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Characterization of States of Infinite Boson Systems

II. On the Existence of the Conditional Reduced Density Matrix

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Abstract. In the present paper we deal with the problem of existence and uniqueness of the conditional reduced density matrix (c.r.d.m.) corresponding to a locally normal state of a boson system. The c.r.d.m. was introduced in [3] (Part I of the present series of papers). In order to characterize the class of states possessing a c.r.d.m. we will introduce the family of conditional states of a locally normal state, and we will discuss the relation between the conditional states, the c.r.d.m. and the conditional distribution of the position distribution of the state.

1. Introduction

In [2, 3] we introduced the position distribution Q_ω and the conditional reduced density matrix k_ω (c.r.d.m.) of a locally normal state ω of a boson system. It was shown that Q_ω and k_ω determine the whole state. In the present paper we will characterize a class of states which possess a c.r.d.m. For that reason we first will introduce the notion of conditional states ω_Λ^φ of a state ω that describe the behaviour of the system inside a bounded region Λ having fixed a configuration φ outside this area. It is shown that the position distribution of the conditional state ω_Λ^φ is just the conditional distribution $Q_\omega(\cdot\,|_{\Lambda^c}\mathfrak{M})(\varphi)$ of the position distribution Q_ω.

Further, we will see that the c.r.d.m. exists if for each $\Lambda \in \mathfrak{B}$ the family (ω_Λ^φ) of conditional states exists and if Q_ω is Σ'_ν-point process. Moreover, we will prove that the c.r.d.m. is a.e.-uniquely determined.

We use the notations and notions given in Part I [3]. We refer to this part by adding I, e.g. I.2.1 means Sect. 2.1 in [3].

As in the previous papers we consider exclusively locally normal states of bosons without spin. The phase space G is assumed to be Polish endowed with a locally finite diffuse measure ν, and the local algebras consist of all bounded linear operators on the Fock space over the bounded regions of the Fock space G.

2. The Results

The idea to describe the infinite system by its behaviour in bounded areas led in quantum statistical mechanics to the introductions of the notion of quasilocal algebras. If one restricts measurements to a bounded region one "forgets" all information about the behaviour outside this set. In analogy to the concept of conditional distributions in probability theory (especially in the classical theory of Gibbs measures) we want to introduce now the conditional states ω_Λ^φ of a locally normal state ω that describe the behaviour of the system in $\Lambda \in \mathfrak{B}$ having fixed a configuration $\varphi \in M_{\Lambda^c}$ (outside of Λ). For basic notations used below cf I.2–I.7.

First, we need a measurability condition for families of states. With respect to further considerations (in Sect. 3) the definition will be slightly more general than needed in this section.

2.1. Definition. Let $\tilde{\mathscr{A}}$ be a $*$-subalgebra of $\mathscr{L}(\mathscr{M})$ and $\tilde{\mathfrak{M}}$ a σ-subalgebra of $\mathfrak{M}$. A family $(\xi^\varphi)_{\varphi \in M}$ of positive linear functionals on $\tilde{\mathscr{A}}$ is called $\tilde{\mathfrak{M}}$-*measurable* if for all $A \in \tilde{\mathscr{A}}$, $A \geqq 0$ the mapping $\varphi \mapsto \xi^\varphi(A)$ from M into $[0, \infty)$ is a $\tilde{\mathfrak{M}}$-measurable function.

2.2. Definition. Let Q be a point process, and let $(\xi_1^\varphi)_{\varphi \in M}$ and $(\xi_2^\varphi)_{\varphi \in M}$ be two $\tilde{\mathfrak{M}}$-measurable families of positive linear functionals on $\tilde{\mathscr{A}}$. We say that $(\xi_1^\varphi)_{\varphi \in M} = (\xi_2^\varphi)_{\varphi \in M}$ *Q-a.s.* if for all $A \in \tilde{\mathscr{A}}$ $\xi_1^\varphi(A) = \xi_2^\varphi(A)$ for *Q-a.a.* φ.

Now, for arbitrary $\Lambda \in \mathfrak{B}$ we set

$$_\Lambda\mathfrak{M}^\perp := \bigcup_{\substack{\Lambda' \in \mathfrak{B} \\ \Lambda \cap \Lambda' = \emptyset}} {}_{\Lambda'}\mathfrak{M}, \qquad {}_l\mathfrak{M} := \bigcup_{\Lambda \in \mathfrak{B}} {}_\Lambda\mathfrak{M}. \tag{2.1}$$

Observe that

$$_\Lambda\mathfrak{M}^\perp = {}_{\Lambda^c}\mathfrak{M} \cap {}_l\mathfrak{M} \qquad (\neq {}_{\Lambda^c}\mathfrak{M}).$$

2.3. Definition. Let ω be a locally normal state on $\mathscr{A}$, and let $\Lambda \in \mathfrak{B}$. A ${}_{\Lambda^c}\mathfrak{M}$-measurable family $({}_\Lambda\omega^\varphi)_{\varphi \in M}$ of states on ${}_\Lambda\mathscr{A}$ is called the *family of conditional states of* ω *on* Λ *if for all* $A \in {}_\Lambda\mathscr{A}$ and $Y \in {}_\Lambda\mathfrak{M}^\perp$

$$\omega(AO_Y) = \int_Y Q_\omega(d\varphi)\, {}_\Lambda\omega^\varphi(A). \tag{2.2}$$

The left side of (2.2) is always well-defined, and we have the following useful relation:

2.4. Lemma. *Let* Λ, $\Lambda' \in \mathfrak{B}$, $\Lambda \cap \Lambda' = \emptyset$, $A \in {}_\Lambda\mathscr{A}$, $Y \in {}_{\Lambda'}\mathfrak{M}$. *Then* $AO_Y \in {}_{\Lambda \cup \Lambda'}\mathscr{A}$, *and we have*

$$AO_Y = S(Y \cap M_{\Lambda^c}, O_{M_\Lambda} A O_{M_\Lambda}). \tag{2.3}$$

2.5. Remark. AO_Y is a combination of the local observable A (from ${}_\Lambda\mathscr{A}$) with a position measurement outside Λ ($Y \in {}_{\Lambda'}\mathfrak{M}$, $\Lambda' \cap \Lambda = \emptyset$). So (2.2) allows an interpretation of ${}_\Lambda\omega^\varphi$ as the conditional state in Λ having fixed the configuration φ outside.

2.6. Proposition. *Let* ω *be a locally normal state,* $\Lambda \in \mathfrak{B}$ *and assume the family* $({}_\Lambda\omega^\varphi)_{\varphi \in M}$ *of conditional states exists. Then we have the following:*

(i) $({}_\Lambda\omega^\varphi)_{\varphi \in M}$ *is* Q_ω*-a.s. uniquely determined.*

(ii) *For* Q_ω*-a.a.* φ ${}_\Lambda\omega^\varphi$ *is a normal state on* ${}_\Lambda\mathscr{A}$.

