcal{B}(\Sigma)} \int ds_{\Gamma} \sum_{\Gamma_f \subset \Gamma} \mathcal{J}! \int (\partial_s^{\Gamma\setminus\Gamma_f} d\mu_s) \partial_s^{\Gamma_f} \cdots. \tag{3.36}$$

A single derivative acting on the measure has the following effect:

$$\partial_{s_b} \int d\mu_s G = \int d\mu_s \int dx dy (\partial_{s_b} C_s)(x, y) \frac{\delta^2}{\delta\phi^*(x)\delta\phi(y)} G. \tag{3.37}$$

We will be able to combine this structure with the fermionic s-derivatives more easily if we adopt the replacement operator notation of [3]. The fields that are differentiated away in (3.37) are replaced with dummy fields φ_γ, using the replacement operator r_γ and the covariance operator $\mathscr{E}_\gamma$:

$$\int dx dy (\partial_s^\gamma C_s)(x, y) \frac{\delta^2}{\delta\phi^*(x)\delta\phi(y)} G = \mathscr{E}_\gamma r_\gamma^* r_\gamma G, \tag{3.38}$$

where

$$r_\gamma \phi(x_1) \cdots \phi(x_m) \phi_1^* \cdots \phi_n^* = \sum_{i=1}^{m} \phi(x_1) \cdots \phi(x_{i-1}) \varphi_\gamma(x_i) \phi(x_{i+1}) \cdots \phi(x_m) \phi_1^* \cdots \phi_n^*, \tag{3.39}$$

$$r_\gamma^* \phi_1 \cdots \phi_m \phi^*(y_1) \cdots \phi^*(y_n) = \sum_{i=1}^{n} \phi_1 \cdots \phi_m \phi^*(y_1) \cdots \phi^*(y_{i-1}) \varphi_\gamma^*(y_i) \phi^*(y_{i+1}) \cdots \phi^*(y_n), \tag{3.40}$$

and

$$\mathscr{E}_\gamma \varphi_\gamma^*(x) \varphi_\gamma(y) = (\partial_s^\gamma C_s)(x, y). \tag{3.41}$$

Then, writing

$$\mathscr{E}(\pi) = \bigotimes_{\gamma\in\pi} \mathscr{E}_\gamma \quad \text{and} \quad \tilde{r}(\pi) = \prod_{\gamma\in\pi} r_\gamma^* r_\gamma,$$

$$Z_{\Sigma, s=1} = \mathscr{I}! \sum_{\Gamma\in\mathscr{B}(\Sigma)} \int ds_\Gamma \sum_{\Gamma_f \subset \Gamma} \sum_{\substack{\text{partitions} \\ \pi_b\in\mathscr{P}(\Gamma\setminus\Gamma_f)}} \int d\mu_s \mathscr{E}(\pi_b) \tilde{r}(\pi_b) \partial_s^{\Gamma_f} \cdots. \tag{3.42}$$

Terms in our expansion arising from differentiating fermionic objects will have the form

$$k!\,\mathrm{Tr}(\wedge^k [1 + K(s)]^{-1} \cdot G) \det{}_3 [1 + K(s)] \equiv \tau_k(G), \tag{3.43}$$

where G is an operator valued distribution on $\wedge^k \mathscr{H}'$ given by antisymmetric products of operators of the type A, E and P defined below. The result of performing one differentiation on $\tau_k(G)$ is then [3]

$$\frac{\partial}{\partial s_b} \tau_k(G) = \tau_k\left(\frac{\partial G}{\partial s_b}\right) + \tau_{k+1}(G \wedge A_b) - \tau_k(G \cdot d\wedge^k E_b), \tag{3.44}$$

where

$$A_b = K^2(s) \frac{\partial K(s)}{\partial s_b}, \quad E_b = (1 - K(s)) \frac{\partial K(s)}{\partial s_b}, \tag{3.45}$$

and the operator $d\wedge^k$ identifies E_b with an element of $\wedge^k \mathscr{H}'$:

$$d\wedge^k E = kE \wedge \mathbf{I}^{k-1}, \quad \mathbf{I}^{k-1} = 1 \wedge \underset{k-1 \text{ times}}{\cdots} \wedge 1. \tag{3.46}$$

Similarly,

$$r_b\tau_k(G) = \tau_k(r_b G) + \tau_{k+1}(G \wedge A^b) - \tau_k(G\cdot d\wedge^k E^b), \tag{3.47}$$

where

$$A^b = K^2(s)r_b K(s), \quad E^b = (1-K(s))r_b K(s), \tag{3.48}$$

where we use superscripts to indicate the effect of a replacement operator and subscripts to indicate an ordinary derivative.

To obtain an overall differentiation formula we need to iterate the above expressions. We expand the derivatives ∂_s^Γ, and then identify where a particular derivative acts by writing Γ as a union of partitions [3]. First we write $\pi\in\mathscr{P}(\Gamma)$ as

$$\pi = \pi_b \cup \pi_f \tag{3.49}$$

to indicate fermion and boson derivatives. Fermionic derivatives are specified more exactly by writing

$$\pi_f = \pi_{f,A} \cup \pi_{f,E} \cup \pi_{f,P} \cup \pi_{f,B} \cup \pi_{f,S} \cup \pi_{f,0}. \tag{3.50}$$

Here A and E sub-partitions produce terms similar to A_b and E_b above, and P terms represent derivatives acting on projection operators $P_j(s)$. We use $\pi_{f,B}$ to indicate a derivative acting on $\mathscr{A}(s)$ (recall $\mathscr{A}(s)$ includes fermionic counterterms) and $\pi_{f,S}$ indicates additional derivatives of such terms. Finally, $\pi_{f,0}$ produces all additional derivatives.

The bosonic case is slightly more complicated, as we must consider both r and r^* terms. Thus we write

$$\pi_b^2 = \pi_b \cup \pi_b^*, \tag{3.51}$$

where each subset of bonds appears twice, one of which is reserved for r terms and one for r^*. Then

$$\pi_b^2 = \pi_{b,A} \cup \pi_{b,E} \cup \pi_{b,B} \cup \pi_{b,S} \cup \pi_{b,0} \cup \pi_{b,\chi} \cup \pi_{b,\Phi} \cup \pi_{b,F}, \tag{3.52}$$

where, e.g. $\pi_{b,A} = \{\gamma_1, \gamma_2, \ldots, \gamma_m, \gamma_1^*, \gamma_2^*, \ldots, \gamma_n^*\}$. The sub-partitions $\pi_{b,\chi}$, $\pi_{b,\Phi}$ and $\pi_{b,F}$ refer to derivatives of characteristic functions, test functions, and the F-terms, respectively.

Putting this all together, we have the following:

Proposition 3.5. *We can write $\partial^\Gamma Z_{\Sigma,s}$ according to the following decomposition:*

$$\begin{aligned}\partial^\Gamma Z_{\Sigma,s} = \sum_{\pi\in\mathscr{P}(\Gamma)} \sum_{\text{decompositions}} &\int d\mu_s(\phi) e^{-\mathscr{A}-F}\mathscr{E}(\pi_b)[\chi^\varphi(\pi_{b,\chi})\Phi(\pi_{b,\Phi})F_{3,4}(\pi_{b,F})\\ &\cdot \partial^{\pi_{f,S}} r(\pi_{b,S}) B(\pi_{f,B}, \pi_{b,B})\\ &\cdot \partial^{\pi_{f,0}} r(\pi_{b,0})\tau_r(P(\pi_{f,P}) \wedge A(\pi_{f,A}, \pi_{b,A})\cdot d\wedge^r E(\pi_{f,E}, \pi_{b,E}))],\end{aligned} \tag{3.53}$$

with the notation explained below.

We have

$$r(\pi_{b,X}) = \prod_{\gamma\in\pi_{b,X}} r_\gamma \prod_{\gamma^*\in\pi_{b,X}} r^*_{\gamma^*}, \tag{3.54}$$

$$\Phi(\pi_{b,\Phi}) = r(\pi_{b,\Phi}) \prod_{i=1}^{\mathscr{I}} (\phi + g)^\#(f_i), \tag{3.55}$$

and

$$\chi^{\varphi}(\pi_{b,\chi}) = r(\pi_{b,\chi}) \prod_{\Delta} \chi_{\Sigma(\Delta)}((\phi + g)_{\Delta}). \tag{3.56}$$

We also have

$$F_{3,4}(\pi_{b,F}) = \prod_{\gamma\in\pi_{b,F}} (-r_{\gamma}F) \prod_{\gamma^*\in\pi_{b,F}} (-r^*_{\gamma^*}F), \tag{3.57}$$

$$B(\pi_{f,B}, \pi_{b,B}) = \prod_{\gamma\in\pi_{f,B}} (-\partial^{\gamma}_{s}\mathscr{A}) \prod_{\gamma\in\pi_{b,B}} (-r_{\gamma}\mathscr{A}) \prod_{\gamma^*\in\pi_{b,B}} (-r^*_{\gamma^*}\mathscr{A}), \tag{3.58}$$

$$A(\pi_{f,A}, \pi_{b,A}) = \bigwedge_{\gamma\in\pi_{f,A}} K^2(s)\partial^{\gamma}_{s}K(s) \bigwedge_{\gamma\in\pi_{b,A}} K^2(s)r_{\gamma}K(s) \bigwedge_{\gamma^*\in\pi_{b,A}} K^2(s)r^*_{\gamma^*}K(s), \tag{3.59}$$

and

$$P(\pi_{f,P}) = \left(\prod_{\gamma\in\pi_{f,P}} \partial^{\gamma}_{s}\right) P_1(s) \wedge \cdots \wedge P_{\mathscr{J}}(s), \tag{3.60}$$

where each ∂^{γ}_{s} acts on a different factor P_i. Additionally,

$$\begin{aligned} d\wedge^r E(\pi_{f,E}, \pi_{b,E}) = & \prod_{\gamma\in\pi_{f,E}} d'\wedge^r(1-K(s))\partial^{\gamma}_{s}K(s) \prod_{\gamma\in\pi_{b,E}} d'\wedge^r(1-K(s))r_{\gamma}K(s) \\ & \cdot \prod_{\gamma^*\in\pi_{b,E}} d'\wedge^r(1-K(s))r^*_{\gamma^*}K(s), \end{aligned} \tag{3.61}$$

where $r = \mathscr{J} + |\pi_{f,A}| + |\pi_{b,A}|$ and the $d'\wedge$ means that terms where E derivatives precede A derivatives (according to an arbitrary ordering of the bonds) are omitted. Finally, we have the convention that $\pi_{f,S}, \pi_{b,S}, \pi_{f,0}$ and $\pi_{b,0}$ derivatives only act on already differentiated terms, one term per derivative.

We will be estimating the terms in (3.53) by fixing all the localization squares in the sum over characteristic functions within the decoupling (Definition 3.3). In addition to the explicit localizations, we implicitly insert partitions of unity into the bosonic integrals defining $\mathscr{A}$ and F in the B and $F_{3,4}$ terms. We indicate such localized terms by writing A_l, B_l, etc., and rewrite Eq. (3.53) to make this explicit:

Lemma 3.6. *The derivative $\partial^{\Gamma}Z_{\Sigma,s}$ may be written as the following sum over localized terms*:

$$\begin{aligned} \partial^{\Gamma}Z_{\Sigma,s} = & \sum_{\pi\in\mathscr{P}(\Gamma)} \sum_{\substack{\text{decomps} \\ \text{localizations}}} \int d\mu_s(\phi) e^{-\mathscr{A}-F}\mathscr{E}(\pi_b)[\chi^{\varphi}(\pi_{b,\chi})\Phi(\pi_{b,\Phi})F_{3,4,l}(\pi_{b,F}) \\ & \cdot \partial^{\pi_{f,S}} r(\pi_{b,S}) B_l(\pi_{f,B}, \pi_{b,B}) \\ & \cdot \partial^{\pi_{f,0}} r(\pi_{b,0}) \tau_r(P_l(\pi_{f,P}) \wedge A_l(\pi_{f,A}, \pi_{b,A}) \cdot d\wedge^r E_l(\pi_{f,E}, \pi_{b,E}))]. \end{aligned} \tag{3.62}$$

As $Z_{\Sigma,s}$ decouples across bonds with $s_b = 0$, we can reorder our summation to show this explicitly. Following [1], we label by Z_i the closures of the connected components of $\mathbb{R}^2$ with the decoupling bonds removed, i.e. the components of $\mathbb{R}^2 \backslash \Gamma$. Then

$$Z_{\Sigma,s} = \pm \prod_i Z_{\Sigma,s}(Z_i), \tag{3.63}$$

where the $\pm$ sign results from possible reordering of the fermionic test functions, and e.g. is $+$ for all regions Z_i that do not intersect with the support of any such function. If we let $\mathbb{Z}_i = \{Z_i, h(\partial Z_i)\}$ be the region Z_i with the phase of its boundary components specified, we can replace $\sum_{\Sigma} \sum_{\Gamma \subset \mathscr{B}(\Sigma)}$ by first summing over admissible $\mathbb{Z}_i$ (we will choose $\mathscr{B}(\Sigma)$ to be the maximal subset of bonds of Λ such that h is constant within a distance L, to be chosen later, of ∂Z_i) and then summing over all Σ and Γ which leads to $\mathbb{Z}_i$:

$$\sum_{\Sigma} \sum_{\Gamma \subset \mathscr{B}(\Sigma)} = \sum_{\{\mathbb{Z}_i\}\,\text{admissible}} \prod_i \sum_{\Sigma_i\,\text{restricted}} \sum_{\substack{\Sigma_\iota \subset \mathscr{B}(\Sigma_i, Z_i) \\ \text{constrained}}}, \tag{3.64}$$

where the restrictions on the summations are described in [1]. Then

$$Z(\{f_i, g_i, h_i\}) = \pm \sum_{\{\mathbb{Z}_j\}} \prod_j \rho(\mathbb{Z}_j), \tag{3.65}$$

where

$$\rho(\mathbb{Z}_i) = \sum_{\Sigma_\iota\,\text{restricted}} \sum_{\substack{\Sigma_i \subset \mathscr{B}(\Sigma_i, Z_i) \\ \text{constrained}}} \int ds_{\Gamma_i} \partial_s^{\Gamma_\iota} Z_{\Sigma_\iota, s}(Z_i), \tag{3.66}$$

and $Z_{\Sigma_i,s}(Z_i)$ is the same as $Z_{\Sigma,s}$ except that all the integrals in $\mathbb{R}^2$ have characteristic functions of Z_i and we replace $\mathscr{J}$ and $\mathscr{I}$ with $\mathscr{J}_{Z_i}$ and $\mathscr{I}_{Z_\iota}$, the number of test functions in Z_i. The activity $\rho(\mathbb{Z}_i)$ depends only on $\mathbb{Z}_i$ (and the test functions contained in Z_i) – specifically it does *not* depend on Σ outside Z_i.

We write our expansion in the form of (3.65)–(3.66) above because now the activity of each region $\mathbb{Z}_i$ is exponentially small in its volume $|Z_i|$. Prior to our resummation our "smallness" depended not on the size of the region but on the size of Γ.

4. Combinatorics

In order to estimate the sums in Lemma 3.6, we will make use of the method of combinatoric factors, as we did in our earlier work [9]. In fact, the discussion here follows the lines of similar discussions in Sect. 4 of [9] which in turn is based on [1, 3]. We refer the reader to these references for details not contained here.

We begin by replacing the sum over decompositions with a supremum over decompositions by including an overall factor of $O(1)^{|\pi|}$. We move on to **localize** (i.e. fix a particular choice of terms from the sum over localization squares) the F, B, P, A and E terms. The F, B, A and E terms are treated as the A terms were treated in [9, 14]: for all $\gamma \in \pi_{b,A} \cup \pi_{b,B} \cup \cdots \cup \pi_{f,E}$ we get a factor

$$O(1)|\gamma|^{O(1)} \prod_k \exp \varepsilon d(\Delta_k, \gamma) \tag{4.1}$$

when we estimate the sum over localizations by the supremum, where the product is over all localization squares in a given term.

Similarly we can localize the P factors with combinatoric factors

$$O(1) \exp \varepsilon [d(\Delta_j, \bar{\Delta}_j) + d(\Delta'_j, \bar{\Delta}'_j)], \tag{4.2}$$

where $\bar{\Delta}_j$ and $\bar{\Delta}'_j$ are the supports of the test functions h_j and g_j.

