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Phase Transition in Gas of Hard Core Spheres

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Abstract. We present a new method of analyzing the gas of hard core spheres. We investigate analytic properties of the thermodynamic function over the circle of convergence of the cluster expansion and describe the way in which phase transition occurs.

1. Introduction

Our aim is to investigate the analytic properties of the thermodynamic function when intensive variables take values on the positive axis not only in the circle of convergence of the Mayer expansion or spectral radius of Kirkwood-Salsburg operator. We use the explicit form of Mayer series coefficients patterned on the cluster expansion [2] and find furthermore their new formulas for the case of hard core gas. We formulate an identity for operator valued generating function and apply it together with an analytic version of Fredholm alternative. Next we explain how the phase transition can occur. We find new regions of analyticity for a thermodynamic function. Our method may be utilized in statistical mechanics and Euclidean field theory.

2. New Form of Mayer Coefficient

In the case when the interaction is a translation-invariant Mayer expansion coefficients have the following form:

$$b_n = \frac{(-\beta)^{n-1}}{n} \sum_{\eta} \int_{[0,1]^{n-1}} d\sigma_{n-1} \int_{\mathcal{R}^{3(n-1)}} d(y)_{n-1} f(\eta, \sigma_{n-1}) \times \prod_{i=1}^{n-1} v(y_{i+1} - y_{\eta(i)}) \exp - \beta W^{(n)}(\sigma_{n-1}) \tag{1}$$

summations are carried out over all η such that $\eta(i) \leqq i$ and $\eta(i)$ is a positive integer for i. Here:

$$W^n(\sigma_{n-1}) = \sum_{1 \leqq i < j \leqq n} s_i \dots s_{j-1} v(y_j - y_i),$$

$$d\sigma_{n-1} = ds_1 \dots ds_{n-1},$$

$$f(\eta, \sigma_{n-1}) = \prod_{i=2}^{n-1} s_{i-1} s_{i-2} \dots s_{\eta(i)}, \qquad n > 2,$$

$$f(\eta, \sigma_1) = 1.$$

Substituting

$$y_k = x_1 + \dots + x_k, \qquad k = 1, \dots, n$$

we obtain:

$$b_n = \frac{(-\beta)^{n-1}}{n} \sum_k \int_{[0,1]^{n-1}} d\sigma_{n-1} \int_{\mathscr{R}^{3(n-1)}} d(x)_{n-1} f(k, \sigma_{n-1})$$
$$\times \prod_{i=1}^{n-1} v(x_i + \dots + x_{i-k(i)}) \exp - \beta W^{(n)}(\sigma_{n-1}), \tag{2}$$

where k is a positive integer value function such that

$$0 \leqq k(i) \leqq i - 1$$

and

$$W^n(\sigma_{n-1}) = \sum_{1 \leqq i \leqq j \leqq n} s_i \dots s_j v(x_i + \dots + x_j),$$

$$f(k, \sigma_{n-1}) = \prod_{i=2}^{n-1} s_{i-1} s_{i-2} \dots s_{k(i)}, \qquad n > 2,$$

$$f(k, \sigma_1) = 1.$$

Let

$$v(x) = v\chi(x), \tag{3}$$

where χ is a characteristic function of the sphere of volume one.

If $v \to \infty$ then the potential will approach the potential of hard core spheres model. For an illustration let us consider the coefficient b_3,

$$b_3 = \tfrac{1}{3}\beta^2 \int_0^1 ds_1 \int_0^1 ds_2 \int d^3x \int d^3y [(v(x)v(y) + s_1 v(x) v(x+y)]$$
$$\times \exp - \beta[s_1 v(x) + s_2 v(y) + s_1 s_2 v(x+y)]. \tag{4}$$

We substitute (3) to b_3 and try to find the limit when v goes to infinity. Let $t_1 = s_1 v$, $t_2 = s_2 v$ in the first term and $t_1 = s_1 v$ in the second term. Then

$$b_3 = \frac{1}{3}\beta^2 \int_0^\infty dt_1 \int_0^\infty dt_2 \int d^3x \int d^3y \chi_{[0,v]}(t_1) \chi_{[0,v]}(t_2)$$
$$\times \chi(x)\chi(y) \exp - \beta \left[t_1 \chi(x) + t_2 \chi(y) + t_1 t_2 \frac{\chi(x+y)}{v} \right]$$
$$+ \frac{1}{3}\beta^2 \int_0^\infty dt_1 \int_0^1 ds_2 \int d^3x \int d^3y \, t_1 \chi(x) \chi(x+y) \chi_{[0,v]}(t_1)$$
$$\times \exp - \beta[t_1 \chi(x) + v s_2 \chi(y) + t_1 s_2 \chi(x+y)]. \tag{5}$$