We thus have that not only the local states ${}_{\Lambda}\omega$ given by

$${}_{\Lambda}\omega(A)=\omega(A) \qquad (A\in{}_{\Lambda}\mathscr{A})$$

are normal ones (by definition) but also the conditional states provided they exist. Observe that by

$$\omega^{\varphi}_{\Lambda}(A):={}_{\Lambda}\omega^{\varphi}(J_{\Lambda}A) \qquad (A\in\mathscr{A}_{\Lambda}) \tag{2.4}$$

we may define an (equivalent) family of normal states on $\mathscr{A}_{\Lambda}$, and (2.2) could be written also in the form

$$\omega(O_Y J_{\Lambda}A)=\int_Y Q_{\omega}(d\varphi)\omega^{\varphi}_{\Lambda}(A). \tag{2.5}$$

In [2] we proved that the position distribution of a locally normal state is locally a Σ^c_v-point process. If the conditional states exist still more is true.

2.7. Proposition. *Let ω be a locally normal state on $\mathscr{A}$ such that for all $\Lambda\in\mathfrak{B}$ the family $({}_{\Lambda}\omega^{\varphi})_{\varphi\in M}$ of conditional states exists. Then Q_{ω} is a Σ^c_v-point process.*

The proof of the above statement is based on the observation that the position distribution of the conditional state ${}_{\Lambda}\omega^{\varphi}$ is just the conditional distribution of the position distribution of ω. Indeed, we have the following connection.

2.8. Proposition. *Let ω be a locally normal state on $\mathscr{A}$ such that for a set $\Lambda\in\mathfrak{B}$ the family $({}_{\Lambda}\omega^{\varphi})_{\varphi\in M}$ of conditional states exists. Then for Q_{ω}-a.a. φ we have*

$${}_{\Lambda}\omega^{\varphi}(O_Y)=Q_{\omega}(Y\,|\,{}_{\Lambda^c}\mathfrak{M})(\varphi) \qquad (Y\in{}_{\Lambda}\mathfrak{M}). \tag{2.6}$$

In Part I of this series we always assumed that Q_{ω} is a Σ'_v-point process. There are some hints for the conjecture that all conditional states exist if Q_{ω} is of the type Σ'_v. However, this could be shown only for normal states.

2.9. Proposition. *Let ω be a normal state on $\mathscr{L}(\mathscr{M})$ such that Q_{ω} is a Σ'_v-point process. Then for all $\Lambda\in\mathfrak{B}$ the family $({}_{\Lambda}\omega^{\varphi})_{\varphi\in M}$ of conditional states of ω on Λ exists.*

In I we introduce the c.r.d.m. and showed that it is a useful tool for the characterization and for the description of the state. States possessing all conditional states are just the states having a c.r.d.m.

2.10. Theorem. *Let ω be a locally normal state on $\mathscr{A}$ such that Q_{ω} is a Σ'_v-point process. Then the c.r.d.m. of ω exists if and only if for all $\Lambda\in\mathfrak{B}$ the family $({}_{\Lambda}\omega^{\varphi})_{\varphi\in M}$ of conditional states exists.*

In Theorem I.7.3 we gave sufficient conditions on a point process Q and a function k ensuring the existence of a (unique) locally normal state ω with position distribution Q and c.r.d.m. k. From Theorem 2.10 we thus obtain that for the state constructed from Q and k all conditional states exist, and we have the following relation:

2.11. Proposition. *Let ω be a locally normal state on $\mathscr{A}$ such that Q_{ω} is a Σ'_v-point process and the c.r.d.m. k exists. Then for all $\Lambda\in\mathfrak{B}$ and Q_{ω}-a.a. φ*

$${}_{\Lambda}\omega^{\varphi}(J_{\Lambda}A)=\frac{1}{\eta^{\varphi_{\Lambda^c}}_{Q_{\omega}}(M_{\Lambda})}\operatorname{Tr}(K^{\varphi_{\Lambda^c}}_{\Lambda}A) \qquad (A\in\mathscr{A}_{\Lambda}), \tag{2.7}$$

where $K^{\varphi_{\Lambda^c}}_{\Lambda}$ is the positive trace-class operator with kernel $k(\cdot,\cdot,\varphi_{\Lambda^c})$.

From the proofs of the above statements we get also that the c.r.d.m. is a.e.-uniquely determined (for normal states this was shown already in I – cf. Theorem I.6.4).

2.12. Proposition. *Let ω be a locally normal state on $\mathscr{A}$ such that the position distribution Q_ω is a Σ'_v-point process, and let k_1, k_2 be two c.r.d.m. of ω. Then for $F \times F \times Q_\omega$-a.a. $(\varphi_1, \varphi_2, \varphi)$ we have $k_1(\varphi_1, \varphi_2, \varphi) = k_2(\varphi_1, \varphi_2, \varphi)$.*

It seems to be obvious that a locally normal state is a normal one if (and only if) Q_ω is a finite point process. However, based on the above result we can show it only for states possessing a c.r.d.m.

2.13. Proposition. *Let ω be a locally normal state such that Q_ω is a Σ'_v-point process and the c.r.d.m. exists. Then ω is a normal state if and only if Q_ω is a finite point process.*

Summarizing, we can give the following characterization:

2.14. Proposition. *Let ω_1, ω_2 be locally normal states such that Q_{ω_1} and Q_{ω_2} are Σ'_v-point processes and the c.r.d.m. $k_{\omega_1}, k_{\omega_2}$ exist. The following statements are equivalent:*
(i) $\omega_1 = \omega_2$,
(ii) $Q_{\omega_1} = Q_{\omega_2}$ *and* $k_{\omega_1} = k_{\omega_2}$ *a.e.*

3. Proofs

3.1. Proof of Lemma 2.4

Observe that $A \in {}_A\mathscr{A} \subseteqq {}_{A \cup A'}\mathscr{A}$, $Y \in {}_{A'}\mathfrak{M} \subseteqq {}_{A \cup A'}\mathfrak{M}$. Thus $AO_Y \in {}_{A \cup A'}\mathscr{A}$. For all $\Psi \in \mathscr{M}$, $\varphi \in M$ we get

$$\begin{aligned} &S(Y \cap M_{A^c}, O_{M_A} A O_{M_A})\Psi(\varphi) \\ &= \sum_{\hat\varphi \subseteqq \varphi} \chi_{Y \cap M_{A^c}}(\hat\varphi)\chi_{M_A}(\varphi - \hat\varphi)(AO_{M_A}\Psi_{\hat\varphi})(\varphi - \hat\varphi) \\ &= \sum_{\hat\varphi \subseteqq \varphi_A} \chi_{Y \cap M_{A^c}}(\hat\varphi + \varphi_{A^c})(AO_{M_A}\Psi_{\hat\varphi + \varphi_{A^c}})(\varphi_A - \hat\varphi) \\ &= \chi_{Y \cap M_{A^c}}(\varphi_{A^c})(AO_{M_A}\Psi_{\varphi_{A^c}})(\varphi_A) = \chi_Y(\varphi)(AO_{M_A}\Psi_{\varphi_{A^c}})(\varphi_A) \\ &= O_Y A \Psi(\varphi). \quad \square \end{aligned} \tag{3.1}$$

3.2. Proof of Proposition 2.6

1°. We fix an $A \in {}_A\mathscr{A}$, $A \geqq 0$. Suppose there are given two versions $(\omega_j^\varphi)_{\varphi \in M}$, $j = 1, 2$ of conditional states of ω on $\mathscr{A}_A$.

