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Simple WZW Currents

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Abstract. A complete classification of simple currents of WZW theories is obtained. The proof is based on an analysis of the quantum dimensions of the primary fields. Simple currents are precisely the primaries with unit quantum dimension; for WZW theories explicit formulae for the quantum dimensions can be derived so that an identification of the fields with unit quantum dimension is possible.

1. Simple Currents

Simple currents are by definition primary fields Φ which upon taking fusion products simply permute the set of primary fields $\{\phi\}$ of a conformal field theory. They play a prominent role in the construction of modular invariant partition functions [1]. An important step in the classification of conformal field theories is therefore the enumeration of simple currents. For WZW theories [2, 3], a large class of simple currents is already known, namely the so-called cominimal fields [4] as well as an exceptional isolated case [5]. In the present paper, we prove that these known examples already exhaust the set of simple currents of WZW theories.

The distinctive property of simple currents which allows for their classification is the special value of their *quantum dimension*. Recall [6] that to any primary field ϕ of a two-dimensional conformal field theory, one can associate a quantum dimension $\mathscr{D}(\phi)$ which, loosely speaking, describes the relative size of the highest weight module of the chiral algebra carried by ϕ as compared to the highest weight module carried by the identity primary field **1**. By a straightforward manipulation of characters, one can express $\mathscr{D}(\phi)$ through the matrix S that implements the modular transformation $\tau \to -\frac{1}{\tau}$ on the set of (characters of) primary fields as

$$\mathscr{D}(\phi) = \frac{S_{\phi 1}}{S_{11}}. \tag{1.1}$$

From elementary properties of the modular matrix S, it follows that quantum dimensions are real numbers, that conjugate primary fields possess the same quantum dimension, $\mathscr{D}(\phi^+) = \mathscr{D}(\phi)$, and that in unitary theories one has $\mathscr{D} \geqq 1$. Moreover, due to the relations between modular transformations and the fusion rule algebra [6, 7, 8], the fusion rules

$$\phi * \phi' = \sum_i \phi_i \tag{1.2}$$

of a conformal field theory satisfy the quantum dimension sum rule

$$\mathscr{D}(\phi)\mathscr{D}(\phi') = \sum_i \mathscr{D}(\phi_i). \tag{1.3}$$

Applying these results to the fusion rule of a simple current Φ with its conjugate field,

$$\Phi * \Phi^+ = \mathbf{1}, \tag{1.4}$$

one learns that simple currents of a (unitary) conformal field theory possess unit quantum dimension,

$$\mathscr{D}(\Phi) = 1. \tag{1.5}$$

Conversely, application of the sum rule (1.3) to $\sum_i (\phi * \phi_i)$ (with the sum extending over all primaries) tells us that $\mathscr{D}(\phi) = 1$ is already sufficient for ϕ to be a simple current.

The classification of simple currents is thus a classification of primary fields with unit quantum dimension. For an arbitrary conformal field theory, this observation is of no great help because usually one doesn't know the modular matrix S. In contrast, for WZW theories the chiral algebra is generated by an untwisted affine Lie algebra g [2] so that the modular behavior of characters is well known [9]. For the quantum dimension (1.1) of a primary field ϕ_Λ (labelled by an integrable highest weight Λ of g at a fixed value k of the level), one then obtains ([10], see also [11, 12])

$$\mathscr{D}(\Lambda) \equiv \mathscr{D}(\phi_\Lambda) = \prod_{\bar{\alpha}>0} \frac{\lfloor(\bar{\alpha} + \bar{\Lambda}, \bar{\rho})\rfloor}{\lfloor(\bar{\alpha}, \bar{\rho})\rfloor}, \tag{1.6}$$

which can be viewed as a quantum analogue of the Weyl dimension formula. Here $\bar{\alpha} > 0$ denote the positive roots of the horizontal subalgebra $\bar{g}$ of g and $\bar{\rho} = \frac{1}{2} \sum_{\bar{\alpha}>0} \bar{\alpha}$ the Weyl vector of $\bar{g}$. $\bar{\Lambda}$ is the horizontal part of Λ, and

$$\lfloor x \rfloor := \sin\left(\frac{\pi x}{k+h}\right) \tag{1.7}$$

with h the dual Coxeter number of $\bar{g}$. Also, we have normalized the inner product $(,)$ of $\bar{g}$ such that the highest root $\bar{\theta}$ has length squared two, $(\bar{\theta}, \bar{\theta}) = 2$.

Our theorem which classifies the simple WZW currents is presented in Sect. 2. The key to the proof of the theorem is the formula (1.6) for the quantum dimensions. As a first part of the proof we show in Sect. 3 that, as a function of λ, absolute minima of $\mathscr{D}(\lambda)$ on the fundamental affine Weyl chamber can only appear at its corners. One then has to investigate the behavior of $\mathscr{D}$ at these corners as a function of the level k. Explicit expressions for these quantities are obtained in Sect. 4. In contrast to the considerations in Sect. 3 which use only very general properties of the $\bar{g}$-root system, the investigation of these formulae necessitates a case by case analysis and consequently is technical and rather lengthy (although straightforward). Therefore we present the manipulations only to an extent which we think will enable the reader to work out the full details if desired.