This results in

Proposition 4.1.

$$|\partial_s^\Gamma Z_{\Sigma,s}(Z)| = O(1)^{|\Gamma|} \sum_\pi \sup_{\substack{\text{decomps,}\\ \text{localizations}}} \prod_{j=1}^{\mathscr{J}_Z} O(1) \exp \varepsilon [d(\Delta_j, \bar{\Delta}_j) + d(\Delta'_j, \bar{\Delta}'_j)]$$

$$\prod_{\substack{\gamma \in \pi_{b,A} \cup \pi_{f,A} \cup \pi_{b,B} \cup \pi_{f,B} \\ \cup \pi_{b,F} \cup \pi_{b,E} \cup \pi_{f,E} \cup \pi_{b,P} \cup \pi_{f,P}}} \left(O(1) |\gamma|^{O(1)} \prod_k \exp \varepsilon d(\Delta_k, \gamma) \right)$$

$$\cdot \left| \int d\mu_s(\phi) e^{-\mathscr{A} - F} \mathscr{E}(\pi_b) [\chi^\varphi(\pi_{b,\chi}) \Phi(\pi_{b,\Phi}) F_{3,4,l}(\pi_{b,F}) \partial^{\pi_{f,S}} r(\pi_{b,S}) B_l(\pi_{f,B}, \pi_{b,B}) \right.$$

$$\left. \cdot \partial^{\pi_{f,O}} r(\pi_{b,O}) \tau_r(P_l(\pi_{f,P}) \wedge A_l(\pi_{f,A}, \pi_{b,A}) \cdot d \wedge^r E_l(\pi_{f,E}, \pi_{b,E}))] \right|.$$

We will use additional combinatoric factors to restrict the internal structure of many of our terms, as well as additional derivatives $(\pi_{b,O} \cdots \pi_{f,S})$. These estimates all make use of exponential pinning, similarly to those on p. 301 of [1].

We begin by bounding the exterior derivative – reducing the sum over terms to the supremum of such terms. To count the number of terms, let $e_L(\Delta)$ be the number of E terms with left-most localization square Δ. Then the number of terms in $d' \wedge^r E_l$ is bounded by

$$2^{\mathscr{J}_Z + |\Gamma|} \prod_\Delta e_L(\Delta)! \leqq 2^{\mathscr{J}_Z} O(1)^{|\Gamma|} \prod_{\gamma \in \pi_{b,E} \cup \pi_{f,E}} \exp \varepsilon d(\gamma, \Delta_L), \tag{4.3}$$

where Δ_L is the left-most localization square corresponding to the E factor being differentiated (by γ).

Similarly we can restrict the sum generating P derivatives. There are $\mathscr{J}_Z$ factors and $|\pi_{f,P}|$ derivatives, so there are less than $(2\mathscr{J}_Z)^{|\pi_{f,P}|}$ possibilities. We have the exponential pinning bound

$$(2\mathscr{J}_Z)^{|\pi_{f,P}|} \leqq O(1)^{|\pi_{f,P}| + \mathscr{J}_Z} \prod_{\gamma \in \pi_{f,P}} \exp [\varepsilon \min \{ d(\gamma, \Delta_l), d(\gamma, \bar{\Delta}_l) \}], \tag{4.4}$$

where Δ_l is one of the localization squares corresponding to γ chosen via (4.2), and $\bar{\Delta}_l$ is the corresponding test function square. The distribution of "additional" derivatives is also restricted. For $\partial^{\pi_{f,O}} r(\pi_{b,O})$ there are less than $[O(1)(\pi_{f,A} \cup \pi_{b,A} \cup \pi_{f,E} \cup \pi_{b,E})]^{|\pi_{f,O} \cup \pi_{b,O}|}$ factors, and for $\partial^{\pi_{f,S}} r(\pi_{b,S})$ there are less than $[O(1)(\pi_{f,B} \cup \pi_{b,B})]^{|\pi_{f,S} \cup \pi_{b,S}|}$ factors, for which we have estimates similar to (4.4):

$$[O(1)(\pi_{f,A} \cup \pi_{b,A} \cup \pi_{f,E} \cup \pi_{b,E})]^{|\pi_{f,O} \cup \pi_{b,O}|}$$
$$\leqq O(1)^{|\pi|} \prod_{\gamma \in \pi_{f,O} \cup \pi_{b,O}} \exp [\varepsilon \min \{ d(\gamma, \Delta_l), d(\gamma, \Delta'_l) \}], \tag{4.5}$$

$$[O(1)(\pi_{f,B} \cup \pi_{b,B})]^{|\pi_{f,S} \cup \pi_{b,S}|} \leqq O(1)^{|\pi|} \prod_{\gamma \in \pi_{f,S} \cup \pi_{b,S}} \exp [\varepsilon \min \{ d(\gamma, \Delta_l), d(\gamma, \Delta'_l) \}], \tag{4.6}$$

where Δ_l and Δ'_l are the localization squares chosen via (4.1) or its equivalent that surround the factor being differentiated by γ.

We also need to fix our choice of derivatives generating χ and Φ terms. As in [1], pp. 306–307, the number of terms making up $\chi^\varphi(\pi_{b,\chi})$ is bounded by

$$\prod_{\gamma \in \pi_{b,\chi}} O(l^2) \exp [\varepsilon d(\gamma, \Delta_L)], \tag{4.7}$$

and the number of terms making up $\Phi(\pi_{b,\Phi})$ is bounded by

$$O(1)^{\mathscr{I}_Z} \prod_{\gamma\in\pi_{b,\Phi}} O(l^2)\exp[\varepsilon d(\gamma, \Delta_L)]. \tag{4.8}$$

Finally, because each $K(s)$ has three decouplings each term can actually be differentiated a number of ways, so that the partitions $\pi_{b,A}, \pi_{f,A}, \ldots$, must be divided into sub-partitions. However, this simply gives a factor of $O(1)^{|\Gamma|}$, and we will leave our notation unchanged for the time being.

The above discussion can be summarized by the following proposition:

Proposition 4.2.

$$|\partial_s^\Gamma Z_{\Sigma,s}(Z)| \leqq O(1)^{|\Gamma|+\mathscr{I}_Z+\mathscr{I}_Z} \sum_{\pi} \sup_{\substack{\text{decomps,}\\ \text{localizations}}} \sup_{\text{derivatives}} \prod_{j=1}^{\mathscr{I}_Z} O(1)\exp\varepsilon[d(\Delta_j, \bar{\Delta}_j) + d(\Delta'_j, \bar{\Delta}'_j)]$$
$$\cdot \prod_{\gamma\in\pi_{f,E}\cup\pi_{b,E}} \exp\varepsilon d(\gamma, \Delta_L) \prod_{\gamma\in\pi_{f,P}} \exp[\varepsilon \min\{d(\gamma, \Delta_l), d(\gamma, \bar{\Delta}_l)\}]$$
$$\cdot \prod_{\substack{\gamma\in\pi_{b,A}\cup\pi_{f,A}\cup\pi_{b,B}\cup\pi_{f,B}\\ \cup\pi_{b,F}\cup\pi_{b,E}\cup\pi_{f,E}\cup\pi_{b,P}\cup\pi_{f,P}}} \left(O(1)|\gamma|^{O(1)} \prod_k \exp\varepsilon d(\Delta_k, \gamma)\right)$$
$$\cdot \prod_{\gamma\in\pi_{b,\chi}\cup\pi_{b,\Phi}} O(l^2)\exp[\varepsilon d(\gamma, \Delta_L)]$$
$$\cdot \left|\int d\mu_s(\phi) e^{-\mathscr{A}-F} \mathscr{E}(\pi_b)[\chi^{\varphi'}(\pi_{b,\chi})\Phi'(\pi_{b,\Phi})F'_{3,4,l}(\pi_{b,F})\partial^{\pi_{f,S'}} r'(\pi_{b,S})B'_l(\pi_{f,B}, \pi_{b,B})\right.$$
$$\left.\cdot \partial^{\pi_{f,0'}} r'(\pi_{b,0})\tau_r(P'_l(\pi_{f,P}) \wedge A'_l(\pi_{f,A}, \pi_{b,A})\cdot d\wedge^r E'_l(\pi_{f,E}, \pi_{b,E}))]\right|.$$

where the primes on χ^φ, Φ, A_l, $d'\wedge^r E_l$, etc., *indicate that we take only one term form each summation.*

5. More Estimates

We now wish to simplify and estimate the terms appearing in Proposition 4.2. First of all, the τ_r expression now consists of exactly one term, as the primes and l's indicate. Thus we write

$$\tau_r(\mathbb{Q}) \equiv \partial^{\pi_{f,0'}} r'(\pi_{b,0})\tau_r(P'_l(\pi_{f,P}) \wedge A'_l(\pi_{f,A}, \pi_{b,A})\cdot d\wedge^r E'_l(\pi_{f,E}, \pi_{b,E})), \tag{5.1}$$

where

$$\mathbb{Q} = Q_1 \wedge Q_2 \wedge \cdots \wedge Q_r \tag{5.2}$$

and Q_i is of the form

$$Q_i = A_l^{\pi_1} E_l^{\pi_2} E_l^{\pi_3} \cdots E_l^{\pi_{k_i}} \tag{5.3}$$

or

$$Q_i = P_l^{\pi_1} E_l^{\pi_2} E_l^{\pi_3} \cdots E_l^{\pi_{k_i}}. \tag{5.4}$$

The expression A_l^π may result from fermion derivatives:

$$K_l^2(s)\partial_s^\gamma K_l(s), \quad K_l(s)\partial_s^{\gamma_2} K_l(s)\partial_s^{\gamma_1} K_l(s), \quad \partial_s^{\gamma_3} K_l(s)\partial_s^{\gamma_2} K_l(s)\partial_s^{\gamma_1} K_l(s), \tag{5.5}$$

or from the action of the replacement operators:

$$K_l^2(s)r_\gamma K_l(s), \quad K_l(s)r_{\gamma_2}K_l(s)r_{\gamma_1}K_l(s), \quad r_{\gamma_3}K_l(s)r_{\gamma_2}K_l(s)r_{\gamma_1}K_l(s), \tag{5.6}$$

or mixed, for example:

$$K_l(s)\partial_s^{\gamma_2}K_l(s)r_{\gamma_3}\partial_s^{\gamma_1}K_l(s). \tag{5.7}$$

It represents the original derivative in $\pi_{f,A}$ or $\pi_{b,A}$ as well as those from $\pi_{f,0}$ and $\pi_{b,0}$. The expressions P_l^π and E_l^π are similar.

For convenience we introduce the following notation:

$$\chi^{\varphi'}(\pi_{b,\chi}) = \Phi^\chi(\pi_{b,\chi})\chi(\pi_{b,\chi}) \tag{5.8}$$

with

$$\Phi^\chi(\pi_{b,\chi}) = \prod_{\gamma\in\pi_{b,\chi}} (\varphi_\gamma)_{\Delta_\gamma} \prod_{\gamma^*\in\pi_{b,\chi}} (\varphi_{\gamma^*}^*)_{\Delta_{\gamma^*}}. \tag{5.9}$$

We split $\chi^{\varphi'}(\pi_{b,\chi})$ in this fashion so that $\chi(\pi_{b,\chi})$ can be associated with the action while $\Phi^\chi(\pi_{b,\chi})$ is associated with the $\Phi'(\pi_{b,\Phi})$ and $F_{3,4,l}(\pi_{b,F})$ terms.

We also note that the replacement covariances $\mathscr{E}(\pi_b)$ can be written (in two different ways) as a linear combination of measures, as in [1, 3]. Thus

$$\mathscr{E}(\pi_b) = \bigotimes_{\gamma\in\pi_b} \sum_{\rho} \pm\, d\mu^{\gamma,\rho}(\varphi_\gamma). \tag{5.10}$$

The sum producing the linear combination of measures can be eliminated with a combinatoric factor $O(1)^{|\Gamma|}$. Thus we can write

Proposition 5.1.

$$\begin{aligned}
|\partial_s^\Gamma Z_{\Sigma,s}(Z)| \leqq O(1)^{|\Gamma|+\mathscr{I}_Z+\mathscr{I}_Z} \sum_{\pi} \sup_{\substack{\text{decomps,}\\ \text{localizations}}} \sup_{\text{derivatives}} \prod_{j=1}^{\mathscr{I}_Z} O(1)\exp\varepsilon[d(\Delta_j,\bar{\Delta}_j)+d(\Delta'_j,\bar{\Delta}'_j)] \\
\cdot \prod_{\gamma\in\pi_{f,E}\cup\pi_{b,E}} \exp\varepsilon d(\gamma,\Delta_L) \prod_{\gamma\in\pi_{f,P}} \exp[\varepsilon\min\{d(\gamma,\Delta_l), d(\gamma,\bar{\Delta}_l)\}] \\
\cdot \prod_{\substack{\gamma\in\pi_{b,A}\cup\pi_{f,A}\cup\pi_{b,B}\cup\pi_{f,B}\\ \cup\pi_{b,F}\cup\pi_{b,E}\cup\pi_{f,E}\cup\pi_{b,P}\cup\pi_{f,P}}} \left(O(1)|\gamma|^{O(1)}\prod_k \exp\varepsilon d(\Delta_k,\gamma)\right) \\
\cdot \prod_{\gamma\in\pi_{b,\chi}\cup\pi_{b,\Phi}} O(l^2)\exp[\varepsilon d(\gamma,\Delta_L)] \\
\cdot \left| \int d\mu_s(\phi)\otimes \bigotimes_{\gamma\in\pi_b} d\mu^{\gamma,\rho}(\varphi_\gamma)[\chi(\pi_{b,\chi})e^{-\mathscr{A}-F}\Phi^\chi(\pi_{b,\chi})\Phi'(\pi_{b,\Phi})F'_{3,4,l}(\pi_{b,F}) \right. \\
\left. \cdot\, \partial^{\pi_{f,S'}}r'(\pi_{b,S})B'_l(\pi_{f,B},\pi_{b,B})\tau_r(\mathbb{Q})] \right| .
\end{aligned}$$

Since we have expressed $\mathscr{E}(\pi_b)$ as a linear combination of measures, we can bound the above expression using inequalities for measures. Specifically, let $\|\cdot\|_{L^q}$ represent an $L^q\left(d\mu_s(\phi)\otimes\bigotimes_\gamma d\mu^{\gamma,\rho}(\varphi_\gamma)\right)$ norm. (More accurately it represents the geometric mean of the two different ways of expressing $\mathscr{E}$ as a combination of measures, as in [3].) Then we can use Hölder's inequality to get

$$\Big| \int d\mu_s(\phi) \otimes \bigotimes_{\gamma\in\pi_b} d\mu^{\gamma,\rho}(\varphi_\gamma)[\chi(\pi_{b,\chi})e^{-\mathscr{A}-F}\Phi^\chi(\pi_{b,\chi})\Phi'(\pi_{b,\Phi})F'_{3,4,l}(\pi_{b,F})$$

$$\cdot\,\partial^{\pi_{f,S'}}r'(\pi_{b,S})B'_l(\pi_{f,B},\pi_{b,B})\tau_r(\mathbb{Q})]\Big|$$

$$\begin{aligned}&\leqq \|\chi(\pi_{b,\chi})e^{-\mathscr{A}-F}\tau_r(\mathbb{Q})\|_{L^p}\|\partial^{\pi_{f,S'}}r'(\pi_{b,S})B'_l(\pi_{f,B},\pi_{b,B})\|_{L^q}\\ &\quad\cdot\|\Phi^\chi(\pi_{b,\chi})\|_{L^q}\|\Phi'(\pi_{b,\Phi})\|_{L^{q/2}}\|F'_{3,4,l}(\pi_{b,F})\|_{L^q}\end{aligned} \tag{5.11}$$

for some $p>1$ and $\frac{1}{p}+\frac{5}{q}\leqq 1$.

5.1. Gaussian Integration Estimates. Let

$$G_1(\gamma,\delta)\equiv \sum_{\sigma\in S_{|\gamma|}} \exp -\delta|l_\sigma(\gamma)|, \tag{5.12}$$

where S_n denotes the permutation group on n elements, and where the "size" $|l_\sigma(\gamma)|$ of the linear ordering of γ determined by the permutation $\sigma\in S_{|\gamma|}$ is defined in [3], p. 12. The G_1 factor gives the primary decay in the number of derivatives.