We thus get for all $Y \in {}_A\mathfrak{M}^\perp$

$$\int_Y Q_\omega(d\varphi)\omega_1^\varphi(A) = \int_Y Q_\omega(d\varphi)\omega_2^\varphi(A). \tag{3.2}$$

Since ${}_A\mathfrak{M}^\perp \subseteqq {}_{A^c}\mathfrak{M}$ and for each $Y \in {}_{A^c}\mathfrak{M}$ there exists an increasing sequence $(Y_n) \subseteqq {}_A\mathfrak{M}^\perp$ with $\lim_{n \to \infty} Y_n = Y$ (3.2) holds for all $Y \in {}_{A^c}\mathfrak{M}$.

The mapping $\varphi \mapsto \omega_j^\varphi(A)$ was assumed to be ${}_{A^c}\mathfrak{M}$-measurable. This implies $\omega_1^\varphi(A) = \omega_2^\varphi(A)$ for Q_ω-*a.a.* φ. Since each operator from ${}_A\mathscr{A}$ may be expressed in a unique way as a sum $A_1 - A_2 + iA_3 - iA_4$ of four positive operators from ${}_A\mathscr{A}$ we get for all $A \in {}_\Lambda\mathscr{A}$ $\omega_1^\varphi(A) = \omega_2^\varphi(A)$ for Q_ω-*a.a.* φ. This proves assertion (i) of Proposition 2.6.

2°. ${}_\Lambda\omega^\varphi$ is a normal state on ${}_\Lambda\mathscr{A}$ if and only if ω_Λ^φ defined by (2.4) is a normal state on $\mathscr{A}_\Lambda$. It is known that a state on $\mathscr{A}_\Lambda$ is normal if it achieves its norm already on compact operators from $\mathscr{A}_\Lambda$ (cf. [1, Theorem 2.6.14]), i.e. we have to prove that for Q_ω-*a.a.* φ

$$\sup\{|\omega_\Lambda^\varphi(A)|: A \in \mathscr{A}_\Lambda,\ A \text{ compact},\ \|A\| = 1\} = 1. \tag{3.3}$$

Consequently, it suffices to show that there exists a sequence $(A_n)_{n \geqq 0}$ of compact operators from $\mathscr{A}_\Lambda$ with $\|A_n\| = 1$ such that for Q_ω-*a.a.* φ $\lim_{n\to\infty} \omega_\Lambda^\varphi(A_n) = 1$. Let $(\Psi_n)_{n \geqq 0}$ be an orthonormal base in $\mathscr{M}_\Lambda$ and denote by A_n the (finite rank) projector onto the span of $\{\Psi_0, \ldots, \Psi_n\}$, $n \geqq 0$. We have $\|A_n\| = 1$, $0 \leqq A_n \leqq A_{n+1} \leqq O_{M_\Lambda}$ for all $n \geqq 0$. Thus the limit $\lim_{n\to\infty} \omega_\Lambda^\varphi(A_n)$ exists for all φ and we have

$$0 \leqq \lim_{n\to\infty} \omega_\Lambda^\varphi(A_n) \leqq 1. \tag{3.4}$$

$(A_n)_{n \geqq 0}$ converges to O_{M_Λ} in the σ-weak topology on $\mathscr{A}_\Lambda$. So $((J_\Lambda A_n) O_Y)_{n \geqq 0}$ converges σ-weakly in $\mathscr{L}(\mathscr{M})$ to O_Y for all $Y \in \mathfrak{M}$. Since the local states are continuous in the σ-weak topology we get for each $Y \in {}_\Lambda\mathfrak{M}^\perp$,

$$\lim_{n\to\infty} \omega((J_\Lambda A_n \mathrm{O}_Y) = \omega(O_Y) = Q_\omega(Y). \tag{3.5}$$

Using Lebesgue's dominated convergence theorem which can be applied because of (3.4) we obtain for all $Y \in {}_\Lambda\mathfrak{M}^\perp$

$$\begin{aligned} Q_\omega(Y) &= \lim_{n\to\infty} \int_Y Q_\omega(d\varphi)\, {}_\Lambda\omega^\varphi(J_\Lambda A_n) = \lim_{n\to\infty} \int_Y Q_\omega(d\varphi)\, \omega_\Lambda^\varphi(A_n) \\ &= \int_Y Q_\omega(d\varphi) \lim_{n\to\infty} \omega_\Lambda^\varphi(A_n) = \int_Y Q_\omega(d\varphi) \cdot 1. \end{aligned}$$

Since $({}_\Lambda\omega^\varphi)_{\varphi \in M}$ was assumed to be ${}_{\Lambda^c}\mathfrak{M}$-measurable we finally get that for Q_ω-*a.a.* φ $\lim_{n\to\infty} \omega_\Lambda^\varphi(A_n) = 1$. Thus, for Q_ω-*a.a.* φ ω_Λ^φ is a normal state on $\mathscr{A}_\Lambda$, i.e. ${}_\Lambda\omega^\varphi$ is a normal one on ${}_\Lambda\mathscr{A}$. □

3.3. Proof of Proposition 2.8

For all $Y_1 \in {}_\Lambda\mathfrak{M}$, $Y_2 \in {}_\Lambda\mathfrak{M}^\perp$ we get from the definition of the conditional state

$$\omega(O_{Y_1} O_{Y_2}) = \int_{Y_2} Q_\omega(d\varphi)\, {}_\Lambda\omega^\varphi(O_{Y_1}). \tag{3.6}$$

On the other side, from $O_{Y_1} O_{Y_2} = O_{Y_1 \cap Y_2}$ we conclude

$$\omega(O_{Y_1} O_{Y_2}) = \omega(O_{Y_1 \cap Y_2}) = Q_\omega(Y_1 \cap Y_2). \tag{3.7}$$

(3.6) and (3.7) lead to the relation

$$\int_{Y_2} Q_\omega(d\varphi) \chi_{Y_1}(\varphi) = \int_{Y_2} Q_\omega(d\varphi)\, {}_\Lambda\omega^\varphi(O_{Y_1}) \qquad (Y_1 \in {}_\Lambda\mathfrak{M},\ Y_2 \in {}_\Lambda\mathfrak{M}^\perp). \tag{3.8}$$