It would certainly be interesting to extend the analysis of this paper to the case of other conformal field theories. As already mentioned, the modular matrix S is then generically not known so that one cannot use (1.1) to compute the quantum dimensions of primary fields. However, the quantum dimensions also possess another interpretation [13, 14, 15]: to the primary fields of any conformal field theory, one can associate irreducible modules of the quantum group that underlies [7, 14] the theory; the quantum dimensions of the primaries are then equal to appropriately defined quantum deformations of the dimensionalities of these modules. (Even more generally, quantum dimensions possess an interpretation as the statistical dimensions in the sense of algebraic quantum field theory [16].) In the case of WZW theories, the underlying quantum group is [14] the quantum universal enveloping algebra $\mathbf{U}_q(\bar{g})$, with the deformation parameter q related to the level of g by

$$q = \exp\left(\frac{2\pi i}{k+h}\right). \tag{1.8}$$

A natural way to proceed is to search for the simple currents of coset theories since presumably any rational conformal field theory can be realized as a (possibly orbifoldized) coset theory. The fusion rules of a coset theory G/H are approximately the tensor products of the fusion rules of the G and H theories so that the quantum dimensions of the coset theory should be expressible as

$$\mathscr{D}_{G/H} = \mathscr{D}_G \cdot \mathscr{D}_H . \tag{1.9}$$

This would lead to the conclusion that the simple currents of the G/H theory are just tensor products of the simple currents of the G and H WZW theories. However, in fact the G/H fusion rules are *not* precisely given as tensor products, due to the subtleties arising in the coset construction that go under the name of field identification [17, 18]. For the simple currents this implies, first, that some of them are projected out because they get identified with other simple currents. This does not pose a serious problem because it is not too difficult to find the relevant identifications [19], and anyway the number of simple currents can only become smaller than naively expected. However, if the field identification procedure possesses fixed points, a more severe problem arises, since a resolution of the fixed points is necessary, and this resolution might produce new simple currents [18]. As a consequence, at least if fixed points are present, the enumeration of simple currents of coset theories is still an open problem.

2. The Classification

Consider a WZW theory with underlying affine Lie algebra g. A primary field ϕ_Λ corresponds[1] to the highest weight Λ of an irreducible integrable highest weight

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[1] To be precise, the primary fields are labelled by a pair of highest weights (Λ_l, Λ_r) referring to the left and right halves of the symmetry algebra $g \oplus g$; for the subject of our interest, it is however sufficient to consider one chiral half of the theory

module of g, with the level of g fixed to some positive integral value k. Thus the allowed highest weights Λ are the integral weights in the (closure of the) fundamental affine Weyl chamber P_k of g at level k. In terms of the horizontal part $\bar{\Lambda}$ of Λ this means

$$\bar{\Lambda} \in \mathbb{Z}^r \cap \bar{P}_k \tag{2.1}$$

with $\bar{P}_k$ the horizontal projection of P_k,

$$\bar{P}_k = \{\bar{\lambda} = (\lambda^1, \lambda^2, \dots, \lambda^r) | \lambda^i \geqq 0 \, \forall i = 1, \dots, r; (\bar{\lambda}, \bar{\theta}) \leqq k\}. \tag{2.2}$$

Here r is the rank of g, and the Dynkin labels $\lambda^i \in \mathbb{R}$ of $\bar{\lambda}$ are the coefficients of the expansion of $\bar{\lambda}$ in the basis $\{\bar{\Lambda}_{(i)}\}$ of fundamental weights, i.e. $\bar{\lambda} = \sum_{i=1}^{r} \lambda^i \bar{\Lambda}_{(i)}$. We also need to introduce the Coxeter labels $a_i, i = 1, \dots, r$, of $\bar{g}$, defined via the expansion of the highest $\bar{g}$-root $\bar{\theta}$ in the basis $\{\bar{\alpha}^{(i)}\}$ of simple roots, $\bar{\theta} = \sum_{i=1}^{r} a_i \bar{\alpha}^{(i)}$.

Having established the necessary notation, we are now in a position to state the classification theorem. In any conformal field theory, the identity field **1** is a simple current; for notational simplicity, this trivial example of a simple current will be neglected from now on. The (nontrivial) simple currents are then as follows.

A primary field ϕ_Λ is a simple current of the WZW *theory with underlying affine Lie algebra g at level k if and only if the following conditions are fulfilled:*
Either

$$\bar{\Lambda} = k\bar{\Lambda}_{(i)} \quad \text{and} \quad a_i = 1 \tag{2.3}$$

for some $i \in \{1, 2, \dots, r\}$ (with $\bar{g}$ and k arbitrary), or else

$$\bar{g} = E_8, \quad k = 2 \quad \text{and} \quad \bar{\Lambda} = \bar{\Lambda}_{(7)}. \tag{2.4}$$

The fundamental weights $\bar{\Lambda}_{(i)}$ such that $a_i = 1$ are known as cominimal fundamental weights (they are precisely those fundamental weights for which the associated node in the extended Dynkin diagram is related to the node of the zero weight by a diagram automorphism; for an enumeration of the relevant values of i for given $\bar{g}$, see Table 2 below). Consequently, the primary fields in the infinite series (2.3) were termed *cominimal* fields in [4]. The fact that they are simple currents was deduced in [4] by an analysis of four point functions. Namely, any four point correlation function containing at least one simple current is just a product of powers (which can be expressed through the conformal dimensions of the fields involved) of the distances between the fields, and combining the null vector equations of [2] and [3], one finds that – modulo a certain naturalness assumption – any primary field with this property must be cominimal. The naturalness property that was assumed in [4] was that two components of a four point function cannot be proportional up to powers in the coordinate differences unless this property is enforced by the symmetries of the theory, i.e. by the null vector equations. The existence of the exceptional solution (2.4) which was found in [5] implies that the naturalness condition can be violated, but the theorem just stated shows that this counter-intuitive situation does not occur more often.

