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# Scalings in Circle Maps II

**F. M. Tangerman**[1] **and J. J. P. Veerman**[2]

[1] Mathematics Department, SUNY at Stony Brook, Stony Brook, NY 11794, USA
[2] Institute for the Mathematical Sciences, SUNY at Stony Brook, Stony Brook, NY 11794, USA

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**Abstract.** In this paper we consider one parameter families of circle maps with nonlinear flat spot singularities. Such circle maps were studied in [Circles I] where in particular we studied the geometry of closest returns to the critical interval for irrational rotation numbers of constant type. In this paper we apply those results to obtain exact relations between scalings in the parameter space to dynamical scalings near parameter values where the rotation number is the golden mean. Then results on [Circles I] can be used to compute the scalings in the parameter space. As far as we are aware, this constitutes the first case in which parameter scalings can be rigorously computed in the presence of highly nonlinear (and nonhyperbolic) dynamics.

## 0. Introduction

In this paper we consider one parameter families of circle maps with nonlinear flat spot singularities. Such circle maps were studied in [Circles I] where in particular we investigated the geometry of closest returns to the critical interval for irrational rotation numbers of constant type. In this paper we apply those results to relate scalings in the parameter space to dynamical scalings near parameter values where the rotation number has constant type. That one should be able to establish such a relation is part of the renormalization philosophy for one-dimensional dynamical systems. The case we study here constitutes the first example in the presence of nonlinear singularities where such relations can be rigorously established. We restrict ourselves to the case where the rotation number is the golden mean.

Let $f_0$ be a circle map with a flat spot singularity which has bounded nonlinearity on the left side of the singular interval and has a power-law $(x \to x^{\nu_r})$ singularity on the right side. Assume that $f_0$ has golden mean rotation number. Let $f_t$ be a one parameter family of such maps such that $\left.\frac{d}{dt}\right|_{t=0} f_t(x)$ is nonvanishing. Such families occur naturally as truncations of families of smooth bimodal maps.

Denote by $I_n$ the set of parameter values where $f_t$ has rotation number $p_n/q_n$, the $n^{\text{th}}$ continued fraction approximant to the golden mean. The length $|I_n|$ of the interval $I_n$ tends to zero as $n$ tends to infinity. Define the parameter scaling $\delta(n)$ as

$$\delta(n) := \frac{|I_n|}{|I_{n-2}|}.$$

Define the collection of dynamical scalings $\{\sigma(n)\}$ for the map $f_0$ as:

$$\sigma(n) := \frac{d(q_n, q_{n+2})}{d(q_{n-2}, q_n)}.$$

Here $d(q_n, q_{n+2})$ denotes the distance between the $q_n{}^{\text{th}}$ and the $q_{n+2}{}^{\text{nd}}$ iterate of the flat spot. These scalings were studied in [Circles I]. To state the main theorem of this paper we introduce some notation:

Let $\{a(n)\}$ and $\{b(n)\}$ be sequences. We write

$$a(n) \cong b(n) \quad \text{iff} \quad \lim_{n\to\infty} \frac{a(n)}{b(n)} = 1$$

and

$$a(n) \geqq \leqq b(n) \quad \text{iff} \quad \left\{\left|\ln \frac{a(n)}{b(n)}\right|\right\}$$

is bounded.

**Main Theorem.** *If $f_0$ has golden mean rotation number then:*

$$\delta(n) \cong [\sigma(n-1)]^{\nu},$$

*where $\nu = 1$ if $\sigma(n-1)$ is a scaling on the bounded nonlinearity (left) side of the singularity and $\nu = \nu_r$ when $\sigma(n-1)$ is a scaling on the powerlaw (right) side of the singularity.*

Comparing this with Theorem 6.2 of [Circles I] we see that

**Corollary.**

$$\delta(2n) \cong \sigma(2n-1) \geqq \leqq \exp(-C\lambda^n\},$$

$$\delta(2n+1) \cong [\sigma(2n)]^{\nu_r} \geqq \leqq \exp\{-\nu_r C(\lambda-1)\lambda^n\},$$

*where*

$$\lambda = \frac{1}{2}\left(1 + \frac{2}{\nu_r} + \left(1 + \frac{4}{\nu_r^2}\right)^{1/2}\right),$$

*and $C$ is a constant depending on the first iterates of the critical orbit.*

In more pictorial terms, the meaning of $\delta(n)$ can be explained as follows. In Fig. 0.1 we have drawn, the parameter plane for a "typical" two parameter family of circle maps (for example:

$$f_{t,k}: x \to x + t + k \sin 2\pi x (\text{mod}\, 1),$$

$t$ in $[0,1]$ and $k$ fixed and greater than one, satisfies all hypotheses of Sect. 1). Denote by $T_{p_n/q_n}$ the regions where $f_{t,k}$ has a well-ordered orbit of rotation number