We begin to estimate the L^q norms on the terms in Eq. (5.11). We will make use of the following result (essentially from [1]):

Lemma 5.2. *There exists $\delta>0$ such that for all $q:1<q<\infty$,*

$$\|\varphi_\gamma(\chi_\Delta f)\|_{L^q}\leqq\|\chi_\Delta f\|_{L^2}O(l^{4/q})O(1)^{|\gamma|}\exp\left\{-\frac{\delta}{2}m_c d(\gamma,\tilde{\Delta})\right\}G_1(\gamma,\delta/4), \tag{5.13}$$

where $\tilde{\Delta}$ is the l-block containing the unit-lattice block Δ.

The above inequality allows us to prove the following lemmas:

Lemma 5.3. *Let $\Delta_{\gamma^\#}$ be the localization square corresponding to the differentiation at $\gamma^\#$. Then*

$$\|\Phi^\chi(\pi_{b,\chi})\|_{L^q}\leqq\prod_{\gamma^\#\in\pi_{b,\chi}}O(1)^{|\gamma^\#|}O(l)\exp\{-\tfrac{1}{2}\delta m_c d(\gamma^\#,\tilde{\Delta}_{\gamma^\#})\}G_1(\gamma^\#,\delta/4). \tag{5.14}$$

Proof. The norm can written

$$\begin{aligned}\|\Phi^\chi(\pi_{b,\chi})\|_{L^q}&=\Big\|\prod_{\gamma\in\pi_{b,\chi}}(\varphi_{\gamma^\#})_{\Delta_{\gamma^\#}}\Big\|_{L^q}\\ &\leqq\prod_{\gamma\in\pi_{b,\chi}}\|(\varphi_{\gamma^\#})_{\Delta_{\gamma^\#}}\|_{L^{2q}},\end{aligned} \tag{5.15}$$

due to Hölder's inequality, as each field appears at most twice. Now by (5.13),

$$\|(\varphi_{\gamma^\#})_{\Delta_{\gamma^\#}}\|_{L^{2q}}\leqq O(l^{2/q})O(1)^{|\gamma^\#|}\exp\{-\tfrac{1}{2}\delta m_c d(\gamma^\#,\tilde{\Delta}_{\gamma^\#})\}G_1(\gamma^\#,\delta/4), \tag{5.16}$$

and the proof is complete. ■

Lemma 5.4. *Let $\mathscr{I}(\Delta)$ be the number of test functions localized in the square Δ. Then*

$$\|\Phi'(\pi_{b,\Phi})\|_{L^{q/2}} \leqq O(\lambda^{-1})^{\mathscr{I}_Z} \prod_{i=1}^{\mathscr{I}_Z} \|f_i\|_{L^2} \prod_{\Delta\subset Z} (\mathscr{I}(\Delta)!)^{1/2}$$
$$\cdot \prod_{\gamma^\#\in\pi_{b,\Phi}} O(1)^{|\gamma^\#|} O(l) \exp\{-\tfrac{1}{2}\delta m_c d(\gamma^\#, \tilde{\Delta}_{\gamma^\#})\} G_1(\gamma^\#, \delta/4). \tag{5.17}$$

Proof. The operator $\Phi'(\pi_{b,\Phi})$ has both original and replacement fields:

$$\Phi'(\pi_{b,\Phi}) = \prod_i{}' (\phi+g)^\#(f_i) \prod_i{}'' \varphi^\#_{\gamma_i^\#}(f_i). \tag{5.18}$$

Thus we have

$$\|\Phi'(\pi_{b,\Phi})\|_{L^{q/2}} \leqq \left\|\prod_i{}' (\phi+g)^\#(f_i)\right\|_{L^q} \left\|\prod_i{}'' \varphi^\#_{\gamma_i^\#}(f_i)\right\|_{L^q}. \tag{5.19}$$

The replacement field term is bounded as in Lemma 5.3:

$$\left\|\prod_i{}'' \varphi^\#_{\gamma_i^\#}(f_i)\right\|_{L^q} \leqq \prod_{\gamma^\#\in\pi_{b,\Phi}} O(1)^{|\gamma^\#|} O(l) \exp\{-\tfrac{1}{2}\delta m_c d(\gamma^\#, \tilde{\Delta}_{\gamma^\#})\} G_1(\gamma^\#, \delta/4) \prod_i{}'' \|f_i\|_{L^2}. \tag{5.20}$$

For the original field, we make use of checkerboard, Hölder and hypercontractivity estimates (as was done in [1]). Thus

$$\begin{aligned}
\left\|\prod_i{}' (\phi+g)^\#(f_i)\right\|_{L^q} &\leqq \prod_{\Delta\subset Z} \left\|\prod_{\omega=1}^{\mathscr{I}(\Delta)} (\phi+g)^\#(f_{i_\omega})\right\|_{L^{q'}} \\
&\leqq \prod_{\Delta\subset Z} \prod_{\omega=1}^{\mathscr{I}(\Delta)} \|(\phi+g)^\#(f_{i_\omega})\|_{L^{\mathscr{I}(\Delta)q'}} \\
&\leqq \prod_{\Delta\subset Z} \prod_{\omega=1}^{\mathscr{I}(\Delta)} [O(\mathscr{I}(\Delta))^{1/2} \|\phi^\#(f_{i_\omega})\|_{L^2} + |g^\#(f_{i_\omega})|] \\
&\leqq \prod_{\Delta\subset Z} (\mathscr{I}(\Delta)!)^{1/2} O(\lambda^{-1})^{\mathscr{I}(\Delta)} \prod_{\omega=1}^{\mathscr{I}(\Delta)} \|\phi^\#(f_{i_\omega})\|_{L^2}
\end{aligned} \tag{5.21}$$

Combining (5.20) and (5.21) we complete the proof. ∎

Lemma 5.5. *Choose $K > 0$; let λ be sufficiently small. Then*

$$\|F'_{3,4,l}(\pi_{b,F})\|_{L^q} \leqq \prod_{\gamma^\#\in\pi_{b,F}} O(1)^{|\gamma^\#|} \lambda^K \exp\{-\tfrac{1}{3}\delta m_c d(\gamma^\#, \tilde{\Delta}_{\gamma^\#})\} G_1(\gamma^\#, \delta/4). \tag{5.22}$$

Proof. $F'_{3,4,l}$ is a product of terms of the form

$$-2\,\mathrm{Re} \int_{\tilde{\Delta}_{\gamma^*}} \varphi^*_{\gamma^*}[(\eta(g-h) - \Delta g)\chi_\Lambda + (1-\eta)(g-h)(1-\chi_\Lambda)]\,dx. \tag{5.23}$$

Thus

$$\begin{aligned}
\|F'_{3,4,l}(\pi_{b,F})\|_{L^q} &\leqq \prod_{\gamma^\#\in\pi_{b,F}} \left\| \int_{\tilde{\Delta}_{\gamma^\#}} \varphi^\#_{\gamma^\#}[(\eta(g-h) - \Delta g)\chi_\Lambda + (1-\eta)(g-h)(1-\chi_\Lambda)]^{\bar{\#}}\,dx \right\|_{L^{2q}} \\
&\leqq \prod_{\gamma^\#\in\pi_{b,F}} \|\chi_{\tilde{\Delta}_{\gamma^\#}}[(\eta(g-h) - \Delta g)\chi_\Lambda + (1-\eta)(g-h)(1-\chi_\Lambda)]\|_{L^2} \\
&\quad \cdot O(l^{2/q}) O(1)^{|\gamma^\#|} \exp\{-\tfrac{1}{2}\delta m_c d(\gamma^\#, \tilde{\Delta}_{\gamma^\#})\} G_1(\gamma^\#, \delta/4)].
\end{aligned} \tag{5.24}$$

Let Z^0 be the set of unit squares in Z within a distance L of a phase boundary, i.e. such that within a distance L the configuration Σ is not a constant. The functions within the L^2 norm are identically zero unless

$$d(\tilde{\Delta}_{\gamma^\#}, Z \backslash Z^0) \geqq \tfrac{1}{2}L \geqq \tfrac{1}{2}|\log \lambda|^2 \tag{5.25}$$

if we take $L = O(|\log \lambda|^2)$. Since $\mathscr{B}(\Sigma)$ does not intersect $Z^0, d(\gamma^\#, \tilde{\Delta}_{\gamma^\#}) > \frac{1}{2}|\log \lambda|^2$ and

$$\exp\{-\tfrac{1}{2}\delta m_c d(\gamma^\#, \tilde{\Delta}_{\gamma^\#})\} \leqq \lambda^{K'} \exp\{-\tfrac{1}{3}\delta m_c d(\gamma^\#, \tilde{\Delta}_{\gamma^\#})\}. \tag{5.26}$$

for any $K' > 0$, if λ is sufficiently small.

The L^2 norm is $O(l\lambda^{-1})$; we bound

$$O(l\lambda^{-1})O(l^{2/q})\lambda^{K'} \leqq \lambda^K, \tag{5.27}$$

which completes the proof. ∎

5.2. Estimates for Derivatives of the Action. $\partial^{\pi_{f,S'}} r'(\pi_{b,S}) B_l'(\pi_{f,B}, \pi_{b,B})$ is a product of terms, each of which is of one of the forms below:

$$r(\{\gamma^\#\}) \int_{\Delta_1 \times \Delta_2} :\mathscr{W}_j((\phi+g)(x))\mathscr{W}_j^*((\phi+g)(y)): \partial_s^\gamma \sigma_j(s)(x,y)\,dx\,dy \tag{5.28}$$

or

$$r(\{\gamma^\#\}) \int_{\Delta} :|\mathscr{W}_j((\phi+g)(x))|^2: dx \tag{5.29}$$

(as well as boundary corrections which are bounded by const. $\times \|\chi_\Delta \partial\alpha\|_2^2$). The polynomials $\mathscr{W}_j$ have the property that if we expand them around $\phi = 0$, away from the phase boundary (i.e. where $g = h$) we obtain the small factors which are needed to ensure convergence of our expansion. For the purely bosonic terms like (5.29), we get a factor λ (or ϖ) since if g (approximately) minimizes the potential, the constant, linear and quadratic terms (almost) cancel, leaving behind terms of order $\lambda^{-2}|\lambda\phi|^{n'} + O(\varpi)$ for $n' \geqq 3$. Near a phase boundary we can get a big factor – $O(\lambda^{-2})$ – which is controlled, however, by exponential decay factors e^{-1/λ^2}.

Counterterm Estimates. Obtaining small factors from the counterterms is a little more delicate (their finiteness was demonstrated in [10] for the fully-coupled case and in [19] for the decoupled case). However, we can simplify matters by only considering those terms which are field independent – every field is associated with a factor λ and the polynomials arise from $\partial^{n'} W_\lambda$ for $n' \geqq 2$ and thus have coefficients $O(1)$ or smaller. Thus only terms containing, e.g. $W_\lambda''(g)$ are important for this discussion.

First consider the fully coupled case ($\mathbf{s} = \mathbf{1}$). We need to examine terms of the form

$$\operatorname{Tr} S\gamma_0 X S \gamma_0 X, \tag{5.30}$$

where one or both of the propagators has a cutoff and $X = X_1 + i\gamma_5 X_2$ represents $\Upsilon(W_\lambda''(g)) - 1$ or $e^{2i\alpha\gamma_5}\Upsilon(W_\lambda''(g)) - 1$.

We wish to commute X through S in order to obtain a local expression. We neglect the commutator of X with a cutoff; since X is smooth this vanishes as $\kappa \to \infty$. Similarly $[X, S]$ introduces terms ∂X which are boundary terms. Taking the Dirac structure into account we get (as the local part of (5.30))

$$\operatorname{Tr}\left[\frac{-1}{p^2+1}(X_1^2+X_2^2)+\frac{1}{(p^2+1)^2}2X_1^2\right]. \tag{5.31}$$

Therefore the relevant counterterm contribution is

$$|W_\lambda''(g)-1|^2\int\frac{d^2p}{(2\pi)^2}\frac{\tilde{\Xi}_\kappa^2(p)}{p^2+1}+2\operatorname{Re}(W_\lambda''(g)-1)\int\frac{d^2p}{(2\pi)^2}\frac{\tilde{\Xi}_\kappa(p)}{p^2+1} \tag{5.32}$$

from the bosons,

$$\begin{aligned}&-2\operatorname{Re}(W_\lambda''(g)-1)\int\frac{d^2p}{(2\pi)^2}\frac{\tilde{\Xi}_\kappa(p)}{p^2+1}-|W_\lambda''(g)-1|^2\int\frac{d^2p}{(2\pi)^2}\frac{\tilde{\Xi}_\kappa^2(p)}{p^2+1}\\&+2(\operatorname{Re}(W_\lambda''(g)-1))^2\int\frac{d^2p}{(2\pi)^2}\frac{\tilde{\Xi}_\kappa^2(p)}{(p^2+1)^2}\end{aligned} \tag{5.33}$$

from the initial fermions, and

$$\begin{aligned}&2\operatorname{Re}(W_\lambda''(g)-e^{2i\alpha}W_\lambda''(g))\int\frac{d^2p}{(2\pi)^2}\frac{\Xi_{\kappa'}(p)}{p^2+1}\\&+(|W_\lambda''(g)-1|^2-|e^{2i\alpha}W_\lambda''(g)-1|^2)\int\frac{d^2p}{(2\pi)^2}\frac{\Xi_{\kappa'}(p)}{p^2+1}\\&-2[(\operatorname{Re}(W_\lambda''(g)-1))^2-(\operatorname{Re}(e^{2i\alpha}W_\lambda''(g)-1))^2]\int\frac{d^2p}{(2\pi)^2}\frac{\Xi_{\kappa'}(p)}{(p^2+1)^2}\end{aligned} \tag{5.34}$$

from our unitary transformation.

Adding (5.32), (5.33), (5.34) gives us

$$2(\operatorname{Re}(W_\lambda''(g)-1))^2\int\frac{d^2p}{(2\pi)^2}\frac{\tilde{\Xi}_\kappa^2(p)-\tilde{\Xi}_{\kappa'}(p)}{(p^2+1)^2}+2(\operatorname{Re}(e^{2i\alpha}W_\lambda''(g)-1))^2\int\frac{d^2p}{(2\pi)^2}\frac{\Xi_{\kappa'}(p)}{(p^2+1)^2}. \tag{5.35}$$

As $\kappa, \kappa' \to \infty$ the first term of (5.35) goes to zero, and the last term is either $O(\varpi)$ or a boundary term since $e^{2i\alpha}\tilde{W}_\lambda''(g)=1$ far from $\partial\Sigma$. ∎

Now consider the case of partially decoupled counterterms. The bosons are not affected by this procedure. For the fermions, the decoupling does not affect diagonal terms – only terms in different l-blocks. The off-diagonal parts are more regular than the diagonal parts; thus e.g. $\operatorname{Tr}(K^2(s)-K^2(s=1))$ has a limit as we remove the cutoff (cf. Lemmas 6.4, 6.5 so that the difference in cutoffs between $\tilde{R}_\kappa$ (2.16) and R' (3.9) is irrelevant (as $\kappa\to\infty$). Therefore our formal cancellation (3.14) properly describes the behavior of the off-diagonal terms; since $\zeta(g)$ is $O(\varpi)$ except near phase boundaries we are done.

The kernels $\partial_s^\gamma\sigma_j(s)(x,y)$ all come from the counterterms, and are in $L^{p_\sigma}(\Delta_1\times\Delta_2)$ for some $p_\sigma>1$ as in [10] and decay exponentially. The derivative ∂_s^γ will give an added factor of

$$e^{O(l)}O(1)^{|\gamma|}\sum_{\gamma_1\cup\cdots\cup\gamma_6=\gamma}\prod_{i=1}^{6}\exp\{-\delta m_c d(\gamma_i,x,y)\}G_1(\gamma_i,\delta) \tag{5.36}$$

for m_c sufficiently small, as $\partial_s^\gamma \sigma_j$ arises from terms like $\operatorname{Tr} \partial_s^\gamma K^2(s)$, i.e.

$$\partial_s^\gamma \prod_{i=1}^{6} H(s, \Delta_i, \Delta_{i+1}) \chi_{\Delta_1} \mathscr{U} \chi_{\Delta_2} D^{-1/2} \chi_{\Delta_3} \zeta D^{-1/2} \chi_{\Delta_4} \mathscr{U} \chi_{\Delta_5} D^{-1/2} \chi_{\Delta_6} \zeta D^{-1/2} \chi_{\Delta_7}, \tag{5.37}$$

and $\partial_s^\gamma H$ has the decay properties indicated by Lemma 3.2. (These terms are trace class although $K^2(s)$ is not since $\partial_s^\gamma H(s, \Delta, \Delta) = 0$; see the computation of $\mathscr{E}_B$ in the proof of Lemma 6.5.)