The integrands of these two terms are dominated by

$$\begin{gathered} \chi(x)\chi(y)\exp-\beta[t_1+t_2], \\ t_1\chi(x)\chi(x+y)\exp-2\beta t_1. \end{gathered} \tag{6}$$

Therefore the assumptions of Lebesgue's Dominated Convergence Theorem are fulfilled.

In the general case we first make the change $s_1 \to t_1$. Next in the expression

$$\prod_{i=1}^{n-1} v(x_i+\ldots+x_{i-k(i)}) \tag{7}$$

we identify a potential with the smallest index i such that $i-k(i)>1$ and change $s_i \to t_i$. Then we look for the new smallest index i' such that $i'-k(i')>i$ and make a change $s_{i'} \to t_{i'}$ and so on.

To justify the correctness of the limiting procedure for b_n we should show integrability of the dominant function. We sketch the proof.

Let j be the greatest index for which we have changed $s_j \to t_j$ in some term of b_n. We can write part of integrand of this term which depends on x_j and t_j in the following way:

$$\begin{aligned} &\ldots \chi(x_{j-k_1}+x_j)\ldots\chi(x_{j-k_l}+\ldots+x_{j+l-1}) \\ &\times(s_{j-k_1}\ldots s_{j-1})\ldots(s_{j-k_l}\ldots s_{j-1}t_js_{j+1}\ldots s_{j+l-2}) \\ &\times\exp-\beta[\chi(x_{j-k_1}+\ldots+x_j)s_{j-k_1}\ldots s_{j-1}t_j+\ldots \\ &+\chi(x_{j-k_l}+\ldots+x_{j+l-1})s_{j-k_l}\ldots s_{j-1}t_js_{j+1}\ldots s_{j+l-1}]. \end{aligned}$$

Integrating by t_j we obtain

$$\begin{aligned} &\ldots[\chi(x_{j-k_1}+\ldots+x_j)\ldots\chi(x_{j-k_l}+\ldots+x_{j+l-1})(s_{j-k_1}\ldots s_{j-1})\ldots \\ &\ldots(s_{j-k_l}\ldots s_{j-1}s_{j+1}\ldots s_{j+l-2})] \\ &\times[s_{j-k_1}\ldots s_{j-1}+\ldots+s_{j-k_l}\ldots s_{j-1}s_{j+1}\ldots s_{j+l-1}]^{1-l}. \end{aligned} \tag{8}$$

We can estimate this factor depending on s by a constant integrating this with respect to $s_{j+l-1}\ldots s_{j+1}$ successively. We can repeat this consideration for the next parts of the term.

3. Analytic Continuation of Mayer Series

In this section we consider analytic continuation of Mayer series.

Let k-component denote one of the terms labelled by k in Eq. (2).

Lemma 1. *Assume we have changed variables with indices $i \in \{p_1, \ldots, p_l\}$ in a k-component and let the potential $v(x_i+\ldots+x_{k(i)})$ belong to the same k-component. Let at least two indices from the set $\{p_1, \ldots, p_l\}$ be contained in the interval $[i-k(i), i]$, then this component vanishes for $v \to \infty$.*

Proof. After the change of variables in the component of b_n we get

$$v^{-(k_1+\ldots+k_{n-1}-(n-1))},$$

where k_i denotes the number of indices in each potential for which this change was effected. If at least one k_i is greater than one, then the integral tends to zero for $v \to \infty$.