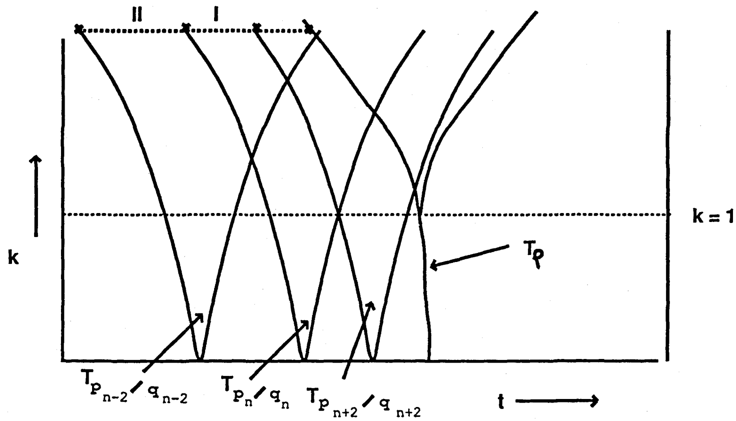


Fig. 0.1 $\delta(n) = \frac{I}{II}$

$p_n/q_n$, and let $p_n/q_n$ be the continued fraction approximants of the golden mean. In effect, the $\delta(n)$ are the rates at which the boundaries of the left boundaries of $T_{p_n/q_n}$ converge to the left boundary of $T_{\text{golden mean}}$ (the region where $f_{t,k}$ has a well-ordered orbit with rotation number golden mean). When one changes $k$, the new convergence rates will be determined by the corollary but with different $C$ (increasing $k$ corresponds to increasing $C$).

*Remarks.* If the rotation number of $f_0$ is irrational and of constant type, one can, using the same method, also obtain asymptotic relations between $\delta(n)$ and various dynamical scalings. These relations depend on the combinatorics and are algebraically very awkward (see [Circles I]). Work in progress is to find the right scalings so that such relations become more transparent. In particular we expect to find simple relations in the case where the continued fraction expansion is periodic.

The theorem above should be compared with the renormalization theory for critical circle maps with golden mean rotation number. In that case there is (assuming the convergence of renormalization) also a relation between parameter scalings and geometrical data: the parameter scaling $\delta(n)$ is the reciprocal of the spectral radius of the renormalization operator which is defined in terms of the geometry of the "fixed point" of renormalization, and $\sigma(n)$ is another eigenvalue of that operator. For the singularities we discuss, the renormalization theory is more non-degenerate, in particular, the associated operator is unbounded.

## I. Assumptions and Notations

We consider one parameter families $\{f_t\}_{t\in J}$ of circle maps with flat spot singularities, where $J$ is a closed interval in $\mathbb{R}$. We make the following assumptions.
1. For each $t$ the map $f_t: S^1 \to S^1$ has degree one.
2. There exists an interval $U=[l,r]$ such that $f_t(U)$ is a point and $f_t$ is strictly monotone in the complement of $U$.

3. In the complement of the flat spot $U, f_t$ is $C^3$ and has negative Schwarzian derivative.
4. In a small left neighborhood of $l$, $f_t$ has bounded nonlinearity; in a small right neighborhood of $r$ we have $f_t = A_t \circ x^{v_r}$, where $A_t$ has bounded nonlinearity and the power $v_r \geqq 1$.
5. The map $t \to f_t(x)$ is $C^1$ and $\Delta f_t(x) := \frac{d}{dt} f_t(x)$ is $C^1$ in $x$ and positive.

*Remark.* The conditions that $f_t$ be $C^3$ and $Sf_t < 0$ can be relaxed to the requirement that $f_t$ be $C^2$, at the cost of extra technicalities.

For most of the notation we refer to [Circles I].

1. Itineraries: denote $f_t^n(U)$ by $n(t)$ for $n \geqq 0$ (thus $0(t) = U$ and $1(t)$ is the critical value); denote the interval $f_t^{-n}(U)$ by $U_t(n)$.
2. $d(a, b)$ denotes the distance from $a$ to $b$. If $b$ is a set then $d(a, b)$ denotes $\inf_{x \in b} d(a, x)$.
3. Scalings: $\sigma(n, t) := \dfrac{d(q_n(t), q_{n+2}(t))}{d(q_{n-2}(t), q_n(t))}$.
4. We denote spatial derivatives of maps by $D: Df_t^n(x)(t) = \frac{d}{dx} f_t^n(x)$.
5. We denote spatial derivatives evaluated at the critical value by: $D(n)(t) = Df_t^n(1(t))$.
6. We denote derivatives with respect to the parameter by $\Delta: \Delta f_t^n(x) := \frac{d}{dt} f_t^n(x)$.
7. We denote derivatives with respect to the parameter at the critical interval $U$ as: $\Delta(n)(t) := \Delta f_t^n(U) = \frac{d}{dt} n(t)$.
8. If it is clear that $t = 0$, we will often write $D(n)$ or $\Delta(n)$ instead of $D(n)(t)$ or $\Delta(n)(t)$.