With the above discussion in mind, we have the following lemma:

Lemma 5.6. *Let q be a positive integer; let λ, ϖ be sufficiently small. Let $M_B(\Delta)$ be the number of subterms localized in the square Δ arising from the derivatives of the action. Then*

$$\begin{aligned} &\| \partial^{\pi_{f,S'}} r'(\pi_{b,S}) B_l'(\pi_{f,B}, \pi_{b,B}) \|_{L^q} \\ &\quad \leqq \prod_{\Delta} \exp[(nM_B(\Delta))^{9/8} O(\log q)] \\ &\quad \quad \prod_{\gamma^\# \in \pi_{f,B} \cup \pi_{b,B} \cup \pi_{f,S} \cup \pi_{b,S}} \lambda e^{O(l)} O(1)^{|\gamma^\#|} \exp\{-\tfrac{1}{3} \delta m_c d(\gamma^\#, \Delta_{\gamma^\#}^1, \Delta_{\gamma^\#}^2)\} G_6(\gamma^\#, \delta/4), \end{aligned} \tag{5.38}$$

where

$$G_6(\gamma, \delta) = \sup_{\gamma_1 \cup \cdots \cup \gamma_6 = \gamma} \prod_{i=1}^{6} G_1(\gamma_i, \delta). \tag{5.39}$$

Proof. Using Theorem 8.5.5 of [4], and making use of our discussion about the counterterms to pull factors of λ from the kernels $\partial_s^\gamma \sigma_j$ we have

$$\begin{aligned} \| \partial^{\pi_{f,S'}} r'(\pi_{b,S}) B_l'(\pi_{f,B}, \pi_{b,B}) \|_{L^q}^q &\leqq \left\| \prod_{\gamma^\# \in \pi_{b,B} \cup \pi_{b,S}} \chi_{\Delta_{\gamma^\#}} n \varphi_{\gamma^\#}^\# \right\|_{L^q}^q O(\lambda)^{q|\{\Delta_{\gamma^\#} \notin Z^0\}|} \\ &\quad \cdot \left\| \prod_{\gamma \in \pi_{f,B} \cup \pi_{f,S}} \chi_{\Delta_\gamma^1} \partial_s^\gamma \sigma_j(s) \chi_{\Delta_\gamma^2} \right\|_{L^{p_\sigma}(\mathbb{R}^2 \times \cdots \times \mathbb{R}^2)}^q \\ &\quad \cdot O(\lambda^{-1})^{q|\{\Delta_{\gamma^\#} \in Z^0\}|} \prod_{\Delta} (nqM_B(\Delta))!. \end{aligned} \tag{5.40}$$

The term $\| \prod \chi_{\Delta_{\gamma^\#}} n \varphi_{\gamma^\#}^\# \|_{L^q}$ is bounded using Lemma 5.2 and Hölder's inequality (each field appears at most twice):

$$\begin{aligned} \| \prod \chi_{\Delta_{\gamma^\#}} n \varphi_{\gamma^\#}^\# \|_{L^q} &\leqq \| \prod \chi_{\Delta_{\gamma^\#}} n \varphi_{\gamma^\#}^\# \|_{L^q} \\ &\leqq \prod n O(l^{2+4/q}) O(1)^{|\gamma^\#|} \exp\{-\tfrac{1}{2} \delta m_c d(\gamma^\#, \Delta_{\gamma^\#})\} G_1(\gamma^\#, \delta/4). \end{aligned} \tag{5.41}$$

We also have

$$\begin{aligned} &\| \prod \chi_{\Delta_\gamma^1} \partial_s^\gamma \sigma_j(s) \chi_{\Delta_\gamma^2} \|_{L^{p_\sigma}(\mathbb{R}^2 \times \cdots \times \mathbb{R}^2)} \\ &\quad \leqq \prod \| \chi_{\Delta_\gamma^1} \partial_s^\gamma \sigma_j(s) \chi_{\Delta_\gamma^2} \|_{L^{p_\sigma}(\Delta_\gamma^1 \times \Delta_\gamma^2)} \\ &\quad \leqq \prod e^{O(l)} O(1)^{|\gamma|} \sum_{\gamma_1 \cup \cdots \cup \gamma_6 = \gamma} \prod_{i=1}^{6} \exp\{-\delta m_c d(\gamma_i, \Delta_{\gamma^1}, \Delta_{\gamma^2})\} G_1(\gamma_i, \delta), \end{aligned} \tag{5.42}$$

using our estimates on the kernels $\partial_s^\gamma \sigma_j$. As in (5.25)–(5.26), we only get factors of λ^{-1}, where $d(\gamma^\#, \tilde{\Delta}_{\gamma^\#}) > \frac{1}{2} |\log \lambda|^2$ so these bad factors are cancelled by our exponential decay:

$$\begin{aligned}&\|\partial^{\pi_{f,S}'}r'(\pi_{b,S})B_l'(\pi_{f,B},\pi_{b,B})\|_{L^q}\\&\leqq\left[\prod_{\Delta}(nqM_B(\Delta))!\right]^{1/q}\prod_{\gamma^{\#}\in\pi_{f,B}\cup\pi_{b,B}\cup\pi_{f,S}\cup\pi_{b,S}}\lambda e^{O(l)}O(1)^{|\gamma^{\#}|}\\&\cdot\exp\{-\tfrac{1}{3}\delta m_c d(\gamma^{\#},\Delta^1_{\gamma^{\#}},\Delta^2_{\gamma^{\#}})\}\,G_6(\gamma^{\#},\delta/4).\end{aligned}\tag{5.43}$$

Using Stirling's formula to bound the factorial completes the proof. ■

5.3. Estimates for Wedge Products. We need to estimate the operator $\mathbb{Q}$; we begin with the following lemma:

Lemma 5.7. *Let $M_{Q_i}(\Delta)$ be the number of subterms localized in the square Δ from the term Q_i; let G_9 be defined analogously to G_6. Then for m_c sufficiently small,*

$$\begin{aligned}\|\,\|Q_i\|_1\,\|_{L^q}&\leqq\prod_{\Delta}\exp[(4nM_{Q_i}(\Delta))^{9/8}O(\log q)]\prod_{\gamma^{\#}\text{ causing }Q_i}\lambda e^{O(l)}O(1)^{|\gamma^{\#}|}\\&\cdot\exp[-\tfrac{1}{3}\delta m_c d(\gamma^{\#},\Delta^1_{\gamma^{\#}},\Delta^2_{\gamma^{\#}})]\,G_9(\gamma^{\#},\delta/4)\prod_i\exp[-\delta d(\Delta^i_{\gamma^{\#}},\Delta^{i+1}_{\gamma^{\#}})]\\&\cdot\{\lambda^{-1}\|g_j\|_{\mathscr{H}}\|h_j\|_{\mathscr{H}}\exp[-\delta[d(\Delta^1_i,\Delta^2_i)+d(\Delta^3_i,\Delta^4_i)]]\},\end{aligned}\tag{5.44}$$

where the term in braces is present only if Q_i contains a projection operator (P term).

Proof. We bound $\|Q_i\|_1$ by

$$\|A_l^{\pi_1}\|_1\|E_l^{\pi_2}\|\cdots\|E_l^{\pi_{k_i}}\|\tag{5.45}$$

or

$$\|P_l^{\pi_1}\|_1\|E_l^{\pi_2}\|\cdots\|E_l^{\pi_{k_i}}\|.\tag{5.46}$$

The A term is a product of three K terms, so we can write

$$\begin{aligned}\|A_l^{\pi}\|_1&\leqq\sum_{\cup\gamma_i^j=\pi}\prod_{i=1}^{3}\|r_{\gamma_i^1}\partial_s^{\gamma_i^2}K_l(s)\|_3\\&\leqq\sum_{\cup\gamma_i^j=\pi}\prod_{i=1}^{3}\|r_{\gamma_i^1}K_l\|_3\sum_{\cup\gamma_{ik}^2=\gamma_i^2}\prod_{k=1}^{3}|\partial_s^{\gamma_{ik}^2}H(s,\Delta_i^k,\Delta_i^{k+1})|.\end{aligned}\tag{5.47}$$

We estimate the field dependent piece, assuming here that all the characteristic functions are near each other. If they are not, we obtain similar estimates with exponential decay in the distance between the blocks through the use of Lemma A.2,

$$\begin{aligned}\|r_\gamma K_l\|_3&=\|\chi_{\Delta_1}\mathscr{U}\chi_{\Delta_2}D^{-1/2}\chi_{\Delta_3}r_\gamma\zeta D^{-1/2}\chi_{\Delta_4}\|_3\\&\leqq\|\chi_{\Delta_2}D^{-1/4}\|_{12}\|D^{-1/4}\chi_{\Delta_3}r_\gamma\zeta D^{-1/2}\chi_{\Delta_4}\|_4.\end{aligned}\tag{5.48}$$

The fourth power of the 4-norm appearing in (5.48) may be written as

$$\int_{\Delta_3\times\Delta_3\times\Delta_3\times\Delta_3}\mathscr{W}(x_1,\dots,x_4)(r_\gamma\zeta)(x_1)(r_\gamma\zeta)^*(x_2)(r_\gamma\zeta)(x_3)(r_\gamma\zeta)^*(x_4)dx_1\cdots dx_4,\tag{5.49}$$

where $\mathscr{W}\in L^p((\Delta_3)^4)$ for some $p>1$. We obtain our needed factors of λ since each ζ is $O(\lambda)$ unless it is localized near a phase boundary. In fact it is the requirement that terms like (5.48) be small that forced us to implement our unitary rotation which produced ζ from $\Upsilon(W_\lambda'')-1$.

E_l^π is of the form

$$(1-K_l(s))r_{\gamma_1}\partial_s^{\gamma_2}K_l(s) \tag{5.50}$$

or

$$r_{\gamma_1}\partial_s^{\gamma_2}K_l(s)r_{\gamma_3}\partial_s^{\gamma_4}K_l(s). \tag{5.51}$$

In either case we can bound this as we did with the A terms, with terms like

$$\int \mathscr{W}(x_1,\ldots,x_4)(r_{\gamma_j}\zeta)(x_1)(r_{\gamma_j}\zeta)^*(x_2)(r_{\gamma_j}\zeta)(x_3)(r_{\gamma_j}\zeta)^*(x_4)dx_1\cdots dx_4 \cdot\prod_k |\partial_s^{\gamma_k}H(s,\Delta_k,\Delta_{k+1})|. \tag{5.52}$$

Finally the projection operator P_l^π is clearly bounded:

$$\begin{aligned}\|P_l^\pi\| &\leqq \|\chi_{\Delta_1^1}D^{-1/2}\tilde{g}_j\chi_{\Delta_1^2}\|_{L^2(\mathbb{R}^2)}\|\chi_{\Delta_2^1}D^{1/2}S\tilde{h}_j\chi_{\Delta_2^2}\|_{L^2(\mathbb{R}^2)}\prod_{i=1}^{2}|\partial_s^{\gamma_i}H(s,\Delta_i^1,\Delta_i^2)|\\ &\leqq \|g_j\|_{\mathscr{H}}\|h_j\|_{\mathscr{H}}\exp[-\delta[d(\Delta_1^1,\Delta_1^2)+d(\Delta_2^1,\Delta_2^2)]]\\ &\quad\cdot\left\{e^{O(l)}\prod_{i=1}^{2}O(1)^{|\gamma_i|}\exp[-\delta m_c d(\gamma_i,\Delta_i^1,\Delta_i^2)]G_1(\gamma^i,\delta)\right\},\end{aligned} \tag{5.53}$$

where $\pi=\gamma_1\cup\gamma_2$; the term in braces is absent if π is empty. When we combine this with reasoning identical to the proof of Lemma 5.6 we complete the proof. ■

For more than one Q_i, we have the obvious generalization:

Corollary 5.8. *Let $M_{\mathbb{Q}}(\Delta)$ be the number of P, A or E subterms localized in the square Δ. Then*

$$\begin{aligned}\|\,\|\mathbb{Q}\|_1\|_{L^p} \leqq \frac{1}{r!}\prod_{\Delta}\exp[(4nM_{\mathbb{Q}}(\Delta))^{9/8}O(\log q)] &\prod_{\substack{\gamma^\#\in\pi_{b,A}\cup\pi_{b,E}\cup\pi_{b,0}\cup\\ \pi_{f,A}\cup\pi_{f,E}\cup\pi_{f,0}\cup\pi_{f,P}}} e^{O(l)}O(1)^{|\gamma^\#|}\\ &\cdot\exp\{-\tfrac{1}{3}\delta m_c d(\gamma^\#,\Delta_{\gamma^\#}^1,\Delta_{\gamma^\#}^2)\}G_9(\gamma^\#,\delta/4)\prod_i\exp\{-\delta d(\Delta_{\gamma^\#}^i,\Delta_{\gamma^\#}^{i+1})\}\\ &\cdot\lambda^{|\pi_{b,A}\cup\pi_{b,E}\cup\pi_{b,0}\cup\pi_{f,A}\cup\pi_{f,E}\cup\pi_{f,0}|}\prod_{j=1}^{\mathscr{I}_Z}\|g_j\|_{\mathscr{H}}\|h_j\|_{\mathscr{H}}\\ &\cdot\exp\{-\delta[d(\Delta_i^1,\Delta_i^2)+d(\Delta_i^3,\Delta_i^4)]\}.\end{aligned} \tag{5.54}$$

6. Vacuum Energy Bound

We wish to calculate an $L^p(d\mu)$ norm of

$$e^{-\mathscr{A}-F}\chi(\pi_{b,\chi})\tau_r(\mathbb{Q}), \tag{6.1}$$

uniformly as $\kappa,\Lambda\to\infty$. In this section we will estimate the L^p norm of

$$z^r \equiv e^{-\mathscr{A}-F}\chi(\pi_{b,\chi})\|\wedge^r[1+K(s)]^{-1}\det_3[1+K(s)]\|, \tag{6.2}$$

which is sufficient to bound (6.1) as we can estimate

$$|\tau_r(\mathbb{Q})|\leqq r!\|\mathbb{Q}\|_1\|\wedge^r[1+K(s)]^{-1}\det_3[1+K(s)]\| \tag{6.3}$$

using operator and trace norms, and the trace norm in (6.3) is estimated in Sect. 5.3.

Let $\mathscr{Z}^r \equiv \| z^r \|$ be the operator norm of (6.2). Let n_Δ be the number of times the characteristic function $\chi_{\Sigma(\Delta)}((\phi + g)_\Delta)$ is differentiated, and let Z' be the union of the blocks where $n_\Delta > 0$. We will prove the following:

Proposition 6.1. *There exists $p > 1$ such that for λ, ϖ sufficiently small,*

$$\| \mathscr{Z}^r \|_{L^p} \leqq O(1)^r \left[\prod_{\Delta} \exp(O(1) n_\Delta)(n_\Delta!)^{O(1)} \right] \exp[O(1)\lambda |Z|] \exp[-O(1)\lambda^{-2} |Z^0 \cup Z'|]. \tag{6.4}$$

The basic idea behind the proof of Proposition 6.1 is that the fermionic determinant is relatively unimportant compared to the bosonic terms in the action. Following [1, 3, 10, 17, 19] we can bound the determinant by terms which have small coefficients and are of lower degree in the field ϕ than the bosonic potential; the result then follows from standard results of $\mathscr{P}(\phi)_2$ models [4–8]. Compare this with the trivially obtained bounds on the determinant in [9].