3. Quantum Dimensions and Their Minima

Of course, (1.6) possesses the interpretation as the quantum dimension of a primary WZW field only if Λ is an integrable weight. However, nothing prevents us from using (1.6) as the definition of a real function $\mathscr{D}(\lambda)$ defined for $\lambda \in \mathbb{R}^r$ an arbitrary point in the weight space. To enter the proof of our theorem, we consider $\mathscr{D}(\lambda)$ for fixed level k as a function on the weight space of $\bar{g}$ and look for its minima on $\bar{P}_k$. We will show that these minima can only occur at the corners of $\bar{P}_k$, i.e. for

$$\bar{\lambda} = \frac{k}{a_i^\vee} \bar{\Lambda}_{(i)} \tag{3.1}$$

for some $i = 1, \ldots, r$, with $a_i^\vee = \frac{1}{2}(\bar{\alpha}^{(i)}, \bar{\alpha}^{(i)}) a_i$ the dual Coxeter labels of $\bar{g}$. To this end, we first investigate the derivatives of $\mathscr{D}(\lambda)$ with respect to the Dynkin labels $\lambda^i, i = 1, \ldots, r$. One finds

$$\frac{\partial}{\partial \lambda^i} \mathscr{D}(\lambda) = \frac{\pi}{k+h} \mathscr{D}(\lambda) \mathscr{E}_i(\lambda), \tag{3.2}$$

$$\frac{\partial^2}{\partial \lambda^i \partial \lambda^j} \mathscr{D}(\lambda) = \frac{\pi}{k+h} \mathscr{D}(\lambda) \left[\frac{\pi}{k+h} \mathscr{E}_i(\lambda) \mathscr{E}_j(\lambda) + \frac{\partial}{\partial \lambda^j} \mathscr{E}_i(\lambda) \right], \tag{3.3}$$

with

$$\mathscr{E}_i(\lambda) = \sum_{\bar{\alpha} > 0} \bar{\alpha}_i \cot \left(\frac{\pi}{k+h} (\bar{\lambda} + \bar{\rho}, \bar{\alpha}) \right); \tag{3.4}$$

here $\bar{\alpha}_i$ are the components of $\bar{\alpha}$ in the basis of simple coroots, $\bar{\alpha} = \sum_{i=1}^{r} \bar{\alpha}_i \bar{\alpha}^{(i)\vee}$. Because of $\bar{\alpha}_i \geqq 0$ and $(\bar{\rho}, \bar{\alpha}) \leqq (\bar{\rho}, \bar{\theta}) = h - 1$ for $\bar{\alpha} > 0$, and $\bar{\theta}_i = a_i^\vee > 0$, it follows immediately that, for $\bar{\lambda} \in \bar{P}_k$,

$$\mathscr{D}(\lambda) > 0 \tag{3.5}$$

and

$$\frac{\partial}{\partial \lambda^j} \mathscr{E}_i(\lambda) < 0 \tag{3.6}$$

for all $i, j = 1, \ldots, r$. As a consequence, one learns from (3.2) and (3.3) that, on $\bar{P}_k$,

$$\frac{\partial}{\partial \lambda^i} \mathscr{D}(\lambda) = 0 \ \Rightarrow \ \frac{\partial^2}{\partial \lambda^i \partial \lambda^j} \mathscr{D}(\lambda) < 0 \quad \forall j = 1, \ldots, r \tag{3.7}$$

for all $i = 1, \ldots, r$. This means in particular that there are no local minima of $\mathscr{D}(\lambda)$ on $\bar{P}_k$. Thus any minimum of $\mathscr{D}(\lambda)$ on $\bar{P}_k$ has to lie on its boundary $\partial \bar{P}_k$. This boundary is the union of $r+1$ hyperplanes,

$$\partial \bar{P}_k = \bigcup_{i=0}^{r} \bar{P}_k^{(i)}, \tag{3.8}$$

with

$$\bar{P}_k^{(i)} = \bar{P}_k \cap \{\lambda^i = 0\} \quad \text{for} \quad i = 1, \ldots, r, \tag{3.9}$$

$$\bar{P}_k^{(0)} = \bar{P}_k \cap \{(\bar{\lambda}, \bar{\theta}) = k\}. \tag{3.10}$$

On $\bar{P}_k^{(l)}, l=1,\ldots,r$, the formulae (3.2) to (3.4), and hence also (3.7) are still valid for i, j restricted to $\{1,\ldots,r\}\backslash\{l\}$; on $\bar{P}_k^{(0)}$, an additional contribution to $\mathscr{E}_i$ in (3.4) arises so that $\partial\mathscr{E}_i/\partial\lambda^j$ may be nonnegative, but for $j=i$ it is still negative. As a consequence, one can conclude that there are no local minima of $\mathscr{D}(\lambda)$ on any of the $\bar{P}_k^{(i)}, i=0,1,\ldots,r$, so that the global minima of $\mathscr{D}(\lambda)$ must lie on the boundaries $\partial\bar{P}_k^{(i)}=\bigcup_{j=0}^{r}\bar{P}_k^{(i,j)}$, with the hyperplanes $\bar{P}_k^{(i,j)}$ $(i\neq j)$ defined in an obvious manner. Analogously, one deduces that local minima are absent from the hyperplanes $\bar{P}_k^{(i,j)}$, and so on. After iterating the argument r times, one is then left with the corners (3.1) of $\bar{P}_k$. It follows that, as claimed, any minimum of $\mathscr{D}(\lambda)$ on $\bar{P}_k$ must lie on one of the corners of $\bar{P}_k$.