In the rest of this article we will make use of a result which has been proved in the course of proving Theorem 7.1 in [Circles I]. Let $f_0$ satisfy the above hypotheses and suppose further that it has irrational rotation number of bounded type. The set $S^1 - \bigcup U_0(i)$ consists of $q_n + 1$ closed intervals denoted by $A_i^n$, $i \in \{0, \ldots, q_n\}$. The one interval whose boundary contains the critical point $(r)$ is called $A_0^n$, all the others will be denoted by $\underline{A}_i^n$.

**Lemma 1.1.** *For all $j$ such that $A_j^n \subset A_i^{n-2}$, we have* $\lim_n \dfrac{|A_j^n|}{|A_i^{n-2}|} = 0$ *(uniform in i).*

## II. Calculation of Derivatives

Consider a one parameter family $f_t$ as defined in Sect. I. We define natural scalings in the parameter space and show how they are related to scalings in the "configuration-space." The crucial part is the relation between configuration scalings and the long term derivative of the critical value.

Suppose that at parameter value $t = 0$ the rotation number of $f_t$ is the golden mean. Denote by the sequence $\{q_n\}$ the successive denominators of the continued fraction convergents to the golden mean. Recall that $q_n = a_{n-1} q_{n-1} + q_{n-2}$, where $a_i = 1$ for all $i$. The $a_i$ are called the continued fraction coefficients. Consider the following interval in parameter space:

$$I_n = \{t \mid f_t \text{ has rotation number } p_n/q_n\}.$$

One observes that this interval has a subinterval $I'_n$

$$I'_n = \{t \mid q_n(t) \in U\} = \{t \mid q_{n-2}(t) \in U_t(q_{n-1})\},$$

where a periodic point is contained in the critical interval. Finally we consider an interval $I''_n$ which contains $I_n$

$$I''_n = [t_1, 0],$$

where $f_0$ has rotation number golden mean and $f_{t_1}$ has rotation number whose continued fraction coefficients are given by

$$a_i = 1 \quad \text{for all} \quad i \neq n-2,$$

$$a_{n-2} = 2.$$

(Note that we have assumed that $n$ is even so that $t_1 < 0$.) Figure 2.1 shows a sketch of the location of various points and intervals at time $t=0$. The fat arrows indicate the direction of movement as the parameter increases.

Because $\frac{d}{dt} f_t > 0$, upon decreasing (increasing) the parameter, inverse images of $U$ move to the right (left), forward images move to the left (right). In particular for $t$ in $I_n$, any point in the forward orbit of the critical interval and any interval in the backward orbit of the critical interval such that the sum of indices is smaller than $q_n$, do not intersect, because otherwise one would have periodic points of period lower than $q_n$. Therefore for $k < n-2$, the points $q_k(t)$ are sandwiched between the intervals $U_t(q_{k+1})$ and $U_t(q_{k-1})$. In [Circles I] we proved that there is very little space between these intervals. As the point $q_k(t)$ also moves in the opposite direction to that of these adjacent intervals without intersecting them, the actual length of the trajectory of $q_k(t)$ is somewhat smaller than the length of the gap at time $t=0$.

The concern of this section is to prove that for $t \in I''_n$ the velocity at which $q_{n-2}(t)$ travels is essentially constant and equals, up to a constant, a spatial derivative. These results are stated in Theorem 2.4 and Proposition 2.5.

The spatial derivative $D(n)(t)$ has been considered extensively in [Circles I]. We recall the main relations (established in Sects. IV, V, and VI). Recall that for $n$ even $q_n$ is to the right of $U$, for $n$ odd $q_n$ is to the left of $U$. These relations are evaluated at $t=0$:

Relation 1: $$\sigma(n) \cong \frac{d(q_n, U)}{d(q_{n-2}, U)}.$$

Relation 2: $$D(q_n) \cong \frac{v_l v_r}{\sigma(n)} = \frac{v_r}{\sigma(n)} \quad (\text{since } v_l = 1).$$

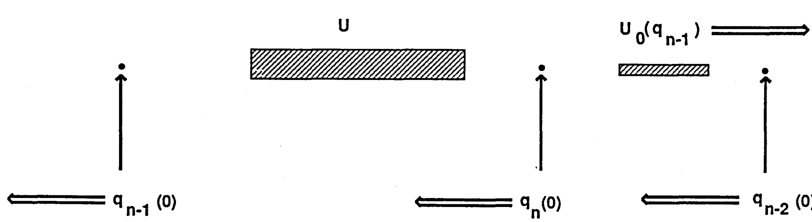


Fig. 2.1

Relation 3: $[\sigma(2n+2)]^{\nu_r} \cong \frac{\sigma(2n+1)\sigma(2n)}{\nu_r}, \quad \sigma(2n+1) \cong \frac{\sigma(2n)\sigma(2n-1)}{\nu_r}$

One observes that Relation 3 implies that the scalings $\{\sigma(n)\}$ tend to zero at least exponentially fast [see also Circles I. Sect. VI].