We begin by splitting $K(s)$ into its diagonal and off-diagonal parts. We write

$$K(s) = A + B, \tag{6.5}$$

where

$$A = \sum_{\Delta \subset Z} \chi_\Delta \mathscr{U} \chi_\Delta D^{-1/2} \chi_\Delta \zeta D^{-1/2} \chi_\Delta, \tag{6.6}$$

$$B = \sum_{\substack{\Delta_1 \cdots \Delta_4 \\ \Delta_1 \neq \Delta_2 \text{ or } \Delta_2 \neq \Delta_3 \\ \text{or } \Delta_3 \neq \Delta_4}} H(s, \Delta_1, \ldots, \Delta_4) \chi_{\Delta_1} \mathscr{U} \chi_{\Delta_2} D^{-1/2} \chi_{\Delta_3} \zeta D^{-1/2} \chi_{\Delta_4} \tag{6.7}$$

and

$$H(s, \Delta_1, \ldots, \Delta_4) = H(s, \Delta_1, \Delta_2) H(s, \Delta_2, \Delta_3) H(s, \Delta_3, \Delta_4); \quad 0 \leqq H(s, \Delta_1, \ldots, \Delta_4) \leqq 1. \tag{6.8}$$

Using this decomposition, we have a determinant inequality:

Lemma 6.2. *There exist constants $a, c > 0$ such that*

$$\| \wedge^r [1 + K(s)]^{-1} \det_3 [1 + K(s)] \| \leqq \exp[\tfrac{1}{4} \operatorname{Tr}(A + A^*)^2 + a \| AB \|_1 + \| B \|_2^2 + cr]. \tag{6.9}$$

Proof. Using Lemma V.5 of [1], there exist $a', c > 0$ such that

$$\begin{aligned} &\| \wedge^r [1 + K(s)]^{-1} \det_3 [1 + K(s)] \| \\ &\quad \leqq (\det_3 [1 + O_A^+])^{1/2} \exp[\tfrac{1}{2} \operatorname{Re} \operatorname{Tr} B^2 + \operatorname{Re} \operatorname{Tr} AB - \operatorname{Re} \operatorname{Tr} A^* A^2 \\ &\quad\quad - \tfrac{1}{4} \operatorname{Tr}(A^* A)^2 + a' \| AB \|_1 + \tfrac{1}{2} \| B \|_2^2 + cr]. \end{aligned} \tag{6.10}$$

Here

$$O_A = A + A^* + A^* A, \tag{6.11}$$

and O_A^+ is its positive part. Using standard determinant inequalities, we have

$$\ln \det_3 [1 + O_A^+] \leqq \tfrac{1}{2} \operatorname{Tr}(O_A^+)^2 \leqq \tfrac{1}{2} \operatorname{Tr}(O_A)^2 = \tfrac{1}{2} \operatorname{Tr}(A + A^*)^2 + \tfrac{1}{2} \operatorname{Tr}(A^* A)^2 + 2 \operatorname{Re} \operatorname{Tr} A^* A^2. \tag{6.12}$$

Thus we can replace $(\det_3[1+O_A^+])^{1/2}$ in (6.10) yielding

$$\begin{aligned}&\| \wedge^r[1+K(s)]^{-1}\det_3[1+K(s)]\| \\ &\quad \leqq \exp[\tfrac{1}{4}\mathrm{Tr}(A+A^*)^2+\tfrac{1}{2}\mathrm{Re}\,\mathrm{Tr}\,B^2+\mathrm{Re}\,\mathrm{Tr}\,AB+a'\|AB\|_1+\tfrac{1}{2}\|B\|_2^2+cr].\end{aligned} \tag{6.13}$$

Setting $a=a'+1$ yields (6.9). ■

Ordinarily an inequality involving a trace quadratic in A would be useless, as A is not a Hilbert–Schmidt operator (almost everywhere). However, $(A+A^*)$ is Hilbert–Schmidt (a.e.) as the singularities cancel. We see this by writing

$$(A+A^*)^2=\sum_i(K_i+K_i^*)^2-\sum_i[B_iA_i+B_iB_i+A_iB_i+B_i^*A_i+B_i^*B_i+A_i^*B_i+\text{adjoint}], \tag{6.14}$$

where

$$A_i=\chi_i\mathscr{U}\chi_iD^{-1/2}\chi_i\zeta D^{-1/2}\chi_i, \tag{6.15}$$

$$B_i=\sum_{\substack{j,k,l\\ j\neq k \text{ or } k\neq i\\ \text{or } i\neq l}}\chi_j\mathscr{U}\chi_kD^{-1/2}\chi_i\zeta D^{-1/2}\chi_l. \tag{6.16}$$

Then

$$K_i=\mathscr{U}D^{-1/2}\chi_i\zeta D^{-1/2} \tag{6.17}$$

is of the form of the finite volume K from [10], where it is shown that the singularities cancel. The remaining terms are sufficiently regular since at least one pair of characteristic functions must be off-diagonal.

We wish to estimate the determinant by a product over blocks of operators that only depend on the field within that block. This will allow us to prove Proposition 6.1. The A and K terms are already of this form, but we have to show that the B and AB terms can be estimated in this fashion:

Proposition 6.3. *For $\varepsilon>0$ there exists $p>1$ such that (almost everywhere)*

$$\| \wedge^r[1+K(s)]^{-1}\det_3[1+K(s)]\|\leqq c_1^r\prod_\Delta E_\Delta(\phi), \tag{6.18}$$

where $E_\Delta(\phi)$ only depends upon ϕ within Δ. Furthermore

$$E_\Delta(\phi_k)\leqq\exp\text{const.}\times l^{O(\varepsilon)}[\|\chi_\Delta(W_\lambda''(\phi_\kappa+g)e^{2i\alpha}-1)\|_{2+\varepsilon}^2+\|\chi_\Delta\partial\alpha\|_{2+\varepsilon}^2] \tag{6.19}$$

and we can write

$$\begin{aligned}E_\Delta(\phi)=\exp\Bigg[&\int_{\Delta\times\Delta}dx\,dy\,\mathscr{E}(x,y)(W_\lambda''(\phi+g)e^{2i\alpha}-1)(x)(W_\lambda''(\phi+g)e^{2i\alpha}-1)^*(y)\\ &+\text{const.}\times l^{O(\varepsilon)}\|\chi_\Delta\partial\alpha\|_{2+\varepsilon}^2\Bigg],\end{aligned} \tag{6.20}$$

where $\mathscr{E}(\cdot)\in L^p(\Delta\times\Delta)$.

Although the proof of Proposition 6.3 is quite long, it is actually just a straightforward application of the regularity of the bosonic covariance. The difficulty arises in transferring regularity across characteristic functions. The techniques to do this are taken from [1], but must be refined due to the non-Gaussian nature of the interaction.

To derive (6.18), we need to estimate $\|AB\|_1$ and $\|B\|_2^2$ by a sum over blocks. We begin by moving some regularity from B to A. Let $\nu>0$ be small. Then

$$\|AB\|_1 = \|AD^{-\nu}D^{\nu}B\|_1 \leqq \|AD^{-\nu}\|_2\|D^{\nu}B\|_2 \leqq \|AD^{-\nu}\|_2^2 + \|D^{\nu}B\|_2^2. \tag{6.21}$$

Now

$$\|AD^{-\nu}\|_2^2 = \operatorname{Tr}\sum_i D^{-\nu}A_i^*A_iD^{-\nu} = \sum_i \|A_iD^{-\nu}\|_2^2, \tag{6.22}$$

while

$$\|D^{-\nu}B\|_2^2 = \operatorname{Tr} \sum_{\substack{ijklmnp\\ i\neq j \text{ or } j\neq k \text{ or } k\neq l\\ l\neq m \text{ or } m\neq n \text{ or } n\neq p}} D^{2\nu}\chi_i\mathcal{U}\chi_jD^{-1/2}\chi_k\zeta D^{-1/2}\chi_lD^{-1/2}\chi_m\zeta^*D^{-1/2}\chi_n\mathcal{U}^*\chi_p$$

$$\leqq \sum_{km}\left\| \sum_{\substack{ijlnp\\ i\neq j \text{ or } j\neq k \text{ or } k\neq l\\ l\neq m \text{ or } m\neq n \text{ or } n\neq p}} D^{2\nu}\chi_i\mathcal{U}\chi_jD^{-1/2}\chi_k\zeta D^{-1/2}\chi_lD^{-1/2}\chi_m\zeta^*D^{-1/2}\chi_n\mathcal{U}^*\chi_p\right\|_1. \tag{6.23}$$

If $d(\Delta_k,\Delta_m)\leqq\sqrt{2}l$, we bound the norm in (6.23) by $\|D^{2\nu}B_k\|_2\|B_m\|_2$. If this is not the case, then either Δ_k,Δ_l or Δ_l,Δ_m don't touch. Choose the pair with the greatest separation. Then using Lemma A.2 (essentially Lemma 2.2 of [17]),

$$\left\|\sum_{\substack{ijlnp\\ i\neq j \text{ or } j\neq k \text{ or } k\neq l\\ l\neq m \text{ or } m\neq n \text{ or } n\neq p}} \cdots\right\|_1 \leqq \text{const.}\times\sum_l \|\tilde{B}_k\|_2\|D^{2\nu}B_m\|_2 e^{-c'd(\Delta_k,\Delta_l)}$$

$$\leqq \text{const.}\times\|\tilde{B}_k\|_2\|D^{2\nu}B_m\|_2 e^{-cd(\Delta_k,\Delta_m)}, \tag{6.24}$$

where

$$\tilde{B}_i = \mathcal{U}D^{-1/2}\chi_i\zeta D^{-2}. \tag{6.25}$$

Thus

$$\|D^{\nu}B\|_2^2 \leqq \sum_{\substack{ij\\ d(\Delta_i,\Delta_j)\leqq\sqrt{2}l}} \|D^{2\nu}B_i\|_2\|B_j\|_2 + \text{const.}\times\sum_{ij}\|\tilde{B}_i\|_2\|D^{2\nu}B_j\|_2e^{-cd(\Delta_i,\Delta_j)}$$

$$\leqq \text{const.}\times\sum_i(\|D^{2\nu}B_i\|_2^2+\|B_i\|_2^2+\|\tilde{B}_i\|_2^2)$$

$$\leqq \text{const.}\times\sum_i(\|D^{2\nu}B_i\|_2^2+\|\tilde{B}_i\|_2^2). \tag{6.26}$$

Of course (6.23), (6.24), (6.26) hold for $\nu=0$, so $\|B\|_2^2$ is bounded in the same way. So combining (6.9), (6.14), (6.21), (6.22) and (6.26), we have

$$\|\wedge^r[1+K(s)]^{-1}\det_3[1+K(s)]\|$$

$$\leqq c_1^r\prod_i \exp[\tfrac{1}{4}\operatorname{Tr}(K_i+K_i^*)^2 + a\|A_iD^{-\nu}\|_2^2 + b\|D^{2\nu}B_i\|_2^2 + c\|\tilde{B}_i\|_2^2]$$

$$\equiv c_1^r\prod_i E_i(\phi), \tag{6.27}$$

yielding the block structure we need for (6.18). ■

We now move on to the proof of (6.19), where we bound $E_i(\phi_\kappa)$ in terms of $2+\varepsilon$ norms. The basic structure follows that of Proposition A.I.1 of [1], which we unfortunately cannot use directly as their bounds in terms of 4-norms ($\|\cdot\|_4$) are insufficient for our purposes. We also note that the added Dirac structure resulting from our unitary transformation $e^{i\alpha\gamma_5}$ does not alter the results of [1] and [10] in any significant sense.

We will need the following lemma:

Lemma 6.4. *Choose $\varepsilon>0$ small. Then there exists $v>0$ and constants (depending on ε) $c_A, c_B, \ldots$, such that for $f\in L^{2+\varepsilon}(\mathbb{R}^2)$,*

$$\text{(a)} \qquad \|\chi_i \mathscr{U} \chi_i D^{-1/2}\chi_i f D^{-1/2}\chi_i D^{-v}\|_2 \leqq c_A l^{O(\varepsilon)} \|\chi_i f\|_{2+\varepsilon}. \tag{6.28}$$

If $i\neq j$, $j\neq k$, or $k\neq l$,

$$\text{(b)} \qquad \|D^{2v}\chi_i \mathscr{U}\chi_j D^{-1/2}\chi_k f D^{-1/2}\chi_l\|_2 \leqq c_B l^{O(\varepsilon)}\|\chi_i f\|_{2+\varepsilon}\exp - c_B'[d(\Delta_i,\Delta_j)+d(\Delta_j,\Delta_k)+d(\Delta_k,\Delta_l)], \tag{6.29}$$

and

$$\text{(c)} \qquad \|\mathscr{U}D^{-1/2}\chi_i f D^{-2}\|_2 \leqq c_{\tilde{B}} l^{O(\varepsilon)}\|\chi_i f\|_{2+\varepsilon}. \tag{6.30}$$

Parts (a), (b) and (c) represent bounds for A, B and $\tilde{B}$ terms, respectively. We reserve the proof of Lemma 6.4 for an appendix.

It is now clear that a short computation to separate the terms arising from the Dirac structure yields bounds of the form (6.19) for all the terms in (6.27) except for $\mathrm{Tr}(K_i+K_i^*)^2$. This term, however, was specifically designed so that the bounds of [10] would apply. Thus we essentially duplicate Eq. (36) of [10]:

$$\mathrm{Tr}(K_i(\phi_\kappa)+K_i^*(\phi_\kappa))^2 \leqq \text{const.}\times\int_{\Delta_i}[|W_\lambda''(\phi_\kappa+g)e^{2i\alpha}-1|^2+|\partial\alpha|^2]dx, \tag{6.31}$$

and complete the derivation of (6.19). ∎

To complete the proof of Proposition 6.3 we need to determine the L^p properties of the kernels of the operators making up $E_i(\phi)$. If we didn't have to worry about the characteristic functions within the operators of E_i, such properties would be obvious. Unfortunately, the characteristic functions obscure the behavior of $\mathscr{E}(x,y)$.

We get around this problem in a fashion similar to that used in the appendix, in the proof of Lemma 6.4. We commute the covariance operators through the characteristic functions, and eventually obtain terms with commutators and terms without (extra) characteristic functions. Where we have no characteristic functions the bounds are simple to derive. Where we have commutators we have extra regularity that allows us to obtain the bounds we need.

Let $\mathscr{E}_K(\cdot)$ be the kernel of $\mathrm{Tr}(K_i+K_i^*)^2$, $\mathscr{E}_A(\cdot)$ be the kernel of $\|A_i D^{-v}\|_2^2$, and similarly for $\mathscr{E}_B(\cdot)$ and $\mathscr{E}_{\tilde{B}}(\cdot)$.

Lemma 6.5. *There exists $p>1$ such that*

$$\mathscr{E}_\#(\cdot)\in L^p(\Delta\times\Delta)$$

for $\#=K, A, B$ and $\tilde{B}$.

Proof. The lemma follows directly from the results of [10] for $\mathscr{E}_K$ and is straightforward for $\mathscr{E}_{\tilde{B}}$, we we will concentrate on the other two cases. First

$$\mathscr{E}_A(x,y)=(D^{-1/2}\chi_i D^{-2\nu}\chi_i D^{-1/2})(x,y)(D^{-1/2}\chi_i\mathscr{U}^*\chi_i\mathscr{U}\chi_i D^{-1/2})(x,y)\chi_i(x)\chi_i(y). \quad (6.32)$$

Now

$$D^{-1/2}\chi_i D^{-2\nu}\chi_i D^{-1/2}=D^{-3\nu/2}(D^{-1/2+3\nu/2}\chi_i D^{-2\nu}\chi_i D^{-1/2}), \quad (6.33)$$

where we can clearly see that the operator in parentheses is Hilbert–Schmidt. As $D^{-3\nu/2}$ has a kernel in $L^{4/(4-3\nu)}$, we can use Young's inequality to show that the left-hand side of (6.33) has a kernel in $L^{4/(2-3\nu)}$.