4. Corner Quantum Dimensions

We now proceed to show that the value of $\mathscr{D}(\lambda)$ at a corner (3.1) is larger or equal to one, with equality iff $\bar{\lambda}$ is of the special form (2.3) or (2.4). As a first step, we derive an explicit expression for $\mathscr{D}(\lambda)$ if $\bar{\lambda}=\mu\bar{\Lambda}_{(i)}$ with $\mu\in\mathbb{Z}$ arbitrary. This simply requires the implementation of the detailed structure of the positive root system of $\bar{g}$ into the formula (1.6). To write the result in a readable form, let us introduce the following abbreviations:

$$\langle v;a,b\rangle=\prod_{\substack{j=a\\ \mathrm{mod}\,\mathbb{Z}}}^{b}\frac{\lfloor j+v\rfloor}{\lfloor j\rfloor},\quad \langle\langle v;a,b\rangle\rangle=\prod_{\substack{j=a\\ \mathrm{mod}\,\mathbb{Z}/s}}^{b}\frac{\lfloor j+v\rfloor}{\lfloor j\rfloor},\tag{4.1}$$

where s is the relative length squared of the long and short roots of $\bar{g}$. Then for $\bar{g}$ one of the classical algebras, the resulting[2] formulae read

$$A_r\colon\ \mathscr{D}(\mu\Lambda_{(i)})=\prod_{j=1}^{i}\langle r-i+1;j,j+\mu-1\rangle\quad\text{for}\quad i=1,\ldots,r,\tag{4.2}$$

$$B_r\colon\ \mathscr{D}(\mu\Lambda_{(r)})=\left\langle\frac{\mu}{2};\frac{1}{2},r-\frac{1}{2}\right\rangle\prod_{j=1}^{[r/2]}\langle\mu;2j,2r-2j\rangle,\tag{4.3}$$

$$\begin{aligned}C_r\colon\ \mathscr{D}(\mu\Lambda_{(i)})=&\prod_{j=1}^{\min\{i,r-i\}}\left(\left\langle\left\langle\frac{\mu}{2};\frac{j}{2},\frac{r-j}{2}\right\rangle\right\rangle\left\langle\left\langle\frac{\mu}{2};\frac{r+j-i+1}{2},r-\frac{j+m-1}{2}\right\rangle\right\rangle\right)\\ &\cdot\prod_{j=1}^{[(i+1)/2]}\langle\langle\mu;r+j-i,r-j+1\rangle\rangle\quad\text{for}\quad i=1,\ldots,r,\end{aligned}\tag{4.4}$$

$$D_r\colon\ \mathscr{D}(\mu\Lambda_{(r-1)})=\mathscr{D}(\mu\Lambda_{(r)})=\prod_{j=1}^{[r/2]}\langle\mu;2j-1,2r-2j-1\rangle,\tag{4.5}$$

[2] For $\bar{g}=A_r$, the result can also be obtained via the Young tableau for $\bar{\Lambda}$ with the help of the q-deformed hook formula [15]

Table 1. Quantum dimensions for $\bar{\lambda} = \mu \bar{\Lambda}_{(i)}$, $\bar{g}$ exceptional

$\bar{g}$	i	$\mathscr{D}(\mu\Lambda_{(i)})$
E_6	1, 5	$\langle\mu;1,11\rangle\langle\mu;4,8\rangle$
	2, 4	$\prod_{j=1}^{4}\langle\mu;j,9-j\rangle\langle 2\mu;7,11\rangle$
	3	$\prod_{j=1}^{3}\langle\mu;j,7-j\rangle\prod_{j=2}^{3}\langle\mu;j,7-j\rangle\prod_{j=5}^{7}\langle 2\mu;j,14-j\rangle\langle 3\mu;10,11\rangle$
	6	$\prod_{j=1,3,4}\langle\mu;j,11-j\rangle\langle 2\mu;11,11\rangle$
E_7	1	$\prod_{j=1,4,6}\langle\mu;j,17-j\rangle\langle 2\mu;17,17\rangle$
	2	$\prod_{j=1}^{5}\langle\mu;j,11-j\rangle\prod_{j=7,9,11}\langle 2\mu;j,22-j\rangle\langle 3\mu;16,17\rangle$
	3	$\prod_{j=1}^{4}\langle\mu;j,8-j\rangle\prod_{j=2}^{3}\langle\mu;j,8-j\rangle\prod_{j=5}^{7}\langle 2\mu;j,16-j\rangle\langle 2\mu;7,9\rangle\prod_{j=10}^{11}\langle 3\mu;j,24-j\rangle\langle 4\mu;15,17\rangle$
	4	$\prod_{j=1}^{5}\langle\mu;j,10-j\rangle\langle\mu;3,7\rangle\prod_{j=7}^{9}\langle 2\mu;j,20-j\rangle\langle 3\mu;13,17\rangle$
	5	$\prod_{j=1,2,4,5}\langle\mu;j,13-j\rangle\prod_{j=9,13}\langle 2\mu;j,26-j\rangle$
	6	$\prod_{j=1,5,9}\langle\mu;j,18-j\rangle$
	7	$\prod_{j=1,3,4,5,7}\langle\mu;j,14-j\rangle\langle 2\mu;11,17\rangle$
E_8	1	$\prod_{j=1,6,10}\langle\mu;j,29-j\rangle\langle 2\mu;29,29\rangle$
	2	$\prod_{j=1,2,5,6,9}\langle\mu;j,19-j\rangle\prod_{j=11,15,19}\langle 2\mu;j,38-j\rangle\langle 3\mu;28,29\rangle$
	3	$\prod_{j=1}^{6}\langle\mu;j,14-j\rangle\prod_{j=9,10,11,13}\langle 2\mu;j,28-j\rangle\prod_{j=16,19}\langle 3\mu;j,42-j\rangle\langle 4\mu;27,29\rangle$
	4	$\prod_{j=1}^{5}\langle\mu;j,11-j\rangle\prod_{j=3}^{4}\langle\mu;j,11-j\rangle\prod_{j=7}^{11}\langle 2\mu;j,22-j\rangle\langle 2\mu;9,13\rangle\prod_{j=13}^{16}\langle 3\mu;j,33-j\rangle$ $\prod_{j=19,21}\langle 4\mu;j,44-j\rangle\langle 5\mu;26,29\rangle$
	5	$\prod_{j=1}^{4}\langle\mu;j,9-j\rangle\prod_{j=2}^{3}\langle\mu;j,9-j\rangle\prod_{j=5}^{9}\langle 2\mu;j,18-j\rangle\langle 2\mu;7,11\rangle\prod_{j=10}^{13}\langle 3\mu;j,27-j\rangle$ $\prod_{j=15}^{17}\langle 4\mu;j,36-j\rangle\prod_{j=21}^{22}\langle 5\mu;j,45-j\rangle\langle 6\mu;25,29\rangle$
	6	$\prod_{j=1}^{6}\langle\mu;j,13-j\rangle\prod_{j=7,9,10,11,13}\langle 2\mu;j,26-j\rangle\prod_{j=16}^{17}\langle 3\mu;j,39-j\rangle\langle 4\mu;23,29\rangle$
	7	$\prod_{j=1,4,6,7,10}\langle\mu;j,23-j\rangle\prod_{j=17,23}\langle 2\mu;j,46-j\rangle$
	8	$\prod_{j=1,3,4,5,6,7}\langle\mu;j,17-j\rangle\prod_{j=11,13,15,17}\langle 2\mu;j,34-j\rangle\langle 3\mu;22,29\rangle$
F_4	1	$\langle\mu;1,7\rangle\langle\langle\mu;\tfrac{5}{2},\tfrac{11}{2}\rangle\rangle\langle 2\mu;8,8\rangle$
	2	$\langle\mu;1,4\rangle\langle\langle\mu;\tfrac{3}{2},\tfrac{7}{2}\rangle\rangle\langle\langle\mu;2,3\rangle\rangle\langle 2\mu;4,6\rangle\langle\langle 2\mu;\tfrac{9}{2},\tfrac{11}{2}\rangle\rangle\langle 3\mu;7,8\rangle$
	3	$\langle\langle\tfrac{1}{2}\mu;\tfrac{1}{2},3\rangle\rangle\langle\langle\mu;2,5\rangle\rangle,\langle\mu;3,4\rangle\langle\langle\tfrac{3}{2}\mu;5,\tfrac{11}{2}\rangle\rangle\langle 2\mu;6,8\rangle$
	4	$\langle\langle\tfrac{1}{2}\mu;\tfrac{1}{2},1\rangle\rangle\langle\langle\tfrac{1}{2}\mu;2,\tfrac{7}{2}\rangle\rangle\langle\langle\tfrac{1}{2}\mu;\tfrac{9}{2},5\rangle\rangle\langle\mu;3,8\rangle\langle\mu;\tfrac{11}{2},\tfrac{11}{2}\rangle$
G_2	1	$\langle\langle\mu;1,2\rangle\rangle\langle 2\mu;3,3\rangle$
	2	$\langle\tfrac{1}{3}\mu;\tfrac{1}{3},\tfrac{4}{3}\rangle\langle\tfrac{2}{3}\mu;\tfrac{5}{3},\tfrac{5}{3}\rangle\langle\mu;2,3\rangle$