As in the first part of the proof of Proposition 2.6 one can see that the equality in (3.8) holds for all $Y_2 \in {}_{\Lambda^c}\mathfrak{M}$. Since $\varphi \mapsto {}_\Lambda\omega^\varphi(O_{Y_1})$ is ${}_{\Lambda^c}\mathfrak{M}$-measurable we get that for all $Y \in {}_\Lambda\mathfrak{M}$

$${}_\Lambda\omega^\varphi(O_Y) = Q_\omega(Y|_{\Lambda^c}\mathfrak{M})(\varphi) \qquad (Q_\omega\text{-}a.a.\ \varphi). \quad \square \tag{3.9}$$

3.4. Proof of Proposition 2.7

We fix a set $\Lambda \in \mathfrak{B}$. From Proposition 2.6 we know that for Q_ω-*a.a.* φ ω_Λ^φ [defined by (2.4)] is a normal state on $\mathscr{A}_\Lambda$. This implies that the position distribution of ω_Λ^φ (which we will denote by Q_Λ^φ) is a finite Σ_v^c-point process concentrated on M_Λ (cf. [2, Proposition 3.1]). The ${}_{\Lambda^c}\mathfrak{M}$-measurability of $({}_\Lambda\omega^\varphi)_{\varphi \in M}$ causes that for Q_ω-a.a. φ $Q_\Lambda^\varphi = Q_\Lambda^{\varphi_{\Lambda^c}}$.

From [6, Theorem 2.13] we conclude that for all $\varphi \in M_{\Lambda^c}$ there exists a measurable function $g_{\Lambda,\varphi}: M \to \mathbb{R}$ such that

$$Q_\Lambda^\varphi(Y) = \int_Y F_\Lambda(d\hat{\varphi}) g_{\Lambda,\varphi}(\hat{\varphi}) \qquad (Y \in \mathfrak{M}_\Lambda). \tag{3.10}$$

From Proposition 2.8 and (3.10) we finally get that for all $\Lambda \in \mathfrak{B}$, $Y \in {}_\Lambda\mathfrak{M}$ and Q_ω-*a.a.* φ,

$$\begin{aligned} Q_\omega(Y|_{\Lambda^c}\mathfrak{M})(\varphi) &= {}_\Lambda\omega^\varphi(O_Y) = \omega_\Lambda^\varphi(O_{v_\Lambda Y}) = Q_\Lambda^\varphi(v_\Lambda Y) \\ &= \int_{v_\Lambda Y} F_\Lambda(d\hat{\varphi}) g_{\Lambda,\varphi_{\Lambda^c}}(\hat{\varphi}). \end{aligned} \tag{3.11}$$

Condition (3.11) is necessary and sufficient for Q_ω to be a Σ_v^c-point process [5, Theorem 2.11]. □

3.5. Proof of Proposition 2.9

There exists an orthonormal sequence $(\Psi^n)_{n \geq 0}$ from $\mathscr{M}$ and a sequence $(\alpha_n)_{n \geq 0}$, $\alpha_n \geq 0$, $\sum_{n=0}^{\infty} \alpha_n = 1$ such that

$$\omega(A) = \mathrm{Tr}(\varrho A) \qquad (A \in \mathscr{L}(\mathscr{M})),$$

where ϱ is a density matrix given by

$$\varrho = \sum_{n=0}^{\infty} \alpha_n (\Psi^n, \cdot) \Psi^n, \tag{3.12}$$

and Q_ω can be written in the form

$$Q_\omega(Y) = \int_Y F(d\varphi) D(\varphi) \qquad (Y \in \mathfrak{M})$$

with

$$D(\varphi) = \sum_{n=0}^{\infty} \alpha_n |\Psi^n(\varphi)|^2 \qquad (\varphi \in M). \tag{3.13}$$

One easily shows that (if Q_ω is of the type Σ'_v) the density $D(\varphi)$ has the property that

$$D(\varphi + \hat{\varphi}) > 0 \quad \text{implies} \quad D(\varphi) > 0 \qquad (F \times F\text{-}a.a.\ (\varphi, \hat{\varphi})) \tag{3.14}$$

and the conditional intensity measure $\eta_{Q_\omega}^\varphi$ of Q_ω is given by

$$\eta_Q^\varphi(Y) = \int_Y F(d\hat{\varphi}) \frac{D(\varphi + \hat{\varphi})}{D(\varphi)}. \tag{3.15}$$

For all $\Lambda \in \mathfrak{B}$ and all $\varphi \in M$ we set

$$_{\Lambda}\omega^{\varphi}(A) = \frac{1}{\eta_{Q_\omega}^{\varphi_{\Lambda^c}}(M_\Lambda) D(\varphi_{\Lambda^c})} \sum_{n=0}^{\infty} \alpha_n(\Psi^n_{\varphi_{\Lambda^c}}, O_{M_\Lambda} A O_{M_\Lambda} \Psi^n_{\varphi_{\Lambda^c}}) \qquad (A \in {}_{\Lambda}\mathscr{A}). \quad (3.16)$$

The right side of (3.16) is well-defined because for each $\Psi \in \mathscr{M}$, $\chi_{M_\Lambda} \Psi_{\varphi_{\Lambda^c}} \in \mathscr{M}_\Lambda$ (Lemma I.2.7) and

$$\sum_{n=0}^{\infty} (\Psi^n_{\varphi_{\Lambda^c}}, O_{M_\Lambda} \mathbb{1} O_{M_\Lambda} \Psi^n_{\varphi_{\Lambda^c}}) = \int_{M_\Lambda} F(d\hat{\varphi}) D(\hat{\varphi} + \varphi_{\Lambda^c})$$
$$= \eta_{Q_\omega}^{\varphi_{\Lambda^c}}(M_\Lambda) D(\varphi_{\Lambda^c}). \quad (3.17)$$

Since for Q_ω-*a.a.* φ $0 < \eta_{Q_\omega}^{\varphi_{\Lambda^c}}(M_\Lambda) < \infty$ from (3.16) and (3.17) we get that $({}_{\Lambda}\omega^{\varphi})_{\varphi \in M}$ is a ${}_{\Lambda^c}\mathfrak{M}$-measurable family of states.