For $t \in I''_n$ and $k < n-2$, we can define a $t$-dependent scaling:

$$\sigma(k,t) \equiv \frac{d(q_k(t), q_{k+2}(t))}{d(q_{k-2}(t), q_k(t))}.$$

**Lemma 2.1.** *For* $i \in \{1, \ldots, q_{k-2}-1\}$ *we have*

$$Df_0^{q_{k-2}-i}((q_{k-1}+i)(0)) \cong Df_0^{q_{k-2}-i}((i)(0)),$$

*uniformly in i.*

*Proof.*

In Fig. 2.2 we have drawn the case in which $k$ is even. (But the proof is independent of the parity of $k$.) The inverse of $f^{q_{k-2}-i}$ has positive Schwarzian derivative on the interval $(q_{k-3}, q_{k-4})$. Theorem 5.3 of [Circles I] states that scalings decrease to zero, and Theorem 6.2 states that they do so very fast. Applying the "one-sided Koebe principle" (3.7) of [Circles I] then gives us that

$$\left| \ln \frac{Df_0^{q_{k-2}-i}((q_{k-i}+i)(0))}{Df_0^{q_{k-2}-i}((i)(0))} \right|$$

is uniformly small in $i$. □

**Lemma 2.2.** *There is a* $C > 0$ *such that*

$$|I''_n| \leqq C d(q_{n-2}(0), U).$$

*Proof.* By definition of $I''_n$, we have

$$|I''_n| \leqq \left[\min_{I''_n} \Delta(q_{n-2})(t)\right]^{-1} \cdot d(q_{n-2}(0), U).$$

By using the chainrule, one obtains

$$\Delta f_t^{q_{n-2}}(0(t)) = \sum_{i=1}^{q_{n-2}} Df_t^{q_{n-2}-i}(i(t)) \Delta f_t(i-1(t)).$$

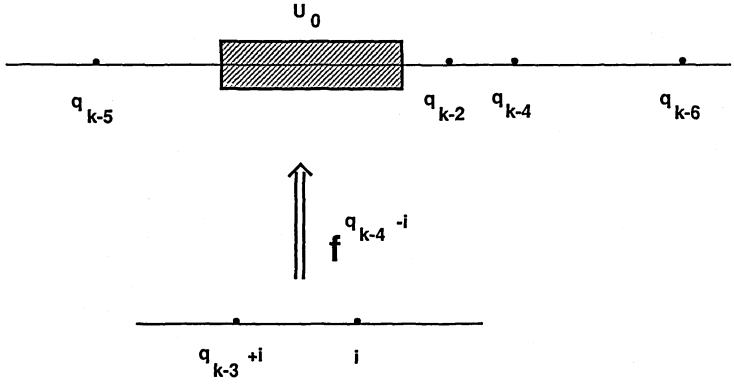


Fig. 2.2

Recall that all derivatives are strictly positive, so that $\Delta f_t^{q_{n-2}}(0(t))$ is bounded from below. □

**Lemma 2.3.** *For* $t \in I''_n$, $i \in \{1, \ldots, q_{n-2} -\}$

$$Df_t^{q_{n-2}-i}(i(t)) \cong Df_0^{q_{n-2}-i}(i(0))$$

*uniformly in i and t.*

*Proof.*

$$\frac{Df_t^{q_{n-2}-i}(i(t))}{Df_0^{q_{n-2}-i}(i(0))} = \prod_{k=1}^{q_{n-2}-i-1} \frac{Df_t((i+k)(0))}{Df_0((i+k)(0))} \cdot \prod_{k=1}^{q_{n-2}-i-1} \frac{Df_t((i+k)(t))}{Df_t((i+k)(0))}.$$

To evaluate the first of these two terms (denoted by $I$) we use the mean value theorem together with Assumption 5,

$$|I| \leqq \prod_{j=1}^{q_{n-2}-i} \left(1 + t \max_{t \in I''_n} \frac{\Delta Df_t(j(0))}{Df_0(j(0))}\right).$$

By Assumption 4, the quotient $\dfrac{\Delta Df_t(x)}{Df_0(x)}$ is bounded. Since $t \in I''_n$, Lemma 2.2 together with the estimates for $d(q_{n-2}(0), U)$ of [Circles I] and Corollary A.2 (see the appendix) imply that this is (uniformly) close to one.