as well as

$$B_r, D_r\colon\ \mathcal{D}(\mu\Lambda_{(i)}) = \left\langle \mu; \frac{n}{2} - i, \frac{n}{2} - 1 \right\rangle \prod_{j=1}^{\min\{i, n-2i-1\}} \langle \mu; j, n-i-j-1 \rangle$$
$$\cdot \prod_{j=1}^{[i/2]} \langle 2\mu; n-2i+2j-1, n-2j-1 \rangle \quad \text{for} \quad i = 1, \ldots, \left[\frac{n-3}{2}\right], \tag{4.6}$$

where $[x]$ stands for the integer part of x, and where in the last equation $n = 2r+1$ or $2r$ depending on whether $B_r \cong so(2r+1)$ or $D_r \cong so(2r)$ is considered. For $\bar{g}$ exceptional, the analogous results are displayed in Table 1.

Next we have to put $\mu = k/a_i^\vee$ in these formulae (of course only the case $k/a_i^\vee \in \mathbb{Z}$ is relevant). Afterwards we can use the identity

$$\lfloor x \rfloor = \lfloor k+h-x \rfloor, \tag{4.7}$$

which is an immediate consequence of the definition (1.7) of $\lfloor x \rfloor$. Then many of the terms cancel between the numerator and denominator of $\mathcal{D}$, and in order to obtain the desired result, one simply has to take a close look at the remaining factors. The situation is particularly simple in the case of the cominimal fields: after the implementation of (4.7), no terms are left at all and so $\mathcal{D} = 1$. More explicitly, for $\mu = k$ the relevant formulae above can be rewritten as displayed in Table 2, where we use the notation

$$\{a,b\} = \prod_{\substack{j=a \\ \bmod \mathbb{Z}}}^{b} \frac{\lfloor k+h-j \rfloor_q}{\lfloor j \rfloor_q}, \quad \{\{a,b\}\} = \prod_{\substack{j=a \\ \bmod \mathbb{Z}/s}}^{b} \frac{\lfloor k+h-j \rfloor_q}{\lfloor j \rfloor_q}, \tag{4.8}$$

with

$$\lfloor x \rfloor_q = \sin\left(-\frac{i}{2}\ln q \cdot x\right). \tag{4.9}$$

Table 2. Quantum dimensions of cominimal fields for q arbitrary

$\bar{g}$	i	$\mathcal{D}(k\Lambda_{(i)})$
A_r	$1,\ldots,r$	$\prod_{j=1}^{k} \{j, i+j-1\}$
B_r	1	$\{1, 2r-2\}\,\{r-\frac{1}{2}, r-\frac{1}{2}\}$
C_r	r	$\prod_{j=1}^{[(r+1)/2]} \{\{j, r-j+1\}\}$
D_r	1	$\{1, 2r-3\}\{r-1, r-1\}$
	$r-1, r$	$\prod_{j=1}^{[r/2]} \{2j-1, 2r-2j-1\}$
E_6	1, 5	$\{1, 11\}\{4, 8\}$
E_7	6	$\{1, 17\}\{5, 13\}\{9, 9\}$

This notation has been adapted so as to apply also to the quantum dimensions for $\mathbf{U}_q(\bar{g})$ with q arbitrary.[3] Of course, in the case of our interest, q is fixed by (1.8) to a root of unity, leading to (4.7) and hence to

$$\{a,b\}\equiv 1,\quad \{\{a,b\}\}\equiv 1. \tag{4.10}$$

Thus indeed the quantum dimension of a cominimal field is equal to one.