Theorem 1. *Assume that for a certain component of the sum* (2) *which is different from zero there exists j such, that* $1 \leqq j < n-1$.

$$k(j)=0, \quad k(j+1)=0.$$

Then

$$\lim_{v\to\infty} \int_{[0,1]^{n-1}} d\sigma_{n-1} \int_{\mathscr{R}^{3(n-1)}} d(x)_{n-1} f(k,\sigma_{n-1}) \prod_{i=1}^{n-1} v(x_i+\ldots+x_{i-k(i)})$$
$$\times \exp -\beta W^{(n)}(\sigma_{n-1})$$
$$=\lim_{v\to\infty} \int_{[0,1]^j} d\sigma_j \int_{\mathscr{R}^{3j}} d(x)_j f(k,\sigma_j) \prod_{i=1}^{j} v(x_i+\ldots+x_{i-k(i)})$$
$$\times \exp -\beta W^{(j)}(\sigma_{j-1}) \cdot \lim_{v\to\infty} \int_{[0,1]^{n-j-3}} d\sigma_{n-j-3} \int_{\mathscr{R}^{3(n-j-3)}} d(x)_{n-j-3}$$
$$\times f(k,\sigma_{n-j-3}) \prod_{i=1}^{n-j-3} v(x_i+\ldots+x_{i-k(i)}) \exp -\beta W^{(n-j-3)}(\sigma_{n-j-4}). \tag{9}$$

Proof. Under our assumptions we should change variables $s_j \to t_j$, s_{j+1}, t_{j+1}. Then in the exponent all expressions which contain s_j and s_{j+1} will tend to zero for $v\to\infty$ because of

$$vs_js_{j+1}=\frac{t_jt_{j+1}}{v}.$$

If the expression on the left-hand side of (9) includes a potential depending on x_j and x_{j+1} simultaneously, then by Lemma 1 this expression equals zero so we can separate variables which completes the proof.

We introduce a new notation. For $i,k \geqq 1$,

$$a_0^{(i)}=v(x_i+x_{i+1}+\ldots+x_{i+k})s_{i+1}\ldots s_{i+k}$$
$$\times \exp -\beta[v(x_{i+k})s_{i+k}+v(x_1+\ldots+x_{i+k})s_1\ldots s_{i+k}]$$

and for $k=0$, $i\geqq 1$,

$$a_0^{(j)}=v(x_i)\exp -\beta[v(x_i)s_i+\ldots+v(x_1+\ldots+x_i)s_1\ldots s_i]. \tag{10}$$

h denotes the integral over all variables x_i, t_i, s_i after limiting procedure. Then:

$$h(a_{l(0)}^{(i)}a_{l(1)}^{(i+1)}\ldots a_{l(n)}^{(i+n)})=h(a_{l(0)}^{(1)}a_{l(1)}^{(2)}\ldots a_{l(n)}^{(n+1)}) \quad \forall i$$

so we can omit indices (i).

In this notation we can rewrite Theorem 1 in following way:

$$h(a_{l(0)}a_{l(1)}\ldots a_{l(n)}a_0a_0a_{p(1)}\ldots a_{p(n)})$$
$$=h(a_{l(0)}a_{l(1)}\ldots a_{l(n)}a_0)\cdot h(a_{p(0)}a_{p(1)}\ldots a_{p(n)}). \tag{11}$$

We notice that

$$(-\beta)^n nb_n=h[a_0(a_0+a_1)\ldots(a_0+\ldots+a_n)]. \tag{12}$$

Let us formulate the following theorem:

Theorem 2. *Let* $\mathscr{A}=\{a_k\}_{k=0}$ *denote a free nonabelian algebra over* $\mathscr{C}$ *and* $h:\mathscr{A}\to\mathscr{R}^+$ *be a linear operator with an additional property, let* $u_i=0$ *and* $u_{i+1}=0$, *then*

$$h(a_{u_1}\ldots a_{u_i}a_{u_{i+1}}\ldots a_{u_n})=h(a_{u_1}\ldots a_{u_i})\cdot h(a_{u_{i+1}}\ldots a_{u_n}).$$

Then we have the following identity in the circle of convergence

$$z+h[z^2a_0+z^3a_0(a_0+a_1)+\ldots+z^{k+1}a_0(a_0+a_1)\ldots(a_0+\ldots+a_{k-1})+\ldots]$$
$$=\frac{\sum\limits_{n=1} z^n c'_n}{1-\sum\limits_{n=1} z^n c_n},$$

where

$$\begin{aligned} c_n&=\sum_l h(a_{l(0)}a_{l(1)}\ldots a_{l(n-2)}a_0), \quad n\geqq 3,\\ c'_n&=\sum_l h(a_{l(0)}a_{l(1)}\ldots a_{l(n-2)}), \quad n\geqq 3,\\ c_1&=h(a_0), \quad c'_1=1, \quad c_2=c'_2=0. \end{aligned} \tag{13}$$

We sum up over all functions l such that $l(i)\leqq i$, $l(i)$ is natural number and there is no such j that $l(j)=0$, $l(j+1)=0$.