The second term (denoted by $II$) requires a bit more work. Recall that by the monotonicity of $f$, for $k < q_{n-2} - i$ and $t \in I''_n$, we have (see Sect. 1 for definitions of $A_i^n$, $\underline{A}_i^n$, and $A_0^n$):

$$(i+k)(t) \in [r(-q_{n-2}+i+k)(0), (i+k)(0)] \subset \underline{A}_j^{n-2},$$

for some $j$. So

$$|II| \leqq \left|\exp\left\{\sum_{k=1}^{q_{n-2}-i-1} [\ln Df_t((i+k)(t)) - \ln Df_t((i+k)(0))]\right\}\right|$$

$$\leqq \exp\left\{\int_{\bigcup_i \underline{A}_i^{n-2}} \left|\frac{D^2 f_t(x)}{Df_t(x)}\right| dx\right\}.$$

Denote the distance (measured in the positive direction from the critical point) by $z$. Then by using Assumption 4 and bounded non-linearity away from the critical point, we obtain that there is a constant $C$ such that:

$$\left|\frac{D^2 f_t(z)}{Df_t(z)}\right| \leqq \left|\frac{C}{z}\right|.$$

Thus

$$|II| \leqq \exp\left\{C \int_{\bigcup_i \underline{A}_i^{n-2}} |dz/z|\right\}.$$

The integral in this expression has two contributions. One comes from the integral over a small subset of $\underline{A}_i^{n-4}$ $\left(\text{namely, } \left(\bigcup_i \underline{A}_i^{n-2}\right) \cap \left(\bigcup_i \underline{A}_i^{n-4}\right)\right)$. Lemma 1.1 implies that this part decreases rapidly to zero. The other contribution comes from the integral over

$$\left(\bigcup \underline{A}_i^{n-2}\right) \cap A_0^{n-4}.$$

This set consists of finitely many intervals associated with closest returns studied in [Circles I]. The fact that all the relevant scalings tend to zero implies that this contribution tends to zero as well.

Thus the second term (*II*) tends to one.

**Theorem 2.4.** *There exists a* $K>0$ *such that for all* $t\in I_n''$ *and* $k\leqq n-2$ *but large*

$$\frac{\Delta(q_k)(0)}{D(q_k-1)(0)}\cong K\,.$$

*Proof.* In this proof $t=0$ and we omit any reference to it. From the definition of $\Delta$ we obtain (chainrule):

$$\Delta f^k(x_0)=\sum_{i=1}^{k} Df^{k-i}(x_i)\cdot \Delta f(x_{i-1}),$$

and so

$$\Delta(k+l)=\Delta f^l(k)+Df^l(k)\cdot \Delta(k)\,.$$

We then obtain a recursion relation similar to the one defining the Fibonacci sequence ($q_k=q_{k-2}+q_{k-1}$):

$$\begin{aligned}\frac{\Delta(q_k)}{D(q_k-1)}&=\frac{D(q_{k-2}-1)}{D(q_k-1)}\cdot\frac{\Delta f^{q_{k-2}}(q_{k-1})}{D(q_{k-2}-1)}+\frac{Df^{q_{k-2}}(q_{k-1})}{Df^{q_{k-2}}(q_{k-1})}\,\frac{\Delta(q_{k-1})}{D(q_{k-1}-1)}\\&=\frac{D(q_{k-2}-1)}{D(q_k-1)}\cdot\frac{\Delta f^{q_{k-2}}(q_{k-1})}{D(q_{k-2}-1)}+\frac{\Delta(q_{k-1})}{D(q_{k-1}-1)}\,.\end{aligned}\qquad(*)$$

Now observe that all terms on the right are positive. In particular the sequence $\left\{\frac{\Delta(q_k)}{D(q_k-1)}\right\}$ is positive and increasing with $k$. Use the chainrule as before to write

$$\Delta f^{q_{k-2}}(q_{k-1})=\sum_{i=1}^{q_{k-2}} Df^{q_{k-2}-i}(q_{k-1}+i)\cdot \Delta f(q_{k-1}+i-1)\,.$$

Note that $\max_i d(q_{k-1}+i-1,i-1)$ converges to zero (uniformly in $i$) as $k$ goes to infinity according to Lemma 1.1. Therefore, by Proposition A.3 and A.4 (appendix) we obtain

$$\Delta f^{q_{k-2}}(q_{k-1})\cong\sum_{i=1}^{q_{k-2}} Df^{q_{k-2}-i}(i)\cdot \Delta f(i-1)=\Delta f^{q_{k-2}}(0)=\Delta(q_{k-2})\,.$$

Thus we obtain from (*) that

$$\frac{\Delta(q_k)}{D(q_k-1)}\cong\frac{D(q_{k-2}-1)}{D(q_k-1)}\cdot\frac{\Delta(q_{k-2})}{D(q_{k-2}-1)}+\frac{\Delta(q_{k-1})}{D(q_{k-1}-1)}\,.$$