For non-cominimal fields, inspection of the previous formulae for $\mathscr{D}$ shows that after the implementation of (4.7) always some nontrivial terms are left over. Moreover, the minimal remaining factor in the numerator is $\geqq \lfloor (k+1)/a_i^\vee \rfloor$ while the maximal remaining factor in the denominator is $\leqq \lfloor (\bar{\rho},\bar{\theta}) \rfloor = \lfloor h-1 \rfloor$, and hence for $k \geqq (h-1)a_i^\vee$ the quantum dimension is manifestly larger than one. Further inspection shows that one can easily do better than that: writing $\mathscr{D}(\lambda)=\prod_j(\lfloor n_j \rfloor/\lfloor m_j \rfloor)$ with $n_j, m_j \leqq \frac{1}{2}(k+h)$ (this can be achieved with the help of (4.7)) as well as $n_j \geqq n_{j-1}$ and $m_j \geqq m_{j-1}$ for all j, one readily checks that for $k \geqq h$ one also has $m_j < n_j$ for all j, and hence the identity

$$\frac{\lfloor x \rfloor}{\lfloor y \rfloor} > 1 \quad \text{for} \quad 0 < y < x \leqq \frac{k+h}{2} \tag{4.11}$$

implies $\mathscr{D} > 1$. For definiteness, let us present two examples: for $\bar{g}=E_8$ and $i=5$ (corresponding to the maximal possible value of $a_i^\vee$) one has

$$\begin{aligned}\mathscr{D}\left(\frac{k}{6}\Lambda_{(5)}\right) = {} & \left\lfloor \frac{k}{6}+1 \right\rfloor \left\lfloor \frac{k}{6}+2 \right\rfloor^3 \left\lfloor \frac{k}{6}+3 \right\rfloor^5 \left\lfloor \frac{k}{6}+4 \right\rfloor^6 \left\lfloor \frac{k}{6}+5 \right\rfloor^6 \left\lfloor \frac{k}{6}+6 \right\rfloor^6 \left\lfloor \frac{k}{6}+7 \right\rfloor^5 \\ & \cdot \left\lfloor \frac{k}{6}+8 \right\rfloor^3 \left\lfloor \frac{k}{6}+9 \right\rfloor \left\lfloor \frac{k}{3}+5 \right\rfloor \left\lfloor \frac{k}{3}+6 \right\rfloor^2 \left\lfloor \frac{k}{3}+7 \right\rfloor^4 \left\lfloor \frac{k}{3}+8 \right\rfloor^5 \left\lfloor \frac{k}{3}+9 \right\rfloor^7 \\ & \cdot \left\lfloor \frac{k}{3}+10 \right\rfloor^7 \left\lfloor \frac{k}{3}+11 \right\rfloor^7 \left\lfloor \frac{k}{3}+12 \right\rfloor^5 \left\lfloor \frac{k}{3}+13 \right\rfloor^4 \left\lfloor \frac{k}{3}+14 \right\rfloor^2 \left\lfloor \frac{k}{3}+15 \right\rfloor \\ & \cdot \left\lfloor \frac{k}{2}+10 \right\rfloor \left\lfloor \frac{k}{2}+11 \right\rfloor^2 \left\lfloor \frac{k}{2}+12 \right\rfloor^3 \left\lfloor \frac{k}{2}+13 \right\rfloor^5 \left\lfloor \frac{k}{2}+14 \right\rfloor^6 \left\lfloor \frac{k}{2}+15 \right\rfloor^3 \Big/ \\ & \lfloor 2 \rfloor^2 \lfloor 3 \rfloor^4 \lfloor 4 \rfloor^5 \lfloor 5 \rfloor^6 \lfloor 6 \rfloor^7 \lfloor 7 \rfloor^7 \lfloor 8 \rfloor^6 \lfloor 9 \rfloor^6 \lfloor 10 \rfloor^6 \lfloor 11 \rfloor^6 \lfloor 12 \rfloor^5 \\ & \lfloor 13 \rfloor^5 \lfloor 14 \rfloor^4 \lfloor 15 \rfloor^4 \lfloor 16 \rfloor^4 \lfloor 17 \rfloor^4 \lfloor 18 \rfloor^3 \lfloor 19 \rfloor^3 \lfloor 20 \rfloor^2 \lfloor 21 \rfloor^2 \\ & \lfloor 22 \rfloor^2 \lfloor 23 \rfloor^2 \lfloor 24 \rfloor \lfloor 25 \rfloor \lfloor 26 \rfloor \lfloor 27 \rfloor \lfloor 28 \rfloor \lfloor 29 \rfloor, \end{aligned} \tag{4.12}$$

and for E_8 and $i=7$ the formula reads

$$\begin{aligned}\mathscr{D}\left(\frac{k}{2}\Lambda_{(7)}\right) = {} & \left\lfloor \frac{k}{2}+1 \right\rfloor \left\lfloor \frac{k}{2}+2 \right\rfloor \left\lfloor \frac{k}{2}+3 \right\rfloor \left\lfloor \frac{k}{2}+4 \right\rfloor^2 \left\lfloor \frac{k}{2}+5 \right\rfloor^2 \left\lfloor \frac{k}{2}+6 \right\rfloor^3 \left\lfloor \frac{k}{2}+7 \right\rfloor^4 \\ & \cdot \left\lfloor \frac{k}{2}+8 \right\rfloor^5 \left\lfloor \frac{k}{2}+9 \right\rfloor^5 \left\lfloor \frac{k}{2}+10 \right\rfloor^6 \left\lfloor \frac{k}{2}+11 \right\rfloor^7 \left\lfloor \frac{k}{2}+12 \right\rfloor^7 \left\lfloor \frac{k}{2}+13 \right\rfloor^8 \end{aligned}$$