Finally, for all $\Lambda \in \mathfrak{B}$, $A \in {}_{\Lambda}\mathscr{A}$, and $Y \in {}_{\Lambda}\mathfrak{M}^{\perp}$ we get using Lemma I.2.5 and the fact that $Y \in {}_{\Lambda^c}\mathfrak{M}$,

$$\begin{aligned}
&\int_Y Q_\omega(d\varphi)\, {}_{\Lambda}\omega^{\varphi}(A) \\
&= \int_{Y \cap M_{\Lambda^c}} Q_\omega(d\varphi) \eta_{Q_\omega}^{\varphi}(M_\Lambda)\, {}_{\Lambda}\omega^{\varphi}(A) \\
&= \int_{Y \cap M_{\Lambda^c}} F(d\varphi) \eta_{Q_\omega}^{\varphi}(M_\Lambda) D(\varphi)\, {}_{\Lambda}\omega^{\varphi}(A) \\
&= \int_{Y \cap M_{\Lambda^c}} F(d\varphi) \sum_{n=0}^{\infty} \alpha_n(\Psi^n_\varphi, O_{M_\Lambda} A O_{M_\Lambda} \Psi^n_\varphi) \\
&= \int_{M_{\Lambda^c}} F(d\varphi) \int_{M_\Lambda} F(d\varphi_1) \chi_Y(\varphi) \sum_{n=0}^{\infty} \alpha_n \overline{\Psi^n_\varphi(\varphi_1)} O_{M_\Lambda} A O_{M_\Lambda} \Psi^n_\varphi(\varphi_1) \\
&= \int F(d\varphi) \chi_Y(\varphi_{\Lambda^c}) \sum_{n=0}^{\infty} \alpha_n \overline{\Psi^n_{\varphi_{\Lambda^c}}(\varphi_\Lambda)} O_{M_\Lambda} A O_{M_\Lambda} \Psi^n_{\varphi_{\Lambda^c}}(\varphi_\Lambda) \\
&= \int F(d\varphi) \chi_Y(\varphi) \sum_{n=0}^{\infty} \alpha_n \overline{\Psi^n(\varphi)} A \Psi^n(\varphi) = \sum_{n=0}^{\infty} \alpha_n(\Psi^n, O_Y A \Psi^n) \\
&= \omega(A O_Y). \quad \square
\end{aligned}$$

3.6. Proof of Theorem 2.10

1°. We first assume that for each $\Lambda \in \mathfrak{B}$ the family $({}_{\Lambda}\omega^{\varphi})_{\varphi \in M}$ of conditional states exists. From Proposition 2.6 we have that for all $\Lambda \in \mathfrak{B}$ and Q_ω-*a.a.* φ ${}_{\Lambda}\omega^{\varphi}$ is a normal state on ${}_{\Lambda}\mathscr{A}$. Thus ω_Λ^φ defined by (2.4) is a normal state on $\mathscr{A}_\Lambda$. Now, we fix a set $\Lambda \in \mathfrak{B}$. Without loss of generality we may assume that ω_Λ^φ is a normal state for all $\varphi \in M$. Since $({}_{\Lambda}\omega^{\varphi})_{\varphi \in M}$ is assumed to be ${}_{\Lambda^c}\mathfrak{M}$-measurable we have, of course, $\omega_\Lambda^\varphi = \omega_\Lambda^{\varphi_{\Lambda^c}}$ for all $\varphi \in M$. Denote by Q_Λ^φ the position distribution of ω_Λ^φ. Q_Λ^φ is a finite Σ'_ν-point process on $[M_\Lambda, \mathfrak{M}_\Lambda]$ [2, Proposition 3.1]. We will show that Q_Λ^φ is even a Σ'_ν-point process. Indeed, from Proposition 2.8 and Lemma I.5.3(iii) we obtain for all $Y \in \mathfrak{M}_\Lambda$ and all $\varphi \in M$,

$$\begin{aligned}
Q_\Lambda^\varphi(Y) &= \omega_\Lambda^\varphi(O_Y) = {}_{\Lambda}\omega^{\varphi}(v_\Lambda^{-1} Y) = Q_\omega(v_\Lambda^{-1} Y \mid {}_{\Lambda^c}\mathfrak{M})(\varphi) \\
&= (\eta_{Q_\omega}^{\varphi_{\Lambda^c}}(M_\Lambda))^{-1} \eta_{Q_\omega}^{\varphi_{\Lambda^c}}(Y) \\
&= (\eta_{Q_\omega}^{\varphi_{\Lambda^c}}(M_\Lambda))^{-1} \int_Y F_\Lambda(d\hat{\varphi}) \kappa(\hat{\varphi}, \varphi_{\Lambda^c}),
\end{aligned} \quad (3.18)$$

where κ is a version of $dC_{Q_\omega}^{(\infty)}/d(F \times Q_\omega)$.

For all $\Lambda' \in \mathfrak{B} \cap \Lambda$, $Y \in \mathfrak{M}_\Lambda$, and $\varphi \in M$ we get

$$\begin{aligned} C^{(1)}_{Q^\varphi_\Lambda}(\Lambda' \times Y) &= \int Q^\varphi_\Lambda(d\hat{\varphi}) \int_{\Lambda'} \hat{\varphi}(dx)\chi_Y(\hat{\varphi} - \delta_x) \\ &= (\eta^{\varphi_{\Lambda^c}}_{Q_\omega}(M_\Lambda))^{-1} \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \nu^n(d\underline{x}^n)\kappa(\delta_{\underline{x}^n}, \varphi_{\Lambda^c}) \\ &\quad \times \sum_{j=1}^{n} \chi_{\Lambda'}(x_j)\chi_Y(\delta_{\underline{x}^n} - \delta_{x_j}) \\ &= (\eta^{\varphi_{\Lambda^c}}_{Q_\omega}(M_\Lambda))^{-1} \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \int_{\Lambda^{n-1}} \nu^{n-1}(d\underline{x}^{n-1}) \\ &\quad \times \int_{\Lambda'} \nu(dx)\chi_Y(\delta_{\underline{x}^{n-1}})\kappa(\delta_{\underline{x}^{n-1}} + \delta_x, \varphi_{\Lambda^c}) \\ &= (\eta^{\varphi_{\Lambda^c}}_{Q_\omega}(M_\Lambda))^{-1} \int_Y F_\Lambda(d\hat{\varphi}) \int_{\Lambda'} \nu(dx)\kappa(\hat{\varphi} + \delta_x, \varphi_{\Lambda^c}). \end{aligned} \tag{3.19}$$

Since Q_ω is a Σ'_ν-point process we have

$$\kappa(\varphi_1 + \varphi_2, \varphi) = \kappa(\varphi_1, \varphi)\kappa(\varphi_2, \varphi + \varphi_1) \quad (F \times F \times Q_\omega\text{-a.a. } (\varphi_1, \varphi_2, \varphi))$$