The other factor making up $\mathscr{E}_A$ can be written

$$\begin{aligned} D^{-1/2}\chi_i\mathscr{U}^*\chi_i\mathscr{U}\chi_i D^{-3/10}D^{-2/10} &= D^{-1/2}\chi_i\mathscr{U}^*\chi_i\mathscr{U}D^{-3/10}\chi_i D^{-2/10} \\ &\quad +(D^{-1/2}\chi_i\mathscr{U}^*\chi_i\mathscr{U}[\chi_i,D^{-3/10}]D^{-1/10})D^{-1/10}. \end{aligned} \quad (6.34)$$

Using Lemma A.1, the term in parentheses is easily seen to be Hilbert–Schmidt, so its convolution with $D^{-1/10}$ is in $L^{2.1}$. For the other term, we commute some more, giving

$$\begin{aligned} D^{-1/2}\chi_i\mathscr{U}^*\chi_i\mathscr{U}D^{-3/10}\chi_i D^{-2/10} &= D^{-1/2}\chi_i\mathscr{U}^*\mathscr{U}D^{-3/10}\chi_i D^{-2/10} \\ &\quad + D^{-1/2}\chi_i\mathscr{U}^*[\chi_i,\mathscr{U}D^{-3/10}]\chi_i D^{-2/10} \\ &= D^{-1/2}\chi_i D^{-3/10}\chi_i D^{-2/10} \\ &\quad +(D^{-1/2}\chi_i\mathscr{U}^*[\chi_i,\mathscr{U}D^{-3/10}]\chi_i D^{-1/10})D^{-1/10}. \end{aligned} \quad (6.35)$$

With Young's inequality we can show that $D^{-1/2}\chi_i D^{-3/10}\chi_i D^{-2/10}$ has a kernel in $L^{2-\varepsilon'}$ for any $\varepsilon'>0$, while as the parenthesized term is once again Hilbert–Schmidt, we once again determine that the right-most term is in $L^{2.1}$.

Combining the above results we see that the kernel of $D^{-1/2}\chi_i\mathscr{U}^*\chi_i\mathscr{U}\chi_i D^{-3/10}D^{-10}$ is in $L^{2-\varepsilon'}$ for any $\varepsilon'>0$. If we also use Hölder's inequality with the bound on (6.33), we finally see that $\mathscr{E}_A\in L^{1+3\nu/4}$.

We now move on to the kernel of $\|D^{2\nu}B_i\|_2^2$:

$$\begin{aligned} \mathscr{E}_B(x,y)= \sum_{\substack{jklmn \\ j\neq i \text{ or } i\neq k \text{ or } k\neq l \\ m\neq n \text{ or } n\neq i \text{ or } j\neq i}} &(\chi_i D^{-1/2}\chi_j D^{-1/2}\chi_i)(x,y) \\ &\cdot(\chi_i D^{-1/2}\chi_k\mathscr{U}^*\chi_l D^{2\nu}\chi_m\mathscr{U}\chi_n D^{-1/2}\chi_i)(x,y). \end{aligned} \quad (6.36)$$

First let us consider $\chi_i D^{-1/2}\chi_j D^{-1/2}\chi_i$. For $i=j$,

$$(\chi_i D^{-1/2}\chi_j D^{-1/2}\chi_i)(\cdot)\in L^{2-\varepsilon'} \quad (6.37)$$

for any $\varepsilon'>0$. If $i\neq j$,

$$\begin{aligned} \chi_i D^{-1/2}\chi_j D^{-1/2}\chi_i &= \chi_i D^{-1/2}[\chi_j,D^{-1/2}]\chi_i \\ &= \chi_i D^{-1/10}(D^{-1/2}D^{1/10}[\chi_j,D^{-1/2}]). \end{aligned} \quad (6.38)$$

The parenthesized term is Hilbert–Schmidt so that similarly to the case for $\mathscr{E}_A$ we obtain an $L^{2.1}$ bound for (6.38). Furthermore, if i and j do not touch, $\chi_j D^{-1/2}\chi_i$ has an L^∞-norm which is exponentially small in the separation between i and j.

Thus

$$\left(\sum_{j:i\neq j}\chi_i D^{-1/2}\chi_j D^{-1/2}\chi_i\right)(\cdot)\in L^{2.1}. \tag{6.39}$$

We move on to the other half of $\mathscr{E}_B$. If all the characteristic functions are the same (in which case $i\neq j$),

$$\begin{aligned}
&\chi_i D^{-1/2}\chi_k\mathscr{U}^*\chi_l D^{2\nu}\chi_m\mathscr{U}\chi_n D^{-1/2}\chi_i\\
&\quad=\chi_i D^{-1/2}\chi_k\mathscr{U}^*\chi_l D^{2\nu}\chi_m\mathscr{U}D^{-1/2}\chi_i+\chi_i D^{-1/2}\chi_k\mathscr{U}^*\chi_l D^{2\nu}\chi_m\mathscr{U}[\chi_n,D^{-1/2}]\chi_i\\
&\quad=\chi_i D^{-1/2}\chi_k\mathscr{U}^*\chi_l D^{2\nu}\mathscr{U}D^{-1/2}\chi_i+\chi_i D^{-1/2}\chi_k\mathscr{U}^*\chi_l D^{2\nu}[\chi_m,\mathscr{U}D^{-1/2}]\chi_i\\
&\qquad+\chi_i D^{-1/2}\chi_k\mathscr{U}^*\chi_l D^{2\nu}\chi_m\mathscr{U}[\chi_n,D^{-1/2}]\chi_i\\
&\quad\vdots\\
&\quad=\chi_i D^{-1+2\nu}\chi_i+\text{sum of commutator terms.}
\end{aligned} \tag{6.40}$$

The kernels of the commutator terms are all in (say) $L^{2.1}$ (if ν is small enough), similarly to (6.38). Additionally, $(\chi_i D^{-1+2\nu}\chi_i)(\cdot)\in L^{2-4\nu}$.

If the characteristic functions are *not* all the same, we essentially repeat the above procedure, except that there will be no non-commutator term left over. We can also get exponential decay as we did before, which results in

$$\left(\sum_{\substack{klmn\\ j\neq i\text{ or }i\neq k\text{ or }k\neq l\\ m\neq n\text{ or }n\neq i\text{ or }j\neq i}}\chi_i D^{-1/2}\chi_k\mathscr{U}^*\chi_l D^{2\nu}\chi_m\mathscr{U}\chi_n D^{-1/2}\chi_i\right)(\cdot)\in\begin{cases}L^{2.1}, & i=j\\ L^{2-4\nu}, & i\neq j\end{cases}, \tag{6.41}$$

and complementary bounds for $\chi_i D^{-1/2}\chi_j D^{-1/2}\chi_i$. Together, these yields (for ν small enough)

$$\mathscr{E}_B(\cdot)\in L^{1.02}, \tag{6.42}$$

completing the proof of Lemma 6.5 and therefore Proposition 6.3 as well. ■

With a bound for the determinant in terms of a product over blocks, we are readily to proceed with the proof of Proposition 6.1. We notice that

$$\begin{aligned}
l^{O(\varepsilon)}\|\chi_i W''_\lambda(\phi_\kappa+g)\|^2_{2+\varepsilon}&\leqq l^{O(\varepsilon)}(\|\chi_i|W''_\lambda(\phi_\kappa+g)|^{2+\varepsilon}\|_1+1)\\
&\leqq l^{O(\varepsilon)}O(\|\chi_i|\lambda(\phi_\kappa+g)|^{2n-2}\|_1)
\end{aligned} \tag{6.43}$$

if ε is sufficiently small (recall that n is the degree of the superpotential), so that for large ϕ_κ the terms arising from the determinant are bounded by the leading term of the bosonic potential. Thus we have Wick lower bounds which are just the integrated versions of those in [8], derived in exactly the same way, provided that our perturbation ϖ is small enough:

$$\begin{aligned}
&\int_\Delta[:|W'_\lambda(\phi_\kappa+g)|^2:-\eta:|\phi_\kappa+g-h|^2:]dx-\ln E_\Delta(\phi_\kappa)-\ln\chi_{\Sigma(\Delta)}((\phi+g)_\Delta)\\
&\quad\leqq-\text{const.}\times l^{2+O(\varepsilon)}(\ln\kappa)^{n-1},
\end{aligned} \tag{6.44}$$

and for $n_\Delta>0$, there exists K such that

$$\begin{aligned}
&\int_\Delta[:|W'_\lambda(\phi_\kappa+g)|^2:-\eta:|\phi_\kappa+g-h|^2:]dx-\ln E_\Delta(\phi_\kappa)-\ln\chi^{(n_\Delta)}_{\Sigma(\Delta)}((\phi+g)_\Delta)\\
&\quad\leqq\text{const.}\times l^{2+O(\varepsilon)}(a(\eta)\lambda^{-2}-\ln Kn_\Delta!-(\ln\kappa)^{n-1}).
\end{aligned} \tag{6.45}$$

This lower bound, combined with the L^p estimates on $\mathscr{E}_{\#}$, satisfy the hypotheses of Theorem 8.5.3 of [4], so that letting

$$\Delta M_\kappa = \ln E_\Delta(\phi) - \ln E_\Delta(\phi_\kappa) + \int_\Delta [:|W'_\lambda(\phi_\kappa+g)|^2: - :|W'_\lambda(\phi+g)|^2: - \eta:|\phi_\kappa+g-h|^2: + \eta:|\phi+g-h|^2:]\,dx, \tag{6.46}$$

where

$$\int d\mu_s(\phi)(\Delta M_\kappa)^2 \leqq \text{const.} \times \kappa^{-\omega} \tag{6.47}$$

uniformly in s, λ and ϖ for some $\omega > 0$.

Using hypercontractivity,

$$\|\Delta M_\kappa\|_p^p \leqq \text{const.} \times \kappa^{-p\omega}(p-1)^{n-1} \tag{6.48}$$

for all $p \geqq 2$. Then, by Proposition 8.6.4 of [4], there exists $b > 0$ such that

$$\int_{|\Delta M_\kappa| \geqq 1} d\mu_s(\phi) \leqq e^{-b\kappa^{\omega/n}}, \tag{6.49}$$

and thus

$$E_\Delta(\phi)\chi^{(n_\Delta)}_{\Sigma(\Delta)}((\phi+g)_\Delta)\exp\left(-\int_\Delta [:|W'_\lambda(\phi+g)|^2: - \eta:|\phi+g-h|^2:]\,dx\right) \in L^p(d\mu_s) \tag{6.50}$$

uniformly in s, λ and ϖ.

We now use multiple reflection bounds in order to take our blockwise bounds and get bounds on the whole volume. Theorem 1.1 of [17] states that

$$\left\|\prod_i \mathcal{O}_i\right\|_{L^p} \leqq \prod_i \|\mathcal{O}_i\|_{L^{\beta p}}, \tag{6.51}$$

where $\beta = 4/(1-e^{-l})^2$. Thus we have

Lemma 6.6.

$$\left\| \chi(\pi_{b,\chi})\| \Lambda^r[1+K(s)]^{-1}\det{}_3[1+K(s)]\| \cdot \exp - \int_Z [:|W'_\lambda(\phi+g)|^2: - \eta:|\phi+g-h|^2:]\,dx \right\|_{L^p}$$
$$\leqq c_1^r\left[\prod_\Delta \exp(O(1)n_\Delta)(n_\Delta!)^{O(1)}\right]\exp[O(1)|Z|]\exp[-O(1)\lambda^{-2}|Z'|]. \tag{6.52}$$

So far we have neglected the counterterms R. However, the counterterms obey exactly the same types of bounds as the B terms. As with the A terms, we can write our counterterms as a sum of terms like those in [10] plus off-diagonal terms. The terms from [10] obey the proper bounds, as do the B-like terms. The only concerns are the terms involving $\not{\partial}\alpha$, which are field independent and can be shown to be finite, and overall factors of λ, which we demonstrated in Sect. 5.2. Thus Lemma 6.6 remains valid if we include the counterterms on the left-hand side.

We have also neglected e^{-F}, which is responsible for the decay in the size of the phase boundary. Recall that $F_1 = \eta\int_\Lambda |h-g|^2 dx + \int_\Lambda |\nabla g|^2 dx$, and that at a phase boundary h changes by $O(\lambda^{-1})$. Thus either $|h-g|^2$ or $|\nabla g|^2$ is large ($O(\lambda^{-2})$). As in [7, 8], F_1 dominates F and we have

Lemma 6.7. *For p sufficiently close to* 1,

$$\left\| \exp\left[-F + (1-\eta)\int_Z :|\phi|^2: dx \right] \right\|_{L^p} \leqq \text{const.} \times \exp - O(1)\lambda^{-2}|Z^0|. \tag{6.53}$$

Combining Lemmas 6.6 and 6.7, we have (using Hölder's inequality)

$$\begin{aligned} &\| e^{-\mathscr{A}-F}\chi(\pi_{b,\chi}) \| \, \| \Lambda^r [1+K(s)]^{-1} \det{}_3[1+K(s)] \| \, \|_{L^p} \\ &\leqq c_1^r \left[\prod_\Delta \exp(O(1)n_\Delta)(n_\Delta!)^{O(1)} \right] \exp[O(1)|Z|] \exp[-O(1)\lambda^{-2}|Z^0 \cup Z'|]. \end{aligned} \tag{6.54}$$

The final step in the proof of Proposition 6.1 is to show that the vacuum energy is $O(\lambda)$, not $O(1)$. This is done, as in (say) [1], by inserting additional decoupling factors $t(\Delta)$ into the interaction wherever we would expect to get a factor of λ. By essentially repeating the cluster expansion procedure that got us this far, we obtain factors of λ for all blocks in $Z \backslash (Z^0 \cup Z')$. Thus we can replace the $O(1)|Z|$ in (6.54) with $O(\lambda)|Z|$, and we have proven Proposition 6.1. ■

7. Convergence of the Expansion

7.1. Decay of Cluster Activities. In this section we will combine the results of the previous sections to arrive at the following estimate:

Proposition 7.1. *Let l be sufficiently large; let λ, ϖ be sufficiently small. Then there exists $\delta > 0$ such that*

$$\begin{aligned} |\rho(\mathbb{Z})| \leqq (\mathscr{I}_Z!)^{1/2} \exp\left[O(\mathscr{I}_Z + \mathscr{J}_Z)l + O(\lambda)|Z| - \frac{\delta}{l}(|Z| - l^2) \right] \\ \cdot \prod_{i=1}^{\mathscr{I}_Z} O(\lambda^{-1}) \|f_i\|_{L^2} \prod_{j=1}^{\mathscr{J}_Z} \|g_j\|_{\mathscr{H}} \|h_j\|_{\mathscr{H}}. \end{aligned}$$

(Note that l^2 is the size of the smallest cluster.)

We begin with a lemma:

Lemma 7.2. *Let l be sufficiently large; let λ, ϖ be sufficiently small. Then there exists $\delta_2 > 0$ such that*

$$\begin{aligned} |\partial_s^\Gamma Z_{\Sigma,s}(Z)| \leqq O(1)^{l(\mathscr{I}_Z + \mathscr{J}_Z)} (\mathscr{I}_Z!)^{1/2} \prod_{j=1}^{\mathscr{J}_Z} \|g_j\|_{\mathscr{H}} \|h_j\|_{\mathscr{H}} \prod_{i=1}^{\mathscr{I}_Z} \lambda^{-1} \|f_i\|_{L^2} \\ \cdot \exp[O(1)\lambda|Z| - \delta_2 l|\Gamma| - O(1)\lambda^{-2}|Z^0|]. \end{aligned}$$

Proof. Combining Proposition 5.1 and the inequalities (5.11) and (6.3), we have

$$\begin{aligned} |\partial_s^\Gamma Z_{\Sigma,s}(Z)| \leqq O(1)^{|\Gamma| + \mathscr{I}_Z + \mathscr{J}_Z} \sum_\pi \sup \prod_{j=1}^{\mathscr{J}_Z} \exp \varepsilon[d(\Delta_j, \bar{\Delta}_j) + d(\Delta'_j, \bar{\Delta}'_j)] \\ \cdot \prod_{\gamma \in \pi_{f,E} \cup \pi_{b,E}} \exp \varepsilon d(\gamma, \Delta_L) \prod_{\gamma \in \pi_{f,P}} \exp[\varepsilon \min\{d(\gamma, \Delta_l), d(\gamma, \bar{\Delta}_l)\}] \end{aligned}$$

$$\prod_{\substack{\gamma\in\pi_{b,A}\cup\pi_{f,A}\cup\pi_{b,B}\cup\pi_{f,B}\\ \cup\pi_{b,F}\cup\pi_{b,E}\cup\pi_{f,E}\cup\pi_{b,P}\cup\pi_{f,P}}}\left(|\gamma|^{O(1)}\prod_k \exp\varepsilon d(\Delta_k,\gamma)\right)$$
$$\cdot\prod_{\gamma\in\pi_{b,\chi}\cup\pi_{b,\Phi}} O(l^2)\exp[\varepsilon d(\gamma,\Delta_L)]$$
$$\cdot r!\,\|\mathscr{Z}^r\|_{L^p}\|\,\|\mathbb{Q}\|_1\,\|_{L^q}\|\partial^{\pi_{f,S'}}r'(\pi_{b,S})B'_l(\pi_{f,B},\pi_{b,B})\|_{L^q}$$
$$\cdot\|\Phi^\chi(\pi_{b,\chi})\|_{L^q}\|\Phi'(\pi_{b,\Phi})\|_{L^{q/2}}\|F'_{3,4,l}(\pi_{b,F})\|_{L^q} \tag{7.1}$$

for some $p>1$ and $\frac{1}{p}+\frac{6}{q}\leqq 1$.