Proof. The theorem comes out from the following identity:

$$\frac{1}{1-\sum\limits_{n=1} z^n c_n}=\sum_{k=0}\left(\sum_{n=1} z^n c_n\right)^k.$$

We will consider instead of the pressure $p(z)$, a density $\varrho(z)$

$$\varrho(z)=z\frac{d}{dz}p(z).$$

By Theorems 1, 2 for the gas of hard core spheres we obtain

$$\varrho(z)=-\frac{\sum\limits_{n=1}(-z)^n c'_n}{1-\sum\limits_{n=1}(-z)^n c_n}. \tag{14}$$

Therefore we have shown that $\varrho(z)$ has an isolated singular point for $z_0<0$ and $|z_0|$ is equal to the radius of convergence for the series $\varrho(z)$. It is known [5] that $1\geqq|z_0|\geqq e^{-1}$. But from the physical point of view the singularities on the positive axis only are interesting. It is clear that the function on the right-hand side of (14) is an analytic continuation of the series $\varrho(z)$.

4. Analytic Properties of Density

We should investigate properties of the following series:

$$\sum z^n c'_n, \quad \sum z^n c_n. \tag{15}$$

At first we notice that a function $l(i)$, $0\leqq i\leqq n$ can be written in the following way:

$$l(i)=\begin{cases} l_1(i) & i\leqq k_1\\ l_2(i-k_1) & k_1\leqq i\leqq k_2\\ \vdots & \\ l_p(i-k_{p-1}) & k_p\leqq i\leqq n \end{cases}, \tag{16}$$

where all l_q have the same property as the function $l(i)$. So we can write

$$h(a_{l(0)} \dots a_{l(n)}) = h(a_{l_1(0)} \dots a_{l_1(k_1)} a_{l_2(0)} \dots a_{l_2(k_2-k_i)} \dots a_{l_p(0)} \dots a_{l_p(n-k_{p-1})}).$$

We observe that each function $a_{l_p(0)} \dots a_{l_p(m)}$ depends on some of the variables appearing in the expressions $a_{l_{p+1}(0)} \dots a_{l_{p+1}(m')}$ and $a_{l_{p-1}(0)} \dots a_{l_{p-1}(m'')}$. Thus functions $a_{l_p(0)} \dots a_{l_p(m)}$ are kernels of integral operators.

For an illustration let us consider an explicit example. We can compute:

$$\begin{aligned} h(a_0 a_1 a_0 a_1 a_1 a_1) = & \int_{\mathcal{R}^0} \int_{[0,1]^3} \int_{\mathcal{R}^3} \frac{\chi(x_1)\chi(x_1+x_2)(1-\chi(x_2))}{(1+s_2)^2} dx_1 \\ & \times \int_{\mathcal{R}^3} \int_{\mathcal{R}^3} \frac{\chi(x_3)\chi(x_3+x_4)(1-\chi(x_4))}{(1+s_4+s_2\chi(x_2+x_3))^2} \\ & \times \frac{\chi(x_4+x_5)\chi(x_5+x_6)(1-\chi(x_6))s_4}{(\chi(x_5)+s_4+s_6+\chi(x_4+x_5+x_6)s_4 s_6)^2} \\ & \times dx_3 dx_5 ds_2 ds_4 ds_6 dx_2 dx_4 dx_6 . \end{aligned} \tag{17}$$

Remark 1. The form of a function $a_{l_p(0)} \dots a_{l_p(m)}$ in the limit depends on l_p and l_{p-1}.

Let us denote the integral kernel by

$$a_{l_p(0)} \dots a_{l_p(m)}((xs)_i (xs)_j),$$

where $(xs)_i = x_1 \dots x_i s_1 \dots s_i$ are variables the same as those in

$$a_{l_{p-1}(0)} \dots a_{l_{p-1}(m')}((xs)_k (xs)_i)$$

and $(xs)_j$ in $a_{l_{p+1}(0)} \dots a_{l_{p+1}(m'')}((xs)_j (xs)_r)$.

Let $h(a_{l_p(0)} \dots a_{l_p(m)}((xs)_i (xs)_j)\phi(xs)_j)$ denote the corresponding integral operator which acts on a function $\phi(xs)_j$, where

$$\phi(xs)_j = \phi(x_1 \dots x_j s_1 \dots s_j) \qquad j \geqq 1 .$$

We notice that the quantity of variables j depends on l_p only.