[3] For arbitrary value of the deformation parameter q, the q-deformed dimension of an irreducible highest weight module of the quantum group $\mathbf{U}_q(\bar{g})$ is still expressible [13, 14, 15] as in (1.6), but with $\lfloor x \rfloor$ replaced by $\lfloor x \rfloor_q$

$$\cdot \left\lfloor \frac{k}{2}+14 \right\rfloor^8 \left\lfloor \frac{k}{2}+15 \right\rfloor^4 \Big/$$
$$\lfloor 4 \rfloor \lfloor 5 \rfloor \lfloor 6 \rfloor^2 \lfloor 7 \rfloor^2 \lfloor 8 \rfloor^3 \lfloor 9 \rfloor^3 \lfloor 10 \rfloor^4 \lfloor 11 \rfloor^4 \lfloor 12 \rfloor^4 \lfloor 13 \rfloor^4 \lfloor 14 \rfloor^4$$
$$\lfloor 15 \rfloor^4 \lfloor 16 \rfloor^4 \lfloor 17 \rfloor^4 \lfloor 18 \rfloor^3 \lfloor 19 \rfloor^3 \lfloor 20 \rfloor^2 \lfloor 21 \rfloor^2 \lfloor 22 \rfloor^2 \lfloor 23 \rfloor^2 \lfloor 24 \rfloor$$
$$\lfloor 25 \rfloor \lfloor 26 \rfloor \lfloor 27 \rfloor \lfloor 28 \rfloor \lfloor 29 \rfloor. \tag{4.13}$$

It remains to investigate the quantum dimensions at the corners in the range $k < h$. For such values of k, it is generically not possible to write $\mathscr{D}$ as a product of fractions $\lfloor n_j \rfloor / \lfloor m_j \rfloor$ each of which is individually larger than one; however, it is possible to employ instead identities such as

$$\lfloor x \rfloor \left\lfloor \frac{k+h}{2} - x \right\rfloor = \frac{1}{2} \lfloor 2x \rfloor \quad \text{for} \quad x \leqq \frac{k+h}{2} \tag{4.14}$$

and

$$\frac{\lfloor x \rfloor \lfloor 2y \rfloor}{\lfloor y \rfloor \lfloor 2x \rfloor} > 1 \quad \text{for} \quad 0 < y < x \leqq \frac{k+h}{2} \tag{4.15}$$

to analyze the formulae further. For exceptional $\bar{g}$, the analysis can be shortcut by implementing the formula (1.6) on a computer. The result is as claimed: there are no non-cominimal simple currents except for the isolated case (2.4). That the quantum dimension of the latter field is unity can be read off (4.13): for $k = 2$, the formula reduces upon use of $\lfloor x \rfloor = \lfloor 32 - x \rfloor$ to

$$\mathscr{D}(\Lambda_{(7)}) = \frac{\lfloor 2 \rfloor \lfloor 12 \rfloor \lfloor 14 \rfloor}{\lfloor 4 \rfloor \lfloor 6 \rfloor \lfloor 10 \rfloor} = 1. \tag{4.16}$$

For classical $\bar{g}$, the computer analysis can not provide a proof of the statement.[4] A close inspection of the formulae for $\mathscr{D}$ reveals, however, that upon use of identities like (4.15), one can always derive the desired result. This analysis is lengthy (and not at all illuminating), and for brevity we just present the simplest examples which already involve all manipulations that are needed in the general case. First, for the "spinor" corner $i = r$ of B_r, one obtains

$$\mathscr{D}(k\Lambda_{(r)}) = \prod_{j=1}^{r} \frac{\left\lfloor \frac{k}{2} + j - \frac{1}{2} \right\rfloor}{\lfloor j - \frac{1}{2} \rfloor} \prod_{j=1}^{[r/2]} \frac{\lfloor 2j - 1 \rfloor}{\lfloor 2r - 2j \rfloor}, \tag{4.17}$$

where the factors $\lfloor x \rfloor$ are written in a form ensuring $0 < x_j \leqq (k + 2r - 1)/2$. By rewriting this as

$$\mathscr{D}(k\Lambda_{(r)}) = \prod_{j=1}^{[r/2]} \frac{\left\lfloor \frac{k}{2} + j - \frac{1}{2} \right\rfloor \lfloor 2j - 1 \rfloor}{\lfloor j - \frac{1}{2} \rfloor \lfloor k + 2j - 1 \rfloor} \prod_{j=[r/2]+1}^{r} \frac{\left\lfloor \frac{k}{2} + j - \frac{1}{2} \right\rfloor}{\lfloor j - \frac{1}{2} \rfloor}, \tag{4.18}$$

one sees that $\mathscr{D}(k\Lambda_{(r)}) > 1$ due to (4.11) and (4.15).