(cf. [4, Chap. 9] or also (6.10) in I). Thus we may continue (3.19)

$$\begin{aligned} C^{(1)}_{Q^\varphi_\Lambda}(\Lambda' \times Y) &= (\eta^{\varphi_{\Lambda^c}}_{Q_\omega}(M_\Lambda))^{-1} \int_Y F_\Lambda(d\hat{\varphi})\kappa(\hat{\varphi}, \varphi_{\Lambda^c}) \\ &\quad \times \int_{\Lambda'} \nu(dx)\kappa(\delta_x, \hat{\varphi} + \varphi_{\Lambda^c}) \\ &= \int_Y Q^\varphi_\Lambda(d\hat{\varphi}) \int_{\Lambda'} \nu(dx)\kappa(\delta_x, \hat{\varphi} + \varphi_{\Lambda^c}). \end{aligned}$$

This proves that $C^{(1)}_{Q^\varphi_\Lambda} \ll \nu \times Q^\varphi_\Lambda$ and

$$\frac{dC^{(1)}_{Q^{\varphi_\Lambda}}}{d(\nu \times Q^\varphi_\Lambda)}(x, \hat{\varphi}) = \frac{dC^{(1)}_{Q_\omega}}{d(\nu \times Q_\omega)}(x, \hat{\varphi} + \varphi_{\Lambda^c}) \quad (x \in \Lambda,\ \hat{\varphi} \in M_\Lambda). \tag{3.20}$$

From (3.20) we easily get that for all $\Lambda \in \mathfrak{B}$, $\varphi \in M$,

$$\frac{dC^{(\infty)}_{Q^{\varphi_\Lambda}}}{d(F_\Lambda \times Q^\varphi_\Lambda)}(\varphi_1, \varphi_2) = \frac{d\eta^{\varphi_2 \varphi}_{Q^\varphi_\Lambda}}{dF_\Lambda}(\varphi_1) = \kappa(\varphi_1, \varphi + \varphi_{\Lambda^c}) \quad (\varphi_1, \varphi_2 \in M_\Lambda). \tag{3.21}$$

2°. Above we thus have shown that for all $\Lambda \in \mathfrak{B}$ and all $\varphi \in M$ ω^φ_Λ is a normal Σ'_ν-state on $\mathscr{A}_\Lambda$. Applying Theorem I.6.4 we obtain that there exists the (a.e. uniquely determined) c.r.d.m. k^φ_Λ of ω^φ_Λ having the following properties:

For all $Y \in \mathfrak{M}_\Lambda$ and integral operators $A \in \mathscr{A}_\Lambda$ such that $S_\Lambda(Y, A) \in \mathscr{A}_\Lambda$ we have

$$\omega^\varphi_\Lambda(S_\Lambda(Y, A)) = \int_Y Q^\varphi_\Lambda(d\varphi_1) \int F_\Lambda(d\varphi_2) k_A \underset{\Lambda}{*} k^\varphi_\Lambda(\varphi_2, \varphi_1) \tag{3.22}$$

and for $F_\Lambda \times Q^\varphi_\Lambda$-a.a. (φ_1, φ_2),

$$k^\varphi_\Lambda(\varphi_1, \varphi_1, \varphi_2) = \kappa(\varphi_1, \varphi_2 + \varphi_{\Lambda^c}). \tag{3.23}$$

Now, for all $\Lambda \in \mathfrak{B}$ we put

$$k_\Lambda(\varphi_1, \varphi_2, \varphi) = k^{\varphi_{\Lambda^c}}_\Lambda(\varphi_1, \varphi_2, \varphi) \quad (\varphi_1, \varphi_2 \in M_\Lambda,\ \varphi \in M). \tag{3.24}$$

We will show now that for all $\Lambda, \Lambda' \in \mathfrak{B}$ $\Lambda \subsetneqq \Lambda'$ one has

$$k_\Lambda(\varphi_1, \varphi_2, \varphi) = k_{\Lambda'}(\varphi_1, \varphi_2, \varphi) \quad (F_\Lambda \times F_\Lambda \times Q_\omega\text{-a.a. } (\varphi_1, \varphi_2, \varphi).$$

We fix $\Lambda, \Lambda' \in \mathfrak{B}$, $\Lambda \subseteqq \Lambda'$. Let $Y_1 \in {}_\Lambda\mathfrak{M}^f$, $Y_2 \in {}_{\Lambda' \setminus \Lambda}\mathfrak{M}^f$, $Y_3 \in {}_{\Lambda'}\mathfrak{M}^\perp$. Then $Y_1 \cap Y_2 \in {}_{\Lambda'}\mathfrak{M}^f$, and for all integral operators $A \in \mathscr{A}^f_\Lambda$ $(\subseteqq \mathscr{A}^f_{\Lambda'})$ we have $S(Y_1 \cap Y_2, A) \in {}_{\Lambda'}\mathscr{A}$ (Proposition I.3.4) and consequently we get using (3.22),

$$\begin{aligned}
&\omega(S(Y_1 \cap Y_2, A) O_{Y_3}) \\
&= \int_{Y_3} Q_\omega(d\varphi)\omega^\varphi_{\Lambda'}(S_{\Lambda'}(v_{\Lambda'}(Y_1 \cap Y_2), A)) \\
&= \int_{Y_3} Q_\omega(d\varphi) \int_{v_{\Lambda'}(Y_1 \cap Y_2)} Q^\varphi_{\Lambda'}(d\varphi_1) \int F_{\Lambda'}(d\varphi_2) k_A \underset{\Lambda'}{*} k^\varphi_{\Lambda'}(\varphi_2, \varphi_1) \\
&= \int_{Y_3} Q_\omega(d\varphi) \int_{v_{\Lambda'}(Y_1 \cap Y_2)} F_{\Lambda'}(d\varphi_1) \kappa(\varphi_1, \varphi_{\Lambda'^c}) \frac{1}{\eta^{\varphi(\Lambda')^c}_{Q_\omega}(M_{\Lambda'})} \\
&\quad \times \int F_\Lambda(d\varphi_2) k_A \underset{\Lambda}{*} k^\varphi_{\Lambda'}(\varphi_2, \varphi_1) \\
&= \int_{v_{\Lambda'^c} Y_3} Q_\omega(d\varphi) \int_{v_{\Lambda'}(Y_1 \cap Y_2)} F_{\Lambda'}(d\varphi_1) \kappa(\varphi_1, \varphi) \\
&\quad \times \int F_\Lambda(d\varphi_2) k_A \underset{\Lambda}{*} k^\varphi_{\Lambda'}(\varphi_2, \varphi_1) \\
&= \int_{Y_1 \cap Y_2 \cap Y_3} Q_\omega(d\varphi) \int F_\Lambda(d\varphi_2) k_A \underset{\Lambda}{*} k^{\varphi(\Lambda')^c}_{\Lambda'}(\varphi_2, \varphi_{\Lambda'}) \\
&= \int_{Y_1 \cap Y_2 \cap Y_3} Q_\omega(d\varphi) \int F_\Lambda(d\varphi_2) k_A \underset{\Lambda}{*} k_{\Lambda'}(\varphi_2, \varphi) .
\end{aligned} \tag{3.26}$$