Remark 2. We have the following inequality:

$$a_{l_p(0)} \dots a_{l_p(m)}((xs)_i (xs)_j) \geqq a_{l_p(0)} \dots a_{l_p(m)}((xs)_{i'} (xs)_j) \tag{18}$$

if $i \leqq i'$.

Let $\mathbf{E}$ denote the space of sequence $\hat{\phi} = \{\phi(xs)_j\}_{j \geqq 0}$, where $\phi(xs)_i$, $1 \leqq i \leqq n$ is a measurable complex-valued function on $\mathcal{R}^{3i} \times [0,1]^i$, $\phi_0 \in \mathscr{C}$ and the norm

$$\|\hat{\phi}\|_{\mathbf{E}} = \sup_{n \geqq 0} \frac{1}{\zeta^n} \sup_{(xs)_n \in \mathcal{R}^{3n} \times [0,1]^n} |\phi(xs)_n| , \qquad \zeta > 1 \tag{19}$$

is finite, then $\mathbf{E}$ is a Banach space.

Let us define the operator

$$\hat{h}(a_{l_p(0)} \dots a_{l_p(m)}((xs)_i (xs)_j)) : \mathbf{E} \to L^\infty(\mathcal{R}^{3i} \times [0,1]^i)$$

in the following way:

$$\hat{h}(a_{l_p(0)} \dots a_{l_p(m)}((xs)_i (xs)_j))\hat{\phi} = h(a_{l_p(0)} \dots a_{l_p(m)}((xs)_i (xs)_j)\phi(xs)_j) , \tag{20}$$

and the operator $\hat{h}(a_{l_0(0)} \dots a_{l_p(m)}) : \mathbf{E} \to \mathbf{E}$ as follows:

$$\hat{h}(a_{l_p(0)} \dots a_{l_p(m)})\hat{\phi} = \{\hat{h}(a_{l_p(0)} \dots \alpha_{l_p(m)}((xs)_i (xs)_j)\hat{\phi}\}_{i \geqq 0} . \tag{21}$$

Theorem 3. *The operator* $\hat{h}(a_{l_p(0)} \dots a_{l_p(m)})$ *is compact in* **E**.

Proof. We can write the operator

$$h(a_{l_p(0)} \dots a_{l_p(m)}((xs)_i(xs)_j)) : L^\infty(\mathscr{R}^{3j} \times [0,1]^j) \to L^\infty(\mathscr{R}^{3i} \times [0,1]^i) \tag{22}$$

as a sum of operators which are the tensor product of the operators K_p, S_p, $p = 1 \dots r$.

$$\begin{aligned}(K_p f)(x_{p-u_q}, x_{p-u_q+1}, \dots, x_{p-1}) \\ = \int \chi(x_{p-u_q} + x_{p-u_q+1} + \dots + x_p) \dots \chi(x_{p-u_1} + \dots + x_p)\chi(x_p) \dots \\ \dots \chi(\dots + x_{p+j-1} + x_{p+j}) f(x_{p+1}, \dots, x_{p+j}) dx_p \dots dx_{p+j},\end{aligned}$$

where $1 \leqq u_1 < u_2 \dots < u_q = i$, and

$$(S_p g)(s_{p-u_q}, s_{p-u_q+1}, \dots, s_{p-1})$$

$$= \int_{[0,1]^j} \frac{s_{p+1}^{w_{p+1}} s_{p+2}^{w_{p+2}} \dots s_{p+j}^{w_{p+j}}}{(s_{p-u_q} s_{p-u_q+1} \dots s_{p-1} + \dots + s_{p-u_1} \dots s_{p-1} + 1 + \dots)^{r_1} \dots (\dots + \dots s_{p+j-1} s_{p+j})^{r_\alpha}}$$

$$\times g(s_{p+1} \dots s_{p+j}) ds_{p+1} \dots ds_{p+j},$$

where w_k are natural numbers.

We illustrate this on an explicit example. We multiply (17) by

$$[1 - \chi(x_2 + x_3)] + \chi(x_2 + x_3),$$
$$[1 - \chi(x_5)] + \chi(x_5),$$
$$[1 - \chi(x_4 + x_5 + x_6)] + \chi(x_4 + x_5 + x_6),$$

and we get desired representation.