We now study the coefficient in this recursion relation. Using Relation 2 we obtain that for all $k$,

$$D(q_k-1)=\frac{D(q_k)}{Df(q_k)}\cong\frac{v_r}{\sigma(k)\cdot Df(q_k)}\,.$$

Therefore,

$$\frac{D(q_{k-2}-1)}{D(q_k-1)} \cong \frac{\sigma(k)}{\sigma(k-2)} \frac{Df(q_k)}{Df(q_{k-2})} \cong C \cdot \sigma(k-1)$$

(using Relation 3) tends to zero very fast. Now apply Corollary A.2 (see Appendix) and we obtain that $\left\{\frac{\Delta(q_k)}{D(q_k-1)}\right\}$ is also bounded from above. Since the sequence is increasing we obtain that $K := \lim_{k\to\infty} \frac{\Delta(q_k)}{D(q_k-1)}$ is finite. □

**Proposition 2.5.** *For* $t \in I''_n$,

$$\Delta(q_{n-2})(t) \cong \Delta(q_{n-2})(0).$$

*Proof.* We have

$$\Delta f_t^{q_{k-2}}(0(t)) = \sum_{i=1}^{q_{k-2}} Df_t^{q_{k-2}-i}(i(t)) \cdot \Delta f_t((i-1)(t)).$$

Recall that $\Delta f$ is uniformly continuous. In view of Lemma 2.3, we can now apply Propositions A.3 and A.4 (Appendix) to obtain the result. □

## III. Parameter Scalings

From Proposition 2.5 we conclude that $q_{n-2}(t)$, $t \in I''_n = [t_1, 0]$, travels at roughly constant velocity to the left as the parameter decreases. From the topological considerations at the beginning of Sect. 2, we know that the endpoint of the journey, $q_{n-2}(t_1)$, is situated to the right of $r(0)$ (see Fig. 3.1). On its journey $q_{n-2}(t)$ traverses the entire interval $U_t(q_{n-1})$ which itself moves to the right. Notice that at $t = t_1$, the number $q_{n-1} = q_{n-2} + q_{n-3}$ is not a continued fraction denominator of the rotation number of the map $f_t$ anymore. By definition of the interval $I''_n$, the continued fraction denominator now is equal to

$$Q_{n-1} = 2q_{n-2} + q_{n-3}.$$

(In Fig. 3.1, only "new" continued fraction denominators are used.)

From this information we will now deduce that the distance travelled by $q_{n-2}(t)$ for $t \in I''_n$ is roughly equal to $d(r(0), q_{n-2}(0))$. In other words: $U_t(q_{n-1})$ does not move appreciably. That result together with Proposition 2.5 is sufficient to prove our main result about parameter scalings (Theorem 3.4).

To have a convenient notation we define:

$$x(t) = l(-q_{n-3}(t)) \quad \text{if } n \text{ is even},$$

$$x(t) = r(-q_{n-3}(t)) \quad \text{if } n \text{ is odd}.$$

We will assume that $n$ is even, the analysis for $n$ odd is analogous.

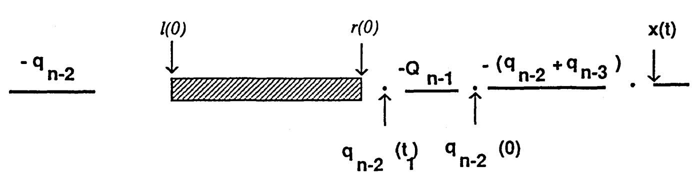


Fig. 3.1

**Lemma 3.1.** *For* $t \in I''_n$, $\lim_n \dfrac{\Delta x(t)}{\Delta(q_{n-2})(t)} = 0$.

*Proof.* Since $f_t^{q_n-3}(x(t)) = l(0)$ (is constant), we obtain by differentiation with respect to $t$:

$$\Delta f_t^{q_n-3}(x(t)) + Df_t^{q_n-3}(x(t)) \cdot \Delta x(t) = 0.$$

So

$$\Delta x(t) = \frac{1}{Df(x(t))} \cdot \frac{-\Delta f^{q_n-3}(x(t))}{Df^{q_n-3-1}(f(x(t)))}.$$

We now claim that for $t \in I''_n$,

$$Df_t^{q_n-3-i}(f^i(x(t))) \cong Df_t^{q_n-3-i}(i(t)) \cong Df_0^{q_n-3-i}(i(0)),$$

uniformly in $i$ and $t$. To prove the claim, observe first that

$$f_t^{q_n-3-i}(f^i(x(t))) = l(0), \quad f_t^{q_n-3-i}(i(t)) = q_{n-3}(t),$$

and that $f_t^{q_n-3-i}$, $i \geqq 1$, is invertible on $(q_{n-4}(t), q_{n-5}(t))$. The reasoning of Lemma 2.1 can now be applied to yield the first equality of the claim. The second equality is directly implied by Lemma 2.3.