On the other side, it is easy to check that

$$S(Y_1 \cap Y_2, A) = S(Y_1, A) O_{Y_2} .$$

Consequently, we get $(S(Y_1, A) \in {}_\Lambda\mathscr{A}$, $O_{Y_1 \cap Y_3} \in {}_\Lambda\mathfrak{M}^\perp)$

$$\begin{aligned}
\omega(S(Y_1 \cap Y_2, A) O_{Y_3}) &= \omega(S(Y_1, A) O_{Y_1 \cap Y_3}) \\
&= \int_{Y_1 \cap Y_3} Q_\omega(d\varphi)\omega^\varphi_\Lambda(S_\Lambda(v_\Lambda Y_1, A)) \\
&= \int_{Y_1 \cap Y_3} Q_\omega(d\varphi) \int_{Y_2} Q^\varphi_\Lambda(d\varphi_1) \int F_\Lambda(d\varphi_2) k_A \underset{\Lambda}{*} k^\varphi_\Lambda(\varphi_2, \varphi_1) \\
&= \int_{Y_1 \cap Y_2 \cap Y_3} Q_\omega(d\varphi) \int F_\Lambda(d\varphi_2) k_A \underset{\Lambda}{*} k^{\varphi_{\Lambda^c}}_\Lambda(\varphi_2, \varphi_\Lambda) \\
&= \int_{Y_1 \cap Y_2 \cap Y_3} Q_\omega(d\varphi) \int F_\Lambda(d\varphi_2) k_A \underset{\Lambda}{*} k_\Lambda(\varphi_2, \varphi) .
\end{aligned} \tag{3.27}$$

Since ${}_\Lambda\mathfrak{M}^f \cap {}_{\Lambda' \setminus \Lambda}\mathfrak{M}^f \cap {}_{\Lambda'}\mathfrak{M}^\perp$ generate $\mathfrak{M}$ and the integral operators from $\mathscr{A}_\Lambda$ are dense in this space we conclude from (3.26) and (3.27) the relation (3.25). Now, we choose an increasing sequence $(\Lambda_n)_{n \geqq 0}$, $\Lambda_n \in \mathfrak{B}$ such that $\lim_{n \to \infty} \Lambda_n = G$. For all $\varphi_1, \varphi_2 \in M^f$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geqq n_0$ $\varphi_1, \varphi_2 \in M_{\Lambda_n}$. Thus, from (3.25) we get that

$$k_\omega(\varphi_1, \varphi_2, \varphi) = \lim_{n \to \infty} k_{\Lambda_n}(\varphi_1, \varphi_2, \varphi) \tag{3.28}$$

exists for $F \times F \times Q_\omega$-*a.a.* $(\varphi_1, \varphi_2, \varphi)$, and defines a measurable function from $M^f \times M^f \times M$ into $\mathbb{C}$. Setting $k_\omega(\varphi_1, \varphi_2, \varphi) = 0$ for $\varphi_1 \in M \setminus M^f$ or $\varphi_2 \in M \setminus M^f$ we obtain a well-defined measurable function from M^3 into $\mathbb{C}$.

3°. We show that k_ω is the c.r.d.m. of ω, i.e. we must show that k_ω has the properties given in Definition I.7.1.

From (3.28), (3.23), (3.24), and (3.25) we obtain for all $\Lambda \in \mathfrak{B}$ and $F_\Lambda \times Q_\omega$-*a.a.* (φ_1, φ),

$$k_\omega(\varphi_1, \varphi_1, \varphi_{\Lambda^c}) = k_\Lambda(\varphi_1, \varphi_1, \varphi_{\Lambda^c}) = k_\Lambda^{\varphi_{\Lambda^c}}(\varphi_1, \varphi_1, \mathbb{0})$$
$$= \kappa(\varphi_1, \varphi_{\Lambda^c}) = \frac{d\eta_{Q_\omega}^{\varphi_{\Lambda^c}}}{dF}(\varphi_1)$$

what proves (i).

For $\Lambda \in \mathfrak{B}$ and $\varphi \in M$ we set

$$K_\Lambda^\varphi \Psi(\varphi_1) = \int F_\Lambda(d\varphi_2) k_\omega(\varphi_1, \varphi_2, \varphi_{\Lambda^c}) \Psi(\varphi_2) \quad (\Psi \in \mathcal{M}_\Lambda,\ \varphi_1 \in M_\Lambda). \tag{3.29}$$

Since for $A \in \mathscr{A}_\Lambda$ $A = S_\Lambda(\{\mathbb{0}\}, A)$ we get from (3.22),

$$\omega_\Lambda^\varphi(A) = Q_\Lambda^\varphi(\{\mathbb{0}\}) \int F_\Lambda(d\varphi_1) k_A \underset{\Lambda}{*} k_\Lambda^\varphi(\varphi_1, \mathbb{0})$$
$$= Q_\Lambda^\varphi(\{\mathbb{0}\}) \int F_\Lambda(d\varphi_1) k_A \underset{\Lambda}{*} k_\omega(\varphi_1, \varphi_{\Lambda^c})$$
$$= Q_\Lambda^\varphi(\{\mathbb{0}\}) \operatorname{Tr}_\Lambda(K_\Lambda^\varphi A). \tag{3.30}$$

Since ω_Λ^φ is a normal state K_Λ^φ is a positive trace-class operator on $\mathcal{M}_\Lambda$.

Finally, we have to show that k_ω fulfills (I.7.1). For all $\Lambda \in \mathfrak{B}$, $Y \in {}_\Lambda\mathfrak{M}$ and all integral operators $A \in \mathscr{A}_\Lambda$ such that $S(Y, A) \in {}_\Lambda\mathscr{A}$ we get (using (3.22) and the fact that $M \in {}_\Lambda\mathfrak{M}^\perp$)

$$\omega(S(Y, A)) = \omega(S(Y, A) O_M) = \int Q_\omega(d\varphi) \omega_\Lambda^\varphi(S_\Lambda(v_\Lambda Y, A))$$
$$= \int_{M_{\Lambda^c}} Q_\omega(d\varphi) \eta_{Q_\omega}^\varphi(M_\Lambda) \int_{v_\Lambda Y} Q_\Lambda^\varphi(d\varphi_1) \int F_\Lambda(d\varphi_2) k_A \underset{\Lambda}{*} k_\Lambda^\varphi(\varphi_2, \varphi_1)$$
$$= \int_Y Q_\omega(d\varphi) \int F_\Lambda(d\varphi_1) k_A \underset{\Lambda}{*} k_\Lambda^{\varphi_{\Lambda^c}}(\varphi_2, \varphi_1)$$
$$= \int_Y Q_\omega(d\varphi) \int F_\Lambda(d\varphi_1) k_A \underset{\Lambda}{*} k_\omega(\varphi_2, \varphi).$$

4°. We still have to show the converse, and we assume now that the locally normal state ω possesses a c.r.d.m. Then we use (3.29) and (3.30) to define a family $(\omega_\Lambda^\varphi)_{\varphi \in M}$ of positive linear functionals on $\mathscr{A}_\Lambda$, i.e. we put

$$Q_\Lambda^\varphi(Y) = Q_\omega(v_\Lambda^{-1} Y \mid {}_{\Lambda^c}\mathfrak{M})(\varphi) \quad (Y \in \mathfrak{M}_\Lambda,\ \varphi \in M)$$

and

$$\omega_\Lambda^\varphi(A) = Q_\Lambda^\varphi(\{\mathbb{0}\}) \operatorname{Tr}(K_\Lambda^\varphi A) \quad (A \in \mathscr{A}_\Lambda), \tag{3.31}$$

where K_Λ^φ is defined by (3.29).