The factors involving $M_{\mathbb{Q}}(\Delta)$ in Corollary 5.8 and $n_\Delta!$ in Proposition 6.1 are bounded using exponential pinning, as we did in (4.3). This introduces additional factors of the form

$$O(1)^{|\Gamma|}\prod_\gamma \exp\varepsilon d(\gamma,\Delta). \tag{7.2}$$

These new factors of $\exp\varepsilon d(\gamma,\Delta)$, as well as existing factors, are all controlled by decay factors $e^{-\delta m_c d(\gamma,\Delta)}$ and $e^{-\delta d(\Delta^i,\Delta^{i+1})}$ arising in Lemmas 5.3–5.6 and Corollary 5.8, if ε is sufficiently small (requiring l large). Similarly the factors of $\exp\varepsilon d(\Delta,\bar{\Delta})$ are beaten by the factors $e^{-\delta[d(\Delta^1,\Delta^2)+d(\Delta^3,\Delta^4)]}$ in Corollary 5.8. This gives us, when combined with Proposition 6.1,

$$|\partial_s^\Gamma Z_{\Sigma,s}(Z)|\leqq O(1)^{|\Gamma|+\mathscr{I}_Z+\mathscr{I}_Z}(\mathscr{I}_Z!)^{1/2}\sum_\pi \sup\prod_{j=1}^{\mathscr{I}_Z}\|g_j\|_{\mathscr{H}}\|h_j\|_{\mathscr{H}}\prod_{i=1}^{\mathscr{I}_Z}\lambda^{-1}\|f_i\|_{L^2}$$
$$\cdot\prod_{\gamma^\#\in\pi}O(1)^l G_9(\gamma^\#,\delta/4)\exp\{-\delta' m_c d(\gamma^\#,\Delta_{\gamma^\#})\}$$
$$\cdot\lambda^{|\pi\setminus(\pi_{b,\chi}\cup\pi_{b,\Phi}\cup\pi_{f,P})|}\exp[O(1)\lambda|Z|]\exp[-O(1)\lambda^{-2}|Z^0\cup Z'|]. \tag{7.3}$$

The factor $\exp-O(1)\lambda^{-2}|Z'|$ gives us smallness in the number of differentiated characteristic functions, but we would prefer an estimate in the number of differentiations. However,

$$\prod_{\gamma^\#\in\pi_{b,\chi}}\exp\{-\delta' m_c d(\gamma^\#,\Delta_{\gamma^\#})-O(1)\lambda^{-2}|Z'|\}\leqq\prod_{\gamma^\#\in\pi_{b,\chi}}\lambda O(1)^l \tag{7.4}$$

(for l large and λ small) by exponential pinning. Thus

$$|\partial_s^\Gamma Z_{\Sigma,s}(Z)|\leqq O(1)^{|\Gamma|+\mathscr{I}_Z+\mathscr{I}_Z}(\mathscr{I}_Z!)^{1/2}\prod_{j=1}^{\mathscr{I}_Z}\|g_j\|_{\mathscr{H}}\|h_j\|_{\mathscr{H}}\prod_{i=1}^{\mathscr{I}_Z}\lambda^{-1}\|f_i\|_{L^2}$$
$$\cdot\sum_{\pi\in\mathscr{P}(\Gamma)}\prod_{\gamma^\#\in\pi}O(1)^l G_9(\gamma^\#,\delta/4)$$
$$\cdot\lambda^{|\pi\setminus(\pi_{b,\Phi}\cup\pi_{f,P})|}\exp[O(1)\lambda|Z|-O(1)\lambda^{-2}|Z^0|]. \tag{7.5}$$

Now $G_9(\gamma^\#,\delta/4)\leqq O(1)^l G_9(\gamma^\#,\delta_1)e^{-\delta_2 l|\gamma^\#|}$, and

$$\sum_{\pi\in\mathscr{P}(\Gamma)}\prod_{\gamma^\#\in\pi}G_9(\gamma^\#,\delta_1)\leqq O(1)^{|\Gamma|}. \tag{7.6}$$

Thus

$$|\partial_s^\Gamma Z_{\Sigma,s}(Z)| \leqq O(1)^{|\Gamma|+\mathscr{I}_Z+\mathscr{I}_Z}(\mathscr{I}_Z!)^{1/2} \prod_{j=1}^{\mathscr{I}_Z} \|g_j\|_{\mathscr{H}} \|h_j\|_{\mathscr{H}} \prod_{i=1}^{\mathscr{I}_Z} \lambda^{-1} \|f_i\|_{L^2}$$
$$\cdot \sup_{\pi \in \mathscr{P}(\Gamma)} \prod_{\gamma^\# \in \pi} O(1)^l e^{-\delta_2 l |\gamma^\#|}$$
$$\cdot \lambda^{|\pi \setminus (\pi_{b,\Phi} \cup \pi_{f,P})|} \exp[O(1)\lambda|Z| - O(1)\lambda^{-2}|Z^0|]. \tag{7.7}$$

Using the factors of λ to beat the factors $O(1)^l$, we have the lemma. ■

To complete the proof of Proposition 7.1, recall from Eq. (3.66) that

$$|\rho(\mathbb{Z})| \leqq \sum_{\Sigma \text{ restricted}} \sum_{\Gamma \text{ constrained}} |\partial_s^\Gamma Z_{\Sigma,s}(Z)|. \tag{7.8}$$

The sum over Γ is constrained so that Z cannot be separated into separate decoupled regions. Thus for Z consisting of more than one block, $l^2|\Gamma|$ is $O(|Z \setminus Z^0|)$, and $O(1)\lambda|Z| < \delta_2 l|\Gamma| + O(1)\lambda^{-2}|Z^0|$. Since the number of possible Γ's is $\exp O(|Z|/l^2)$,

$$|\rho(\mathbb{Z})| \leqq O(1)^{l(\mathscr{I}_Z+\mathscr{I}_Z)}(\mathscr{I}_Z!)^{1/2} \sum_{j=1}^{\mathscr{I}_Z} \|g_j\|_{\mathscr{H}} \|h_j\|_{\mathscr{H}} \prod_{i=1}^{\mathscr{I}_Z} \lambda^{-1} \|f_i\|_{L^2}$$
$$\cdot \exp -\frac{\delta}{l}|Z| \sum_{\Sigma \text{ restricted}} \exp[-O(1)\lambda^{-2}|Z^0|]. \tag{7.9}$$

Finally, the number of block-spin configurations is bounded:

$$\sum_{\Sigma \text{ restricted}} \exp[-O(1)\lambda^{-2}|Z^0|] \leqq O(1)^{|Z|/l^2}, \tag{7.10}$$

and the proposition follows. ■

7.2. Lower Bound. In order to obtain useful information from the cluster expansion, we need to bound the single square vacuum cluster activity from below. We obtain the following bound:

Proposition 7.3. *Let λ, ϖ be sufficiently small, and let $\mathbb{Z}$ be a single-phase vacuum cluster (no test functions) of size l^2. Then*

$$|\rho(\mathbb{Z})| - 1 \leqq O(\lambda l^2).$$

Proof. Write

$$\rho(\mathbb{Z}) = \int \chi_{\Sigma(Z)} e^{-\mathscr{A}-F} d\mu_Z$$
$$= 1 + \int \chi_{\Sigma(Z)}(e^{-\mathscr{A}-F} - 1) d\mu_Z + \int (\chi_{\Sigma(Z)} - 1) d\mu_Z. \tag{7.11}$$

Following [8], the last term is $\exp -\mathcal{O}(\lambda^{-2})$, while the second term is bounded by $O(\lambda l^2)$ through a repeat of our vacuum energy calculation. ■

7.3. Calculation of the Free Energy. It remains to prove the supersymmetric aspect of Theorem 1, the vanishing of the free energy. In order to make use of the machinery of [2], we need a cluster expansion for the model with periodic boundary conditions, not just for the Dirichlet-like condition we have imposed. We will indicate roughly how to proceed.

Let Λ be a rectangle in the $\mathbb{Z}^2$ lattice, with edges $\vec{l}, \vec{m}$. We begin with the periodic covariance

$$C_P(x,y) = \chi_\Lambda(x)\chi_\Lambda(y) \sum_{t \in \mathbb{Z}^2} C(x, y + t(l,m)), \tag{7.12}$$

where

$$t(l,m) = t_1 \vec{l} + t_2 \vec{m}. \tag{7.13}$$

Given a set γ of bonds of $\mathbb{Z}^2 \cap \Lambda$, we introduce the periodic covariance with Dirichlet conditions on γ as follows: let $\tilde{\gamma}$ be the set of bonds in $\mathbb{Z}^2$ which are translates of γ by $i\vec{l} + j\vec{m}$, $i,j \in \mathbb{Z}$. Let $C^{\tilde{\gamma}}$ be the ordinary covariance with Dirichlet conditions on the bonds of $\tilde{\gamma}$, and let

$$C_P^\gamma(x,y) = \chi_\Lambda(x)\chi_\Lambda(y) \sum_{t \in \mathbb{Z}^2} C^{\tilde{\gamma}}(x, y + t(l,m)). \tag{7.14}$$

The covariance C_P^γ is the periodic covariance with Dirichlet conditions on the bonds of γ. We may interpolate between C_P^γ and C_P in the usual way by introducing the covariance

$$C_P(s) = \sum_{\Gamma \subset \mathscr{B}(\Sigma)} \left[\prod_{b \in \Gamma} s_b \prod_{n \in \mathscr{B}(\Sigma) \setminus \Gamma} (1 - s_b) \right] C_P^{\mathscr{B}(\Sigma) \setminus \Gamma}. \tag{7.15}$$

This covariance gives rise to a measure $d\mu_P(s)$ which satisfies the usual integration by parts formulae. Furthermore, the covariance $C_P(s)$ and its s-derivative are easily shown to satisfy all standard bounds satisfied by C_s, provided the distance are replaced by the torus distance

$$d_T(x,y) = \min_{t \in \mathbb{Z}^2} d(x, y + t(l,m)). \tag{7.16}$$

This fact follows immediately from the "sum of translates" representation (7.14) for $C_P(s)$.

We have a similar representation for the Fermi determinant. The determinant appearing in the period partition function is

$$\det{}_3 [1 + S_P \Upsilon(\dot{W}_\lambda''(\phi) - 1)], \tag{7.17}$$

where S_P is the periodic fermionic covariance and has an expression in terms of translates similar to (7.12),

$$S_P(x,y) = \sum_{t \in \mathbb{Z}^2} S(x, y + t(l,m)), \quad x, y \in \Lambda. \tag{7.18}$$

The unitary transformation method of [9, 14] again shows that it suffices to consider

$$\det{}_3 [1 + S_P \zeta] \tag{7.19}$$

as in Lemma 3.1. The determinants appearing in (7.17) and (7.19) must be considered as determinants on the Hilbert space $\mathscr{H}^{\mathrm{per}}_{-1/2}(\Lambda) \oplus \mathscr{H}^{\mathrm{per}}_{-1/2}(\Lambda)$ of functions on the torus Λ; however, by examining the Fredholm series for the determinant, we may note that

$$\det{}_3 [1 + S_P \zeta] |_{\mathscr{H}^{\mathrm{per}}_{-1/2}(\Lambda) \oplus \mathscr{H}^{\mathrm{per}}_{-1/2}(\Lambda)} = \det{}_3 [1 + \tilde{S}_P \tilde{\zeta}] |_{\mathscr{H}}, \tag{7.20}$$

where the operator $\tilde{S}_P$ is given by the kernel

$$\tilde{S}_P(x,y)=\sum_{t\in\mathbb{Z}^2} S(x,y+t(l,m))\chi_\Lambda(y) \tag{7.21}$$

and $\tilde{\zeta}=\zeta$, considered as a function on $\mathbb{R}^2$ supported in $\Lambda\subset\mathbb{R}^2$. Then, as in Sect. 3, we have

$$\det{}_3[1+\tilde{S}_P\tilde{\zeta}]|_{\mathscr{H}}=\det{}_3[1+D^{1/2}\tilde{S}D^{-1/2}]|_{\mathscr{H}'}. \tag{7.22}$$

Consider the translation operator $T_t:\mathscr{H}'\to\mathscr{H}'$ defined by

$$(T_t f)(x)=f(x+t) \tag{7.23}$$

for $t\in\mathbb{R}^2$. Then

$$D^{1/2}\tilde{S}_P\tilde{\zeta}D^{-1/2}=\sum_{t\in\mathbb{Z}^2} D^{1/2}SD^{1/2}T_{t(l,m)}D^{-1/2}\zeta D^{-1/2}. \tag{7.24}$$

We produce a decoupling of this operator as in (3.31):

$$K_P(s)=\left(\sum_{t\in\mathbb{Z}^2} D^{1/2}SD^{1/2}T_{t(l,m)}\right)_s(D^{-1/2})_s\zeta(D^{-1/2})_s. \tag{7.25}$$

We note that this operator is *not* periodic. However, the last three terms in (7.25) are precisely those appearing in the definition of $K(s)$, (3.31). As for the first term, we note that it is equal to

$$\left(\sum_{t\in\mathbb{Z}^2}\mathscr{U}T_{t(l,m)}\right)_s=\sum_{t\in\mathbb{Z}^2}\sum_{\Delta,\Delta'\subset\Lambda} H(s,\Delta,\Delta')\chi_\Delta\mathscr{U}\chi_{\Delta'+t(l,m)}T_{t(l,m)}. \tag{7.26}$$

Since $\|T_{t(l,m)}\|=1$, the norms of the operators appearing on the right-hand side of (7.26) may be estimated just as in the nonperiodic case, using the estimates proved in the appendix, which yield the necessary exponential decay. The correction term $\exp -R$ may be treated similarly.

In view of the above remarks it is clear that the periodic partition function Z_P^Λ (and its corresponding Green's functions) satisfies a cluster expansion of the type satisfied by Z^Λ.

The clusters of this periodic cluster expansion differ from those of the ordinary expansion in two ways. First, there are the clusters which "wrap around" the torus Λ, which do not appear at all in the nonperiodic expansion. Second, there is the effect of the boundary on the individual clusters due to the differences between periodic and nonperiodic covariances. Recalling (7.14) and (7.25), these differences can be expressed as a sum of translates, which will produce corrections that are exponentially small in the distance to the boundary, and hence will not affect any of our basic estimates.

With this in mind, we sketch the arguments from [2] that lead to our main results.

The index theorem of [11, 13] relates the partition function of the model with *periodic* boundary conditions to the index of the supercharge Q of (1.2), namely

$$Z_P^\Lambda=\operatorname{ind}Q=n-1. \tag{7.27}$$

Therefore the free energy f_P of the periodic model is

$$f_P = -\lim_{\Lambda \nearrow \mathbb{R}^2} \frac{1}{|\Lambda|} \log Z_P^\Lambda = 0. \tag{7.28}$$

We compare this exact calculation of the free energy to the free energy as computed via our polymer expansion, which is convergent by Propositions 7.1 and 7.3. By examining the form of the expansion, we see that the expansion can be reorganized as a sum over all stable phases (which contribute equally to the periodic partition function), and a correction term for the large clusters (where there is no natural choice of the correct phase) which vanishes in the thermodynamic limit so that [2]

$$Z_P^\Lambda \sim \sum_{q\,\text{stable}} e^{-f_P^q |\Lambda|}, \tag{7.29}$$

where f_P^q is the contribution to the periodic free energy from contours in the q^{th} phase. Equations (7.27) and (7.29) clearly imply that $f_P^q = 0$ for $q = 1, \ldots, n-1$.