Using the claim and Theorem 2.4, one easily derives that

$$|\Delta x(t)| \cong \frac{K}{Df(x(t))} \lesssim \frac{K}{Df_0(q_{n-2}(0))},$$

where for the inequality we have used Assumption 4. Thus using Proposition 2.5 and Theorem 2.4, one finds:

$$\left|\frac{\Delta x(t)}{\Delta(q_{n-2})(t)}\right| \lesssim \frac{1}{D(q_{n-2}-1)(0) \cdot Df_0(q_{n-2}(0))} = \frac{1}{D(q_{n-2})(0)} \cong \sigma(n-2)(0). \quad \square$$

**Lemma 3.2**

$$\frac{d(q_{n-2}(0), q_{n-2}(t_1))}{d(q_{n-2}(0), r(0))} \cong 1.$$

*Proof.* Since the rotation number of $f_{t_1}$ is an irrational number of bounded type, Theorem 5.3 and Lemmas 4.3, 4.4, and 5.2 of [Circles I] apply (these results say that scalings converge to zero). Thus the interval $U_t(q_{n-2}+q_{n-3})$ occupies almost all of the interval $(q_{n-2}(t_1), x(t_1))$. We have (see Fig. 3.1)

$$d(q_{n-2}(t_1), x(t_1)) \cong d(r(0), x(t_1)),$$

which implies:

$$\begin{aligned} &d(q_{n-2}(t_1), q_{n-2}(0)) + d(q_{n-2}(0), x(0)) + d(x(0), x(t_1)) \\ &\cong d(r(0), q_{n-2}(0)) + d(q_{n-2}(0), x(0)) + d(x(0), x(t_1)). \end{aligned}$$

We divide both sides of the equation by $d(r(0), q_{n-2}(0))$. The lemma is now obtained by noting that Lemma 3.1 implies that

$$\lim_n \frac{d(x(0), x(t_1))}{d(q_{n-2}(0), q_{n-2}(t_1))} = 0,$$

and that Lemma 4.4 in [Circles I] gives:

$$\lim_{n} \frac{d(q_{n-2}(0), x(0))}{d(q_{n-2}(0), r(0))} = 0. \quad \square$$

**Proposition 3.3.**

$$|I'_n| \cong |I_n| \cong |I''_n| \cong \frac{d(q_{n-2}(0), U)}{KD(q_{n-2}-1)}.$$

*Proof.* From Proposition 2.5, we deduce that $q_{n-2}(t)$ moves at roughly constant velocity through almost all of the interval $[\partial U, q_{n-2}(0)]$. Therefore the previous lemma implies

$$|I''_n| \cong \frac{d(q_{n-2}(0), U)}{\Delta(q_{n-2})(0)} \cong \frac{d(q_{n-2}(0), U)}{KD(q_{n-2}-1)}.$$

While $q_{n-2}(t)$ moves from $q_{n-2}(0)$ towards $\partial U$, almost all of its time is spent in the interval $U_t(q_{n-1})$. This then shows that $|I'_n| \cong |I''_n|$. Since $|I'_n| \leqq |I_n| \leqq |I''_n|$, the claim then follows. $\square$

Define the scalings $\{\delta(n)\}$ in the parameter space as

$$\delta(n) = \frac{|I_n|}{|I_{n-2}|}.$$

**Theorem 3.4.** *Let $v = v_r$ if $n$ is odd; $v = 1$ if $n$ is even. Then*

$$\delta(n) \cong \{\sigma(n-1)\}^{v}.$$

*Proof.* These scalings can be expressed in terms of dynamical scalings for the map $f = f_0$. We will express the value of the scaling in derivatives and scalings taken at $t = 0$, and omit reference to $t$. Using the previous lemmas and the relations between the scalings and derivatives mentioned in the beginning of Sect. II, we obtain:

$$\begin{aligned}
\delta(n+2) &\cong \frac{|I''_{n+2}|}{|I''_n|} \overset{\text{Proposition 3.3}}{\cong} \frac{d(q_n, U)}{D(q_n - 1)} \cdot \frac{D(q_{n-2}-1)}{d(q_{n-2}, U)} \\
&\cong \frac{d(q_n, U)}{d(q_{n-2}, U)} \cdot \frac{D(q_{n-2})}{D(q_n)} \cdot \frac{Df(q_n)}{Df(q_{n-2})} \\
&\cong \frac{d(q_n, U)^{v'}}{d(q_{n-2}, U)^{v'}} \frac{\sigma(n)}{\sigma(n-2)} \cong \sigma(n)^{v'} \frac{\sigma(n)}{\sigma(n-2)}.
\end{aligned}$$