Since $\kappa(\mathbb{0}, \varphi) = 1$ and

$$Q_\Lambda^\varphi(Y) = (\eta_{Q_\omega}^{\varphi_{\Lambda^c}}(M_\Lambda))^{-1} \int_Y F(d\hat{\varphi}) \kappa(\hat{\varphi}, \varphi) \tag{3.32}$$

we get

$$Q_\Lambda^\varphi(\{\mathbb{0}\}) = (\eta_{Q_\omega}^{\varphi_{\Lambda^c}}(M_\Lambda))^{-1}. \tag{3.33}$$

Further, it is easy to see that

$$\omega_\Lambda^\varphi(O_{M_\Lambda}) = Q_\Lambda^\varphi(\{\mathbb{0}\}) \operatorname{Tr}(K_\Lambda^\varphi O_{M_\Lambda}) = Q_\Lambda^\varphi(\{\mathbb{0}\}) \eta_{Q_\omega}^{\varphi_{\Lambda^c}}(M_\Lambda) = 1.$$

Thus, $(\omega_\Lambda^\varphi)_{\varphi \in M}$ is a ${}_{\Lambda^c}\mathfrak{M}$-measurable family of normal states on $\mathscr{A}_\Lambda$, and by

$${}_\Lambda\omega^\varphi(A) = \omega_\Lambda^\varphi(O_{M_\Lambda} A O_{M_\Lambda}) \quad (A \in {}_\Lambda\mathscr{A},\ \varphi \in M),$$

there is defined a ${}_{\Lambda^c}\mathfrak{M}$-measurable family of normal states on ${}_{\Lambda}\mathscr{A}$. We finally have to show that these states are just the conditional states of ω. Let $A \in \mathscr{A}_{\Lambda}$, $Y \in {}_{\Lambda'}\mathfrak{M}$, $\Lambda \cap \Lambda' = \emptyset$, $\Lambda, \Lambda' \in \mathfrak{B}$. From Lemma 2.4 we know that $J_{\Lambda} A O_Y \in {}_{\Lambda \cap \Lambda'}\mathscr{A}$ and $J_{\Lambda} A O_Y = S(Y \cap M_{\Lambda^c}, A)$. Consequently, if A is an integral operator we get from the definition of the c.r.d.m.,

$$\begin{aligned}
&\omega(J_{\Lambda} A O_Y)\\
&= \omega(S(Y \cap M_{\Lambda^c}, A))\\
&= \int_{Y \cap M_{\Lambda^c}} Q_{\omega}(d\varphi) \int F_{\Lambda \cup \Lambda'}(d\varphi_1) \int F_{\Lambda \cup \Lambda'}(d\varphi_2) k_{\Lambda}(\varphi_1, \varphi_2) k_{\omega}(\varphi_2, \varphi_1, \varphi)\\
&= \int_{Y \cap M_{\Lambda^c}} Q_{\omega}(d\varphi) \int F_{\Lambda}(d\varphi_1) \int F_{\Lambda}(d\varphi_2) k_{\Lambda}(\varphi_1, \varphi_2) k_{\omega}(\varphi_2, \varphi_1, \varphi)\\
&= \int_{Y \cap M_{\Lambda^c}} Q_{\omega}(d\varphi) \operatorname{Tr}(K_{\Lambda}^{\varphi} A) = \int_{Y \cap M_{\Lambda^c}} Q_{\omega}(d\varphi) \eta_{Q_{\omega}}^{\varphi}(M_{\Lambda}) Q_{\Lambda}^{\varphi}(\{\mathbb{0}\}) \operatorname{Tr}(K_{\Lambda}^{\varphi} A)\\
&= \int_{Y} Q_{\omega}(d\varphi) Q_{\Lambda}^{\varphi_{\Lambda^c}}(\{\mathbb{0}\}) \operatorname{Tr}(K_{\Lambda}^{\varphi_{\Lambda^c}} A) = \int_{Y} Q_{\omega}(d\varphi) \omega_{\Lambda}^{\varphi}(A)\\
&= \int_{Y} Q_{\omega}(d\varphi)\, {}_{\Lambda}\omega^{\varphi}(J_{\Lambda} A). \quad \square
\end{aligned}$$

3.7. *Proof of Proposition 2.11*

The proof follows immediately from the proof of Theorem 2.10 [from (3.31), (3.33), and (2.4)]. □

3.8. *Proof of Proposition 2.12*

Let k_1, k_2 be two c.r.d.m. of ω. As in the proof of Theorem 2.10 [(3.26), (3.27)] one easily shows that

$$k_1(\varphi_1, \varphi_2, \varphi) = k_2(\varphi_1, \varphi_2, \varphi) \qquad (F_{\Lambda} \times F_{\Lambda} \times Q_{\omega} - a.a.\ (\varphi_1, \varphi_2, \varphi)).$$

Since F is concentrated on $M^f = \bigcup_{\Lambda \in \mathfrak{B}} M_{\Lambda}$ this implies immediately that the c.r.d.m. is a.e. uniquely determined. □

3.9. *Proof of Proposition 2.13*

If ω is normal it follows from Proposition I.6.1 that Q_{ω} is a finite point process. Now, let Q_{ω} be a finite Σ_v'-point process. It is easy to see that the c.r.d.m. k_{ω} of ω fulfills the assumptions of Theorem I.6.8. Consequently, there exists a normal Σ_v'-state $\tilde{\omega}$ such that $Q_{\tilde{\omega}} = Q_{\omega}$ and $k_{\tilde{\omega}} = k_{\omega}$ a.e. Since a normal state is also a locally normal one we conclude from Proposition 2.12 that $\tilde{\omega} = \omega$ [more precisely, ω may be extended in a unique way to a normal state on $\mathscr{L}(\mathscr{M})$]. □

3.10. *Proof of Proposition 2.14*

The implication (i) ⇒ (ii) is a consequence of Proposition 2.12 and of the uniqueness of the position distribution [2, Theorem 3.3]. The implication (ii) ⇒ (i) follows from Remark I.7.5, 5°.

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