Now we consider the non-periodic case. Since we have a convergent polymer expansion, with activities that are exponentially small in the size of the polymers, the boundary conditions cannot affect the bulk properties and $f^q = f_P^q = 0$ for all q. Thus any ratio of partition functions that would appear in the re-summed polymer expansion is less than $e^{O(\partial\Lambda)}$, which is a sufficient condition to construct any of the $n-1$ phases (if we had initially chosen appropriate boundary conditions), and to demonstrate exponential decay of correlations in each of the phases.

Note that this also shows that the infinite volume limit of the periodic model results in a state that is the equal mixture of the stable states.

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Appendix: Proof of Lemma 6.4

We begin with the proof of (6.30), which is the simplest case. It is only slightly complicated by our desire to end up with a $2+\varepsilon$ norm.

Using the Hölder inequality,

$$\| \mathscr{U} D^{-1/2} \chi_i f D^{-2} \|_2 \leqq \| \mathscr{U} D^{-1/2} \chi_i \|_p \| \chi_i f D^{-2} \|_{2+\varepsilon}, \tag{A.1}$$

where $1/p = 1/2 - 1/(2+\varepsilon)$. Now $\| \mathscr{U} D^{-1/2} \chi_i \|_p$ is clearly bounded, and we bound $\| \chi_i f D^{-2} \|_{2+\varepsilon}$ using Lemma 2.1 of [17] (essentially Young's inequality):

$$\| \chi_i f D^{-2} \|_{2+\varepsilon} \leqq \| \chi_i f \|_{2+\varepsilon} \left\| \frac{1}{p^2+1} \right\|_{2+\varepsilon}. \tag{A.2}$$

$(p^2+1)^{-2(2+\varepsilon)}$ is integrable, so we arrive at Lemma 6.4c:

$$\| \mathscr{U} D^{-1/2} \chi_i f D^{-2} \|_2 \leqq c_{\tilde{B}} l^{O(\varepsilon)} \| \chi_i f \|_{2+\varepsilon} \quad \blacksquare$$

We now move on to (6.28). As $\chi_i \mathscr{U} \chi_i$ is a bounded operator (with norm $\leqq 1$),

$$\| \chi_i \mathscr{U} \chi_i D^{-1/2} \chi_i f D^{-1/2} \chi_i D^{-\nu} \|_2 \leqq \| D^{-1/2} \chi_i f D^{-1/2} \chi_i D^{-\nu} \|_2. \tag{A.3}$$

In [16] we learn that $D^{\delta}\chi_i D^{-\nu}$ is a bounded operator provided that $\nu > 2\delta$. Thus

$$\begin{aligned}\|D^{-1/2}\chi_i f D^{-1/2}\chi_i D^{-\nu}\|_2 &= \|D^{-1/2}\chi_i f D^{-1/2-\delta}D^{\delta}\chi_i D^{-\nu}\|_2\\ &\leqq \text{const.} \times \|D^{-1/2}\chi_i f D^{-1/2-\delta}\|_2\\ &= \text{const.} \times [\operatorname{Tr} D^{-1/2-\delta}\chi_i f D^{-1}\chi_i f D^{-1/2-\delta}]^{1/2}\\ &= \text{const.} \times [\operatorname{Tr} \chi_i D^{-2\delta} D^{-1}\chi_i f D^{-1}\chi_i f]^{1/2} \qquad \text{(A.4)}\end{aligned}$$

by the cyclicity of the trace. (We treat f as a positive function; clearly our results extend to general f.) Using Hölder's inequality, this is less than

$$\text{const.} \times \|\chi_i D^{-2\delta}\chi_i\|_{2/\varepsilon}^{1/2} \|D^{-1}\chi_i f\|_{2+\varepsilon}, \qquad \text{(A.5)}$$

which we can bound using Lemma 2.1 of [17] by

$$\text{const.} \times \left\|\frac{1}{(p^2+1)^{\delta}}\right\|_{2/\varepsilon}^{1/2} \|\chi_i\|_{2/\varepsilon}^{1/2} \left\|\frac{1}{(p^2+1)^{1/2}}\right\|_{2+\varepsilon} \|\chi_i f\|_{2+\varepsilon}. \qquad \text{(A.6)}$$

If $2\delta > \varepsilon$, then the norms of the p^2+1 terms are bounded, yielding

$$\|\chi_i \mathscr{U}\chi_i D^{-1/2}\chi_i f D^{-1/2}\chi_i D^{-\nu}\|_2 \leqq \text{const.} \times l^{\varepsilon/2} \|\chi_i f\|_{2+\varepsilon}, \qquad \text{(A.7)}$$

which proves Lemma 6.4a. ■

Finally we come to the most difficult part of the proof, the derivation of (6.29). We will need the following two lemmas which are extensions of lemmas in [1] and [17].

This extension of Lemma A.I.4 of [1] and Lemma 2.3 of [17] follows by duplicating the steps of the proof in [17]:

Lemma A.1. *Let n be an integer $\geqq 2$. If $0 \leqq \nu < 2^{-n}$ and $\lambda - 2\nu > 2^{-n}$ then*

$$[D^{-\lambda}, \chi_a]D^{\nu} \in \mathscr{L}_{2^n}.$$

The same is true if $D^{-\lambda}$ is replaced by $\mathscr{U}D^{-\lambda}$.

We also have an extension of Lemma A.I.5 of [1] and Lemma 2.2 of [17], which differs only in that the result is uniform in the blocksize l:

Lemma A.2. *Let Δ, Δ' be non-touching l-blocks. Fix k, a positive integer. Then for $\lambda \geqq 0$ and l sufficiently large we have*

$$\|\mathcal{O}_1\chi_{\Delta}D^{-\lambda}\chi_{\Delta'}\mathcal{O}_2\|_1 \leqq O(1)\exp[-c'd(\Delta,\Delta')]\,\|\mathcal{O}_1\chi_{\Delta}D^{-2k}\|_q \|D^{-2k}\chi_{\Delta'}\mathcal{O}_2\|_{q'}$$

for any complementary q, q'. The same is true if $D^{-\lambda}$ is replaced by $\mathscr{U}D^{-\lambda}$.

We need to estimate

$$\mathfrak{B} \equiv \|D^{\nu}\chi_i \mathscr{U}\chi_j D^{-1/2}\chi_k f D^{-1/2}\chi_l\|_2, \qquad \text{(A.8)}$$

where at least one pair of i,j, j,k and k,l is distinct. The case where no such pair represents touching squares and which yields exponential decay will be dealt with later; first we treat with the three "touching" cases.

Case 1 – $k \neq l$. Recall that $D^{\pm\nu}$ and $\mathscr{U}$ commute, so

$$\mathfrak{B} = \|D^{\nu}\chi_i D^{-3\nu}\mathscr{U}D^{3\nu}\chi_j D^{-9\nu}D^{-1/2+9\nu}\chi_k f D^{-1/2}\chi_l\|_2$$

$$\leqq \|D^{\nu}\chi_i D^{-3\nu}\| \, \|\mathscr{U}\| \, \|D^{3\nu}\chi_j D^{-9\nu}\| \, \|D^{-1/2+9\nu}\chi_k f D^{-1/2}\chi_l\|_2$$
$$\leqq \text{const.} \times \|D^{-1/2+9\nu}\chi_k f^{1/2}\|_{4+\delta}\|f^{1/2}\chi_k D^{-1/2}\chi_l\|_{4-\delta}, \tag{A.9}$$

using Hölder's inequality. Then for $\delta \gg \nu$ we have

$$\mathfrak{B} \leqq \text{const.} \times \|\chi_k f\|_{2+\delta/2}^{1/2}\mathfrak{B}_1, \tag{A.10}$$

where

$$\mathfrak{B}_1 = \|f^{1/2}\chi_k D^{-1/2+\lambda}(\chi_l D^{-\lambda} + [D^{-\lambda}, \chi_l])\|_{4-\delta}$$
$$\leqq \|f^{1/2}\chi_k D^{-1/2+\lambda}\chi_l D^{-\lambda}\|_{4-\delta} + \|f^{1/2}\chi_k D^{-1/2+\lambda}[D^{-\lambda}, \chi_l]\|_{4-\delta}. \tag{A.11}$$

Choose the smallest λ such that $\lambda > \delta/(4-\delta)$ and λ is of the form $\lambda = 5\cdot 2^{-n}/4$. Then for $p = \dfrac{4-\delta}{1-(4-\delta)/2^n}$ we have

$$\mathfrak{B}_1 \leqq \text{const.} \times \|f^{1/2}\chi_k D^{-1/2+\lambda}\chi_l\|_4 + \|f^{1/2}\chi_k D^{-1/2+\lambda}\|_p \|[D^{-\lambda}, \chi_l]\|_{2^n}$$
$$\leqq \text{const.} \times (\|f^{1/2}\chi_k D^{-1/2+\lambda}\chi_l\|_4 + \|f^{1/2}\|_p), \tag{A.12}$$

using Lemma A.1 and Lemma 2.1 of [17]. Now using trace cyclicity,

$$\mathfrak{B}_1 \leqq \text{const.} \times [\|\chi_k f\|_{p/2}^{1/2} + (\mathrm{Tr}(\chi_l D^{-1/2+\lambda} f \chi_k D^{-1/2+\lambda}\chi_l)^2)^{1/4}]$$
$$\leqq \text{const.} \times [\|\chi_k f\|_{p/2}^{1/2} + \|D^{-1/2+\lambda}\chi_l D^{-1/2+\lambda}\chi_k f\|_2^{1/2}]$$
$$= \text{const.} \times [\|\chi_k f\|_{p/2}^{1/2} + (\mathrm{Tr}\, D^{-1/2+\lambda}\chi_l D^{-l+2\lambda}\chi_l D^{-1/2+\lambda}\chi_k f^2)^{1/4}]. \tag{A.13}$$

The operators $D^{-\alpha}$ have positive definite kernels with well-known properties. Thus we can write out the trace in (A.13) as an integral. After discarding a characteristic function (allowed since the integrand is positive) we have

$$\int D^{-3/2+3\lambda}(x,y) D^{-1/2+\lambda}(x,y) f^2(x)\chi_k(x)\chi_l(y) d^2x d^2y. \tag{A.14}$$

Now $D^{-3/2+3\lambda}(x,y)D^{-1/2+\lambda}(x,y)$ has a singularity like $|x-y|^{-2-4\lambda}$. When we integrate d^2y we are left with a singularity of the form $(d(x,\Delta_l))^{-4\lambda}$ which is in L^q for any $q < 1/\lambda$. Thus the entire integral is bounded if, say, $\chi_k f^2 \in L^{1+2\lambda}$, yielding

$$\mathfrak{B}_1 \leqq \text{const.} \times [\|\chi_k f\|_{p/2}^{1/2} + \|\chi_k f\|_{2+4\lambda}^{1/2}]. \tag{A.15}$$

By choosing ν and δ properly, we can finally bounded $\mathfrak{B}$ by a $2+\varepsilon$ norm as desired:

$$\mathfrak{B} \leqq C_B l^{O(\varepsilon)} \|\chi_k f\|_{2+\varepsilon}. \tag{A.16}$$

The l-dependence arises from using Hölder to get our $2+\varepsilon$ norm exactly:

$$\|\chi f\|_q \leqq \|\chi\|_{O(\varepsilon)} \|\chi f\|_{2+\varepsilon}, \tag{A.17}$$

finishing case 1.

Case 2 – $j \neq k$. In a fashion similar to the previous case,

$$\mathfrak{B} = \|D^{\nu}\chi_i D^{-3\nu}\mathscr{U} D^{3\nu}\chi_j D^{-1/2}\chi_k f D^{-1/2}\chi_l\|_2$$
$$\leqq \text{const.} \times \|D^{3\nu}\chi_j D^{-1/2}\chi_k f D^{-1/2}\chi_l\|_2$$
$$\leqq \text{const.} \times \|D^{3\nu}\chi_j D^{-1/2}\chi_k f^{1/2}\|_{4-\delta}\|\chi_k f^{1/2} D^{-1/2}\|_{4+\delta}. \tag{A.18}$$

We commute some regularity through χ_j:

$$\mathfrak{B} \leqq \text{const.} \times \|D^{3\nu}(D^{-\lambda}\chi_j + [\chi_j, D^{-\lambda}])D^{-1/2+\lambda}\chi_k f^{1/2}\|_{4-\delta}\|\chi_k f\|_{2+\delta/2}^{1/2}, \quad (A.19)$$

and are now left with a situation almost identical to case 1.

Case 3 – $i \neq j$.

$$\begin{aligned}\mathfrak{B} &\leqq \|D^\nu \chi_i \mathscr{U} \chi_j D^{-1/2}\chi_k f D^{-1/2}\|_2 \\ &= \|D^\nu \chi_i \mathscr{U}(D^{-\lambda}\chi_j + [\chi_j, D^{-\lambda}])D^{-1/2+\lambda}\chi_k f D^{-1/2}\|_2. \end{aligned} \quad (A.20)$$

Because the operator $\mathscr{U}$ has no regularity by itself, we have to repeat the above procedure:

$$\begin{aligned}\mathfrak{B} \leqq\; &\|D^\nu[\chi_i, \mathscr{U}D^{-\lambda}]\chi_j D^{-1/2+\lambda}\chi_k f D^{-1/2}\|_2 \\ &+ \|D^\nu \chi_i D^{-3\nu}\mathscr{U}D^{3\nu}[\chi_j, D^{-\lambda}]D^{-1/2+\lambda}\chi_k f D^{-1/2}\|_2. \end{aligned} \quad (A.21)$$

Using Hölder and pulling out some bounded operators, we have

$$\begin{aligned}\mathfrak{B} \leqq\; &\text{const.} \times \|\chi_k f^{1/2} D^{-1/2}\|_{4+\delta}(\|D^\nu[\chi_i, \mathscr{U}D^{-\lambda}]\chi_j D^{-1/2+\lambda}\chi_k f^{1/2}\|_{4-\delta} \\ &+ \|D^{3\nu}[\chi_j, D^{-\lambda}]D^{-1/2+\lambda}\chi_k f^{1/2}\|_{4-\delta}), \end{aligned} \quad (A.22)$$

which we can now bound as before since $[\chi, D^{-\lambda}]$ and $[\chi, \mathscr{U}D^{-\lambda}]$ obey the same bounds in Lemma A.1.

Case 4 – Exponential Decay. When at least one pair of i,j; j,k or k,l do not touch, we actually have so much regularity that we can estimate a 1-norm:

$$\mathfrak{B} \leqq \|D^\nu \chi_i \mathscr{U} \chi_j D^{-1/2}\chi_k f D^{-1/2}\chi_l\|_1. \quad (A.23)$$

Thus these estimates are actually simpler than cases 1–3, and we will only briefly sketch their proof.

If i does not touch j we use Lemma A.2 to obtain

$$\mathfrak{B} \leqq O(1)\exp[-c'd(\Delta_i, \Delta_j)]\|D^\nu \chi_i D^{-2}\|_4 \|D^{-2}\chi_j D^{-1/2}\chi_k f D^{-1/2}\chi_l\|_{2+\varepsilon}, \quad (A.24)$$

otherwise we use the bound

$$\mathfrak{B} \leqq \text{const.} \times \|D^{3\nu}\chi_j D^{-1/2}\chi_k f D^{-1/2}\chi_l\|_2 \quad (A.25)$$

as before. Now if j,k and k,l touch or coincide, we bound the right-hand side of (A.24) as before, If, on the other hand, j doesn't touch k (for example)

$$\begin{aligned}&\|D^{3\nu}\chi_j D^{-1/2}\chi_k f D^{-1/2}\chi_l\|_2 \\ &\quad \leqq \text{const.} \times \|D^{3\nu}\chi_j D^{-2}\|_4 \|D^{-2}\chi_k f D^{-1/2}\chi_l\|_2 \exp[-c'd(\Delta_j, \Delta_k)]. \end{aligned} \quad (A.26)$$

We continue this procedure until we have accounted for all possibilities, and obtain exponential decay for all pairs of characteristic functions. Thus, combined with cases 1–3, proves Lemma 6.4b. ■

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