---

[4] It does, however, serve as a useful check of the assertion. We have performed this numerical check for all algebras up to rank 200 (this calculation required roughly 300 h of CPU time)

Next, consider the "adjoint" corner $i=2$ of $so(n)$. In this case after cancellation of numbers appearing both in the numerator and in the denominator one is left with

$$\mathscr{D}(\tfrac{1}{2}kA_{(2)})=\frac{\left\lfloor \frac{k+n}{2}-2\right\rfloor}{\left\lfloor \frac{n}{2}-1\right\rfloor\left\lfloor \frac{n}{2}-2\right\rfloor}\prod_{j=2}^{k/2}\frac{\left\lfloor \frac{k}{2}+j\right\rfloor}{\lfloor j\rfloor}\prod_{j=2}^{k/2+1}\frac{\left\lfloor \frac{k}{2}+j+1\right\rfloor}{\lfloor j\rfloor}\tag{4.19}$$

which is (except for $n=5$, but in this case the result is easily checked numerically) manifestly a product of factors that are larger than one.

As a last example, take $i=1$ for $\bar{g}=C_r$. After some trivial cancellations, one obtains

$$\mathscr{D}(kA_{(1)})=\frac{\lfloor 1\rfloor\left\lfloor \frac{r}{2}\right\rfloor}{\left\lfloor \frac{k}{2}\right\rfloor\left\lfloor \frac{k+r}{2}\right\rfloor}\prod_{j=1}^{k-1}\frac{\left\lfloor \frac{k+j}{2}+1\right\rfloor}{\left\lfloor \frac{j}{2}\right\rfloor}.\tag{4.20}$$

One deduces that this is larger than one, e.g. by substituting $\lfloor\frac{1}{2}(k+r)\rfloor\lfloor\frac{1}{2}\rfloor=\frac{1}{2}\lfloor 1\rfloor$ in the denominator and taking into account that $\lfloor r/2\rfloor\geqq\sin(\pi/5)>\frac{1}{2}$.

This concludes our proof of the classification of simple WZW currents. Note that the proof clarifies to some extent the presence of the exceptional simple current (2.4). Namely, in order to have $\mathscr{D}=1$, the numerator and denominator of (1.6) have to be equal. This happens generically iff each factor in the numerator cancels a factor of the denominator; exceptions from this rule require a rather nontrivial numerical coincidence, and precisely in the case of (2.4) this requirement is satisfied, as made explicit in (4.16).

Let us finally add a few further observations.

1. The quantum dimensions are invariant under the automorphisms ω of the extended Dynkin diagram [15, 11], so that in particular

$$\mathscr{D}\left(\frac{k}{a_i^\vee}A_{(i)}\right)=\mathscr{D}\left(\frac{k}{a_i^\vee}A_{(\omega(i))}\right)\tag{4.21}$$

(recall that $a_{\omega(i)}^\vee=a_i^\vee$). From this symmetry it follows immediately that the cominimal fields (which are precisely the fields that are related to the identity primary field by such an automorphism) have $\mathscr{D}=1$, but also e.g. that $\mathscr{D}(kA_{(i)})=\mathscr{D}(kA_{(r-i)})$, $i=1,\dots,r-1$, for C_r, and $\mathscr{D}(\frac{1}{2}kA_{(i)})=\mathscr{D}(\frac{1}{2}kA_{(r-i)})$, $i=2,\dots,r-2$, for D_r. The latter relations are not manifest in formulae such as (4.4) with $\mu=k$, but by suitable rearrangement of the various factors one can verify that they are indeed fulfilled.

2. The quantum dimensions at the non-cominimal corners are actually larger than two except for some cases at level one and two, namely: $\mathscr{D}=\sqrt{2}$ for $A_{(r)}$ of B_r at level 1, for $A_{(7)}$ of E_7 at level 2, and for $A_{(1)}$ of E_8 at level 2; $\mathscr{D}=2$ for $A_{(i)}$, $i=1,\dots,\left[\frac{n-3}{2}\right]$, of $so(n)$ at level 2, and for $2A_{(r)}$ of B_r at level 2: $\mathscr{D}=2\cos\left(\frac{\pi}{r+2}\right)$ for $A_{(1)}$ and $A_{(r-1)}$ of C_r at level 1; $\mathscr{D}=2\cos(\pi/5)=\frac{1}{2}(1+\sqrt{5})$ for $A_{(1)}$ and $A_{(5)}$ of E_7 at level 2, for $A_{(4)}$ of F_4 at level 1, and for $A_{(2)}$ of G_2 at level

1: $\mathscr{D} = 2\cos(\pi/7)$ for $\Lambda_{(i)}$, $i = 2, 4, 6$ of E_6 at level 2; $\mathscr{D} = 2\cos(\pi/9)$ for $\Lambda_{(1)}$ of G_2 at level 2; $\mathscr{D} = 2\cos(\pi/11)$ for $\Lambda_{(8)}$ of E_8 at level 3, and for $\Lambda_{(1)}$ of F_4 at level 2. (Consider e.g., the formula (4.19): for $k = 2$, one has $\mathscr{D}(\Lambda_{(2)}) = \lfloor 4 \rfloor / \lfloor 2 \rfloor \left\lfloor \frac{n}{2} - 2 \right\rfloor$ which is equal to two due to (4.14).)

3. For $\bar{g} = C_r$, one can show that $\mathscr{D}(k\Lambda_{(i)})$ is monotonically increasing with i for $i = 1, \ldots, \left\lfloor \frac{r}{2} \right\rfloor$. Together with the symmetry property $\mathscr{D}(k\Lambda_{(i)}) = \mathscr{D}(k\Lambda_{(r-i)})$, this may be used to reduce the proof of the inequality $\mathscr{D} > 1$ for $i = 2, \ldots, r-1$ to that for $i = 1$.

4. For fixed algebra and fixed value of $i \in \{1, \ldots, r\}$ corresponding to non-cominimal fields, $\mathscr{D}(k\Lambda_{(i)}/a_i^\vee)$ is monotonically increasing with the level k. We have not been able to find a general proof of this fact, but the numerical evidence is convincing.

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