We show that K_p and S_p are compact, hence the operators (22) are also compact. The kernel of S_p is continuous with respect to the variables $s_{p-u_q} \dots s_{p-1}$ and summable so S_p is compact. We change the variables in the operator $K_p : L^\infty(\mathscr{R}^{3j}) \to L^\infty(\mathscr{R}^{3u_q})$ as follows:

$$\begin{aligned} y_1 &= x_{p-u_1} + \dots + x_{p-1}, \\ y_2 &= x_{p-u_2} + \dots + x_{p-u_1+1}, \\ &\vdots \\ y_q &= x_{p-u_q} + \dots + x_{p-u_{q-1}+1}, \end{aligned}$$

so $K'_p : L^\infty(\mathscr{R}^{3j}) \to L^\infty(\mathscr{R}^{3q})$. We notice that the kernel

$$K(y_1, \dots, y_q, x_{p+1}, \dots, x_{p+j})$$

is of compact support and the convergence of the sequence $\{\phi_n(y_1, \dots, y_q)\}$ in $L^\infty(\mathscr{R}^{3q})$ implies the convergence of the sequence

$$\{\phi_n(x_{p-u_1} + \dots + x_{p-1}, \dots, x_{p-u_q} + \dots + x_{p-u_{q-1}+1})\} \quad \text{in} \quad L^\infty(\mathscr{R}^{3u_q}).$$

Thus it suffices to show the compactness of $K'_p : L^\infty(\Omega^j) \to L^\infty(\Omega^q)$, where Ω is compact and

$$\operatorname{supp} K(y_1, \dots, y_q, x_{p+1}, \dots, x_{p+j}) \subset \Omega^q \times \Omega^j.$$

We have

$$|(K'f)(y)'_q-(K'f)(y)_q|\leqq \|f\|_{L^\infty(\Omega^j)}\|K((y)'_q,x_{p+1},\ldots,x_{p+j})$$
$$-K((y)_q,x_{p+1},\ldots,x_{p+j})\|_{L^1(\Omega^j)},$$

so we should prove that

$$\forall (y)_q\in\Omega^q \lim_{(y)'_q\to(y)_q}\|K((y)'_q,x_{p+1},\ldots,x_{p+j})-K((y)_q,x_{p+1},\ldots,x_{p+j})\|_{L^1(\Omega^j)}=0, \tag{23}$$

then from the Ascoli Theorem we obtain compactness of the operator K'.

Observe that the difference

$$K(y'_1,\ldots,y'_q,x_{p+1},\ldots,x_{p+j})-K(y_1,\ldots,y_q,\ x_{p+1},\ldots,x_{p+j})$$

tends to 0 for $y'_1,\ldots,y'_q\to y_1,\ldots,y_q$ a.e. Hence, from the Lebesgue Dominated Convergence Theorem (23) follows.

Hence $\hat{h}(a_{l_p(0)}\ldots a_{l_p(m)}((xs)_i(xs)_j))$ is compact. Let us consider the bounded sequence $\{\hat{\phi}\}$, $\|\hat{\phi}_l\|\leqq C$. Since the operators

$$\hat{h}(a_{l_p(0)}\ldots a_{l_p(m)}((xs)_i(xs)_j))$$

are compact, by applying the diagonal method we can choose a subsequence $\hat{\phi}_{l'}$ such that sequence

$$\hat{h}(a_{l_p(0)}\ldots a_{l_p(m)}((xs)_i(xs)_j)\hat{\phi}_{l'})$$

is convergent in $L^\infty(\mathscr{R}^{3i}\times[0,1]^i)$ for all i. To finish the proof we should show that for each $\varepsilon>0$ there exists a natural n such that

$$\|\hat{h}(a_{l_p(0)}\ldots a_{l_p(m)})(\hat{\phi}_k-\hat{\phi}_l)\|_{\mathbf{E}}\leqq\varepsilon \quad\text{for all}\quad k,l>n,\ \hat{\phi}_k,\hat{\phi}_l\in\{\hat{\phi}_{l'}\}. \tag{24}$$