Finally, applying Relation 3 twice:

$$\delta(n+2) \cong \sigma(n)^{v'} \frac{\sigma(n)}{\sigma(n-2)} \cong \frac{\sigma(n-1)\sigma(n)}{v_r} \cong \{\sigma(n+1)\}^{v},$$

where $v'$ denotes 1 if $n$ is odd (the linear side of the singularity), $v_r$ if $n$ is even (the powerlaw side); $v$ denotes 1 if $n+1$ is odd, $v_r$ if $n+1$ is even. Therefore $\delta(n+2) \cong \{\sigma(n+1)\}_{v}$. $\square$

## Appendix

The first part of this appendix is concerned with the solutions $\{x_k\}$ to the recursion relation:

$$x_k = a_k x_{k-2} + x_{k-1}.$$

Here the sequence of coefficients $\{a_k\}$ is fixed and positive.

**Proposition A.1.** *If $x_1$ and $x_2$ are positive then the sequence $\{x_k\}$ is positive and increasing. Moreover there exists a constant $C$ so that for all $k$:*

$$x_k < C \cdot \prod_1^k (1 + a_i).$$

*Proof.* The first statements follow from the positivity of the sequence $\{a_k\}$. We prove the inequality by induction. Choose $C$ such that the inequality holds for $k=1$ and $k=2$. Assume that the inequality holds up to $k-1$. Then we obtain for $x_k$:

$$\begin{aligned} x_k &\leqq (a_k + 1 + a_{k-1}) C \cdot \prod_1^{k-2} (1 + a_i) \\ &\leqq (1 + a_{k-1})(1 + a_k) C \cdot \prod_1^{k-2} (1 + a_i) = C \cdot \prod_1^k (1 + a_i). \quad \square \end{aligned}$$

Under the assumptions to the previous proposition we obtain the following corollary.

**Corollary A.2.** *If the sequence $\{a_k\}$ is summable then the sequence $\{x_k\}$ converges.*

*Proof.* Since the sequence $\{a_k\}$ is summable, the sequence $\{x_k\}$ is bounded. Since the sequence $\{x_k\}$ is also increasing, the limit then exists and if finite. □

Let $a_i$ $(i \in \mathbb{N})$ and $b$ be strictly positive continuous functions from a compact metric space $X$ to $\mathbb{R}$. Let $x_\infty = \{x(i)\}_{i \in \mathbb{N}}$ and $y_\infty = \{y(i)\}_{i \in \mathbb{N}}$ be infinite sequences of points in $X$. Let $x_k$ and $y_k$ be sequences of sequences converging uniformly to $x_\infty$ and $y_\infty$, respectively. Define

$$c_k(x, y) = \sum_{i=1}^{k-1} a_{k-i}(x(i)) \cdot b(y(i)).$$

**Proposition A.3.**

$$c_k(x_k, y_k) \cong c_k(x_k, y_\infty).$$

*Proof.*

$$\lim_k \frac{c_k(x_k, y_k)}{c_k(x_k, y_\infty)} - 1 = \lim_k \sum_i \left[ \frac{a_{k-i}(x_k(i)) \cdot b(y_\infty(i))}{\sum_i a_{k-i}(x_k(i)) \cdot b(y_\infty(i))} \cdot \frac{b(y_k(i)) - b(y_\infty(i))}{b(y_\infty(i))} \right].$$

Since $X$ is compact and $b$ is uniformly continuous and bounded from below, we have that $\frac{b(y_k(i)) - b(y_\infty(i))}{b(y_\infty(i))}$ converges to zero uniformly in $i$. The above summation is a weighted average over terms that converge uniformly to zero. Thus

$$\lim_k \frac{c_k(x_k, y_k)}{c_k(x_k, y_\infty)} - 1 = 0. \quad \square$$

**Proposition A.4.** *If* $\frac{a_{k-i}(y_k(i)) - a_{k-i}(y_\infty(i))}{a_{k-i}(y_\infty(i))} \to 0$ *uniformly in* $i$ *as* $k \to \infty$, *then*

$$c_k(x_k, y_\infty) \cong c_k(x_\infty, y_\infty).$$

*Proof.* Using the same method as in Proposition A.3, we obtain that $\frac{c_k(x_k, y_k)}{c_k(x_k, y_\infty)} - 1$ is a weighted average of terms $\frac{a_{k-i}(y_k(i)) - a_{k-i}(y_\infty(i))}{a_{k-i}(y_\infty(i))}$ which converge to zero uniformly. □

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## References

[Circles I] Veerman, J.J.P., Tangerman, F.M.: Scalings in Circle Maps I. Commun. Math. Phys. **134**, 89–107 (1990)

### *Erratum*

In Figs. 4.1 and 7.1 of [Circles I] "$a_{\text{index}-1}$" should be "$a_{\text{index}} - 1$" and "$q_{\text{index}-1}$" should be "$q_{\text{index}} - 1$".

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