Indeed, for each $\varepsilon>0$ there is n_0 such that for $n>n_0$,

$$\frac{1}{\zeta^n}\|\hat{h}(a_{l_p(0)}\ldots a_{l_p(m)}((xs)_i(xs)_j))(\hat{\phi}_k-\hat{\phi}_l)\|_{L^\infty}$$
$$\leqq 2C\frac{1}{\zeta^n}\|\hat{h}(a_{l_p(0)}\ldots a_{l_p(m)}((xs)_i(xs)_j))\|_{L^\infty}\leqq\frac{c'}{\zeta^n}\leqq\varepsilon$$

by Remark 2. So we have

$$\|\hat{h}(a_{l_p(0)}\ldots a_{l_p(m)})(\hat{\phi}_k-\hat{\phi}_l)\|_{\mathbf{E}}$$
$$\leqq\max\left\{\sup_{n\leqq n_0}\frac{1}{\zeta^n}\|\hat{h}(a_{l_p(0)}\ldots a_{l_p(m)}((xs)_i(xs)_j))(\hat{\phi}_k-\hat{\phi}_l)\|_{L^\infty},\varepsilon\right\}.$$

By exploiting the property of convergence of the sequences

$$\hat{h}(a_{l_0(0)}\ldots a_{l_p(m)}((xs)_i(sx)_j)\hat{\phi}_l)$$

we can find such a natural number l_0 that for $k,l>l_0$ (24) is fulfilled.

The operator series

$$K(z)=\sum_{m=1} z^m\sum_{l'}\hat{h}(a_{l'(0)}\ldots a_{l'(m)}), \tag{25}$$

where l' are the functions such that $l'(i)\leqq i$ and there is no such $j>0$ that $l'(j)=0$, are convergent for small z by Remark 2 and the fact that we sum over a subset of functions (13). Hence (25) is an operator-valued function and by Theorem 3 $K(z)$ is a compact operator for z belonging to the domain of analyticity of $K(z)$. We have the following identities for small z:

$$\Pi(1-K(z))^{-1}(1)=1+\sum_{n=1} c_n' z^n,$$

$$\Pi(1-K(z))^{-1}(\psi)=1+\sum_{n=1} c_n z^n,$$

where $\Pi\hat{\phi}=\phi_0$, $(1)=\{1,1,\ldots\}$,

$$(\psi)=\left(1,\frac{1}{1+s_1},\frac{1}{1+s_1+s_1s_2},\frac{1}{1+s_1+s_1s_2+s_1s_2s_3},\ldots\right).$$

From the analytic version of the Fredholm alternative [3, 7], $(1-K(z))^{-1}$ is invertible on the whole domain of analyticity of $K(z)$ but is a discrete set and is a meromorphic function of z because it is invertible for small z. Hence $\Pi(1-K(z))^{-1}(1)$ and $\Pi(1-K(z))^{-1}(\psi)$ are analytic continuations of (15). We can write

$$\varrho(z)=\frac{1-\Pi(1-K(-z))^{-1}(1)}{2-\Pi(1-K(-z))^{-1}(\psi)}. \tag{26}$$

Proposition 1. *If z belongs to the domain of analyticity of $K(z)$, then the density $\varrho(z)$ is an analytic function of z for $z \geqq 0$.*

Proof. The numerator and denominator of (26) are the meromorphic functions of z. Analyticity of $\varrho(z)$ arises from the fact that $\varrho(z)$ is an increasing positive and bounded function of $z>0$ [5] so it cannot be equal to zero and infinity.

In the one-dimensional case we can calculate the exact activity as a function of the density [4]

$$z=\frac{\varrho}{1-\varrho}\exp\frac{\varrho}{1-\varrho},$$

where $v(x)=\infty$ for $|x|\leqq 1$ and $v(x)=0$ for $|x|>1$.

We note that ϱ has an isolated singularity for $z=-e^{-1}$ and ϱ becomes infinite at this point. Hence it is compatible with (14). The density ϱ is an analytic function of z on the positive real axis. In one-dimension many terms in the coefficients c_n and c_n' vanish, for instance $h(a_0a_1a_2a_{l_1}\ldots a_{l_k})=0$ for an arbitrary function l_i. In dimensions greater than one these terms do not vanish. Hence it is expected that analytic properties of $K(-z)$ are different than in the one-dimensional case.

An analysis of the formula (26) may suggest that one of the following alternatives holds:

a) $K(-z)$ has a singularity on the positive axis and it is infinite at this point. Then a phase transition occurs for $\varrho=1/2$.

b) $K(-z)$ has a singularity but it is finite at this point. Then a phase transition can occur for another value of ϱ.

c) $K(-z)$ is analytic at all points of positive axis so any phase transition does not occur as in the one-dimensional case.

From numerical experiments (for example [1]) it seems that the first possibility holds.

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